Trigonometry Approximations (approximations.c)

Technical Note

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Authors: Mark Pedley and Michael Stanley
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1 Introduction

1.1 Summary

This Application Note documents the functions in file approximations.c. These functions implement highly accurate approximations to inverse trigonometry functions to reduce the computational overhead of using the standard C floating point library functions on integer microcontrollers. The result is in degrees rather than radians saving an additional multiplication to convert from radians to degrees.

The benchmarks in the following table were measured on the Freescale FRDM-KL25Z board which uses a 32 bit ARM M0+ integer core running at 48MHz (giving 48 million clock ticks per second).

C99 library function	**approximations.c**
float asinf(float x) 4000-6000 clock ticks	float fasin_deg(float x) 3000-4000 clock ticks
float acosf(float x) 4000-6000 clock ticks	float facos_deg(float x) 3000-4000 clock ticks
float atanf(float x) 3800-4800 clock ticks	float fatan_deg(float x) 1900-3500 clock ticks

1.2 Functions

float fasin_deg(float x); Inverse sine function (deg) range -90 deg to 90 deg
float facos_deg(float x); Inverse cosine function (deg) range 0 deg to 180 deg
float fatan_deg(float x); Inverse arctangent (deg) range -90 deg to 90 deg
float fatan2_deg(float y, float x); Inverse arctangent (deg) range -180 deg to 180 deg
float fatan_15deg(float x); Inverse arctangent for range -15 deg to +15 deg only

2 Mathematics

2.1 Approximation to Inverse Sine (-90 deg to +90 deg)

Function fasin_deg computes the inverse sine of x as the inverse tangent of the new argument $\frac{x}{\sqrt{1-x^2}}$. The overhead of the square root and division is still less than the overhead of the standard C inverse sine function.

Putting $x = sin\theta$ into the definition of the tangent gives:

$$tan\theta = \frac{sin\theta}{cos\theta} = \frac{x}{\sqrt{1-x^2}} \Rightarrow \theta = sin^{-1}x = tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

Eq 2.1.1

2.2 Approximation to Inverse Cosine (0 deg to +180 deg)

Function facos_deg allows the inverse cosine of x to be determined as the inverse tangent of the new argument $\frac{\sqrt{1-x^2}}{x}$. Since the inverse tangent returns an angle in the range -90^o to +90^o, 180^o is added if the argument is negative to give the inverse cosine in the range 0^o to 180^o.

Putting $x = cos\theta$ into the definition of the tangent gives.

$$tan\theta = \frac{sin\theta}{cos\theta} = \frac{\sqrt{1-x^2}}{x} \Rightarrow \theta = cos^{-1}x = tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

Eq 2.2.1

2.3 Approximation to Inverse Tangent (-90 deg to +90 deg)

The general inverse tangent for angles in the range -90^o to +90^o is computed in function fatan_deg which successively maps its argument to the inverse tangent of an angle in the range 0^o to +15^o.

Negative arguments are mapped to positive arguments using the identity:

$tan^{-1}(-x) = -tan^{-1}(x)$ Eq 2.3.1

An argument x greater than 1 (implying an angle above +45^o) is mapped to argument less than 1 (implying an angle below +45^o) using the identity:

$$tan\left(\frac{\pi}{2} - \theta\right) = \frac{sin\left(\frac{\pi}{2} - \theta\right)}{cos\left(\frac{\pi}{2} - \theta\right)} = \frac{cos\theta}{sin\theta} = \frac{1}{tan\theta}$$

Eq 2.3.2

The new argument is then compared with $tan(15deg)$. If the angle is above 15^o (in the range 15^o to 45^o) then it is mapped to the range -15^o to 15^o using the identity:

$$tan(\theta + \phi) = \frac{tan\theta + tan\phi}{1 - tan\theta tan\phi}$$

Eq 2.3.3

Substituting $\phi = \frac{-\pi}{6}$ $(30\ \mathrm{deg})$ gives:

$$tan\left(\theta - \frac{\pi}{6}\right) = \frac{tan\theta - tan\left(\frac{\pi}{6}\right)}{1 + tan\theta tan\left(\frac{\pi}{6}\right)} = \frac{tan\theta - \left(\frac{1}{\sqrt{3}}\right)}{1 + tan\theta\left(\frac{1}{\sqrt{3}}\right)} = \frac{\sqrt{3}tan\theta - 1}{tan\theta + \sqrt{3}}$$

Eq 2.3.4

With the substitution $x = tan\theta$:

$$tan\left(\theta - \frac{\pi}{6}\right) = \frac{x\sqrt{3} - 1}{x + \sqrt{3}} \Rightarrow \theta = \left(\frac{\pi}{6}\right) + tan^{-1}\left(\frac{x\sqrt{3} - 1}{x + \sqrt{3}}\right)$$

Eq 2.3.5

Once the inverse tangent is computed in the range -15^o to 15^o, the manipulations applied in equations 2.3.1 to 2.3.5 are then reversed to give an angle in the range -90^o to 90^o.

2.4 Pade[3, 3] Approximation to Inverse Tangent (-15 deg to +15 deg)

The Pade[3,3] rational approximation to the inverse tangent expanded about $x = 0$ is:

$$tan^{-1}(x) \approx \frac{x + \left(\frac{4}{15}\right)x^3}{1 + \left(\frac{3}{5}\right)x^2} = \frac{x\left\{1 + \left(\frac{4}{15}\right)x^2\right\}}{1 + \left(\frac{3}{5}\right)x^2}\ rad$$

Eq 2.5.1

$$= \frac{x\left\{\left(\frac{180}{\pi}\right) + \left(\frac{180}{\pi}\right)\left(\frac{4}{15}\right)x^2\right\}}{1 + \left(\frac{3}{5}\right)x^2}\ deg$$

Eq 2.5.2

Equation 2.5.2 is anti-symmetric as required. It is used in function fatan_15deg with slightly modified Pade parameters selected to minimize the maximum error in the range -15^o to +15^o.