A Beginner's Guide to Support Theory via Tensor Triangulated Categories

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Introduction

Let $\mathcal{K}$ be a tensor triangulated category, a natural question to ask is can we classify all the thick subcategories of $\mathcal{K}$? This thesis will introduce one possible way via the support theory which is developed by A. Neeman. M. Hopkins, Paul Balmer, and Greg Stevenson. The thesis consists of three sections. The first section reviews the classic theory about the Balmer spectrum of the small tensor triangulated categories. From the second section, we will drop the condition "small" and focus on the case that $\mathcal{K}$ is a rigidly-compactly generated tensor triangulated category. We will study how to generalize the support data of the compact object to new support data of the whole category by using the Rickard idempotent. The last section is based on the theory developed by Greg Stevenson, where we will apply the new support data to the "module" over $\mathcal{K}$, and use it to classify the localizing subcategory of $\mathcal{K}$.

In the end, we will provide two examples: The first example is the derived category of noetherian commutative ring R, where we will give an alternative proof of Thomason's classification theorem of localizing subcategories. Another example is the representation of categories over the commutative noetherian ring. The classification of localization subcategories of such categories is studied by Greg Stevenson and Benjamin Antieau. This machinery could be used to, for example, study the localizing subcategory of the derived category of Dykin quiver.

The thesis will be mostly self-contained, and there will be a reference for those theorem/lemma without proof. The writing style will follow the philosophy of Bourbaki, but we also try to make our readers enjoyable when they're reading. Therefore, we will try to provide some interesting motivations. In addition, there is a special part called **Advertisement** which is used to spread some interesting ideas from other fields that are relevant to the material but won't be used in this thesis. The reader can choose to skip it when they first meet it.

Ackownledgement

Firstly, I would like to thank my advisors Benjamin Briggs and Henrik Holm for their invaluable guidance and support. Their expertise and encouragement have been instrumental in the successful completion of this thesis. I am also grateful to my friends Xiansheng Li, Pengkun Huang, Hongxiang Zhao, Sophus, and Andreas for their useful discussion, support, and encouragement throughout this journey, and I am thankful for the memories we have shared. In the end, it would be impossible for me to finish this thesis without my family, and I owe a debt of gratitude to my family for their unconditional love, support, and understanding.

Notation

In this thesis, we will let $(\mathcal{K}, \otimes, \mathbb{1}, \Sigma)$ be an essentially small symmetric monoidal category with a compatible triangulated structure Σ. More specifically, the category $\mathcal{K}$ will have a monoidal functor $\otimes$ such that $\otimes$ is symmetric (e.g there is a natural isomorphism $a \otimes b \simeq b \otimes a$) and a unit $\mathbb{1}$ that satisfy the the axiom of monoidal category as in [SM97] Section VII.1. and a $\mathcal{K}$ a triangulated category as in [GM02] Section V.1 with suspension functor Σ, such that

(1) There is a natural isomorphism $e_{a,b} : (\Sigma a) \otimes b \simeq \Sigma(a \otimes b)$.

(2) For each object a, the functor $a \otimes (-) \simeq (-) \otimes a : \mathcal{K} \to \mathcal{K}$ is an additive functor.

(3) $a \otimes (-) \simeq (-) \otimes a : \mathcal{K} \to \mathcal{K}$ is an exact functor of triangulated category.

For any morphism $f \in \mathrm{Mor}(\mathcal{K})$, the $\mathrm{cone}(f)$ will be the cone object of f.

We can consider the category of tensor triangulated categories. The morphism is between two objects $\mathcal{K}, \mathcal{L}$ is a *tensor triangulated* functor (e.g it is a triangulated functor and tensor functor [SM97],[GM02] for more details) $F : \mathcal{K} \to \mathcal{L}$.

A tensor triangulated category $\mathcal{K}$ is *closed* if there is a bifunctor $\mathrm{hom}(-,-) : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ with a natural isomorphism

$$\mathrm{Hom}(a \otimes b, c) \simeq \mathrm{Hom}(a, \mathrm{hom}(c, b)) \tag{0.1}$$

so $a \otimes -$ has a right adjoint $\mathrm{hom}(a,)$ for each object $a \in \mathcal{K}$. we call the functor $\mathrm{hom}(-,-)$ *internal hom* of $\mathcal{K}$. We will use $\mathrm{Hom}(a, b)$ and $\mathrm{hom}(a, b)$ as Hom-set and internal hom of any two objects a and b of $\mathcal{K}$ respectively.

In this thesis, the main object we will focus on is the *thick tensor ideal*, and it is defined as follows

Definition 0.1. A subcategory $\mathcal{I}$ of $\mathcal{K}$ is called *tensor-thick ideal* if:

(1) $\mathcal{I}$ is a *triangulated subcategory*: It contains 0, and it is a full subcategory, closed under suspension, and for any distinguished triangle $a \to b \to c \to \Sigma a$ in $\mathcal{K}$, if two out of a, b, c in $\mathcal{I}$, then so is third.

(2) $\mathcal{I}$ is *thick*: If $c \in \mathcal{K}$ splits, i.e. $a \oplus b = c$, then $a, b \in \mathcal{I} \iff c \in \mathcal{I}$.

(3) $\mathcal{I}$ is a *tensor ideal*: if $a \in \mathcal{K}, b \in \mathcal{I}$, then $a \otimes b \in \mathcal{I}$.

We notice the intersection of any family of tensor thick ideal is tensor thick ideal. Given a collection of object $\mathcal{S}$ in $\mathcal{K}$, we denote $\langle \mathcal{S} \rangle \subset \mathcal{K}$ be the smallest tensor ideal of $\mathcal{K}$ contains $\mathcal{S}$ (e.g intersection of all tensor thick ideal contains $\mathcal{S}$). We will also call $\langle \mathcal{S} \rangle$ the *thick closure* of $\mathcal{S}$. Similarly, we denote $\mathrm{Thick}^{\otimes}(S)$ as the smallest tensor thick ideal contains S.

Remark 0.2. The condition (1) will force $\mathcal{I}$ closed under isomorphism: if $a \simeq b \in \mathcal{I}$, then $a \in \mathcal{I}$ by using the following fact: let $f : a \to b$ be the isomorphism, then we have the morphism of the distinguished triangle:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d] & b \arrow[r] \arrow[d, "f^{-1}"] & \mathrm{cone}(f) \arrow[d] \\ a \arrow[r, "\mathrm{id}_a"'] & a \arrow[r] & 0 \end{tikzcd}$$

Now, because the first two vertical arrows are isomorphism, so does the third one, which implies $\mathrm{cone}(f)$ is 0 and so $b \in \mathcal{I}$

Here are some important examples of tensor triangulated category.

Example 0.3. (1) Let R be commutative ring, then the derived category $\mathscr{D}^{\mathrm{perf}}(R)$ is a tensor triangulated category with unit R and tensor product is the total derived tensor product $\otimes_R^{\mathbb{L}}$.

(2) Let Sp^{ω} be the homotopy category of finite Spectra. This is a tensor triangulated category where the unit is sphere spectrum $\mathbb{S}$ and the tensor product is the smash product.

(3) Let G be finite group and k a field. The stable category $\underline{\mathrm{mod}}kG$ of kG consists of finite dimension kG module and the Hom between two module M, N is $\mathrm{Hom}_{kG}(M, N)/(\mathscr{P})$ where $\mathscr{P}$ = homomorphism that factors through a projective object. This is a tensor triangulated category with tensor product $\otimes_k$ and the unit is the trivial representation.

(4) Let G be a finite group, and R be a commutative ring, for a G-set A, we denote $R(A)$ the free R-module with G-action extended R-linearly from its basis A. An $RG-$module is called *permutation* if it is isomorphic to $R(A)$ for some G-set A. It is *finitely generated* is A is a finite set.

The category $\mathrm{Perm}(G, R)$ of permutation module over G is an additive category, but it has been unknown for a long time what the "derived" category is until recent work given by [BM22]. To construct the expected derived category, we need to consider "$G-$equivarant weak equivalence": a morphism $f : X \to Y$ in $K(\mathrm{Perm}(G, R))$ is called *$G-$quasi-isomorphism* if for every open (or equivalently, finite index) subgroup $H \leqslant G$, the morphism between $H-$fixed point $X^H \to Y^H$ is quasi-isomorphism.

We then define the (big-)derived category of permutation module as the Verdier localization

$$\mathrm{DPerm}(G, R) = K(\mathrm{Perm}(G, R))[\{G\text{-quasi-isos}\}^{-1}]$$

This category is tensor triangulated with unit RG and the tensor product is the total tensor product. In particular, the machinery introduced here could be generalized to the profinite group, i.e. a compact, Hausdorff, and totally disconnected topological group (note that every finite group with discrete topology is profinite). When G is an absolute Galois group over a field k, $\mathrm{DPerm}(G, R)$ has a very interesting relation with Voevodsky's triangulated category of $k-$motive. See [BM23] for a survey.

1 Balmer Spectrum

1.1 Motivation

Before we discuss the formal definition of Balmer Spectrum, we would like to start with some examples to motivate the reader for its definition.

Given a triangulated category $\mathcal{K}$, one natural question to ask is whether we can classify all the full subcategories of it. Certainly, it is quite impossible if we want to classify "all" the subcategories, but it is possible to classify a special subcategory, namely, the *thick* subcategory. Furthermore, if $\mathcal{K}$ is a tensor triangulated category, we want to classify all the *tensor-thick* ideal. Indeed, for a tensor triangulated category $\mathcal{K}$, it is possible that every thick subcategory is a tensor thick ideal.

Lemma 1.1. *Let $\mathcal{K}$ be a tensor triangulated category, if the smallest thick category that contains $\mathbb{1}$ is $\mathcal{K}$, then every thick subcategory is a tensor thick ideal.*

Proof. Let $\mathcal{I}$ be thick subcategory, $s \in \mathcal{I}$, consider

$$\mathcal{S} = \{b \in \mathcal{K} \mid b \otimes s \in \mathcal{I}\}$$

we will show that $\mathcal{S}$ is a thick subcategory. Especially, it should contains $\{\mathbb{1}\}$, so by assumption $\mathcal{S} = \mathcal{K}$. Clearly $0 \in \mathcal{S}$, let $a \to b \to c \to \Sigma a$ be a distinguished triangle such that $a, c \in \mathcal{S}$ then $a \otimes s \to b \otimes s \to c \otimes s \to \Sigma a \otimes s$ is a triangle such that $a \otimes s$ and $c \otimes s$ are in $\mathcal{I}$, as $\mathcal{I}$ is triangulated subcategory, $b \otimes s \in \mathcal{I}$. Suppose $a \oplus b \in \mathcal{S}$, which means $(a \oplus b) \otimes s = a \otimes s \oplus b \otimes s \in \mathcal{I}$, as $\mathcal{I}$ is thick , we get $a, b \in \mathcal{S}$. □

If $\mathcal{K} = \mathrm{Sp}^\omega$ is the stable homotopy category of Finite Spectra, we can localize this category at prime number $\mathfrak{p} \in \mathbb{Z}$ and get a new category $\mathrm{Sp}^\omega_{(p)}$. There is a collection of functors $\{K(n) \mid \forall n \in \mathbb{Z}_+\}$, called *Morava-K theory*, such that $K(n) : \mathrm{Sp}^\omega_{(p)} \to V_n$ where V_n is a tensor category. Let $\mathcal{C}_{p,n} = \ker(K(n))$ then the Devinatz-Hopkins-Smith theorem will give us all the tensor ideal (See Orange Book [Orange93] for more details if you like it!)

Theorem 1.2. *The tensor thick ideal of $\mathrm{Sp}^\omega_{(p)}$ are exactly all the kernels*

$$0 = \mathcal{C}_{p,\infty} \subsetneq \cdots \subsetneq \mathcal{C}_{p,n} \subsetneq \mathcal{C}_{p,n-1} \subsetneq \ldots \mathcal{C}_{p,1} \subsetneq \mathcal{C}_{p,0} \subsetneq \mathrm{Sp}^\omega_{(p)}$$

Remark 1.3. (You can skip this remark if you're not familiar with stable homotopy theory) Indeed, Balmer in [PB10] Corollary 9.2 shows that every tensor ideal of $\mathrm{Sp}^\omega_{(p)}$ is *prime*, so this theorem also gives a complete description of the Balmer spectrum of $\mathrm{Sp}^\omega_{(p)}$ (you will see what it means soon)

Let X be a noetherian scheme. If $\mathcal{K} = \mathscr{D}^{\mathrm{perf}}(X)$, the derived category of perfect chain complex over X, Hopkins and Neeman introduce an approach to classify all the tensor ideals via underlying topological space of X.

Definition 1.4. Let X be a scheme, For a $P \in \mathscr{D}^{\mathrm{perf}}(X)$, we define

1. $\operatorname{supp}(P) = \{x \in X \mid P_x \not\simeq 0 \text{ in } \mathscr{D}^{\mathrm{perf}}(\mathscr{O}_{X,x})\}$;

2. Let $Y \subset X$, we let $\mathscr{D}^{\mathrm{perf}}_Y(X) = \{P \in \mathscr{D}^{\mathrm{perf}}(X) \mid \operatorname{supp}(P) \subset Y\}$;

In [MHOP87] and [AM92], we have the following classification theorem

Theorem 1.5. *Let X be a noetherian scheme,* $\mathrm{Thick}(X)$ *be the collection of tensor-thick subcategories of $\mathscr{D}^{\mathrm{perf}}(X)$, and let* $\mathrm{Sp(X)}$ *be the collection of specialization closed subset of X, e.g if $y \in Y$, then $\overline{\{y\}} \subset Y$, then there is an one to one correspondence between* $\mathrm{Thick}(X)$ *and* $\mathrm{Sp}(X)$ *given by*

$$\mathrm{Thick}(X) \to \mathrm{Sp}(X) : J \mapsto \bigcup_{a \in J} \operatorname{supp}(a)$$

$$\mathrm{Sp}(X) \to \mathrm{Thick}(X) : Y \mapsto \mathscr{D}^{\mathrm{perf}}_Y(X)$$

In fact, the assignment $P \mapsto \operatorname{supp}(P)$ together with X gives an example of "support data" $(X, \operatorname{supp}(p))$ of tensor triangulated category $\mathcal{K} = \mathscr{D}^{\mathrm{perf}}(X)$. The Balmer spectrum generalized this idea: it is topological space, denoted by$\mathrm{Spc}(\mathcal{K})$, associated to a tensor triangulated category $\mathcal{K}$ with a support data $(\mathrm{Spc}(\mathcal{K}), \operatorname{supp})$. As we will see in section 1.4, this support data enjoys a "universal property" among all the support data of $\mathrm{Spc}(\mathcal{K})$, and in section 1.6, we will introduce the "Hilbert Nullstellensatz" theorem of $\mathrm{Spc}(\mathcal{K})$ to classify the subcategory of $\mathcal{K}$ by some subspace of $\mathrm{Spc}(\mathcal{K})$ through this universal support data.

1.2 Construction of Balmer Spectrum

Now let us give a precise definition of Balmer Spectrum.

Definition 1.6. A tensor-thick subcategory $\mathcal{P}$ of $\mathcal{K}$ is *prime* if for $a, b \in \mathcal{K}, a \otimes b \in \mathcal{P}$, then a or b in $\mathcal{P}$. By abusing the notation, we will also call it the prime ideal of $\mathrm{Spc}(\mathcal{K})$

Definition 1.7. The Balmer spectrum of $\mathcal{K}$, denoted by $\mathrm{Spc}(\mathcal{K})$ is the set of all prime tensor thick subcategories.

We can equip a topology over $\mathrm{Spc}(\mathcal{K})$

Definition 1.8. For a family of object $\mathcal{S}$ in $\mathcal{K}$, we denote $\mathrm{V}(\mathcal{S}) = \{\mathcal{P} \mid \mathcal{P} \cap \mathcal{S} = \varnothing\}$. Notice $\mathrm{V}(\bigcup \mathcal{S}_i) = \bigcap_{i \in I} \mathrm{V}(\mathcal{S}_i)$ and $\mathrm{V}(\mathcal{S}_1) \cup \mathrm{V}(\mathcal{S}_2) = \mathrm{V}(\mathcal{S}_1 \oplus \mathcal{S}_2)$. Hence, the collection $\{\mathrm{V}(\mathcal{S}) \mid \mathcal{S} \subset \mathcal{K}\}$ define the closed subset of a topology in $\mathrm{Spc}(\mathcal{K})$. In particularly, if $\mathcal{S} = \{a\}$ for an object, we denote the $\operatorname{supp}(a)$ as the closed set

$$\mathrm{V}(\{a\}) = \operatorname{supp}(a)$$

We also define the open complement of $\mathrm{V}(\mathcal{S})$ by

$$\mathrm{U}(\mathcal{S}) = \mathrm{Spc}(\mathcal{K}) - \mathrm{V}(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \cap \mathcal{S} \neq \varnothing\}$$

and this collection will be an open base of the topology of $\mathrm{Spc}(\mathcal{K})$

Remark 1.9. You may notice the open subset and closed subset of the Balmer spectrum are defined in a "reversal" way with respect to the Zariski spectrum of a commutative ring. Indeed, it is still an open question to decide if a Balmer spectrum is a (affine) scheme(see [PB05], section 6). We will see in section 1.5, that, with some "nice" conditions, the Balmer spectrum has a natural continuous map to a Zariski spectrum. In some special cases, this map is a homeomorphism(and even isomorphism of ringed space).

Before we discuss the further properties of the Balmer spectrum, let's first make sure that the topology we define is not empty. A collection $\mathcal{S}$ is called *tensor-multiplicative family of objects* if $\mathbb{1} \in \mathcal{S}$ and if $x, y \in \mathcal{S}$, then $x \otimes y \in \mathcal{S}$.

Lemma 1.10. *Let $\mathcal{K}$ be a non-zero tensor triangulated category, $\mathcal{D} \subset \mathcal{K}$ be a thick tensor ideal, and $\mathcal{S} \subset \mathcal{K}$ be a tensor multiplicative family of objects such that $\mathcal{S} \cap \mathcal{D} = \varnothing$. Then there is a prime ideal $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ such that $\mathcal{D} \subset \mathcal{P}$ and $\mathcal{P} \cap \mathcal{S} = \varnothing$*

Proof. Let $\mathcal{F}$ be a collection of thick tensor ideals $\mathcal{J} \subset \mathcal{K}$ such that

i) $\mathcal{I} \cap \mathcal{S} = \varnothing$;

ii) $\mathcal{D} \subset \mathcal{I}$;

iii) if $s \in \mathcal{S}$ and $a \in \mathcal{K}$ such that $s \otimes a \in \mathcal{I}$, then $a \in \mathcal{I}$

Notice this collection is not empty, because $\mathcal{I}_0 = \{a \in \mathcal{K} \mid \exists s \in \mathcal{S} \text{ such that } a \otimes s \in \mathcal{S}\}$ will be a tensor thick ideal.

If we have a chain $\mathscr{C}$ in $\mathcal{F}$, then the union of all objects in this chain $\mathcal{I} = \bigcup_i \mathcal{I}_i$ will satisfy the condition (1)-(3) 0.1, so it is a tensor thick subcategory and it is in $\mathcal{F}$. Hence, $\mathcal{I}$ is the upper bound of the chain. By Zorn's Lemma, there is a maximal element $\mathcal{Q}$ in $\mathcal{F}$.

We will show that $\mathcal{Q}$ is a prime ideal. Assume $a \otimes b \in \mathcal{Q}$, and $b \notin \mathcal{Q}$, we can check

$$\mathcal{I}_1 = \{c \in \mathcal{K} \mid c \otimes a \in \mathcal{Q}\}$$

is a tensor-thick ideal, and it contains $\mathcal{Q}$ properly. By maximality of $\mathcal{Q}$, we should have $\mathcal{I}_1$ not belong to $\mathcal{F}$, and it is easy to check $\mathcal{I}_1$ satisfy ii) and iii), so it could not satisfy i). Hence, there is a $s \in \mathcal{S}$ such that $s \otimes a \in \mathcal{Q}$, but this implies $a \in \mathcal{Q}$ as $\mathcal{Q}$ is a prime ideal.

□

Corollary 1.11. *Let $\mathcal{K}$ be a non-zero tensor triangulated category, then*

(1) Let $\mathcal{S}$ be a tensor multiplicative collection of objects which does not contain zero. There exists a prime ideal $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ such that $\mathcal{P} \cap \mathcal{S} = \varnothing$

(2) Let $\mathcal{I} \subsetneq \mathcal{K}$ is a proper tensor thick ideal. There exists a maximal proper tensor thick ideal $\mathcal{M} \subsetneq \mathcal{K}$ contains $\mathcal{I}$

(3) Any maximal thick tenor ideal $\mathcal{M}$ is prime.

(4) For any prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$, there is a minimal prime $\mathcal{P}' \subset \mathcal{P}$. In particular, $\mathrm{Spc}(\mathcal{K})$ has a minimal prime.

(5) $\mathrm{Spc}(\mathcal{K})$ *is not empty.*

Proof. Apply the lemma 1.10 for $\mathcal{D} = 0$, we get (1). Choose $\mathcal{S} = \{\mathbb{1}\}$ in lemma 1.10, then we observe that the condition (3) is automatically correct for all prime ideal contains $\mathcal{I}$, so the maximal element is exactly the maximal ideal we want, and this prove (2). For (3), use the lemma 1.10 for maximal ideal and $\mathcal{S} = \{\mathbb{1}\}$.

We now prove the (4). Given a chain $\mathscr{C} \subset \mathrm{Spc}(\mathcal{K})$, then $\mathcal{P}' = \bigcap_{\mathcal{P} \in \mathscr{C}} \mathcal{P}$ is a tensor thick ideal. In particular, it is a prime ideal, because if $a_1, a_2 \notin \mathcal{P}'$, then there is a $\mathcal{P}_i \in \mathscr{C}$ such that $a_i \notin \mathcal{P}_i$ for $i = 1, 2$. Since $\mathscr{C}$ is a chain of inclusion, we may choose the smallest one between $\mathcal{P}_1$ and $\mathcal{P}_2$. Without loss of generality, we assume $\mathcal{P}_1 \subsetneq \mathcal{P}_2$. So $a_i, a_2 \notin \mathcal{P}_1$ and $a_1 \otimes a_2 \notin \mathcal{P}_1 \Rightarrow a_1 \otimes a_2 \notin \mathcal{P}'$. Hence, we get every chain of $\mathrm{Spc}(\mathcal{K})$ has a lower bound. By Zorn's Lemma $\mathrm{Spc}(\mathcal{K})$ has a minimal prime. For a fixed prime ideal $\mathcal{Q}$, apply a similar process for the collection of prime ideal contained by $\mathcal{Q}$, and we get a minimal prime ideal containing it.

The (5) follows immediately by (1). □

1.3 Some Topological Properties

Not only the construction, but the topological property of the Balmer spectrum is also similar to the Zariski spectrum. We will discover some topological properties of the Balmer spectrum, and the reader can find more in [PB05] section 2

Proposition 1.12. *let* $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ *be a prime ideal, then* $\overline{\{\mathcal{P}\}} = \mathrm{V}(\mathcal{K} \backslash \mathcal{P}) = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}) | \mathcal{Q} \subset \mathcal{P}\}$. *In particularly* $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}}$, *then* $\mathcal{P}_1 = \mathcal{P}_2$, *which means the space* $\mathrm{Spc}(\mathcal{K})$ *is* T_0

Proof. Let $\mathcal{S} = \mathrm{Spc}(\mathcal{K}) - \mathcal{P}$. Clearly, we have $\mathcal{P} = \mathrm{V}(\mathcal{S})$. If $\mathcal{P} \in \mathrm{V}(\mathcal{S}')$, then $\mathcal{S}' \subset \mathcal{S}$, so $\mathrm{V}(\mathcal{S}) \subset \mathrm{V}(\mathcal{S}')$. Hence $\mathrm{V}(\mathcal{S})$ is the smallest closed subset contains $\mathcal{P}$, which mean $\mathrm{V}(\mathcal{S}) = \overline{\{\mathcal{P}\}} = \mathrm{V}(\mathcal{P}) = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}) | \mathcal{Q} \subset \mathcal{P}\}$. The second statement is immediate by definition. □

It's also not hard to show that the Balmer spectrum defines a contravariant functor

Proposition 1.13. *Let* $F : \mathcal{K} \to \mathcal{L}$ *be a morphism of tensor triangulated category. The map*

$$\mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$$

$$\mathcal{P} \mapsto F^{-1}(\mathcal{P})$$

is well-defined, and continuous. For all $a \in \mathcal{K}$, *we have*

$$\mathrm{Spc}(F)^{-1}(\mathrm{supp}_{\mathcal{K}} a) = \mathrm{supp}_{\mathcal{L}}(F(a))$$

The assignment $\mathcal{K} \mapsto \mathrm{Spc}(\mathcal{K})$ *defines a contravariant functor from tensor triangulated category to topological space.*

Proof. As F is a tensor triangulated functor, it commutes with the tensor product, translation, and direct sum, which will imply $F^{-1}(\mathcal{Q})$ is a prime ideal of $\mathcal{K}$. It is also not hard to see $\mathrm{Spc}(F)^{-1}(\mathrm{supp}_{\mathcal{K}} a) = \mathrm{supp}_{\mathcal{L}}(F(a))$, so the $\mathrm{Spc}(F)$ is continuous (use the fact that F commute with tensor product, triangle and finite direct sum). □

The following two results are immediate

Corollary 1.14. *Given two tensor triangulated functor $F_1, F_2 : \mathcal{K} \to \mathcal{L}$ such that for any $a \in \mathcal{K}$, we have $\langle F_1(a)\rangle = \langle F_2(a)\rangle$ in $\mathcal{L}$. Then the induced map* $\mathrm{Spc}(F_1) = \mathrm{Spc}(F_2)$.

Proof. Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{L})$, for $i = 1, 2$ if $a \in F_i^{-1}(\mathcal{P})$, then $F_i(a) \in \mathcal{P}$, so $\langle F_i(a)\rangle \subset \mathcal{P}$, and if $\langle F_i(a)\rangle \subset \mathcal{P}$, we easily get $a \in \mathcal{P}$. Hence $a \in F_i^{-1}(\mathcal{P}) \iff \langle F_i(a)\rangle \subset \mathcal{P}$ □

Corollary 1.15. *Given an essentially surjective tensor triangulated functor $F : \mathcal{K} \to \mathcal{L}$, e.g for any object $y \in \mathcal{L}$, there is an object $a \in \mathcal{K}$ such that $F(a) \simeq b$, then $\mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ is injective*

Proof. Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{L})$, we claim $\langle F(F^{-1}(\mathcal{P}))\rangle = \mathcal{P}$. It is easy to $\langle F(F^{-1}(\mathcal{P}))\rangle \subset \mathcal{P}$. Conversely, let $a \in \mathcal{P}$, then there is a $b \in \mathcal{K}$ such that $F(b) \simeq a$, so $F(b) \in \mathcal{P}$ by remark 0.2, which means $b \in F^{-1}(\mathcal{P})$. Then, use the remark 0.2 again, as $a \simeq F(b) \in \langle F(F^{-1}(\mathcal{P}))\rangle$, we have $a \in \langle F(F^{-1}(\mathcal{P}))\rangle$. Hence$\langle F(F^{-1}(\mathcal{P}))\rangle \supset \mathcal{P}$ and $F^{-1}(\mathcal{P}_1) = F^{-1}(\mathcal{P}_2)$ force $\mathcal{P}_1 = \mathcal{P}_2$ □

Given a tensor ideal $\mathcal{I}$, we can "quotient" $\mathcal{I}$ category by $\mathcal{K}$ and get a tensor triangulated category $\mathcal{K}/\mathcal{I}$. On the next proposition, we will identify the Balmer spectrum $\mathrm{Spc}(\mathcal{K}/\mathcal{I})$ as a subspace of $\mathrm{Spc}(\mathcal{K})$. However, unlike the commutative ring, the quotient doesn't construct a closed subspace containing $\mathcal{I}$

Remark 1.16. The quotient $\mathcal{K}/\mathcal{I}$ is called *Verdier quotient*. See [HK10] 4.6 or [AN01] Chapter 9 for an introduction if you aren't familiar with this notation.

We summarize some important properties of the Vardier quotient. Let $\mathcal{I}$ be tensor thick ideal, we will denote $\mathcal{L} = \mathcal{K}/\mathcal{I}$ as the quotient of $\mathcal{K}$ by $\mathcal{P}$, then

1. The object of $\mathcal{L}$ is same as the $\mathcal{K}$. There is a functor, called localization functor $p : \mathcal{K} \to \mathcal{K}/\mathcal{I} = \mathcal{L}$. For any morphism $f \in \mathrm{Mor}(\mathcal{K})$ such that $\mathrm{cone}(f) \in \mathrm{Mor}(\mathcal{I})$, $p(f)$ is a isomorphism in $\mathcal{L}$.

2. $\mathcal{L}$ is a tensor triangulated category, and p is tensor triangulated functor.

3. There is a short exact sequence of the tensor triangulated categories, i.e. $\mathcal{I} = \ker(p)$ and p is essentially surjective

$$0 \to \mathcal{I} \to \mathcal{K} \xrightarrow{p} \mathcal{L} \to 0$$

Proposition 1.17. *Let $\mathcal{I}$ be a tensor thick ideal, $p : \mathcal{K} \to \mathcal{K}/\mathcal{I} = \mathcal{L}$ be the localization functor, then the induced morphism $\mathrm{Spc}(p)$ will give a homeomorphism between $\mathrm{Spc}(\mathcal{L})$ and $V = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{I} \subset \mathcal{P}\}$*

Proof. See [PB05] Proposition 3.11. □

We may expect the Balmer spectrum to be quasi-compact. Indeed, this is correct by the following proposition.

Lemma 1.18. *Let $a \in \mathcal{K}$ be an object, $\mathcal{S}$ be a collection of objects of $\mathcal{K}$. We have $\mathrm{U}(a) \subset \mathrm{U}(\mathcal{S})$, or equivalently $\mathrm{V}(a) \supset \mathrm{V}(\mathcal{S})$ if and only if there exist $b_1, \ldots, b_n \in \mathcal{S}$ such that $b_1 \otimes \cdots \otimes b_n \in \langle a\rangle$.*

Proof. Let $\mathcal{S} = \{\bigotimes_{i=1}^n a_i \mid a_i \in \mathcal{S}\} \cup \{\mathbb{1}\}$ be a collection of the object consisting of the finite product of element in $\mathcal{S}$, then we can check this is a tensor multiplicative family. Clearly, $\mathrm{U}(\mathcal{S}) \subset \mathrm{U}(\mathcal{S}')$. Conversely, if $\mathcal{P} \cap \mathrm{U}(\mathcal{S}') \neq \varnothing$, then there is a finite product $\bigotimes_{i=1}^n a_i \in \mathcal{P}$, but this imply $b_i \in \mathcal{P}$ for some i, so we conclude $\mathrm{U}(\mathcal{S}) = \mathrm{U}(\mathcal{S}')$.

Now the proposition is equivalent to show $\mathrm{U}(a) \subset \mathrm{S}'$ if and only if $\mathrm{U}(\mathcal{S}') \cap \langle a \rangle \neq \varnothing$. If $\mathrm{U}(\mathcal{S}') \cap \langle a \rangle \neq \varnothing$, clearly any $\mathcal{P}$ contains a will meet $\mathcal{S}'$ at an element. Conversely, if $\mathrm{U}(\mathcal{S}') \cap \langle a \rangle = \varnothing$, Let $\langle a \rangle$ be the $\mathcal{D}$ Lemma 1.10, we then get a $\mathcal{P} \subset \mathrm{U}(a)$ such that $\mathcal{P} \notin \mathrm{U}(\mathcal{S}')$. □

Proposition 1.19. *The following statements are true*

(i) For any $a \in \mathcal{K}$, $\mathrm{U}(a)$ is quasi-compact.

(ii) Any quasi-compact open subset of $\mathrm{Spc}(\mathcal{K})$ is of the form $\mathrm{U}(a)$.

Proof. (i): Given an open cover $\{\mathrm{U}(\mathcal{S})_i\}_{i\in I}$ of $\mathrm{U}(a)$. Let $\mathcal{S} = \bigcup_{i\in I} \mathcal{S}_i$, then $\mathrm{U}(a) \subset \mathrm{U}(\mathcal{S})$. By Lemma 1.18, there is $b_1, \dots, b_n \in \mathcal{S} = \bigcup_{i\in I} \mathcal{S}_i$ such that $b_1 \otimes \cdots \otimes b_n \in \langle a \rangle$. But then, we can choose a finite subset of indices $I_0 \subset I$ such that $b_i \in \bigcup_{j\in I_0}, \forall i$. Using Lemma 1.18, we conclude $\mathrm{U}(a) \subset \bigcup_{j\in I_0} \mathcal{S}_j$

(ii): Suppose $\mathrm{U}(\mathcal{S})$ is a quasi-compact open subset for $\mathcal{S} \subset \mathcal{K}$. We have $\mathrm{U}(\mathcal{S}) = \bigcup_{a\in\mathcal{S}} \mathrm{U}(a)$, because of quasi-compactness, we get $\mathrm{U}(\mathcal{S}) = \bigoplus_{i=1}^n \mathrm{U}(a_i)$ for $a_i \in \mathcal{S}$. It is not hard to see $\bigcup_{i=1}^n \mathrm{U}(a_i) = \mathrm{U}(a_1 \otimes \cdots \otimes a_n)$(use the fact that a prime ideal $\mathcal{P}$ is prime), so we complete the proof.

□

Corollary 1.20. *$\mathrm{Spc}(\mathcal{K})$ is quasi-compact. In particular, if $\mathrm{U}(\mathcal{S}) = \mathrm{Spc}(\mathcal{K})$ for $\mathcal{S} \subset \mathcal{K}$, there exists $b_1, \dots, b_n \in \mathcal{S}$ such that $b_1 \otimes \cdots \otimes b_n = 0$*

Proof. Notice $\mathrm{U}(0) = \mathrm{Spc}(\mathcal{K})$. So using Proposition 1.19 (i), we get $\mathrm{Spc}(\mathcal{K})$ is quasi-compact. The second statement is immediately by Lemma 1.18 □

Recall that a topological is called *noetherian* if every open subset is quasi-compact. It is *irreducible* if it can not be written as a union of two proper closed subspace (or equivalently, every two open subsets intersect nonempty). Using the Proposition 1.19, we see

Corollary 1.21. *$\mathrm{Spc}(\mathcal{K})$ is a noetherian topological space if and only if every open subset (or closed subset) is of the form $\mathrm{U}(a)$ (or $\mathrm{supp}(a)$).*

We also get the following equivalent condition for a closed subset of $\mathrm{Spc}(\mathcal{K})$ to be irreducible

Proposition 1.22. *For closed subset $\varnothing \neq Z \subset \mathrm{Spc}(\mathcal{K})$, the following are equivalent*

(1) Z is irreducible.

(2) For all $a, b \in \mathcal{K}$, if $\mathrm{U}(a \oplus b) \cap Z = \varnothing$, then $\mathrm{U}(a) \cap Z = \varnothing$ or $\mathrm{U}(b) \cap Z = \varnothing$.

(3) $\mathcal{Q} = \{a \in \mathcal{K} \mid \mathrm{U}(a) \cap Z \neq \varnothing\}$ *is prime.*

In particular, this implies every non-empty irreducible closed Z has a unique generic point $\mathcal{P}$ such that $Z = \overline{\{\mathcal{P}\}}$.

Proof. (1) $\Rightarrow$ (2): Recall $\mathrm{U}(a \oplus b) = \mathrm{U}(a) \cap \mathrm{U}(b)$, so if $\mathrm{U}(a \oplus b) \cap Z = \varnothing$, then we should have $\mathrm{U}(a) \cap Z = \varnothing$ or $\mathrm{U}(b) \cap Z = \varnothing$ because otherwise we get two open subsets $\mathrm{U}(a) \cap Z$ and $\mathrm{U}(b) \cap Z$ of Z with empty intersection which is contradict to Z is irreducible.

(2) $\Rightarrow$ (3): Let's check all three condition of 0.1 and show that it is prime. First, $\mathrm{U}(0) = \mathrm{Spc}(\mathcal{K})$, so $0 \in \mathcal{Q}$. We also notice that by (2), if $a, b \in \mathcal{Q}$, then $a \otimes b \in \mathcal{Q}$. On other hand, because $\mathrm{U}(\mathbb{1}) = \mathrm{Spc}(\mathcal{K})$, so $\mathbb{1} \notin \mathcal{Q}$.

(i) Suppose there is a triangle $a \to b \to c \to \Sigma a$ with $a, b \in \mathcal{Q}$, then we have $c \in \langle a \oplus b \rangle$, and so $\mathrm{U}(a \oplus b) \subset \mathrm{U}(c)$. Therefore, $Z \cap \mathrm{U}(a \oplus b) \neq \varnothing \Rightarrow Z \cap \mathrm{U}(c) \neq \varnothing$.

(ii) If $a \oplus b \in \mathcal{Q}$, as $\mathrm{U}(a \oplus b) = \mathrm{U}(a) \cap \mathrm{U}(b)$, we get a and b in $\mathcal{Q}$.

(iii) Let $a \in \mathcal{K}, b \in \mathcal{Q}$, as $\mathrm{U}(a \otimes b) = \mathrm{U}(a) \cup \mathrm{U}(b)$, we easily get $a \otimes b \in \mathcal{Q}$

Prime: Let $a \otimes b \in \mathcal{Q}$, then, again, because $\mathrm{U}(a \otimes b) = \mathrm{U}(a) \cup \mathrm{U}(b)$, so $\mathrm{U}(a \otimes b) \cap Z \neq \varnothing$ imply $\mathrm{U}(a)$ or $\mathrm{U}(b)$ intersection Z nontrivial.

(3) $\Rightarrow$ (1) We claim that $Z = \overline{\{\mathcal{Q}\}}$ so we get (3) $\Rightarrow$ (1) and show the existence of generic point. We first need a lemma.

Lemma 1.23. *Let $W \subset \mathcal{K}$ be a subset of the Balmer spectrum, then its closure is*

$$\overline{W} = \bigcap_{a \in \mathcal{K} \text{ s.t } W \subset \mathrm{supp}(a)} \mathrm{supp}(a)$$

Proof. Notice $\mathcal{B} = \{\mathrm{supp}(a) \mid a \in \mathcal{K}\}$ is a base of the closed subset (finite union $\bigcup_{i=1}^{n} \mathrm{supp}(a_i)$ is closed because of Lemma 1.25, SD3), and the closure is the intersection of all $B \in \mathcal{B}$ such that $W \subset B$, which means $\overline{W} = \bigcap_{a \in \mathcal{K} \text{ s.t } W \subset \mathrm{supp}(a)} \mathrm{supp}(a)$. ∎

Let $\mathcal{P} \in Z$. For $a \in \mathcal{Q}$, we have $\mathcal{Q} \in \mathrm{U}(a) \cap Z \neq \varnothing$, so $\mathcal{P} \subset \mathcal{Q}$, that is $Z \subset \overline{\{\mathcal{Q}\}}$ by Proposition 1.12. Conversely, by Lemma 1.23, we can write $Z = \bigcap_{a \in \mathcal{K} \text{ s.t } Z \subset \mathrm{supp}(a)} \mathrm{supp}(a)$. If $a \in \mathcal{K}$ is a object such that $Z \subset \mathrm{supp}(a)$, then $\mathrm{U}(a) \cap Z = \varnothing$, which means $a \notin \mathcal{Q}$, or equivalently, $\mathcal{Q} \in \mathrm{supp}(a)$. Hence $\mathcal{Q} \in \bigcap_{a \in \mathcal{K} \text{ s.t } Z \subset \mathrm{supp}(a)} \mathrm{supp}(a) = \overline{Z}$.

The uniqueness of such a generic point is given by Proposition 1.12 ∎

1.4 Universal Property of $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$

We now tend to the classification theorem of the Balmer spectrum. First, we will formalize the notation of the support date.

Definition 1.24. A *support data* on a tensor triangulated category $(\mathcal{K}, \otimes, \mathbb{1}, \Sigma)$ is a pair (X, σ) consists of the following data:

(SD1) : a topological space X and an assignment σ that associate each object k a closed subset $\sigma(k)$ of X.
(SD2) : $\sigma(\mathbb{1}) = X$ and $\sigma(0) = \varnothing$.
(SD3) : $\sigma(k \oplus l) = \sigma(k) \cup \sigma(l)$.
(SD4) : $\sigma(\Sigma k) = \sigma(k)$.
(SD5) : $\sigma(k \otimes l) = \sigma(k) \cap \sigma(l)$.
(SD6) $\sigma(b) \subset \sigma(a) \cup \sigma(c)$ for any triangle $a \to b \to c \to \Sigma a$

A morphism $f : (X, \sigma) \to (Y, \tau)$ of support data on the same category $\mathcal{K}$, where $f : X \to Y$ is a continuous map such that $\sigma(a) = f^{-1}(\tau(a))$ for all object $a \in \mathcal{K}$. It is an isomorphism if and only f is a homeomorphism.

Lemma 1.25. *The pair* $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ *is a support data.*

Proof. For an object a, $\mathrm{supp}(a)$ is a closed subset of $\mathrm{Spc}(\mathcal{K})$, so SD1 is clear. A prime ideal $\mathcal{P}$ containing $\mathbb{1}$ will equal $\mathcal{K}$, so any proper prime could not contain $\mathbb{1}$ which confirms SD2. Using the fact that the prime ideal is a thick category, we get SD3-SD4 (see 0.1). The property of being prime will show the SD5, and SD6 is given by the fact thick subcategory is a triangulated subcategory, which will complete the proof.

□

The support data $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ will enjoy the following universal property.

Theorem 1.26 (Universal Property of $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$**).** *Let $\mathcal{K}$ be a tensor triangulated category. If (X, σ) is support data on $\mathcal{K}$, then there exist a unique continuous map $f : X \to \mathrm{Spc}(\mathcal{K})$ such that $(\sigma(x)) = f^{-1}(\mathrm{supp}(a))$ for any object $a \in \mathcal{K}$. More precisely, this map f is defined by*

$$x \mapsto \{a \in \mathcal{K} \mid x \notin \sigma(a)\}$$

To prove this theorem, we need the following lemma

Lemma 1.27. *Let X be a set and $f_1, f_2 : X \to \mathrm{Spc}(\mathcal{K})$ be two maps such that $f_1^{-1}(\mathrm{supp}(a)) = f_2^{-1}(\mathrm{supp}(a)), \forall a \in \mathcal{K}$, then $f_1 = f_2$*

Proof. Let $x \in X$, then by assumption, for an object $a \in \mathcal{K}$ the $f_1(x) \in \mathrm{supp}(a)$ is equivalent to $f_2(x) \in \mathrm{supp} a$. This implies

$$\bigcap_{f_1(x) \in \mathrm{supp}(a)} \mathrm{supp}(a) = \bigcap_{f_2(x) \in \mathrm{supp}(a)} \mathrm{supp}(a)$$

but then, by Lemma 1.23, this is equivalent to say $\overline{\{f_1(x)\}} = \overline{\{f_2(x)\}}$ in $\mathrm{Spc}(\mathcal{K})$. Now use Proposition 1.12, we conclude $f_1(x) = f_2(x)$, which complete the proof. □

Lemma 1.28. *Let (X, σ) be a support data on $\mathcal{K}$ and $Y \subset X$ be a subset, then the full subcategory of $\mathcal{K}$ with objects $\{a \in \mathcal{K} \mid \sigma(a) \subset Y\}$ is a tensor thick ideal*

Proof. This can be easily verified by SD3-SD5 of Def 1.24. For example, the SD3 will imply this full subcategory is thick, SD4 implies it is a triangulated subcategory and SD5 implies it is a tensor ideal. □

Proof of theorem 1.26. The uniqueness is given by Lemma 1.27. Now, we show the map f defined in 1.26 is the map we want. Apply Lemma 1.28 to $Y = X - \{x\}$, then we get $f(x) = \{a \in \mathcal{K} \mid a \notin \sigma(x)\}$ is a tensor thick ideal. To this is a prime ideal, let $a \otimes b \in f(x)$, that is $x \notin \sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$, so $x \notin \sigma(a)$ or $x \notin \sigma(b)$, which means a or b in $f(x)$. By definition, we have

$$f(x) \in \operatorname{supp}(a) \Leftrightarrow a \notin f(x) \Leftrightarrow x \in \sigma(a)$$

, so $f^{-1}(\operatorname{supp}(a)) = \sigma(a)$, and this confirm f is continuous as $\operatorname{supp}(a)$ form a base of closed subset of $\operatorname{Spc}(\mathcal{K})$ by Lemma 1.23 □

1.5 Natural Map Between Balmer Spectrum and Zariski Spectrum

Let $\mathcal{K}$ be a tensor triangulated category, we will briefly introduce how to construct a map from the Balmer spectrum of $\mathcal{K}$ to the Zaiski spectrum of its endomorphism of $\mathbb{1}$. Most of the proof can be found in [PB10] sections 3 and 4.

Definition 1.29. A ring R is called *graded commutative* if (1) $R = \bigoplus R_i$ is a graded ring; (2) $\forall a, b \in R, ab = (-1)^{|a||b|}ba$ where $|a|$ is the grade of element. Notice $R_{\text{even}} = \{a \in R \mid |a| \text{ is even}\}$ will be a commutative ring.

Construction 1.30. For a graded commutative ring, we can define the homogeneous spectrum $\operatorname{Spec}(R)$ as the set that consists of all homogeneous prime ideals $\mathfrak{p}$, i.e $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ (this can be checked by homogeneous element only). There is Zariski topology on it, where the closed subset is $\mathrm{V}(I) = \{\mathfrak{p} \in \operatorname{Spc}(R) \mid S \subset \mathfrak{p}\}$ for I a homogenous ideal. The principle open set $\mathrm{U}(a) = \{a \notin \mathfrak{p} \mid \mathfrak{p} \subset \operatorname{Spc}(R)\}$ form a open base, for a a homogeneous element.

Remark 1.31. If R doesn't have the order 2 odd degree element, i.e. $2a = 0$ implies $a = 0$ if a is odd degree, we can have a more precise description of the element in $\operatorname{Spec}(R)$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$, then $\mathfrak{p} \cap R_{\text{even}}$ is a prime ideal of $\operatorname{Spec}(R_{\text{even}})$. On the other hand, if a is an odd-degree element, then $a^2 = -a^2$ will imply $a^2 = 0$. Because every $\mathfrak{p}$ contains 0, so every $\mathfrak{p} \in \operatorname{Spec}(R)$ should contain all the odd degree elements. By the discussion above, $\mathfrak{p} \in \operatorname{Spec}(R)$ will exactly on the form $\mathfrak{q} \cup \{\text{odd degree element}\}$ for $\mathfrak{q} \in \operatorname{Spec}(R_{\text{even}})$. Some people call this kind of graded-commutative ring *strict*. In practice, most of the graded commutative rings we meet will be strict.

Let M, N be two graded modules of R. For each integer n, we write $M[n]$ for graded module where $M[n]^i = M^{i+n}$. We write $\operatorname{Hom}_R^*(M, N)$ for graded homomorphism between M, N where it is a graded abelian group and the degree n component is:

$$\operatorname{Hom}_R^n(M, N) = \operatorname{Hom}_R(M, N[n])$$

The degree zero component is $\operatorname{Hom}_R(M, N)$ and it is a graded module of R

Example 1.32. One of the most fundamental examples is the (singular)cohomology $\mathscr{H}^*(X, \mathbb{Z})$ of topological space X. This graded abelian group has a ring structure given by the cup product, and it is graded commutative. See [Hat01] Chapter 3 for more detail.

Example 1.33. Let R be a noetherian commutative ring and $x_1, \ldots, x_n$ be n element of R. We can think it as a element $\mathbf{x} = (x_1, \ldots, x_n) \in R^n = V$ of free module V, and construct a *Koszul Complex* as

$$K(x_1, \ldots, x_n) : 0 \to R \to \wedge V \xrightarrow{\wedge \mathbf{x}} \wedge^2 V \xrightarrow{\wedge \mathbf{x}} \cdots \to \wedge^{n-1} \xrightarrow{\wedge \mathbf{x}} \wedge^n V \to 0$$

or equivalently, we can first construct a degree 1 Koszul complex

$$K(x_i) = 0 \to R \xrightarrow{\times x_i} R \to 0$$

then $K(x_1, \ldots, x_n) = K(x_1) \otimes \cdots \otimes K(x_n)$. The associated graded abelian group $\bigoplus_{i=1}^n \wedge^i V$ is a graded-commutative ring where the multiplication is the wedge sum.

Advertisement 1.34 (*"Higher Algebra"*)**.** The example 1.32 and 1.33 are two different models for "Higher Algebra" ("Higher" stands for higher homotopy information) where the example 1.32 is $\mathbb{E}_\infty$-ring spectra and 1.33 is the differential graded ring. In higher algebra, we will replace all strong equal "$\cong$" with weak equivalence (or quasi-isomorphism)"$\simeq$" and study the algebraic property "up to homotopy". For example the $\mathbb{E}_\infty$-ring spectra is a "commutative" algebra with "commutativity" up to homotopy. Indeed, we can discuss the graded commutative ring under a "higher setting": we can define it as the commutative algebra object of the (∞-)symmetric monoidal category of $\mathbb{Z}-$graded abelian group with symmetric isomorphism

$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$$

This kind of object is studied by Lars Hesselholt and Piotr Pstragowski in [LarsPiotr23] which they call it "Dirac ring". If you like the idea of higher algebra, CLICK HERE [HTT09], [HA15] TO JOIN!!!

Example 1.35 (*Important Example!!!*)**.** Let $\mathcal{K}$ be a tensor triangulated category, $a, b \in \mathcal{K}$ be two objects, then there is a left action of endomorphism $R = \mathrm{End}(\mathbb{1})$ on $\mathrm{Hom}(a, b)$ defined by

$$\mathrm{End}(\mathbb{1}) \times \mathrm{Hom}(a, b) \to \mathrm{Hom}(a, b)$$

$$(f, g) \mapsto f \otimes g$$

we can also define the right action similarly. By [PB10] Proposition 2.2, the right action and left action are coincident. Especially,

$$\mathrm{End}(\mathbb{1}) \times \mathrm{End}(\mathbb{1}) \to \mathrm{End}(\mathbb{1})$$

$$(f, g) \mapsto f \otimes g$$

define a multiplicative of $R_{\mathcal{K}} = \mathrm{End}(\mathbb{1})$ so it is a commutative ring, and this makes $\mathrm{Hom}(a, b)$ become $R_{\mathcal{K}}$-module.

We define the graded homomorphism of two objects a, b as $\mathrm{Hom}^*_{\mathcal{K}}(a,b) = \bigoplus_i \mathrm{Hom}_{\mathcal{K}}(a, \Sigma^i b)$, then this is a graded abelian group with $\mathrm{Hom}(a,b)$ as degree 0 component, and there is an obvious composition given by

$$\mathrm{Hom}^i(a,b) \times \mathrm{Hom}^j(b,c) \to \mathrm{Hom}^{i+j}(a,b)$$

$$(a \xrightarrow{f} b[i], b \xrightarrow{g} c[j]) \mapsto \begin{tikzcd} a \arrow[rr, "f\times g"] \arrow[dr, "f"'] & & c[i+j] \\ & b[i] \arrow[ur, "\Sigma^j g"'] & \end{tikzcd}$$

In particular, we will denote the $\mathrm{End}^*(\mathbb{1}) = \bigoplus_i \mathrm{Hom}(\mathbb{1}, \Sigma^i \mathbb{1})$ as $R^*_{\mathcal{K}}$. With the composition we defined above, it is a graded ring where the 0 component is the graded ring $R_{\mathcal{K}}$.

Proposition 1.36. *$R^*_{\mathcal{K}}$ is a graded-commutative ring*

Proof. See [PB10] proposition 3.3 for a direct proof or [BKSN20] Corollary 2.3 to prove $R^*_{\mathcal{K}}$ isomorphic to the graded center of $\mathcal{K}$. □

Definition 1.37. Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$, define a map $\rho_{\mathcal{K}} : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}(R^*_{\mathcal{K}})$ as

$$\mathcal{P} \mapsto \{f \in R^{\mathrm{hom}}_{\mathcal{K}} \mid \mathrm{cone}(f) \notin \mathcal{P}\}$$

Notice the image $\rho_{\mathcal{K}}(\mathcal{P})$ "reverse" the inclusion: if $\mathcal{P} \subset \mathcal{Q}$, then $\rho_{\mathcal{K}}(\mathcal{Q}) \supset \rho_{\mathcal{K}}(\mathcal{P})$

Theorem 1.38. *(1) For $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$, $\rho = \rho_{\mathcal{K}}(\mathcal{P})$ is a homogenous prime ideal.*

(2) The map $\rho : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}(R^*_{\mathcal{K}})$ is continuous. Composing with the map

$$f : \mathrm{Spec}(R^*_{\mathcal{K}}) \xrightarrow{\mathfrak{p} \mapsto \mathfrak{p} \cap (R^*_{\mathcal{K}})^0} \mathrm{Spec}(R_{\mathcal{K}})$$

we get a natural map

$$\mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}(R_{\mathcal{K}})$$
$$\mathcal{P} \mapsto \{f \in R_{\mathcal{K}} \mid \mathrm{cone}(f) \notin \mathcal{P}\}$$

The proof can be found in [PB10] Theorem 5.3 and Corollary 5.6. Rather than provide specific proof, we will give an example to convince you this theorem should be correct.

Example 1.39 (*Important Toy Model!!!*). Consider $\mathcal{K} = \mathscr{D}^b(\mathbb{Z})$, the bounded derived category of a finitely generated abelian group. This category is the Hereditary category, e.g. $\mathrm{Ext}^2_{\mathbb{Z}}(-,-)$ vanish, so each chain complex $X \simeq \bigoplus_{i\in\mathbb{Z}} \Sigma^{-i} \mathscr{H}^i(X)$.

This fact implies the smallest thick subcategory contains $\mathbb{Z}$ is $\mathcal{K}$, so by Lemma 1.1, every thick subcategory is tensor ideal. To understand the tensor thick ideal of $\mathscr{D}^b(\mathbb{Z})$, it is enough to understand what happens on the finitely generated abelian group.

We observe that if $p \neq q$ be two distinct prime number, then $\mathbb{F}_q \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_p \simeq 0$. So, let $\mathcal{P} \in \mathrm{Spc}(\mathscr{D}^b(\mathbb{Z}))$, it should contain 0, and this means it should contain almost all the $\mathbb{F}_p$ but a possible $\mathbb{F}_q$. So we can define a map

$$\phi : \mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spc}(\mathcal{K})$$

$$p \mapsto \langle \mathbb{F}_q \mid q \neq p \rangle$$

This is a Prime ideal where the elements are all the chain complex without the p-torsion homology group. Conversely, use the $\rho_{\mathcal{K}}$, we have a map

$$\rho : \mathrm{Spc}(\mathcal{K}) \Rightarrow \mathrm{Spc}(\mathrm{End}^*(\mathbb{Z})) \xrightarrow{\mathfrak{p} \mapsto \mathfrak{p} \cap \mathrm{End}^0(\mathbb{Z})} \mathrm{Spc}(\mathrm{End}(\mathbb{Z}))$$

$$\mathcal{P} \mapsto \{m \mid \mathrm{cone}(\mathbb{Z} \xrightarrow{m} \mathbb{Z}) \notin \mathcal{P}\}$$

The $\mathfrak{p} = \{m \mid \mathrm{cone}(\mathbb{Z} \xrightarrow{m} \mathbb{Z}) \notin \mathcal{P}\}$ is a prime ideal of $\mathbb{Z}$ because: we first notice that $\mathrm{cone}(\xrightarrow{m}) \simeq \mathbb{Z}/m\mathbb{Z}$ in $\mathscr{D}^b(\mathbb{Z})$. Let $mn \in \mathfrak{p}$, and suppose $n, m \notin \mathfrak{p}$. We have a distinguished triangle

$$\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$$

as $\mathcal{P}$ is a triangulated subcategory, this imply $\mathbb{Z}/mn\mathbb{Z} \simeq \mathrm{cone}(\xrightarrow{mn}) \in \mathcal{P}$, which is a contradiction, so we must have n or m in $\mathfrak{p}$. One can check the ϕ is also continuous. Moreover, ϕ and ρ are mutually invertible, which implies $\mathrm{Spc}(\mathscr{D}^b(\mathbb{Z})) \simeq \mathrm{Spc}(\mathbb{Z})$. Hence we get the first example of Balmer spectrum isomorphic to Zariski spectrum. In general, if X is a noetherian scheme, then $X \simeq \mathrm{Spc}(\mathscr{D}^{\mathrm{perf}}(X))$. The interested reader can see [PB05] section 6 for more details and another example of the stable category of finite group(scheme).

Thanks to the specific construction of the map ϕ, we observe that the elements of a prime ideal of $\mathrm{Spc}(\mathcal{K})$ are exactly all chain complex without the p-torsion homology group for $p = \rho(\mathcal{P})$. Hence, $X \in \mathscr{D}^b(\mathbb{Z})/\mathcal{P}$ if and only if the homological support $X_{(p)} \not\simeq 0$. We can also characterize the prime ideal of Balmer spectrum via the functor

$$F_p : \mathscr{D}^b(\mathbb{Z}) \to \mathrm{GrAb}$$

$$X \mapsto \bigoplus_i \mathrm{Ext}^1_{\mathbb{Z}}(\mathscr{H}^i(X), \mathbb{Z}/p\mathbb{Z})$$

For a finitely generated abelian group, $\mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}/p\mathbb{Z})$ detect the p-torsion element of M. Hence the kernel of F_p is exactly the $\phi(p)$, so the kernel of the family of functor $\{F_p \mid p \text{ a prime number}\}$ are exactly the elements of $\mathrm{Spc}(\mathscr{D}^b(\mathbb{Z}))$.

1.6 "Hilbert Nullstellensatz"

In classical algebraic geometry, one of the most fundamental results is *Hilbert Nullstellensatz*, where it states

Theorem 1.40 (*Hilbert Nullstellensatz*)**.** *Let k be a algebraically closed field, $A = [x_1, \dots, x_n]$. For a algebraic closed subset X of $\mathbf{A}^n$, we defined $I(X) = \{f \in A \mid f(x) = 0, \forall x \in X\}$. Then there is a one-to-one inclusion preserving correspondence between algebraic closed subset X of A^n and radical ideal J of A given by*

$$X \mapsto I(X)$$
$$J \mapsto \mathrm{V}(J)$$

In particular, the irreducible algebraic subset (variety) will correspond exactly to the prime ideal of A.

the theorem allows us to freely translate the geometry problem to commutative algebra and implies the prime ideal will encode the geometry data. This idea later motivates the construction of the Zariski spectrum and the development of scheme theory.

For the Balmer spectrum, we would also like to have a "Hilbert Nullstellensatz" which allows us to translate the "radical of the tensor thick ideal of $\mathcal{K}$" to "some subspace of $\mathrm{Spc}(\mathcal{K})$". To begin with, let's first define the radical of the tensor thick ideal.

Definition 1.41. The radical $\sqrt{\mathcal{I}}$ of a tensor thick ideal $\mathcal{I}$ is defined as

$$\{a \in \mathcal{K} \mid \exists n \geqslant 1 \text{ such that } a^{\otimes n} \in \mathcal{I}\}$$

A tensor thick ideal $\mathcal{I}$ is called *radical* if $\sqrt{\mathcal{I}} = \mathcal{I}$.

Lemma 1.42. *$\sqrt{\mathcal{I}}$ is a tensor thick ideal and it is equal to $\bigcap_{\mathcal{I} \subset \mathcal{P}} \mathcal{P}$ for all the primes $\mathcal{P}$ contains $\mathcal{I}$.*

Proof. The intersection of the tensor thick ideal is a tensor thick ideal, so it is enough to show $\bigcap_{\mathcal{I} \subset \mathcal{P}} \mathcal{P} = \sqrt{\mathcal{I}}$. By definition, we have $\sqrt{\mathcal{I}} \subset \mathcal{P}$ if $\mathcal{P}$ contains $\mathcal{I}$. Conversely, let $a \in \mathcal{K}$ be an object such that $a \in \bigcap_{\mathcal{I} \subset \mathcal{P}} \mathcal{P}$. Consider tensor multiplicative family $\mathcal{S} = \{\otimes^n a \mid n \geqslant 1\} \cup \{\mathbb{1}\}$ be collection of finite product of a. It is enough to show $\mathcal{S} \cap \mathcal{I} \neq \varnothing$. Indeed, suppose $\mathcal{S} \cap \mathcal{I} = \varnothing$, then apply Lemma 1.10, then we get a prime ideal $\mathcal{P} \cap \mathcal{S} = \varnothing$ but $\mathcal{I} \subset \mathcal{P}$ which is impossible. □

Definition 1.43. Let $\mathcal{S} \subset \mathcal{K}$ be collection of object of $\mathcal{K}$, the *support* of $\mathcal{S}$ is defined as

$$\mathrm{supp}(\mathcal{S}) = \bigcup_{a \in \mathcal{S}} \mathrm{supp}(a)$$

Lemma 1.44. *Let $\mathcal{S} \subset \mathcal{K}$, then $\mathrm{supp}(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{S} \not\subset \mathcal{P}\}$.*

Proof. If $\mathcal{P} \in \mathrm{supp}(\mathcal{S})$, this imply there is a $s \in \mathcal{S}$ such that $\mathcal{P} \in \mathrm{supp}(s)$, which means $s \notin \mathcal{P}$ by Def 1.24. □

Conversely, we consider the subspace Y of $\mathrm{Spc}(\mathcal{K})$ such that (1) $Y = \bigcup_{i \in I} Y_i$ for Y_i is closed subset and (2) the complement of Y_i is quasi-compact. We call this kind of space *Thomason subset*. Note that every Thomason subset is a specialization closed, e.g. $\overline{\{y\}} \in Y, \forall y \in Y$.

Definition 1.45. We define a subcategory *supported* by Y as a full subcategory

$$\mathcal{K}_Y = \{a \in \mathcal{K} \mid \operatorname{supp}(a) \subset Y\}$$

By Lemma 1.28, $\mathcal{K}_Y$ is tensor thick ideal.

Lemma 1.46. *let $Y \subset \operatorname{Spc}(\mathcal{K})$ be a subset, then $\mathcal{K}_Y$ is equal to the intersection $\bigcap_{\mathcal{P} \notin Y} \mathcal{P}$.*

Proof. Let $a \in \mathcal{K}$, then $a \in \mathcal{K}_Y \Leftrightarrow \operatorname{supp}(a) \subset Y$. Note $\operatorname{supp}(a) \subset Y$ is equivalent to $\forall Prime \notin Y, Y \notin \operatorname{supp}(a)$. $\mathcal{P} \notin \operatorname{supp}(a)$ means $a \in \mathcal{P}$ by defintion. So $a \in \mathcal{K}_Y \Leftrightarrow \forall \mathcal{P} \notin Y, a \in \mathcal{P}$. □

Lemma 1.47. *Let $\mathcal{I} \subset \mathcal{K}$ be a tensor thick ideal, then $\operatorname{supp}_{\operatorname{supp}(\mathcal{I})} = \sqrt{\mathcal{I}}$.*

Proof. By Lemma 1.46, $\mathcal{K}_{\operatorname{supp}(\mathcal{I})} = \bigcap_{\mathcal{P} \notin \mathcal{K}_{\operatorname{supp}(\mathcal{I})}} \mathcal{P}$. By Lemma 1.44, $\mathcal{P} \notin \operatorname{supp}(\mathcal{I}) \Leftrightarrow \mathcal{I} \subset \mathcal{P}$. Applying the Lemma 1.42, we conclude the result.

□

Now we can prove the classification theorem

Theorem 1.48. *.Let $\operatorname{Rad}^{\otimes}(\mathcal{K})$ be the collection of radical tensor thick ideal of $\mathcal{K}$, and Thom be the collection of all Thomason subset of $\operatorname{Spc}(\mathcal{K})$. The assignment*

$$\sigma : \operatorname{Rad}^{\otimes}(\mathcal{K}) \to \operatorname{Thom} : \mathcal{I} \mapsto \operatorname{supp}(\mathcal{I}) = \bigcup_{a \in \mathcal{I}} \operatorname{supp}(a)$$

$$\tau : \operatorname{Thom} \to \operatorname{Rad}^{\otimes}(\mathcal{K}) : Y \mapsto \mathcal{K}_Y$$

gives a bijection between Thom and $\operatorname{Rad}^{\otimes}(\mathcal{K})$.

Remark 1.49. You may feel disappointed with the assumption of "radical tensor ideal" in the theorem. However, in practice, we will study the tensor triangulated category with "nice" properties so that every tensor thick ideal of $\mathcal{K}$ is radical! For example, we will see in section 2.2 this happens if $\mathcal{K}$ is "rigid".

Proof. We first check the map is well defined: Let $a \in \mathcal{K}_Y$, then by Def 1.24 (5), $\operatorname{supp}(a^{\otimes n}) = \operatorname{supp}(a) \cap \cdots \cap \operatorname{supp}(a) = \operatorname{supp}(a)$, so $\mathcal{K}$ is radical. On the other hand, $\operatorname{supp}(\mathcal{I})$ is the union of closed subsets by definition, and by Proposition 1.19, $\operatorname{U}(a) = \operatorname{Spc}(\mathcal{K}) - \operatorname{supp}(a)$ is quasi-compact $\forall a \in \mathcal{I}$. Hence $\operatorname{supp}(\mathcal{I})$ is a Thomason subset.

Now, we check the assignment mutually invertible. By Lemma 1.47, if $\mathcal{I}$ is radical, then $\mathcal{K}_{\operatorname{supp}(\mathcal{I})} = \sqrt{\mathcal{I}} = \mathcal{I}$, so we get $\tau\sigma = \operatorname{id}$. If Y is a Thomason subset. It is straightforward to see $\operatorname{supp}(\mathcal{K}_Y) \subset Y$ by definition. Conversely, if $\mathcal{P} \subset Y$, there exist a closed subset $Y_i \subset$ such that $\mathcal{P} \subset Y_i$ with $\operatorname{Spc}(\mathcal{K}) - Y_i$ quasi-compact. By Proposition 1.19, we $\operatorname{Spc}(\mathcal{K}) - Y_i = \operatorname{U}(a)$ for some $a \in \mathcal{K}$, which imply $Y_i = \operatorname{supp}(a)$. Hence $\mathcal{P} \subset \operatorname{supp}(a) \subset Y$ and this means $\mathcal{P} \in \bigcup_{a \in \mathcal{K}_Y} \operatorname{supp}(a) = \operatorname{supp}(\mathcal{K}_Y)$. Therefore, $Y = \operatorname{supp}(\mathcal{K}_Y)$ and $\sigma\tau = \operatorname{id}$. □

2 Support Theory of Compact Object

So far, we have seen how to use the Balmer spectrum and classify the thick subcategory of a **small tensor triangulated category** $\mathcal{K}$. However, there are many interesting tensor triangulated categories are not **small**. For example the derived category $\mathscr{D}(R)$ of a commutative ring R. Nevertheless, it is still possible to discuss the support data of $\mathcal{K}$ under suitable conditions. In this section, we will study the Balmer spectrum of a small family of objects in $\mathcal{K}$ called *compact object*, and see how to use it to define a support data of the whole category.

2.1 Rigid Tensor Category

As we mention in Remark 1.49, every tensor ideal of a tensor triangulated category $\mathcal{K}$ could be radical if $\mathcal{K}$ has some nice properties. We now introduce one of the possible properties, which means $\mathcal{K}$ is rigid. We first begin with an arbitrary tensor category(without the triangulated structure) $\mathcal{C}$.

Definition 2.1. Let $a \in \mathcal{C}$, an object $a^\vee \in \mathcal{C}$ is called *dual* of a if there exist a morphism $\mathrm{ev_a} : a \otimes a^\vee \to \mathbb{1}$ and $\mathrm{coev_a} : \mathbb{1} \to a \otimes a^\vee$ called *evaluation* and *coevalution* map such that the composition

$$a \xrightarrow{\mathrm{coev}_a \otimes \mathrm{id}_x} (a \otimes a^\vee) \otimes a \xrightarrow{\mathbf{a}_{a,a^\vee,a}} a \otimes (a^\vee \otimes a) \xrightarrow{\mathrm{id}_a \otimes \mathrm{ev}_a} a \tag{2.1}$$

and

$$a^\vee \xrightarrow{\mathrm{id}_{a^\vee} \otimes \mathrm{coev}_a} a^\vee \otimes (a \otimes a^\vee) \xrightarrow{\mathbf{a}^{-1}_{a,a^\vee,a}} (a \otimes a^\vee) \otimes a \xrightarrow{\mathrm{ev}_a \otimes \mathrm{id}_a} a^\vee \tag{2.2}$$

are the identity morphism of a and $a^\vee$ respectly. ($\mathbf{a}_{a,b,c}$ is the natural isomorphism of association $(a \otimes b) \otimes c \simeq a \otimes (b \otimes c)$. A tensor category $\mathcal{C}$ is called *rigid* if every object has a dual.

Proposition 2.2. *Let $a \in \mathcal{C}$, if the dual $a^\vee$ exist, it is unique (up to isomorphism).*

Proof. Let $a \in \mathcal{C}$ and suppose $a_1^\vee$, $a_2^\vee$ are two dual objects of a with the evaluation and coevaluation mape_1, c_1, e_2, c_2. We can define a map $\alpha : a_1^\vee \to a_2^\vee$ as the composition of

$$a_1^\vee \xrightarrow{\mathrm{id}_{a_1^\vee} \otimes c_2} a_1^\vee \otimes (a \otimes a_2^\vee) \xrightarrow{\mathbf{a}^{-1}_{a_1^\vee,a,a_2^\vee}} (a_1^\vee \otimes a) \otimes a_2^\vee \xrightarrow{e_1 \otimes \mathrm{id}_{a_2^\vee}} a_2^\vee$$

similarly, we define $\beta : x_2^\vee \to x_1^\vee$. We claim that $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identity morphism so they're isomorphisms. Consider the following diagram

$$\begin{tikzcd}
a_1^\vee \arrow[r, "\mathrm{id}\otimes c_1"] \arrow[d, "\mathrm{id}\otimes c_2"'] & a_1^\vee \otimes a \otimes a_1^\vee \arrow[d, "\mathrm{id}\otimes c_2\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
a_1^\vee \otimes a \otimes a_2^\vee \arrow[r, "\mathrm{id}\otimes c_1"'] \arrow[d, "e_1\otimes\mathrm{id}"'] & a_2^\vee \otimes a \otimes a_1^\vee \otimes a \otimes a_1^\vee \arrow[r, "\mathrm{id}\otimes e_2\otimes\mathrm{id}"'] \arrow[d, "e_1\otimes\mathrm{id}"] & a_1^\vee \otimes a \otimes a_1^\vee \arrow[d, "e_1\otimes\mathrm{id}"] \\
a_2^\vee \arrow[r, "\mathrm{id}\otimes c_1"'] & a_2^\vee \otimes a \otimes a_1^\vee \arrow[r, "e_2\otimes\mathrm{id}"'] & a_1^\vee
\end{tikzcd}$$

We suppress the association in the diagram. Clearly, the three small squares commute and the triangle in the right upper corner commute because of axiom 2.1 by applying to $a_2^\vee$. Now, the top row is exactly equation 2.2, so it is $\mathrm{id}_{a_1^\vee}$. On the other hand, the composition of the bottom row is $\beta \circ \alpha$. Using the commutativity of the diagram, we can get $\beta \circ \alpha = \mathrm{id}_{a_1}^\vee$. Similarly, we have $\alpha \circ \beta = \mathrm{id}_{a_2^\vee}$. $\square$

Remark 2.3. In fact, the isomorphism α, β is the unique isomorphism between $a_1^\vee$ and $a_2^\vee$ that preserves evaluation and coevaluation map.

Remark 2.4. We can define the "dual object" in arbitrary monoidal category $\mathcal{C}$. However, in this case, we should distinguish between the "right dual" and "left dual". Because in this thesis, tensor category means symmetric monoidal category, there is no difference between right dual and left dual in our case as the dual object is unique. If you need to work with a more general setting, see [EGDO15] Ch 2, 2.10.

Example 2.5 (*Important Example!!!*)**.** Let k be a field, then we note that the category of finite dimensional vector space Vect_k is rigid, where the dual of a vector space V is exactly the dual vector space $V^\vee$, and the evaluation map is defined by

$$\mathrm{ev} : V \otimes_k V^\vee \to k$$

$$a \otimes f \mapsto f(a)$$

To construct the coevluation map, we first notice there is an isomorphism

$$f_{V,W} : V \otimes W^\vee \to \mathrm{Hom}(W, V) \tag{2.3}$$

$$(a \otimes f)(w) \mapsto f(w)a$$

In particular, we have $f_{V,V} : V \otimes V^\vee \to \mathrm{End}(V)$ is a isomorphism of vector space, and this helps us to construct

$$\mathrm{coev} : k \to V \otimes V^\vee \tag{2.4}$$

$$\mathrm{coev}(1) \mapsto f_{V,V}^{-1}(\mathrm{id}_V)$$

Both evaluation and coevaluation maps are very "natural"(it doesn't depend on the choice of the basis element), so we may expect both maps to exist for other $k-$linear categories. This leads to the next example.

Example 2.6. Let $\mathrm{Rep}(-)$ be the finite dimension representation category of finite group G or Lie algebra $\mathfrak{g}$ or finite dimension algebra A over k. This category is also rigid where each object X has a dual $\mathrm{Hom}(X, k)$ because we note the map 2.3 and the map 2.4 is still a well-defined isomorphism of $\mathrm{Rep}(-)$. Hence, we can construct the evaluation map and coevaluation map in the same way.

We observe all the examples we discuss are closed symmetric monoidal categories with

$$V \otimes W^\vee \simeq \mathrm{hom}(W, V)$$

In fact, this is not an conincident.

Lemma 2.7. *Let $\mathcal{C}$ be a rigid tensor category, then it is closed where the internal hom is defined as* $\hom(x, y) = y \otimes x^\vee$

Proof. We need to show

$$\mathrm{Hom}(x \otimes y, c) \simeq \mathrm{Hom}(x, y^\vee \otimes c)$$

Let

$$\rho : \mathrm{Hom}(x \otimes y, c) \to \mathrm{Hom}(x, y^\vee \otimes c)$$

$$f \mapsto (f \otimes \mathrm{id}_{y^\vee}) \circ (\mathrm{id}_x \otimes \mathrm{coev}_y)$$

$$\phi : \mathrm{Hom}(x, y^\vee \otimes c) \to \mathrm{Hom}(x \otimes y, c)$$

$$g \mapsto (\mathrm{ev}_y \otimes \mathrm{id}_c) \circ (g \otimes \mathrm{id}_y)$$

It is easy to check that f and g are mutually invertible, hence we get the wanted isomorphism. □

Remark 2.8. One can use a similar approach to show that $x^\vee \otimes -$ is also a left adjoint of $x \otimes -$. Therefore, both functors preserve colimit and limit by the property of adjunction.

On the other hand, if $\mathcal{C}$ is a closed tensor category, and x is an object of $\mathcal{C}$, we set $x^* = \hom(x, \mathbb{1})$. Using the tensor-hom adjunction 0.1, we have the unit and count map for the adjunction

$$\eta_{k,l} : l \to \hom(k, k \otimes l)$$

$$\epsilon_{k,l} : \hom(k, l) \otimes k \to l$$

There is a natural morphism

$$\gamma_{k,l} : k \otimes l^* \to \hom(l, k)$$

defined by the imagine of id_l of composition of the map

$$\mathrm{Hom}(l, l) \simeq \mathrm{Hom}(l \otimes \mathbb{1}, l) \xrightarrow{\mathrm{Hom}(\mathrm{id}_l \otimes \epsilon_{k,\mathbb{1}}, \mathrm{id}_l)} \mathrm{Hom}(k \otimes k^* \otimes l, l) \simeq \mathrm{Hom}(k^* \otimes l, \hom(k, l))$$

Lemma 2.9. *If the natural morphism $\Gamma_{l,k}$ is an isomorphism for all object $l, k \in \mathcal{C}$, then x^* is a dual of x and therefore $\mathcal{C}$ is rigid.*

Remark 2.10. Because the dual object is unique up to isomorphism, combining with Lemma 2.7, we can always assume the dual of x is on the form $\hom(x, \mathbb{1})$

Proof. See Proposition 2.1 of [ST23]. □

Combine the previous two lemmas, we get

Proposition 2.11. *Let $\mathcal{C}$ be a closed tensor triangulated category. It is rigid if and only if the natural map $\gamma_{k,l} : k \otimes l^* \to \hom(l, k)$ is an isomorphism.*

Proof. If $\mathcal{C}$ is a rigid closed tensor triangulated category. Lemma 2.7 implies $a^\vee \otimes -$ is another right adjoint of $a \otimes -$. As the adjoint functor is unique up to isomorphism, $a^\vee \otimes - \simeq \hom(a, -)$, and so $f_{k,l}$ is isomorphism. Another direction is easy. □

Remark 2.12. Because of Proposition 2.9, sometimes people use the condition "$\gamma_{k,l} : k \otimes l^\vee \to \hom(l, k)$ is an isomorphism for all object k, l" as the definition of being rigid for a closed tensor category.

Remark 2.13. For a rigid tensor category, using the fact from remark 2.10, we can easily prove $(x^\vee)^\vee \simeq x$.

Back to the Balmer spectrum, let $\mathcal{K}$ be a small tensor triangulated category, we will prove the following theorem which we are seeking.

Proposition 2.14. *Let $\mathcal{K}$ be a rigid tensor triangulated category, then every tensor thick ideal is radical.*

The proof of this proposition will based on the following lemma.

Lemma 2.15. *The following statements are equivalents:*

(i) Any tensor thick ideal of $\mathcal{K}$ is radical.

(ii) $a \in \langle a \otimes a \rangle$ for all object $a \in \mathcal{K}$.

Proof. $(i) \Rightarrow (ii)$ is clear because $\langle a \otimes a \rangle$ will be a radical ideal with assumption. Conversely, suppose (ii) holds, and let $\mathcal{J}$ be a tensor thick ideal of $\mathcal{K}$. We must show $a^{\otimes n} \in \mathcal{J} \Rightarrow a \in \mathcal{J}$. To do this, let's precede an induction on n:

$n = 1$: This is trivial

$n - 1 \Rightarrow n$: Suppose $a^{\otimes n} \in \mathcal{J}$, then $a^{\otimes n-1} \otimes a^{\otimes n-1} = a^{\otimes n} \otimes a^{\otimes n-2} \in \mathcal{J}$. Hence, use the (ii), we get $a^{\otimes n-1} \in \langle a^{\otimes n-1} \otimes a^{\otimes n-1} \rangle \subset \mathcal{J}$. By induction, this means $a \in \mathcal{J}$.

□

proof of Proposition 2.14. Let $a \in \mathcal{K}$ be an object, we claim a is a direct summand of $a \otimes a^\vee \otimes a \in \langle a \otimes a \rangle$, so we can apply the previous lemma. Indeed, this is just because of the axiom 2.1

$$a \xrightarrow{\mathrm{id}_{a^\vee} \otimes \mathrm{coev}_a} a^\vee \otimes a \otimes a^\vee \xrightarrow{\mathrm{ev}_a \otimes \mathrm{id}_a} a$$

We ignore the association because of remark 0.2. As the composition is the identity, the map $\mathrm{ev}_a \otimes \mathrm{id}_a$ is the section of $\mathrm{id}_{a^\vee} \otimes \mathrm{coev}_a$. Therefore, we conclude the result. □

Before we end this section, let's look at some examples of the rigid tensor triangulated category.

Example 2.16. Let R be a commutative ring, then $\mathscr{D}^{\mathrm{perf}}(R)$ the derived category of perfect chain complex, i.e the complexes in the derived category that is quasi-isomorphic to a bounded complex of a finitely generated projective module, is a closed tensor triangulated category with tensor product $\otimes^{\mathbb{L}}_R$ and unit R. Moreover, we can easily check this is a rigid category by checking the natural map in Lemma 2.9 is isomorphism.

Example 2.17. The stable category defined in Example 0.3 is also a rigid tensor triangulated category

Example 2.18. Similarly, the stable homotopy category Sp^{ω} given in Example 0.3 is also rigid, but this is not an easy fact to show.

2.2 Rigidly-Compactly Generated Tensor Triangulated Category

Convention 2.19. From now on, we will only let $\mathcal{K}$ be a tensor triangulated category(**not necessarily small**). Any triangulated category $\mathcal{C}$ will admit all small coproducts.

In Remark 2.12, we indicate that a closed tensor category $\mathcal{K}$ is rigid if and only if the natural map Γ is an isomorphism for all objects in $\mathcal{K}$. However, if $\mathcal{K}$ itself is not a rigid tensor category, the isomorphism still possibly holds for a "small family object" of $\mathcal{K}$.

Definition 2.20. Let $\mathcal{C}$ be a triangulated category (**not necessarily small**). We say an object $a \in \mathcal{C}$ is *compact* if $\mathrm{Hom}(a, -)$ preserves arbitrary coproduct, i.e if for any family of the object $\{x_i \mid i \in I\}$, the natural morphism

$$\mathrm{Hom}(a, \coprod_{i \in I} x_i) \to \bigoplus_{i \in I} \mathrm{Hom}(a, x_i)$$

We denote by $\mathcal{C}^c$ the full subcategory of all compact objects of $\mathcal{C}$. A triangulated category $\mathcal{C}$ is called *compactly generated* if there is a set T of compacts of $\mathcal{C}$ such that for any $x \in \mathcal{C}$

$$x = 0 \Leftrightarrow \mathrm{Hom}(g, \Sigma^i x) = 0, \forall g \in T, \forall i \in \mathbb{Z}$$

The objects in T are called *compact generator*, and T is called *a set of compact generator*

Definition 2.21. Let $\mathcal{C}$ be a compactly generated triangulated category, a subset T of compact objects of $\mathcal{C}$ is called *generating set* if

(1) $x = 0 \Leftrightarrow \mathrm{Hom}(g, \Sigma^i x) = 0, \forall g \in T, \forall i \in \mathbb{Z}$

(2) T is closed under suspension

Remark 2.22. Because finite direct sum commute with $\mathrm{Hom}(-, -)$, $\mathcal{C}^c$ is a thick subcategory of $\mathcal{C}$.

Remark 2.23. A compactly generated triangulated category can have more than one generating set. Indeed, let T be a set of compact generators, $\bigcup_{i \in \mathbb{Z}} \Sigma^i T$ is a generating set; the condition (1) holds as it holds for T. Also, because the compact object is closed under suspension, so $\Sigma^i T$ is a set of compact objects.

Advertisement 2.24 (*"Generator of a Triangulated category"*)**.** The "compact generator" of a triangulated category gives one way to talk about the "generator" of a triangulated category. There is another way to generate a category from a family of objects $\mathcal{S}$ via suspension. To do this, we introduce a new operation: Let $\mathcal{L}$ and $\mathcal{J}$ be two families of the object in $\mathcal{C}$, $\mathcal{J} * \mathcal{I}$ will consist of object $x \in \mathcal{C}$ such that there is a triangle

$$a \to x \to b \to \Sigma a; a \in \mathcal{J}, b \in \mathcal{I}$$

this operation is associative because of the Octahedron axiom. Using this new operation, we generate a triangulated subcategory through the following step:
step 1: Let $\mathcal{S}_0 = \mathcal{S}$
step 2: Let $\mathcal{S}_1$ consist of $\Sigma^n a, \forall a \in \mathcal{S}, \forall n \in \mathbb{Z}$
step 3: Let $\mathcal{S}_r = \mathcal{S}_1 * \cdots * \mathcal{S}_1$ be the product of r copies $\mathcal{S}_1$.
step 4: Let $\mathcal{T} = \bigcup_{i=0}^{i=\infty} \mathcal{S}_i$
Then it is easy to see $\mathcal{T}$ is a triangulated subcategory of $\mathcal{C}$. Moreover, it is the **smallest** one contains $\mathcal{S}$ in $\mathcal{C}$!!!

Now the $\mathcal{S}$ could be thought of as the "generator" of $\mathcal{T}$. Another surprising fact is some triangulated category could be generated within the "finite step": Let $\overline{\mathcal{S}}$ be the smallest subcategory containing $\mathcal{S}$ and closed under finite direct sum, direct summand, and suspension, then it is possible $\mathcal{K} = (\overline{\mathcal{S}})_i = \overline{\mathcal{S}} * \cdots * \overline{\mathcal{S}}, i < \infty$.

The smallest $n \in \mathbb{Z}_+$ such that there is an object $c \in \mathcal{C}$ with $(\overline{\{c\}})_n = \mathcal{C}$ is called the **Rouquier Dimension** of $\mathcal{C}$. This very good invariant encodes many homological properties of $\mathcal{C}$. If you like this idea, CLICK HERE [Rouq08] TO START!!!

Example 2.25. (1) The derived category $\mathscr{D}(R)$ of a commutative ring R is compactely generated and R is a compact generator. The $\mathscr{D}(R)^c$ is exactly $\mathscr{D}^{\mathrm{perf}}(R)$

(2) the stable homotopy category of Spectra Sp is a compactly generated category and $\mathbb{S}$ is a compact generator. The category of compact object is the finite spetra Sp^{ω}

(3) The stable category $\underline{\mathrm{Mod}}kG$ of a group with order divides the character of k is a compact generated triangulated category.

(4) The category $\mathrm{DPerm}(G, R)$ is also compactly generated, and it has been shown in [BM23], Corollary 3.10 that $\mathrm{DPerm}(G, R)^c$ is isomorphic to thick closure of finitely generated permutation module.

All the examples above are also the tensor triangulated category, especially the units are compact objects. This leads to the following definition.

Definition 2.26. A *compactly generated tensor triangulated category* is a tensor triangulated category $(\mathcal{K}, \otimes, \mathbb{1}, \Sigma)$ such that $\mathcal{K}$ is a compactly generated triangulated category. Additionally, we require $\otimes$ to preserve arbitrary coproduct in each variable and $\mathcal{K}^c$ form a tensor subcategory. In particular, $\mathbb{1}$ should be a compact object.

The assumption of "$\otimes$ preserves arbitrary coproduct" is important in the following sense. First A. Neemann in [AN96] Theorem 3.1 shows the following representable theorem

Theorem 2.27 (*Brown Representable Theorem*)**.** *Let $\mathscr{H}$ be the cohomological functor of a compactly generated triangulated category $\mathcal{C}$. If for any small coproduct $\coprod_{i\in I} x_i$ in $\mathcal{C}$, the natural morphism $\mathscr{H}(\coprod_{i\in I} x_i) \to \prod_{i\in I} \mathscr{H}(x_i)$ is an isomorphism. Then $\mathscr{H}$ is representable.*

Now consider the cohomological functor $\mathrm{Hom}(x \otimes -, y)$, because $- \otimes -$ preserve coproduct, and $\mathrm{Hom}(-, y)$ takes coproduct to product. So $\mathrm{Hom}(x \otimes -, y)$ is a cohomological functor that satisfies the condition in theorem 2.27. Therefore, there is a object $Z_{x,y} \in \mathcal{K}$ such that there is an natural morphism f

$$\mathrm{Hom}(a \otimes c, b) \xrightarrow{f} \mathrm{Hom}(c, Z_{a,b})$$

is an isomorphism for any object $c \in \mathcal{K}$. If we set $\mathrm{hom}(a, b) = Z_{a,b}$, then we notice $\mathrm{hom}(a, b)$ is a right adjoint functor of $a \otimes c$ for all object $c \in \mathcal{K}$. This fact implies the $\mathcal{K}$ **is automatically a closed tensor triangulated category**. In addition, the argument above is a special case of the adjointness theorem in Corollary 2.44.

Definition 2.28. A *rigidly-compactly generated tensor triangulated category* $\mathcal{K}$ is a compactly generated tensor triangulated category such that the full subcategory $\mathcal{K}^c$ is *rigid*. More explicitly, $\mathcal{K}^c$ is a tensor subcategory and closed under internal hom(which exists via the previous discussion).

In the remained section, our goal is to prove the following theorem

Theorem 2.29. *Let $\mathcal{K}$ be rigidly-compactly generated tensor triangulated category, then for $t \in \mathcal{K}^c$, the isomorphism of endofunctor of $\mathcal{K}^c$*

$$t^\vee \otimes - \xrightarrow{\gamma} \mathrm{hom}(t, -)$$

extend to an isomorphism of the endofunctor of $\mathcal{K}$, that is for any $x \in \mathcal{K}$, $t^\vee \otimes x \simeq \mathrm{hom}(x, t)$. In particular, $\mathrm{hom}(t, -)$ preserve coproducts, and $t \otimes -$ is left and right adjoin of $t^\vee \otimes -$ if we view them as endofunctor of $\mathcal{K}$

To prove this theorem, we need to introduce an important notation.

Definition 2.30. A triangulated subcategory $\mathcal{L}$ of $\mathcal{K}$ is called *localizing* if it is closed under the arbitrary coproduct of $\mathcal{K}$. Dually, it is *cololizing* if it is closed under arbitrary products in $\mathcal{K}$. A localizing subcategory $\mathcal{L}$ is called *localizing tensor ideal*, if it is also *tensor ideal* (see Def 0.1)

If $\mathcal{S}$ is a family of objects in $\mathcal{K}$, we denote $\mathrm{Loc}(\mathcal{S})$(resp. $\mathrm{Loc}^{\otimes}(\mathcal{S})$) as the smallest localizing category (resp. the smallest localizing tensor ideal) contains $\mathcal{S}$. We shall denote the family of all localizing subcategories of $\mathcal{K}$ by $\mathrm{LOC}(\mathcal{K})$ and all localizing tensor ideal by $\mathrm{LOC}^{\otimes}(\mathcal{K})$.

Lemma 2.31. *Let S be a set of object in $\mathcal{K}$, then* $\mathrm{Loc}(S) = \mathrm{Loc}(\bigcup_{i\in\mathbb{Z}} \Sigma^i S)$

Proof. Clearly, $S \subset \bigcup_{i\in\mathbb{Z}} \Sigma^i S$, so $\mathrm{Loc}(\bigcup_{i\in\mathbb{Z}} \Sigma^i S)$ is localizing subcategory that contains S. Hence, we get $\mathrm{Loc}(S) \subset \mathrm{Loc}(\bigcup_{i\in\mathbb{Z}} \Sigma^i S)$. On the other hand, because $\mathrm{Loc}(S)$ is closed under suspension, $\bigcup_{i\in\mathbb{Z}} \Sigma^i S \subset \mathrm{Loc}(S)$, which implies $\mathrm{Loc}(S) \supset \mathrm{Loc}(\bigcup_{i\in\mathbb{Z}} \Sigma^i S)$ □

Remark 2.32. This lemma allows us to always assume S is "a set of objects that closed under suspension" when we consider $\mathrm{Loc}(S)$. In particular, if S is a set of compact generators, because of the Remark 2.23, we will think $\mathrm{Loc}(S)$ as the smallest localizing subcategory of a generating set S. This fact will be used implicitly in the remained context.

The following proposition is cited directly without proof from [AN01]

Proposition 2.33 (Proposition 1.68 of [AN01])**.** *Let $\mathcal{C}$ be a triangulated category that admits all coproducts. Let x be an object in $\mathcal{C}$, then every idempotent $e : x \to x$; that is $e^2 = e$, will splits in $\mathcal{C}$, i.e. there are morphism f, g*

$$f : x \xrightarrow{f} y \xrightarrow{g} x$$

with $gf = e$ and $fg = \mathrm{id}_x$

Remark 2.34 (*ZFC's Problem*)**.** In the original statement of [AN01], there is a more specific requirement for "admits all coproduct"; it requires all coproduct $\coprod_\Lambda x_\lambda$, with the cardinal of Λ "less" than $\aleph_1$ exists. Because the discussion will become much much much more complicated if we are involved with set-theoretic problems (in fact, it is not very easy to explain how large the $\aleph_1$ number is, where it is relevant to the famous Hilbert 1st question: *Continuum hypothesis*), we will ignore all the set-theoretic problems whenever we face it (this will frequently happen when we use the result from [AN01]).

Lemma 2.35. *Any localizing subcategory $\mathcal{T}$ of a triangulated category $\mathcal{C}$ is thick.*

Proof. Because the localizing subcategory admits all coproduct, this result is immediately by Proposition 2.33. □

Lemma 2.36. *For a set of objects $\mathcal{S}$ of $\mathcal{K}$,* $\mathrm{Loc}(\mathcal{S}) = \mathrm{Loc}(\langle\mathcal{S}\rangle)$

Proof. This is because $\mathrm{Loc}(\mathcal{S})$ is also thick by the previous lemma. □

Therefore, we will also implicitly assume $\mathcal{S}$ is a thick category when we discuss $\mathrm{Loc}(\mathcal{S})$. The following lemma may not be surprising.

Lemma 2.37. *If* $\mathrm{Loc}(\mathbb{1}) = \mathcal{K}$, *then* $\mathrm{LOC}(\mathcal{K}) = \mathrm{LOC}^{\otimes}(\mathcal{K})$

Proof. The idea is same as Lemma 1.1: Let $\mathcal{L}$ be localizing subcategory and $a \in \mathcal{L}$, then we check

$$\mathcal{S} = \{x \in \mathcal{K} \mid a \otimes x \in \mathcal{L}\}$$

is a localizing subcategory (notice $\otimes$ preserve the coproduct!). Especially, because $\mathbb{1} \in \mathcal{S}$, we must have $\mathcal{S} = \mathcal{K}$. We left the details to the reader. □

You may feel the requirement $\mathrm{Loc}(\mathbb{1}) = \mathcal{K}$ is hard to satisfy. Nevertheless, this is in fact common for us through the following lemma

Lemma 2.38 (*Important Lemma*!!!)**.** *Let T be a set of compact objects of $\mathcal{K}$.* $\mathrm{Loc}(T) = \mathcal{K}$ *if and only if the object T is a set of compact generator.*

Hence, (1) and (2) of example 2.25 will satisfy $\mathrm{Loc}(\mathbb{1}) = \mathcal{K}$.

Sketch of proof. Suppose $\mathrm{Loc}(T) = \mathcal{K}$, and $x \in \mathcal{K}$ such that

$$\mathrm{Hom}(t, \Sigma^i x) = 0; \forall i \in \mathbb{Z}, t \in T$$

We want to show $x = 0$. To do this, Let's consider the full subcategory

$$\mathcal{S} = \{y \in \mathcal{K} \mid \mathrm{Hom}(y, \Sigma^i x) = 0, \forall i \in \mathbb{Z}\}$$

We check $\mathcal{S}$ is indeed a localizing subcategory. First it is clearly $0 \in \mathcal{S}$, also, as $\mathrm{Hom}(-,-)$ commute with coproduct, so $\mathcal{S}$ should close under coproduct. Now, given a triangle

$$a \xrightarrow{f} b \xrightarrow{g} c \to \Sigma a$$

and suppose $a, b \in \mathcal{S}$. We claim that $c \in \mathcal{S}$. Applying $\mathrm{Hom}(-, \Sigma^i x)$ for arbitrary integer i to get the diagram between long exact sequence

$$\begin{tikzcd}
\mathrm{Hom}(a, \Sigma^i x) \arrow[r, "f^*"] \arrow[d] & \mathrm{Hom}(b, \Sigma^i x) \arrow[r, "g^*"] \arrow[d] & \mathrm{Hom}(c, \Sigma^i x) \arrow[r, "\Sigma f^*"] \arrow[d] & \mathrm{Hom}(\Sigma a, \Sigma^i x) \arrow[r, "\Sigma g^*"] \arrow[d, "\mathrm{id}"] & \mathrm{Hom}(\Sigma b, \Sigma^i x) \arrow[d, "\mathrm{id}"] \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathrm{Hom}(\Sigma a, \Sigma^i x) \arrow[r, "\Sigma g^*"] & \mathrm{Hom}(\Sigma d, \Sigma^i x)
\end{tikzcd}$$

By our assumption, $a, b \in \mathcal{S}$, which means $f^*, g^* = 0$, and so $\Sigma f^* = 0, \Sigma g^* = 0$. Hence this diagram commute. By five Lemma, $\mathrm{Hom}(c, \Sigma^i x) \simeq 0, \forall i \in \mathbb{Z}$. So $c \in \mathcal{S}$. Use the fact that there is triangle $a \to 0 \to \Sigma a$ for any $a \in \mathcal{S}$, and the claim above, we get $\mathcal{S}$ closed under suspension.

In conclusion, we get $\mathcal{S}$ as a full triangulated subcategory closed under product, which means it is localizing. Especially, $T \subset \mathcal{S}$, so $\mathcal{S} = \mathcal{K}$ by assumption. Therefore, $\mathrm{Hom}(x, x) = 0 \Rightarrow x = 0$

The other direction will based on the following theorem

Theorem 2.39 (Theorem 3.1 of [AN96], assertation (1) and (2))**.** *Let x be an object of $\mathcal{K}$. By thinking $\mathrm{Hom}(-, x)$ as cohomological functor of $\mathrm{Loc}(T)$, there is $y \in \mathrm{Loc}(T)$ such that the natural transformation*

$$\mathrm{Hom}_{\mathcal{K}}(-, x) \to \mathrm{Hom}_{\mathrm{Loc}(T)}(-, y)$$

is an isomorphism over $\mathrm{Loc}(T)$

Now apply this proposition for an arbitrary $x \in \mathcal{K}$. By Yoneda Lemma, this implies there is a morphism $f : x \to y$ such that $\forall z \in \mathrm{Loc}(T)$ the induced natural transformation $f^* : \mathrm{Hom}(z, x) \to \mathrm{Hom}(z, y)$ is an isomorphism. Complete the f to the triangle

$$x \to y \to \mathrm{cone}(f)$$

and let s be an arbitrary object in $\mathrm{Loc}(\mathcal{S})$, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(s,x) \arrow[r, "f^*"] \arrow[d, "\mathrm{id}"] & \mathrm{Hom}(s,y) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Hom}(s,\mathrm{cone}(f)) \arrow[r] \arrow[d] & \mathrm{Hom}(s,\Sigma x) \arrow[r, "\Sigma f^*"] \arrow[d, "\mathrm{id}"] & \mathrm{Hom}(s,\Sigma y) \arrow[d, "\mathrm{id}"] \\
\mathrm{Hom}(s,x) \arrow[r, "f^*"] & \mathrm{Hom}(s,y) \arrow[r] & 0 \arrow[r] & \mathrm{Hom}(s,\Sigma x) \arrow[r, "\Sigma f^*"] & \mathrm{Hom}(s,\Sigma y)
\end{tikzcd}$$

The bottom row is an exact sequence because f^* and Σf^* are isomorphisms. Using the five lemma, we conclude the middle arrow should be an isomorphism, which means

$$\mathrm{Hom}(s, \Sigma^i \mathrm{cone}(f)) = 0, \forall s \in \mathrm{Loc}(T), \forall i \in \mathbb{Z}$$

. But T is the generating set. By definition, we must have $\mathrm{cone}(f) = 0$. As $\mathrm{Loc}(T)$ is a full triangulated subcategory, we get $\mathrm{Hom}(s, \mathrm{cone}(f)) = 0, \forall s \in \mathcal{K}$. Applying $\mathrm{Hom}_{\mathcal{K}}(s, -)$ to the triangle $x \to y \to \mathrm{cone}(f)$, we get $\mathrm{Hom}_{\mathcal{K}}(-, x) \simeq \mathrm{Hom}_{\mathcal{K}}(-, y)$, so the Yoneda Lemma implies $x \simeq y \Rightarrow x \in \mathrm{Loc}(T)$. □

Proof of Theorem 2.29. Using the natural map Γ, we get a natural transformation $\mathrm{hom}(t, -) \to t^\vee \otimes -$ of an endofunctor $\mathcal{K} \to \mathcal{K}$. We define a new subcategory $\mathcal{L}$ as follow:

$$\mathcal{L} = \{x \in \mathcal{K} \mid \gamma_x : t^\vee \otimes x \to \mathrm{hom}(t, x) \text{ is an isomorphism}\}$$

As both $t^\vee \otimes -$ and $\mathrm{hom}(t, -)$ preserve suspension and coproduct(Remark 2.8), and γ is compatible with suspension and coproduct, we have $\mathcal{L}$ is closed under suspension and coproduct. Given a triangle

$$x \to y \to z \to \Sigma x$$

with $x, z \in \mathcal{L}$. The naturality of γ gives the following commutative diagram

$$\begin{tikzcd}
t^\vee \otimes x \arrow[r] \arrow[d, "\gamma_x"] & t^\vee \otimes y \arrow[r] \arrow[d, "\gamma_y"] & t^\vee \otimes z \arrow[r] \arrow[d, "\gamma_z"] & t^\vee \otimes \Sigma x \arrow[d, "\gamma_{\Sigma x}"] \\
\mathrm{hom}(t,x) \arrow[r] & \mathrm{hom}(t,y) \arrow[r] & \mathrm{hom}(t,z) \arrow[r] & \mathrm{hom}(t,\Sigma x)
\end{tikzcd}$$

By our assumption, γ_x, γ_z are isomorphisms, by the property morphism between distinguished triangles, we get γ_y isomorphism. Hence we get $\in \mathcal{L}$ and $\mathcal{L}$ is a localising subcategory contains $\mathcal{K}^c$(by the rigidity of $\mathcal{K}^c$). Using the Lemma 2.38, we conclude the $\mathcal{L} = \mathcal{K}$, which proves the first assertion. The remained part is immediate. □

2.3 Localization and Smashing Sequence

The terminology of "Localizing subcategory" is not an imaginative name! They exactly correspond to the localization of a category. More precisely, they will be the kernel of localizing functors (see [HK10] section 2 for an introduction if you don't what it means), and hence we can construct a Verdier localization. To elaborate more details, we need the idea of localization sequence.

Definition 2.40. Let $\mathcal{C}$ be a compactly generated triangulated category. A *localization sequence* is a diagram

$$\mathcal{L} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{C} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{T}$$

where $(i_*, i^!)$ and (j^*, j_*) are two adjoint pairs. Both i_* and j_* are fully faithful so embed $\mathcal{L}$ and $\mathcal{T}$ as a localizing and colocalizing subcategory respectively. Furthermore, we have the equality

$$(i_*\mathcal{L})^\perp = j_*\mathcal{T} \text{ and } {}^\perp(j_*\mathcal{T}) = i_*\mathcal{L}$$

where=

$$(i_*\mathcal{L})^\perp = \{y \in \mathcal{K} \mid \mathrm{Hom}_{\mathcal{K}}(i_*l, y) = 0, \forall l \in \mathcal{L}\}$$

and

$${}^\perp(j_*\mathcal{T}) = \{x \in \mathcal{K} \mid \mathrm{Hom}(x, j_*t) = 0, \forall t \in \mathcal{T}\}$$

We also call $i_*i^!$ the *acyclization functor* and j_*j^* the *localization functor* of this localization sequence.

We cite the following important proposition of localization sequence here, and suggest the reader to refer [BF11] theorem 2.6 for a proof

Proposition 2.41. *Suppose we have the localization sequence defined above*

(1) The composition of functor j^*i_* and $i^!j_*$ are 0. Moreover, the kernel of j^* is exactly $\mathcal{L}$.

(2) The composition

$$\mathcal{T} \xrightarrow{j_*} \mathcal{C} \to \mathcal{C}/\mathcal{L}$$

is an equivalence. In particular, the Verdier quotient $\mathcal{C}/\mathcal{L}$ is locally small and the canonical projective $\mathcal{C} \to \mathcal{C}/\mathcal{T}$ has a right adjoint.

(3) For every $x \in \mathcal{C}$, there is a triangle

$$i_*i^!x \to x \to j_*j^*x \to \Sigma i_*i^!x$$

which is unique in the sense that if there is another triangle

$$x' \to x \to x'' \to \Sigma x'$$

with $x' \in \mathcal{L}$ and $x'' \in \mathcal{T}$. There are unique isomorphisms such that $i_*i^!x \simeq x'$ and $j_*j^*x \simeq x''$

(4) the localization sequence is completely determined by either of the pairs of adjoint functor $(i_*, i^!), (j_*, j^*)$

Remark 2.42. In particular, we note that if $x \in \mathcal{L}$, because the triangle

$$x \xrightarrow{\mathrm{id}} x \to 0$$

satisfy $x \in \mathcal{L}, 0 \in \mathcal{T}$, so by the uniqueness of (3), $i_*i^!x \simeq x$. Similarly, for $y \in \mathcal{T}$, one has

$$0 = \Sigma^{-1}0 \to y \xrightarrow{\mathrm{id}} y$$

is a triangle and so $j_*j^*y \simeq y$.

Next, we show that for any localization subcategory of $\mathcal{C}$, there exists a localization sequence corresponding to it.

Theorem 2.43. *Let $\mathcal{C}$ be a compactly generated triangulated subcategory, let $L \subset \mathcal{C}$ be a set of objects, and set $\mathcal{L} = \mathrm{Loc}(L)$. Then the inclusion $i_* : \mathcal{L} \to \mathcal{C}$ admits a right adjoint $i^!$ that fit into the following localization sequence*

$$\mathcal{L} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{C} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{C}/\mathcal{L}$$

The proof of this theorem is relatively technical so we only sketch the idea of the proof. The following proposition plays a key role in the proof.

Corollary 2.44 ([AN96] Theorem 4.1)**.** *Let $\mathcal{C}$ be a compactly generated triangulated category, and $\mathcal{D}$ be an arbitrary triangulated category. Suppose there is a triangulated functor $F : \mathcal{C} \to \mathcal{D}$ such that for any coproduct $\coprod_{\lambda\in\Lambda} s_\lambda$, the natural map*

$$F(s_\lambda) \to F(\coprod_{\lambda\in\Lambda} s_\lambda)$$

makes $F(\coprod_{\lambda\in\Lambda} s_\lambda)$ be a coproduct of $\mathcal{D}$, then F has a right adjoint functor $G : \mathcal{D} \to \mathcal{C}$.

Proof. Let $X \in \mathcal{D}$ be an object, consider the functor

$$\mathscr{H}(s) = \mathrm{Hom}_{\mathcal{C}}(F(s), X)$$

This is a cohomological functor, and it takes coproduct to the product:

$$\mathrm{Hom}(F(\coprod_{\lambda\in\Lambda} s_i), X) = \mathrm{Hom}(\coprod_{\lambda\in\Lambda} F(s_\lambda), X) \text{ (by assumption)}$$

$$= \prod_{\lambda\in\Lambda} \mathrm{Hom}(F(s_\lambda), X) \text{(by property of Hom)}$$

Hence, by the Theorem 2.27, $\mathscr{H}$ is representable; there is an object $Y_s \in \mathcal{D}$ such that

$$\mathrm{Hom}(F(s), X) \simeq \mathrm{Hom}(s, Y_s)$$

set $G : s \mapsto Y_s$, we get the wanted right adjoint functor G. □

Sketch of Proof of Theorem 2.43. The existence of adjoint functor is now the same as to show both i_* and j^* preserve coproduct. This is clearly for i_*, and for j^*, this is confirmed by [AN01], Lemma 3.2.10. The right adjoint j_* is then a *Bausfield localization functor*. It is fully faithful embedded because of [AN01] Lemma 9.1.7 which says that the unit $\mathrm{id} \to j_*j^*$ is an isomorphism.

By [AN01], Theorem 9.1.16, the existence of j_* implies $(i_*\mathcal{L})^\perp$ is isomorphic to $\mathcal{C}/\mathcal{L}$ under the restriction of j^*. Hence, $j_*(\mathcal{C}/\mathcal{L}) = (i_*\mathcal{L})^\perp$. Also, we note that $(i_*\mathcal{L})^\perp$ closed under product so $\mathcal{C}/\mathcal{L}$ is colocalizing. To see ${}^\perp(j_*\mathcal{C}/\mathcal{L}) = i_*\mathcal{L}$, we note that the condition

$$\mathrm{Hom}(x, j_*y) = 0, \forall y \in \mathcal{C}/\mathcal{L}$$

is equivalent to

$$\mathrm{Hom}(j^*x, y) = 0, \forall y \in \mathcal{C}/\mathcal{L}$$

. This happens if and only if $j^*x = 0$ as $\mathrm{End}(j^*x) = 0$, and because $i_*\mathcal{L}$ is exactly the kernel of j^*, we get our conclusion.

□

By definition, both i_*, j^* preserve coproduct because they're left adjoint. However, in practice, we can also see many examples that the right adjoint $i^!, j_*$ preserves coproduct. If such a condition holds, the localization sequence will be updated to a powerful tool called *smahsing sequence*.

Definition 2.45. A localization sequence

$$\begin{tikzcd} \mathcal{L} \arrow[r, shift left, "i_*"] & \mathcal{C} \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^*"] & \mathcal{T} \arrow[l, shift left, "j_*"] \end{tikzcd}$$

is called *smashing* if $i^!$ (or equivalently j_*) preserve the coproducts. In this case, we called $\mathcal{L}$ a *smashing subcategory* of $\mathcal{C}$, and if $\mathcal{L}$ is also tensor ideal, we call it *smashing tensor-ideal*.

The following proposition provides a standard machinery to construct a smashing subcategory.

Proposition 2.46. *Let $\mathcal{C}$ be a compactly generated triangulated category and $L \subset \mathcal{C}^c$ be a set of compact objects. Then $\mathcal{L} = \mathrm{Loc}(L)$ is a smashing subcategory, i.e. the inclusion $i_* : \mathcal{L} \to \mathcal{C}$ admits a coproduct preserve right adjoint $i^!$ that fixes to a smashing sequence.*

Proof. The existence of such $i^!$ is guaranteed by Theorem 2.43. The only thing we need to do is to show it preserves coproduct. We need the following theorem.

Theorem 2.47 ([AN96], Theorem 5.1)**.** *Let $\mathcal{C}$ be a compactly generated category triangulated category with S a generating set and $\mathcal{D}$ be a triangulated category. If $F : \mathcal{C} \to \mathcal{D}$ is a triangulated functor that preserves arbitrary coproduct. Then its right adjoint G (exists by Corollary 2.44) preserve coproduct if and only if $F(s)$ is compact object $\forall s \in S$*

Proof. Suppose G preserves coproduct, and $s \in S$, then

$$\begin{aligned} \mathrm{Hom}(F(s), \coprod_{\lambda\in\Lambda} x_\lambda) &\simeq \mathrm{Hom}(s, G(\coprod_{\lambda\in\Lambda} x_\lambda) \\ &\simeq \mathrm{Hom}(s, \coprod_{\lambda\in\Lambda} G(x_\lambda)) \\ &\simeq \coprod_{\lambda\in\Lambda} \mathrm{Hom}(s, G(x_\lambda)) \\ &\simeq \coprod_{\lambda\in\Lambda} \mathrm{Hom}(F(s), x_\lambda) \end{aligned}$$

So, $F(s)$ is compact object.

Conversely, suppose $F(s)$ is compact for all $s \in S$, let $\coprod_{\lambda\in\Lambda} x_\lambda$ be an coproduct, then for any $s \in S$, we have

$$\begin{aligned} \mathrm{Hom}(s, G(\coprod_{\lambda\in\Lambda} x_\lambda)) &\simeq \mathrm{Hom}(F(s), \coprod_{\lambda\in\Lambda} x_\lambda) \\ &\simeq \coprod_{\lambda\in\Lambda} \mathrm{Hom}(F(s), x_\lambda) \text{ (as } s \text{ is compact)} \\ &\simeq \coprod_{\lambda\in\Lambda} \mathrm{Hom}(s, G(x_\lambda) \end{aligned}$$

In other words, the natural map

$$\phi : \coprod_{\lambda\in\Lambda} G(x_\lambda) \to G(\coprod_{\lambda\in\Lambda} x_\lambda)$$

induce an natural transformation

$$\phi^* : \mathrm{Hom}(-\coprod_{\lambda\in\Lambda} G(x_\lambda)) \to \mathrm{Hom}(-G(\coprod_{\lambda\in\Lambda} x_\lambda))$$

such that $\phi^*(s)$ is an isomorphism for all $s \in S$. But then, apply $\mathrm{Hom}(s, -)$ to the triangle

$$\coprod_{\lambda\in\Lambda} G(x_\lambda) \to G(\coprod_{\lambda\in\Lambda} x_\lambda) \to \mathrm{cone}(\phi^*) \to \Sigma \coprod_{\lambda\in\Lambda} G(x_\lambda)$$

we get $\mathrm{Hom}(s, \mathrm{cone}(\phi)) = 0, \forall s \in S$ (use the same trick in the "converse" part of the proof of Lemma 2.38). Because S is the set of compact generators, so $\mathrm{cone}(\phi) = 0$. Hence ϕ is an isomorphism. □

To finish the proof, we need to show that $\mathrm{Loc}(L)$ is compactly generated, First, it is easy to see $L \subset \mathrm{Loc}(L)^c$, but then Lemma 2.38, will then imply L is a set of the compact generator.

Lemma 2.48. $\mathrm{Loc}(L)^c = L$

Note that we actually implicitly assume L is a thick subcategory by Lemma 2.36 so this lemma makes sense (remember that compact objects of a category are a thick subcategory).

Proof. It is easy to see $L \subset \mathrm{Loc}(L)^c$. The proof of the other direction is a bit out of our scope, and the reader who's interested in this can read [AN92] Lemma 2.2 to see proof. □

In conclusion, we get L is the generating set of $\mathrm{Loc}(L)$, and clearly, the inclusion functor i_* sends any object in L to a compact object in $\mathcal{C}$. Hence, we get $i^!$ preserves coproduct □

2.4 Tenor Idempotent and New Support Data

Let $\mathcal{K}$ be a rigid-compactly generated tensor triangulated category, by the general result of the Balmer spectrum, we can easily construct a support data in $\mathrm{Spc}(\mathcal{K}^c)$ and classify all the thick subcategories of $\mathcal{K}^c$. We now use the $\mathrm{Spc}(\mathcal{K}^c)$ to construct a "generalized support" for the arbitrary object in $\mathcal{K}$. To simplify our discussion, we will assume in addition that $\mathrm{Spc}(\mathcal{K}^c)$ is a *noetherian* space. Recall this implies every open subset is noetherian, and hence, the Thomason subsets are exactly all the specialization closed subset. The most significant example to keep in mind is $\mathscr{D}(R)$ for a noetherian ring R.

The main idea of the construction of the new support data is to introduce a new object called *Rickard idempotent*. We first show how to produce it via the smashing sequence.

Remark 2.49. The assumption "notherian" on $\mathrm{Spc}(\mathcal{K}^c)$ could be dropped, but it will make the discussion more complicated. One can see in [BF11] how to generalize the discussion below to non-noetherian assumption, and [GS14] to see we can always recover the support of compact objects via the new support information.

Lemma 2.50. *Let $\mathcal{K}$ be a rigidly-compactly generated tensor triangulated category, $S \subset \mathcal{C}^c$ be a set of compact objects, then*

$$\mathrm{Loc}^{\otimes}(S) = \mathrm{Loc}(\mathrm{Thick}^{\otimes}(S))$$

Proof. Because of Lemma 2.35, $\mathrm{Loc}^{\otimes}(S)$ is a tenor-thick ideal contains $\mathcal{S}$, so $\mathrm{Thick}^{\otimes}(S) \subset \mathrm{Loc}^{\otimes}(S)$. As $\mathrm{Loc}^{\otimes}(S)$ is a localizing subcategory, we conclude $\mathrm{Loc}(\mathrm{Thick}^{\otimes}(S)) \subset \mathrm{Loc}^{\otimes}(S)$.

Conversely, we want to show $\mathcal{T} = \mathrm{Loc}(\mathrm{Thick}^{\otimes}(S))$ is a tensor ideal. Consider the following full subcategory

$$\mathcal{D} = \{x \in \mathcal{T} \mid x \otimes s \in \mathcal{T}, \forall s \in \mathcal{K}\}$$

then we note:

1. It is closed under coproduct: Let $\coprod_{\Lambda} x_\lambda$ be abitrary coproduct with $x_\lambda \in \mathcal{D}$, then $s \otimes \coprod_{\lambda \in \Lambda} x_\lambda \simeq \coprod_{\lambda \in \Lambda}(s \otimes x_\lambda)$ (by definition 2.26, we require $s \otimes -$ preserve coproduct). Because $x_\lambda \in \mathcal{D}$ and $\mathcal{T}$ admits all coproduct, we have $s \otimes \coprod_{\lambda \in \Lambda} x_\lambda \in \mathcal{T}, \forall s \in \mathcal{K}$.

2. If $a \to b \to c \to \Sigma a$ is a triangle with $a, c \in \mathcal{D}$, then for the triangle

$$a \otimes s \to\to b \otimes s \to c \otimes s \to \Sigma a \otimes s$$

   $a \otimes s$ and $c \otimes s$ in $\mathcal{T}$ implies $b \otimes s \in \mathcal{T}$, so $b \in \mathcal{D}$.

3. Clearly, it contains 0 and closed under suspension (as $\Sigma(- \otimes -) \simeq (\Sigma -) \otimes -$)

Hence $\mathcal{D}$ is a localising subcategory contained by $\mathcal{T}$, and it is easily to see $\mathrm{Thick}^{\otimes}(S) \subset \mathcal{D}$. By the smallest of $\mathcal{T}$, we conclude $\mathcal{D} = \mathcal{T}$. So $\mathcal{T}$ is a tensor localising subcategory contains S, which imply $\mathcal{T} = \mathrm{Loc}(\mathrm{Thick}^{\otimes}(S)) \supset \mathrm{Loc}^{\otimes}(S)$. □

Thanks to this lemma, we now know that $\mathrm{Loc}^{\otimes}(S)$ is a localizing subcategory generated by compact objects because

Lemma 2.51. $\mathrm{Loc}(\mathrm{Thick}^{\otimes}(S)) = \mathrm{Loc}(c \otimes s \mid c \in \mathcal{C}^c, s \in S)$

Proof. Clearly $c \otimes s \in \mathrm{Thick}(S), \forall c \in \mathcal{C}^c, s \in S$ because the property of tensor ideal, so $\mathrm{Loc}(\mathrm{Thick}^{\otimes}(S)) \supset \mathrm{Loc}(c \otimes s \mid c \in \mathcal{C}^c, s \in S)$. For the other direction, we note that it is enough to show that $\mathrm{Loc}(c \otimes s \mid c \in \mathcal{C}^c, s \in S)$ is a tensor ideal, which then will imply it contains $\mathrm{Thick}(S)$.

Given an arbitrary $x \in \mathcal{C}$, by Lemma 2.38, $x \in \mathrm{Loc}(\mathcal{C}^c)$. By Lemma 3.9(where the action is just $\otimes$), we have

$$\mathrm{Loc}(c \otimes s \mid c \in \mathcal{C}^c, s \in S) = \mathrm{Loc}(x \otimes s \mid x \in \mathrm{Loc}(\mathcal{C}^c), s \in \mathrm{Loc}(S))$$

, hence $x \otimes c \otimes s \in \mathrm{Loc}(c \otimes s \mid c \in \mathcal{C}^c, s \in S), \forall s \in S, c \in \mathcal{C}^c$, which implies $\mathrm{Loc}(c \otimes s \mid c \in \mathcal{C}^c, s \in S)$ is a tensor ideal. □

According to Proposition 2.33, we can discuss the associated smashing sequence.

Proposition 2.52. *Let* $\mathcal{S} = \mathrm{Loc}^{\otimes}(S)$*, consider the corresponding smashing sequence*

$$\begin{tikzcd} \mathcal{S} \arrow[r, shift left, "i_*"] & \mathcal{K} \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^*"] & \mathcal{S}^{\perp} = \mathcal{K}/\mathcal{S} \arrow[l, shift left, "j_*"] \end{tikzcd}$$

Then the following statements are true

(i) $\mathcal{S}^{\perp}$ *is a localising tensor ideal.*

(ii) *there are isomorphism of functors*

$$i_* i^! \mathbb{1} \otimes - \simeq i_* i^!(-) \quad \text{and} \quad j_* j^* \mathbb{1} \otimes - \simeq j_* j^*(-)$$

(iii) *the object* $i_* i^! \mathbb{1}$ *and* $j_* j^* \mathbb{1}$ *satisfy*

$$i_* i^! \mathbb{1} \otimes i_* i^! \mathbb{1} \simeq i_* i^! \mathbb{1}; j_* j^* \mathbb{1} \otimes j_* j^* \mathbb{1} \simeq j_* j^* \mathbb{1}; i_* i^! \mathbb{1} \otimes j_* j^* \mathbb{1} \simeq 0$$

Hence, they are idempotent and mutually orthogonal.

Proof. (i) We consider the following full subcategory

$$\mathcal{D} = \{x \in \mathcal{K} \mid x \otimes \mathcal{S}^{\perp} \subset \mathcal{S}^{\perp}\}$$

We left the reader to check(as we have done many times before) that $\mathcal{D}$ is a localizing subcategory by the fact: $- \otimes -$ preserve coproduct and exact, and $\mathcal{S}^{\perp}$ is localizing. Now let $t \in \mathcal{K}^c$, $y \in \mathcal{S}^{\perp}$, then for any $z \in \mathcal{S}$, we have

$$\mathrm{Hom}(z, t \otimes y) \simeq \mathrm{Hom}(z, t^{\vee} \otimes y)$$

$$\simeq \mathrm{Hom}(z \otimes t^\vee, y) = 0$$

where the isomorphism comes from Theorem 2.29, the second is because $\mathcal{K}$ is closed, and the final equality is because $\mathcal{S}$ is a tensor ideal, together with the fact $y \in \mathcal{S}^\perp$. In conclusion, we have shown that $\mathcal{K}^c \subset \mathcal{D}$, and by Lemma 2.38, we get $\mathcal{D} = \mathcal{K}$ so $\mathcal{S}^\perp$ is tensor ideal.

(ii) We now show (ii). Consider the triangle given in Proposition 2.41 (3)

$$i_* i^! \mathbb{1} \to \mathbb{1} \to j_* j^* \mathbb{1} \to \Sigma i_* i^! \mathbb{1} \tag{2.5}$$

Give $x \in \mathcal{K}$ and tensor it with the triangle

$$i_* i^! \mathbb{1} \to \mathbb{1} \otimes x \to j_* j^* x \to \Sigma i_* i^! \mathbb{1} \otimes x$$

because both $\mathcal{S}$ and $\mathcal{S}^\perp$ are tensor ideal, so $i_* i^! \mathbb{1} \otimes x \in \mathcal{S}$ and $j_* j^* \mathbb{1} \otimes x \in \mathcal{S}^\perp$. The functorial and uniqueness of the localization triangle in Proposition 2.41(c) implies we must have

$$i_* i^! x \simeq i_* i^! \mathbb{1} \otimes x \text{ and } j_* j^* x \simeq j_* j^* \mathbb{1} \otimes x$$

(iii) By Proposition 2.41 (1) and (ii), we get $i_* i^! \mathbb{1} \otimes j_* j^* \mathbb{1} \simeq i_* i^! j_* j^* \mathbb{1} = 0$. Tensoring the $i_* i^! \mathbb{1}$ to the localizing triangle (2.5), we get

$$i_* i^! i_* i^! \mathbb{1} \to i_* i^! \mathbb{1} \to 0$$

but this implies $\mathrm{cone}(i_* i^! i_* i^! \mathbb{1} \to i_* i^! \mathbb{1}) = 0$, so this morphism is an isomorphism (see the last part of the proof in Lemma 2.38). Similarly, we get $j_* j^* \mathbb{1} \otimes j_* j^* \mathbb{1} \simeq j_* j^* \mathbb{1}$ □

Let $\mathcal{V}$ be a specialization subset of $\mathrm{Spc}(\mathcal{K}^c)$, we recall from Theorem 1.6 that the associated tensor thick ideal is defined as

$$\tau(\mathcal{V}) = \{x \in \mathcal{K}^c \mid \mathrm{supp}(x) \subset \mathcal{V}\}$$

we set

$$\mathit{\Gamma}_{\mathcal{V}} \mathcal{K} = \mathrm{Loc}(\tau(\mathcal{V}))$$

By Proposition 2.46, there is corresponding smashing localization sequence

$$\mathit{\Gamma}_{\mathcal{V}} \mathcal{K} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{K} \underset{j_*}{\overset{j^*}{\rightleftarrows}} L_{\mathcal{V}} \mathcal{K}$$

Indeed, $\mathit{\Gamma}_{\mathcal{V}} \mathcal{K}$ is not just a smashing subcategory but a smashing tensor ideal because of the following theorem

Theorem 2.53 (Theorem 4.1 of [BF11]). *Let $\mathcal{I}$ be a tensor thick ideal of $\mathcal{K}^c$, then*

(1) $\mathrm{Loc}(\mathcal{I})$ is a tensor smashing ideal of $\mathcal{K}$ and $\mathrm{Loc}(\mathcal{I})^c = \mathcal{K} \cap \mathrm{Loc}(\mathcal{I}) = \mathcal{I}$.

(2) $\mathcal{K}/\mathrm{Loc}(\mathcal{I})$ has all small hom-set and is a compactly generated tensor triangulated category.

(3) *$\mathcal{K}^c/\mathcal{I}$ can be fully faithful embed into the compact object of $\mathcal{K}/\mathrm{Loc}(\mathcal{I})$. Furthermore, $\langle \mathcal{K}^c/(\mathcal{I})\rangle = (\mathcal{K}/\mathrm{Loc}(\mathcal{I}))^c$*

Hence, the Proposition 2.52 yields the tensor idempotent $\mathit{\Gamma}_{\mathcal{V}}\mathbb{1} = i_*i^!\mathbb{1}$ and $L_{\mathcal{V}}\mathbb{1} = j_*j^*\mathbb{1}$.

Definition 2.54. For any $\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)$, we define

$$\mathrm{Z}(\mathcal{P}) = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}^c) \mid \mathcal{P} \not\subset \mathcal{Q}\}$$

Remark 2.55. Recall from Proposition 1.12 $\overline{\{\mathcal{P}\}} = \mathrm{V}(\mathcal{P}) = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}^c) \mid \mathcal{Q} \subset \mathcal{P}\}$. Note that both $\mathrm{V}(\mathcal{P})$ and $\mathrm{Z}(\mathcal{P})$ are specialization closed, and hence Thomason because $\mathrm{Spc}(\mathcal{K}^c)$ is noetherian. In particular, we have

$$\mathrm{V}(\mathcal{P}) - \mathrm{Z}(\mathcal{P}) \cap \mathrm{V}(\mathcal{P}) = \{\mathcal{P}\}$$

Remark 2.56. By Lemma 1.46, $\tau(\mathrm{Z}(\mathcal{P})) = \bigcap_{\mathcal{P}' \notin \mathrm{Z}(\mathcal{P})} \mathcal{P}' = \bigcap_{\mathcal{P} \subset \mathcal{P}'} \mathcal{P}' = \mathcal{P}$, so the classification theorem 1.6 implies $\mathrm{Z}(\mathcal{P}) = \mathrm{supp}(\mathcal{P}) = \bigcup_{x \in \mathcal{P}} \mathrm{supp}(x)$

Definition 2.57. Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)$, we define *Rickard tensor idempotent*.

$$\mathit{\Gamma}_{\mathcal{P}}\mathbb{1} = \mathit{\Gamma}_{\mathrm{V}(\mathcal{P})}\mathbb{1} \otimes L_{\mathrm{Z}(\mathcal{P})}\mathbb{1}$$

Definition 2.58. For $x \in \mathcal{K}$, we define a new support

$$\mathrm{Supp}(x) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c) \mid x \otimes \mathit{\Gamma}_{\mathcal{P}}\mathbb{1} \neq 0\}$$

Lemma 2.59 (Lemma 7.8 of [BF11]). *For $\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)$, the following are equivalent*

(i) *$\mathrm{Spc}(\mathcal{K}^c) - \mathcal{P}$ is quasi-compact*

(ii) *there is a $s \in \mathcal{K}^c$ such that $\mathrm{supp}(s) = \overline{\{\mathcal{P}\}}$*

(iii) *$\overline{\{\mathcal{P}\}}$ is a Thomason subset.*

Proposition 2.60 (Proposition 7.17 of [BF11]). *For any $x \in \mathcal{K}^c$, $\mathrm{supp}(x) = \mathrm{Supp}(x)$. So $\mathrm{Supp}(x)$ will satisfy all the properties in definition 1.24 for all compact objects x.*

Proof. We want to show $x \in \mathcal{P} \Leftrightarrow x \otimes \mathit{\Gamma}_{\mathcal{P}}\mathbb{1} = 0$. Suppose $x \in \mathcal{P}$, then $x \in \mathrm{Loc}(\mathcal{P})$. By remark 2.56, this imply $x \in \mathrm{Loc}(\sigma(\mathrm{Z}(\mathcal{P})))$. Because $x \otimes L_{\mathrm{V}(\mathcal{P})}\mathbb{1} = 0$ by the fact $x \in \ker(j^*)$ and $x \otimes L_{\mathrm{V}}\mathbb{1} \simeq j_*j^*x$, we get $x \otimes \mathit{\Gamma}_{\mathcal{P}}\mathbb{1} = 0$.

Conversely, suppose $x \otimes \mathit{\Gamma}_{\mathcal{P}}\mathbb{1} = 0$. As $\overline{\{\mathcal{P}\}}$ is Thomason, Lemma 2.4 implies there is $s \in \mathcal{K}^c$ such that $\mathrm{supp}(s) = \overline{\{\mathcal{P}\}} \Rightarrow s \in \tau(\mathrm{V}(\mathcal{P}))$, which yields

$$0 = x \otimes s \otimes \mathit{\Gamma}_{\mathcal{P}}\mathbb{1} = x \otimes s \otimes \mathit{\Gamma}_{\mathrm{V}(\mathcal{P})}\mathbb{1} \otimes L_{\mathrm{Z}(\mathcal{P})}\mathbb{1} \simeq a \otimes s \otimes L_{\mathrm{Z}(\mathcal{P})}$$

where the third equality comes from Remark 2.42. It is not hard to check $\{x \in \mathcal{K} \mid x \otimes L_{\mathrm{Z}(\mathcal{P})}\mathbb{1} = 0\} = \mathrm{Loc}(\tau(\mathrm{Z}(\mathcal{P}))) = \mathrm{Loc}(\mathcal{P})$. Hence, we get $a \otimes s \in \mathrm{Loc}(\mathcal{P}) \cap \mathcal{K}$, then Theorem 2.53 (1) implies $\mathrm{Loc}(\mathcal{P}) \cap \mathcal{K} = \mathcal{P}$. Because $\mathcal{P}$ is a prime, if $a \notin \mathcal{P}$, we must have $s \in \mathcal{P}$, but this is impossible because $\mathcal{P} \in \mathrm{supp}(s)$. Therefore, we must have $a \in \mathcal{P}$, which completes the proof. □

Convention 2.61. By abusing the notation, we will use the notation $\mathrm{supp}(x)$ for the new support data in the remained context.

Lemma 2.62. *Let $\mathcal{V}_1, \mathcal{V}_2 \subset \mathrm{Spc}(\mathcal{K}^c)$ be two Thomason subset, then*

$$\mathit{\Gamma}_{\mathcal{V}_1}\mathbb{1} \otimes \mathit{\Gamma}_{\mathcal{V}_2}\mathbb{1} \simeq \mathit{\Gamma}_{\mathcal{V}_1 \cap \mathcal{V}_2}\mathbb{1}$$

.

Proof. This is the "Mayer–Vietoris sequence" of tensor idempotent. See [BF11], Theorem 5.18. □

Proposition 2.63. *Let $x \in \mathcal{K}$ and $\mathcal{V} \subset \mathrm{Spc}(\mathcal{K}^c)$ be a Thomason subset, we have*

$$\mathrm{supp}(\mathit{\Gamma}_{\mathcal{V}}\mathbb{1} \otimes x) = \mathrm{supp}(x) \cap \mathcal{V}$$

$$\mathrm{supp}(L_{\mathcal{V}}\mathbb{1} \otimes x) = \mathrm{supp}(x) \cap (\mathrm{Spc}(\mathcal{K}^c) - \mathcal{V})$$

Proof. Suppose $\mathcal{P} \subset \mathcal{V}$, then Lemma 2.62 gives $\mathit{\Gamma}_{\mathrm{V}(\mathcal{P})}\mathbb{1} \otimes \mathit{\Gamma}_{\mathcal{V}}\mathbb{1} \simeq \mathit{\Gamma}_{\mathrm{V}(\mathcal{P})}\mathbb{1}$ because $\mathrm{V}(\mathcal{P})$ is a Thomason subset. This implies $x \otimes \mathit{\Gamma}_{\mathcal{V}}\mathbb{1} \otimes \mathit{\Gamma}_{\mathcal{P}}\mathbb{1} \simeq x \otimes \mathit{\Gamma}_{\mathcal{P}}\mathbb{1}$ for any $x \in \mathcal{K}$. Therefore, we get $\mathrm{supp}(\mathit{\Gamma}_{\mathcal{V}}\mathbb{1} \otimes x) \cap \mathcal{V} = \mathrm{supp}(x) \cap \mathcal{V}$.

Hence, we only need to show $\mathcal{V} \supset \mathrm{Supp}(\mathit{\Gamma}_{\mathcal{V}}\mathbb{1} \otimes x)$. Since $\mathrm{Supp}(x \otimes \mathit{\Gamma}_{\mathcal{V}}\mathbb{1}) \subset \mathrm{supp}(x) \cap \mathrm{supp}(\mathit{\Gamma}_{\mathcal{V}}\mathbb{1})$, it is enough to show $\mathrm{supp}(\mathit{\Gamma}_{\mathcal{V}}\mathbb{1}) \subset \mathcal{V}$. We claim the reverse inclusion holds for their complement, i.e. $\mathrm{supp}(\mathit{\Gamma}_{\mathcal{V}}\mathbb{1})^c \supset \mathcal{V}^c$. Because $\mathcal{V}$ is specialization closed, if $\mathcal{P} \notin \mathcal{V}$, we have $\mathcal{V} \subset \{\mathcal{Q} \mid \mathcal{P} \notin \overline{\{\mathcal{Q}\}}\} = \{\mathcal{Q} \mid \mathcal{P} \not\subset \mathcal{Q}\}$. Then again, by Lemma 2.62, we get $\mathit{\Gamma}_{\mathcal{V}}\mathbb{1} \otimes \mathit{\Gamma}_{\mathrm{Z}(\mathcal{P})}\mathbb{1} \simeq \mathit{\Gamma}_{\mathcal{V}}\mathbb{1}$, and since $\mathit{\Gamma}_{\mathrm{Z}(\mathcal{P})}\mathbb{1} \otimes L_{\mathrm{Z}(\mathcal{P})}\mathbb{1} = 0$, this gives

$$\mathit{\Gamma}_{\mathcal{V}}\mathbb{1} \otimes \mathit{\Gamma}_{\mathcal{P}}\mathbb{1} \simeq \mathit{\Gamma}_{\mathcal{V}}\mathbb{1} \otimes \mathit{\Gamma}_{\mathrm{Z}(\mathcal{P})}\mathbb{1} \otimes \mathit{\Gamma}_{\mathrm{V}(\mathcal{P})}\mathbb{1} \otimes L_{\mathrm{Z}(\mathcal{P})}\mathbb{1} = 0$$

The equation above shows $\mathcal{P} \notin \mathrm{supp}(\mathit{\Gamma}_{\mathcal{V}}\mathbb{1})$ as we want. Thus, we prove the first assertion, and the proof for the second one is similar(the Mayer-Vietoris sequence has a similar statement for $L_{\mathcal{V}}\mathbb{1}$. Again, see [BF11], Theorem 5.18). □

3 Local-to-Global Principal

Let $\mathcal{K}$ be a rigidly-compactly generated tensor triangulated category, the classic support over the Balmer spectrum gives a correspondence between the "radical tensor thick ideal" and "Thomason subset" of $\mathcal{K}^c$. In this section, we will use the new support data to construct a correspondence between "Localizing subcategory of $\mathcal{K}$" and "some subset of $\mathrm{Spc}(\mathcal{K}^c)$". In this section, we will continue to assume that $\mathrm{Spc}(\mathcal{K}^c)$ is noetherian, and $\mathcal{A}$ is a triangulated category admit all small coproduct.

3.1 Action and Module

Definition 3.1. A *left action* of $\mathcal{K}$ on a compactly generated triangulated category $\mathcal{A}$ consist of the following data:

1. A bifunctor
$$(-) * (-) : \mathcal{K} \times \mathcal{A} \to \mathcal{A}$$
which is exact in each variable, together with natural isomorphism
$$\mathbf{a}_{x,y,c} : (x \otimes y) * a \simeq x \otimes (y * a)$$
and
$$\mathbf{l}_a : \mathbb{1} * a \simeq a$$
for all $x, y \in \mathcal{K}$ and $a \in \mathcal{A}$ and compatible with exactness hypothesis on $(-) * (-)$ in the following sense: There are natural isomorphism
$$\lambda : - * \Sigma(-) \to \Sigma(- * -)$$
$$\rho : \Sigma(-) * - \to \Sigma(- * -)$$

2. The association $\mathbf{a}$ makes the following diagram commute for all $x, y, z \in \mathcal{K}$ and $a \in \mathcal{A}$
$$\begin{tikzcd}
 & x * (y * (z * a)) & \\
x * ((y \otimes z) * a) \arrow[ur, "\mathrm{id} * \mathbf{a}_{y,z,a}"] & & (x \otimes y) * (z * a) \arrow[ul, "\mathbf{a}_{x,y,z*a}"'] \\
(x \otimes (y \otimes z)) * a \arrow[u, "\mathbf{a}_{x,y\otimes z,a}"] & & ((x \otimes y) \otimes z) * a \arrow[u, "\mathbf{a}_{x\otimes y,z,a}"'] \arrow[ll, "\mathbf{a}_{x,y,z} * \mathrm{id}"]
\end{tikzcd}$$

3. The unitor $\mathbf{l}$ make the following diagram for every $x \in \mathcal{K}$ and $a \in \mathcal{C}$
$$\begin{tikzcd}
x * (\mathbb{1} * a) \arrow[r, "x * \mathbf{l}_a"] \arrow[d, "\mathbf{a}_{x,\mathbb{1},a}"] & x * a \arrow[d, "\mathrm{id}"] \\
(x \otimes \mathbb{1}) * a \arrow[r] & x * a
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} * (x * a) \arrow[r, "\mathbf{l}_{x*a}"] & x * a \arrow[d, "\mathrm{id}"] \\
(\mathbb{1} \otimes x) * a \arrow[r] \arrow[u, "\mathbf{a}_{\mathbb{1},x,a}"] & x * a
\end{tikzcd}$$

4. For every $a \in \mathcal{A}$ and $r, s \in \mathbb{Z}$, the diagram
$$\begin{tikzcd}
\Sigma^r \mathbb{1} * \Sigma^s a \arrow[r, "\simeq"] \arrow[d, "\rho^r"'] & \Sigma^{r+s} a \arrow[d] \\
\Sigma^r (\mathbb{1} * \Sigma^s a) \arrow[r, "\Sigma^r(\mathbf{l})"'] & \Sigma^{r+s} a
\end{tikzcd}$$
commute, where the top map is the composition
$$\Sigma^r \mathbb{1} * \Sigma^s a \xrightarrow{\lambda^s} \Sigma^s(\Sigma^r \mathbb{1} * a) \xrightarrow{\rho^s} \Sigma^{r+s}(\mathbb{1} * a)$$

5. The functor $*$ distribute with coproducts. More explicitly, let $x \in \mathcal{K}$, $a \in \mathcal{A}$, for coproduct $\coprod_i x_i$ of a family of object $\{x_i\}_{i \in I}$ in $\mathcal{K}$ and $\coprod_j a_j$ in $\mathcal{A}$ of a family of object $\{a_j\}_{j \in J}$, the canonical map
$$\coprod_i (x_i * a) \to (\coprod_i x_i) * a$$
and
$$\coprod_j (x * a_j) \to \coprod x * (\coprod_j a_j)$$
are isomorphisms.

Remark 3.2. Let $f, f' \in \mathcal{K}$ and $g, g' \in \mathcal{A}$ be composable morphism, we can define
$$(f * g)(f' * g') = ff' * gg'$$
by functorial of $(-) * (-)$

Remark 3.3. We also note that the "common sense" $0 * (-) \simeq (-) * 0 \simeq 0$ is correct, and this could be checked by definition: let $x \in \mathcal{K}$ and $a \in \mathcal{A}$, then applying $a * (-)$ to the triangle
$$x \xrightarrow{\text{id}} x \to 0 \to \Sigma x$$
we get the following triangle by exactness of $(-) * (-)$
$$x * a \xrightarrow{\text{id}} x * a \to 0 * a \to \Sigma x * a$$
Hence, we have $0 * a \simeq 0, \forall a \in \mathcal{A}$. Similarly, we have $(-) * a \simeq 0$.

When we equip a compactly generated triangulated category $\mathcal{A}$ and action of $\mathcal{K}$, we think $\mathcal{A}$ as a *module* of $\mathcal{K}$. Therefore, we can start to study the "linear algebra" of $\mathcal{A}$ and one of our most interesting targets in "linear algebra" is the "invariant subspace" of $\mathcal{A}$.

Definition 3.4. Let $\mathcal{L} \subset \mathcal{A}$ be a localizing subcategory of $\mathcal{A}$, then $\mathcal{L}$ is called *localising* $\mathcal{K}-$*submodule* of $\mathcal{A}$ if $\mathcal{L}$ is closed under the action of $\mathcal{K}$, i.e. for any $l \in \mathcal{L}$, one has $x * l \in \mathcal{L}, \forall x \in$ $\forall x \in \mathcal{K}$

Example 3.5. If we let $\mathcal{A} = \mathcal{K}$, where the action is given by the tensor product $(-) \otimes (-)$, then the localising-$\mathcal{K}$ submodule of $\mathcal{K}$ exactly recover the notation of localizing tensor ideal.

Convention 3.6. For a collection of objects $\mathcal{X}$ in $\mathcal{A}$, we denote $\mathrm{Loc}(\mathcal{X})$ be the smallest localizing subcategory contains $\mathcal{X}$ and $\mathrm{Loc}^*(\mathcal{X})$ be the **smallest localizing** $\mathcal{K}-$**submodule** contains $\mathcal{X}$. We will similarly denote $\mathrm{LOC}^*_{\mathcal{K}}(\mathcal{A})$ be the collection of all localizing $\mathcal{K}-$module of $\mathcal{A}$. We will omit the $\mathcal{K}$ when the action is clear.

Given a collection of $\mathcal{S}$ of object of $\mathcal{K}$, we dente
$$\mathcal{S} * \mathcal{X} = \mathrm{Loc}^*(\{x * a \mid x \in \mathcal{S}, a \in \mathcal{X}\})$$
be the localizing submodule generated by the product of $\mathcal{S}$ and $\mathcal{X}$.

We first show some useful lemmas

Lemma 3.7 (Lemma 3.9 of [GS13]). *Suppose that $\mathcal{S}$ is a collection of objects of $\mathcal{A}$ such that $\mathcal{A}$ is stable under the action of $\mathcal{K}$, then $\mathrm{Loc}(\mathcal{S})$ is a localizing $\mathcal{K}-$submodule. Similarly, if $\mathcal{X}$ is a collection of objects of $\mathcal{K}$ and $\mathcal{L}$ is a localizing subcategory of $\mathcal{A}$ that closed under the action of the object of $\mathcal{X}$, then $\mathcal{L}$ is closed under the action of $\mathrm{Loc}(\mathcal{X})$.*

Lemma 3.8. *Suppose $\mathcal{I} \subset \mathcal{K}$ is a localizing tensor ideal and $\mathcal{S}$ is a collection of objects of $\mathcal{A}$, then there is an equality of localizing submodule of $\mathcal{A}$*

$$\mathcal{I} * \mathcal{X} = \mathrm{Loc}(\{x * s \mid x \in \mathcal{I}, s \in \mathcal{X}\})$$

Proof. Because $\mathcal{I}$ is a tensor ideal, given $x * s \in \{x * s \mid x \in \mathcal{I}, s \in \mathcal{S}\}$, for any $a \in \mathcal{K}$, we have

$$a * (x * s) \simeq (a \otimes x) * s \in \{x * s \mid x \in \mathcal{I}, s \in \mathcal{X}\}$$

Therefore, the collection $\{x * s \mid x \in \mathcal{I}, s \in \mathcal{S}\}$ is closed under the action of $\mathcal{K}$. By the previous lemma, $\mathrm{Loc}(\{x * s \mid x \in \mathcal{I}, s \in \mathcal{S}\})$ is localizing $\mathcal{K}-$submodule, and so the equality in the claim follows immediately by the "smallest" of $\mathrm{Loc}^*(-)$. ☐

Lemma 3.9 (Lemma 3.11 of [GS13]). *Given a set of objects $\mathcal{S} \subset \mathcal{K}$ and a set of objects $\mathcal{X}$ of $\mathcal{A}$.*

$$\mathrm{Loc}(\{x * y \mid x \in \mathrm{Loc}(\mathcal{A}), y \in \mathrm{Loc}(\mathcal{X})\}) = \mathrm{Loc}(\{a * c \mid a \in \mathcal{A}, c \in \mathcal{X}\})$$

Lemma 3.10. *Given a collection of object $\mathcal{S}$ of $\mathcal{K}$ and a collection of objects $\mathcal{X}$ of $\mathcal{A}$, we have*

$$\begin{aligned}
\mathrm{Loc}^{\otimes}(\mathcal{S}) * \mathrm{Loc}(\mathcal{X}) &= \mathrm{Loc}(\mathcal{S}) * \mathrm{Loc}(\mathcal{X}) \\
&= \mathcal{S} * \mathcal{X} \\
&= \mathrm{Loc}(\{x * (y * a) \mid x \in \mathcal{K}, x \in \mathcal{S}, a \in \mathcal{X}\})
\end{aligned}$$

Proof. The inclusion $\mathrm{Loc}(\mathcal{S}) * \mathrm{Loc}(\mathcal{X}) \subset \mathrm{Loc}^{\otimes}(\mathcal{S}) * \mathrm{Loc}(\mathcal{X})$ is clear. In the other direction, we observe that

$$\mathrm{Loc}^{\otimes}(\mathcal{S}) = \mathcal{K} \otimes \mathrm{Loc}(\mathcal{S}) = \{x \otimes s \mid x \in \mathcal{K}, s \in \mathrm{Loc}(\mathcal{S})\} = \mathrm{Loc}(s \otimes x \mid x \in \mathcal{K}, s \in \mathcal{S})$$

where the second equality comes from Lemma 3.8 and the third comes from Lemma 3.9. We then see that

$$\begin{aligned}
\mathrm{Loc}^{\otimes}(\mathcal{S}) * \mathrm{Loc}(\mathcal{X}) &= \mathrm{Loc}(x * y \mid x \in \mathrm{Loc}(z \otimes s \mid z \in \mathcal{K}, s \in \mathcal{S}), y \in \mathrm{Loc}(\mathcal{X})) \\
&= \mathrm{Loc}(x * (y * a) \mid x \in \mathcal{K}, y \in \mathcal{S}, a \in \mathcal{X}) \\
&\subset \mathrm{Loc}(\mathcal{S}) * \mathrm{Loc}(\mathcal{X})
\end{aligned}$$

where again the first equality comes from Lemma 3.8 and the second comes from the previous Lemma 3.9 and the associativity of action. This proves the first and third equality in the claim.

For the second equality, we first note

Lemma 3.11. *Let $\mathcal{X}$ be a collection of object in $\mathcal{A}$, then*

$$\mathrm{Loc}^*(\mathcal{X}) = \mathrm{Loc}^*(\mathrm{Loc}(\mathcal{X}))$$

Proof. Because $\mathcal{X}$ contained by a localizing module $\mathrm{Loc}^*(\mathrm{Loc}(\mathcal{X}))$, so $\mathrm{Loc}^*(\mathcal{X}) \subset \mathrm{Loc}^*(\mathrm{Loc}(\mathcal{X}))$. On the other hand, because $\mathrm{Loc}^*(\mathcal{X})$ is a localizing subcategory, so $\mathrm{Loc}(\mathcal{X}) \subset \mathrm{Loc}^*(\mathcal{X})$, and this give $\mathrm{Loc}^*(\mathcal{X}) \supset \mathrm{Loc}^*(\mathrm{Loc}(\mathcal{X}))$ by the smallest of $\mathrm{Loc}^*(-)$ □

Then, combine the Lemma 3.9 and the previous Lemma, we can get the conclusion. □

The following lemma is our "good old friend"

Lemma 3.12. *If $\mathcal{K}$ is generated as a localizing subcategory by the tensor unit $\mathbb{1}$, then every localizing subcategory of $\mathcal{A}$ is $\mathcal{K}-$submodule.*

Proof. This is immediately by Lemma 3.7: every localizing subcategory of $\mathcal{A}$ is invariant under the action of $\mathbb{1}$, so they're invariant under $\mathrm{Loc}(\mathbb{1}) = \mathcal{K}$. □

3.2 Support of Module

We now introduce the support of the module over $\mathcal{K}$. Recall for each $\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)$, we have an Rickard tensor idempotent

$$\varGamma_{\mathcal{P}}\mathbb{1} = \varGamma_{\mathrm{V}(\mathcal{P})}\mathbb{1} \otimes L_{\mathrm{Z}(\mathcal{P})}\mathbb{1}$$

and use it to define a support by

$$\mathrm{supp}(x) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c) \mid x \otimes \varGamma_{\mathcal{P}}\mathbb{1} \neq 0\}$$

We now generalized this construction for $\mathcal{K}-$module. For $\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)$, we denote the essential image of the functor $\varGamma_{\mathcal{P}}\mathbb{1} * (-) : \mathcal{A} \to \mathcal{A}$ by $\varGamma_{\mathcal{P}}\mathcal{A}$. This essential image is a $\mathcal{K}-$module: for any $x \in \mathcal{K}, a \in \varGamma_{\mathcal{P}}\mathcal{A}$, write $a \simeq \varGamma_{\mathcal{P}}\mathbb{1} * a'$ by the fact a is in the essential image, then

$$x * a \simeq x * (\varGamma_{\mathcal{P}}\mathbb{1} * a') \simeq \varGamma_{\mathcal{P}}\mathbb{1} * (x * a')$$

On the other hand, it is not hard to see this category is also localizing: Let $\coprod_\Lambda x_\lambda$ be a coproduct of a family of object in $\mathcal{A}$, then

$$\varGamma_{\mathcal{P}}\mathbb{1} * \coprod_\Lambda x_\lambda \simeq \coprod_\Lambda (\varGamma_{\mathcal{P}} * x_\lambda)$$

so $\varGamma_{\mathcal{P}}\mathcal{A}$ is closed under the coproduct. Also, for a morphism $f : \varGamma_{\mathcal{P}}a \to \varGamma_{\mathcal{P}}a'$, we get a morphism of triangle

$$\begin{tikzcd}
\varGamma_{\mathcal{P}}\mathbb{1} * a \arrow[r] \arrow[d] & \varGamma_{\mathcal{P}}\mathbb{1} * a \arrow[r] \arrow[d] & \mathrm{cone}(f) \arrow[d] \\
\varGamma_{\mathcal{P}}\mathbb{1} * \varGamma_{\mathcal{P}}\mathbb{1} * a \arrow[r] & \varGamma_{\mathcal{P}}\mathbb{1} * \varGamma_{\mathcal{P}}\mathbb{1} * a' \arrow[r] & \varGamma_{\mathcal{P}}\mathbb{1} * \mathrm{cone}(f)
\end{tikzcd}$$

The first two vertical arrows are naturally isomorphism, and so is the last one. Hence $\varGamma_{\mathcal{P}}\mathcal{A}$ is closed under the cone object and so it is a triangulated subcategory.
We will also use the notation$\varGamma_{\mathcal{P}}a$ for shorthand $\varGamma_{\mathcal{P}}\mathbb{1} * a$.

Definition 3.13. Let $a \in \mathcal{A}$, we define the support of a by

$$\operatorname{supp}_{(\mathcal{K},*)}(a) = \{\mathcal{P} \in \operatorname{Spc}(\mathcal{K}^c) \mid \varGamma_{\mathcal{P}}\mathbb{1} * a \neq 0\}$$

We will omit the action ($\mathcal{K}$,*) when there is no ambiguity.

When we think of $\mathcal{K}$ as a module over $\mathcal{K}$ via the tensor product, the support defined here is coincident with the new support data in Definition 2.58. There is also an analogy of Proposition 2.60 and Proposition 2.63.

Proposition 3.14. *The assignment of support* $\operatorname{supp}_{\mathcal{K}}$ *satisfies the following proposition*

(a) for a triangle

$$a \to b \to c \to \Sigma a$$

in $\mathcal{A}$, we have $\operatorname{supp}_{\mathcal{K}}(b) \subset \operatorname{supp}_{\mathcal{K}}(a) \cup \operatorname{supp}_{\mathcal{K}}(c)$

(b) The support is invariant under suspension, i.e. for $a \in \mathcal{A}$

$$\operatorname{supp}_{\mathcal{K}}(a) = \operatorname{supp}_{\mathcal{K}}(\Sigma a)$$

(c) For any coproduct $\coprod_i x_i$ for a family of object $\{x_i\}_{i \in I}$ in $\mathcal{A}$, we have

$$\operatorname{supp}_{\mathcal{K}}(\coprod_i x_i) = \bigcup_i \operatorname{supp}_{\mathcal{K}}(x_i)$$

(d) Let $a \in \mathcal{A}$ and $\mathcal{V} \subset \operatorname{Spc}(\mathcal{K}^c)$ be a Thomason subset, we have

$$\operatorname{supp}_{\mathcal{K}}(\varGamma_{\mathcal{V}}\mathbb{1} * a) = \operatorname{supp}_{\mathcal{K}}(a) \cap \mathcal{V}$$

$$\operatorname{supp}_{\mathcal{K}}(L_{\mathcal{V}}\mathbb{1} * a) = \operatorname{supp}_{\mathcal{K}}(a) \cap (\operatorname{Spc}(\mathcal{K}^c) - \mathcal{V})$$

Proof. (a), (b), (c) directly comes from the fact that $\varGamma_{\mathcal{P}}\mathbb{1} * (-)$ is exact and preserves coproduct. The proof of (d) is similar to the proof of Proposition 2.63, see [GS13] Proposition 5.7 for details.

□

3.3 Local-to-Global Principal

Now we want to generalized the map "τ" and "σ" in the classification theorem 1.6 to the module $\mathcal{A}$ over $\mathcal{K}$ to connect the localising submodule of $\mathcal{A}$ and the subset $\operatorname{Spc}(\mathcal{K}^c)$.

Definition 3.15. We define an order-preserving assignment

$$\{\text{subset of } \mathrm{Spc}(\mathcal{K}^c)\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \mathrm{LOC}^*(\mathcal{A})$$

where both are ordered by inclusion, by

$$\sigma : \mathcal{L} \mapsto \mathrm{supp}(\mathcal{L}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c) \mid \mathit{\Gamma}_{\mathcal{P}}\mathcal{L} \neq 0\}$$

$$\tau : W \mapsto \{a \in \mathcal{A} \mid \mathrm{supp}(a) \subset W\}$$

These maps are well-defined: this is clearly to σ. For τ, one can use the Proposition 3.14 (a), (b), (c) to show $\{a \in \mathcal{A} \mid \mathrm{supp}(a) \subset W\}$ is a localizing subcategory of $\mathcal{A}$. To see it is a $\mathcal{K} - submodule$, we note that if $\mathrm{supp}(a) \subset W$, for any $x \in \mathcal{K}$, if $x * a * \mathit{\Gamma}_{\mathcal{P}}\mathbb{1} \neq 0$, then we must have $a * \mathit{\Gamma}_{\mathcal{P}} \neq 0$, so $\mathrm{supp}(x * a) \subseteq \mathrm{supp}(a) \subset W$.

Definition 3.16. A left action $\mathcal{K} \times \mathcal{A} \xrightarrow{*} \mathcal{A}$ is said to satisfy *local-to-global principle* if for each $a \in \mathcal{A}$,

$$\mathrm{Loc}^*(a) = \mathrm{Loc}^*(\mathit{\Gamma}_{\mathcal{P}}\mathbb{1} * a \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c))$$

The philosophy behind the local-to-global principle is to think a as "global case" and $\mathit{\Gamma}_{\mathcal{P}}\mathbb{1} * a$ is the "local case at $\mathcal{P}$", and if a module satisfies this principle, we can recover the $\mathrm{Loc}^*(a)$ from the local case. The global-to-local principle is very powerful as it can help us to understand the corresponding map τ and σ.

Lemma 3.17. *Suppose the local-to-global principle holds for the action on $\mathcal{A}$ and let W be a subset of $\mathrm{Spc}(\mathcal{K}^c)$. Then*

$$\tau(W) = \mathrm{Loc}^*(\{\mathit{\Gamma}_{\mathcal{P}}\mathcal{A} \mid \mathcal{P} \in W \cap \sigma\mathcal{A}\})$$

Proof. By the local-to-global principle, we have for every $a \in \mathcal{A}$, there is a equation

$$\mathrm{Loc}^*(a) = \mathrm{Loc}^*(\mathit{\Gamma}_{\mathcal{P}}\mathbb{1} * a \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c))$$

Therefore

$$\begin{aligned}
\tau W &= \mathrm{Loc}^*(a \mid \mathrm{supp}(a) \subset W) \\
&= \mathrm{Loc}^*(\mathit{\Gamma}_{\mathcal{P}}a \mid a \in \mathcal{A}, \mathcal{P} \in W) \\
&= \mathrm{Loc}^*(\mathit{\Gamma}_{\mathcal{P}}a \mid a \in \mathcal{A}, \mathcal{P} \in W \cap \sigma\mathcal{A}) \\
&= \mathrm{Loc}^*(\mathit{\Gamma}_{\mathcal{P}}\mathcal{A} \mid \mathcal{P} \in W \cap \sigma\mathcal{A})
\end{aligned}$$

□

Proposition 3.18. *Suppose the local-to-global principle holds for the action of $\mathcal{K}$ on $\mathcal{A}$, and W a subset of $\mathrm{Spc}(\mathcal{K}^c)$, then we have the following equation*

$$\sigma\tau(W) = W \cap \sigma\mathcal{A}$$

In particular, the restriction of τ in the subset of $\sigma\mathcal{A}$ is injective.

Proof. Let $W \subset \mathrm{Spc}(\mathcal{K}^c)$ be a subset, then

$$\begin{aligned}\sigma\tau(W) &= \mathrm{supp}(\tau W)\\ &= \mathrm{supp}(\mathrm{Loc}^*(\Gamma_{\mathcal{P}}\mathcal{A} \mid \mathcal{P} \in W \cap \sigma\mathcal{A}))\end{aligned}$$

where the first equality is the definition and the section is by the previous lemma. Therefore, by the Prospotion 3.14, we have $\sigma\tau \subset W \cap \sigma\mathcal{A}$, and we note that they should be equal because $\mathcal{P} \in W \cap \sigma\mathcal{A}$ if and only if $\Gamma_{\mathcal{P}}\mathcal{A}$ doesn't contain the non-zero object. □

Because the local-to-principle is so powerful, we want our most interesting object $\mathscr{D}(R)$ to satisfy it. To make this "dream" come true, we need to change our viewpoint for $\mathscr{D}(R)$: besides the standard way to construct the $\mathscr{D}(R)$(Using the multiplicative system of quasi-isomorphism of the naive homotopy category $\mathbf{K}(R)$), another way to construct it is via the "model structure" of chain complex $\mathbf{Ch}(R)$. The classic theory of model category then helps us to get a "homotopy category" $\mathbf{Ho}(R)$ which is the same as the $\mathscr{D}(R)$. Moreover, $\mathbf{Ch}(R)$ is a *monoidal model category* which means the tensor product will be compatible with the model structure. The reader can read [MH99] for a brief introduction to the model category.

Remark 3.19. The model structure is indeed a very "soft" structure: It is possible to have more than one model structure in one category but they all have the same homotopy category. For example, $\mathbf{Ch}(R)$ has two standard model structures, one is called projective model structure, and another is called injective model structure, but both of them have the same homotopy category $\mathscr{D}(R)$.

Definition 3.20. A triangulated category is said to have a *model* if it occurs as a homotopy category of a monoidal model category

One of the main reasons for us to care about the homotopy category of the monoidal model category is because it has a very good theory in a new object called "homotopy limit/colimit" that is compatible with the tensor product. It is very natural to think of homotopy limit/colimit in the homotopy category! Because the colimit/limit in the original category may not be "unique up to weak equivalence". See [Kerdon], Tag 0106, Warning 3.4.0.1 for an example that pullback of the simplicial set is not necessarily invariant under weak equivalence.

Lemma 3.21. *Suppose $\mathcal{K}$ has a model. Then for any chain $\{\mathcal{V}_i\}_{i\in I}$ of specialization closed subset of $\mathrm{Spc}(\mathcal{K}^c)$ with union $\mathcal{V}$, there is an isomorphism*

$$\Gamma_{\mathcal{V}}\mathbb{1} \simeq \mathrm{hocolim}\Gamma_{\mathcal{V}_i}\mathbb{1}$$

Proof. Because we failed to introduce the homotopy limit/colimit formally, we choose not to conclude a proof here. The reader can see [GS16], Lemma 3.16 for details. □

Lemma 3.22. *Let $W \subset \mathrm{Spc}(\mathcal{K}^c)$ be a given subset and suppose a is an object $\mathcal{A}$ such that $\Gamma_{\mathcal{P}}a \simeq 0$ for all $a \in \mathrm{Spc}(\mathcal{K}^c) - W$. If $\mathcal{K}$ has a model, then $a \in \mathrm{Loc}(\Gamma_{\mathcal{P}}\mathbb{1} \mid \mathcal{P} \in W)$.*

Proof. Again, the proof involves the technique of homotopy limit, so we suggest the reader see [GS16], Lemma 3.17 for proof. □

Proposition 3.23. *Suppose $\mathcal{K}$ has a model, then the local-to-global principle holds for the action of $\mathcal{K}$ on $\mathcal{A}$*

Proof. Applying the Lemma 3.22 to the action $\mathcal{K} \times \mathcal{K} \xrightarrow{\otimes} \mathcal{K}$ and take $W = \mathrm{Spc}(\mathcal{K}^c)$, then we get $\mathcal{K} = \mathrm{Loc}(\Gamma_{\mathcal{P}}\mathcal{K} \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c))$. Use the Proposition 3.14, we can check that $\Gamma_{\mathcal{P}}\mathcal{A}$ is a localizing tensor ideal contains $\mathrm{Loc}^{\otimes}(\Gamma_{\mathcal{P}}\mathbb{1})$. Because $\Gamma_{\mathcal{P}}\mathcal{A} \subset \mathrm{Loc}^{\otimes}(\Gamma_{\mathcal{P}}\mathbb{1})$, so they should equal. This imply the set of object $\{\Gamma_{\mathcal{P}}\mathbb{1} \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)\}$ generates $\mathcal{K}$ as tensor localizing ideal. By Lemma 3.10, given a object $a \in \mathcal{A}$, we have

$$\mathcal{K} * \mathrm{Loc}(a) = \mathrm{Loc}^{\otimes}(\Gamma_{\mathcal{P}}\mathbb{1} \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)) * \mathrm{Loc}(a) \tag{3.1}$$

but clearly, $\mathrm{Loc}^{\otimes}(\Gamma_{\mathcal{P}}\mathbb{1} \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)) \subset \mathrm{Loc}^{\otimes}(\mathbb{1})$, so we have $\mathcal{K} = \mathrm{Loc}^{\otimes}(\mathbb{1})$. Using Lemma 3.10 again, we get

$$\mathcal{K} * \mathrm{Loc}^{\otimes}(a) = \mathrm{Loc}^{\otimes}(\mathbb{1}) * \mathrm{Loc}(a) = \{\mathbb{1}\} * \{a\} = \mathrm{Loc}^{*}(a) \tag{3.2}$$

Hence, combining the equation (3.1) and (3.2), we conclude

$$\mathrm{Loc}^{*}(a) = \mathrm{Loc}^{*}(\Gamma_{\mathcal{P}}\mathbb{1} \mid \mathcal{P} \subset \mathrm{supp}(a))$$

□

Theorem 3.24 (Theorem 6.9 of [GS13])**.** *Suppose $\mathcal{K}$ is a rigidly-compactly generated tensor triangulated category with a model such that $\mathrm{Spc}(\mathcal{K}^c)$ is noetherian, for any compactly generally triangulated category $\mathcal{A}$ that is module over $\mathcal{K}$, e.g take $\mathcal{A} = \mathcal{K}$, we have*

(i) The action of $\mathcal{K}$ on $\mathcal{A}$ satisfy the local-to-global principle.

(ii) $a \in \mathcal{A} = 0$ if and only if $\mathrm{supp}(a) = \varnothing$.

(iii) For any chain $\{\mathcal{V}_i\}_{i \in I}$ of specialization closed subset of $\mathrm{Spc}(\mathcal{K}^c)$ with union $\mathcal{V}$, there is an isomorphism

$$\Gamma_{\mathcal{V}}\mathbb{1} \simeq \mathrm{hocolim}\,\Gamma_{\mathcal{V}_i}\mathbb{1}$$

Proof. (i) is the previous proposition because it has a model, and (iii) is also clear. Let's prove the (ii). Indeed, (i) will imply (ii): For a module $\mathcal{A}$ over $\mathcal{K}$ with action satisfying local- to-global principle, observe if $\mathrm{supp}(a) = \varnothing$ for $a \in \mathcal{A}$, then local-global principle implies

$$\mathrm{Loc}^{*}(a) = \mathrm{Loc}^{*}(\Gamma_{\mathcal{P}}a \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)) = \mathrm{Loc}^{*}(0)$$

so $a \simeq 0$. □

3.4 Understand the Support Locally

Although we have constructed the map σ and τ between the localizing submodule of $\mathcal{A}$ and subset of $\mathrm{Spc}(\mathcal{K}^c)$, we haven't seen an example that these two maps become bijective; Indeed, as we have seen in Proposition 3.18, the σ is left inverse of τ. This section will illustrate one of the possible ways to see when τ is inverse to σ: It is enough to check this locally. To explain what it means, let's first begin with the following definition.

Convention 3.25. Let $U \subset \mathrm{Spc}(\mathcal{K}^c)$, we let $\mathcal{V}$ be its complement. This is a closed subset and so specialization closed subset. Therefore, we can have a smashing localization sequence as before

$$\Gamma_{\mathcal{V}}\mathcal{K} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{K} \underset{p_*}{\overset{p^*}{\rightleftarrows}} L_{\mathcal{V}}\mathcal{K} = \mathcal{K}(U)$$

We denote the right category by $\mathcal{K}(U)$. The quotient functor p^* is monoidal so we will denote $\mathbb{1}_U$ as tensor unit $p^*\mathbb{1}$ in $\mathcal{K}(U)$. There are some important facts about $\mathcal{K}(U)$

1. $\mathcal{K}(U)$ is a compactly generated tensor triangulate category where the $\mathcal{K}^c(U)$ is exactly the idempotent completion of $\mathcal{K}$ of $\mathcal{K}^c/\mathcal{K}^c_{\mathcal{V}}$, this is part of *Thomason Localization theorem*, the reader can see for example [AN96], Theorem 2.1, 2.1.4. Moreover, $\mathcal{K}^c(U)$ is a rigid tensor tensor category by [PB07], Proposition 2.15, and so $\mathcal{K}(U)$ is a rigidly compactly generated tensor triangulated category.

2. The $\mathrm{Spc}(\mathcal{K}^c(U))$ is naturally isomorphic to U by [BF07] Proposition 1.11. The Thomason subset of $\mathrm{Spc}(\mathcal{K}^c(U))$ is isomorphic to the set of Thomason subset of U. Hence, if we let Z be specialization closed subset of $\mathrm{Spc}(\mathcal{K}^c)$, then $Z \cap U$ is a specialization of $\mathrm{Spc}(\mathcal{K}^c(U)$.

3. Because of (1) and (2), we will denote $\Gamma_{Z\cap U}\mathbb{1}_U, L_{Z\cap U}\mathbb{1}_U$ as the tensor idempotent respectively. In particular, we set $\Gamma_{\mathcal{P}}\mathbb{1}_U = \Gamma_{\mathrm{V}(\mathcal{P})\cap U}\mathbb{1}_U \otimes L_{\mathrm{Z}(\mathcal{P})\cap U}\mathbb{1}_U$

We also recall that $i_*i^!(-) \simeq \Gamma_{\mathcal{V}}\mathbb{1} \otimes (-)$ and $p_*p^*(-) = L_{\mathcal{V}}\mathbb{1} \otimes (-)$.

Lemma 3.26. *Let $Z \subset \mathrm{Spc}(\mathcal{K}^c)$ be a specialization closed subset, then*

$$p^*\Gamma_Z\mathbb{1} \simeq \Gamma_{Z\cap U}\mathbb{1}_U$$

and

$$p^*L_Z\mathbb{1} \simeq L_{Z\cap U}\mathbb{1}_U$$

Proof. This is exactly the restate of Corollary 6.5 of [BF11]. □

Lemma 3.27 (*Projection Formula*)**.** *Suppose $x \in \mathcal{K}$ and $y \in \mathcal{K}(U)$, there is an isomorphism*

$$x \otimes p_*y \simeq p_*(p^*x \otimes y)$$

Proof. Because $y \in \mathcal{K}(U)$, by Remark 2.42, we have $p^*p_*y \simeq y$. Thus,

$$p_*y \simeq p_*p^*p_*y \simeq L_{\mathcal{V}}\mathbb{1} \otimes p_*y$$

From this, we see

$$\begin{aligned}
\Gamma_{\mathcal{V}}\mathbb{1} \otimes x \otimes p_*y &\simeq x \otimes \Gamma_{\mathcal{V}}\mathbb{1} \otimes p_*y \\
&\simeq x \otimes \Gamma_{\mathcal{V}}\mathbb{1} \otimes L_{\mathcal{V}}\mathbb{1} \otimes p_*y \\
&\simeq 0
\end{aligned}$$

Combine the result with the triangle

$$\Gamma_{\mathcal{V}}\mathbb{1} \otimes x \otimes p_*y \to x \otimes p_*y \to L_{\mathcal{V}}\mathbb{1} \otimes x \otimes p_*y$$

because $\Gamma_{\mathcal{V}}\mathbb{1} \otimes x \otimes p_*y \simeq 0$, we have $x \otimes p_*y \simeq L_{\mathcal{V}}\mathbb{1} \otimes x \otimes p_*y$. Therefore, we deduce

$$\begin{aligned}
p_*(p^*x \otimes y) &\simeq p_*(p^*x \otimes p^*p_*y) \\
&\simeq p_*p^*(x \otimes p_*y) \\
&\simeq L_{\mathcal{V}}\mathbb{1} \otimes x \otimes p_*y \\
&\simeq x \otimes p_*y
\end{aligned}$$

as we claimed. □

Proposition 3.28 (Proposition 8.3 of [GS13]). *For all $\mathcal{P} \in U$, there is an isomorphism*

$$p^*\Gamma_{\mathcal{P}}\mathbb{1}_U \simeq \Gamma_{\mathcal{P}}\mathbb{1}$$

Proof. Observe we have

$$p_*\Gamma_{\mathcal{P}}\mathbb{1}_U \simeq p_*(\Gamma_{\mathrm{V}(\mathcal{P})\cap U}\mathbb{1}_U \otimes L_{\mathrm{Z}(\mathcal{P})\cap U}\mathbb{1}_U)$$

Using Lemma 3.26, this is equal to

$$\begin{aligned}
&\simeq p_*(p^*\Gamma_{\mathrm{V}(\mathcal{P})}\mathbb{1}_U \otimes p^*L_{\mathrm{Z}(\mathcal{P})}\mathbb{1}_U) \\
&\simeq p_*p^*(\Gamma_{\mathrm{V}(\mathcal{P})\cap U}\mathbb{1}_U \otimes L_{\mathrm{Z}(\mathcal{P})}\mathbb{1}_U) \\
&\simeq L_{\mathcal{V}}\mathbb{1} \otimes \Gamma_{\mathcal{P}}\mathbb{1} \\
&\simeq \Gamma_{\mathcal{P}}\mathbb{1}
\end{aligned}$$

where for the last isomorphism, we use the fact $\Gamma_{\mathcal{P}}\mathbb{1} \in L_{\mathcal{V}}\mathcal{K} = \mathcal{K}(U)$ (this could be proved by computing the support of $\Gamma_{\mathcal{P}}\mathbb{1}$ via Proposition 3.14, (d)) □

Proposition 3.29. *For all $\mathcal{P} \in U$, the functor p_* induces an equivalence between $\Gamma_{\mathcal{P}}\mathcal{K}$ and $\Gamma_{\mathcal{P}}\mathcal{K}(U)$.*

Proof. Note that for any $x \in \mathcal{K}$

$$\begin{aligned}
p^*(\Gamma_{\mathcal{P}}\mathbb{1} \otimes x) &\simeq p^*\Gamma_{\mathcal{P}}\mathbb{1} \otimes p^*x \\
&\simeq p^*p_*\Gamma_{\mathcal{P}}\mathbb{1}_U \otimes p^*x \\
&\simeq \Gamma_{\mathcal{P}}\mathbb{1}_U \otimes p^*x
\end{aligned}$$

where the second isomorphism comes from Proposition 3.28. From this, we see the essential image of p_* restricted to $\Gamma_{\mathcal{P}}\mathcal{K}$ is $\Gamma_{\mathcal{P}}\mathcal{K}(U)$.

On the other hand, using the projection formula and Proposition 3.28, we have

$$p_*(\Gamma_{\mathcal{P}}\mathbb{1}_U \otimes p^*x) \simeq p_*\Gamma_{\mathcal{P}}\mathbb{1}_U \otimes x \simeq \Gamma_{\mathcal{P}}\mathbb{1} \otimes x$$

This shows that the essential image of p_* restrict to $\Gamma_{\mathcal{P}}\mathcal{K}(U)$ is $\Gamma_{\mathcal{P}}\mathcal{K}$, and $p_*p^* \simeq \mathrm{id}_{\mathrm{Im}p_*}$. On the other hand, because p_* is fully faithful, so $p^*p_* \simeq \mathrm{id}_{\mathcal{K}(U)}$, and from this we get the equivalence we want. □

The Proposition 3.29 shows that the Rickard tensor idempotent could be understood locally, and we now want to generalize these results over a module $\mathcal{A}$ of $\mathcal{K}$. Luckily, through [GS13], Lemma 4.4 and Corollary 4.11, given $\mathcal{V} = \mathrm{Spc}(\mathcal{K}^c) - U$, there is a smashing localization sequence

$$\Gamma_{\mathcal{V}}\mathcal{A} \underset{j^!}{\overset{j_*}{\rightleftarrows}} \mathcal{A} \underset{q_*}{\overset{q^*}{\rightleftarrows}} L_{\mathcal{V}}\mathcal{A} = \mathcal{A}(U)$$

and similarly, we have

$$j_*j^!(-) \simeq \Gamma_{\mathcal{V}}\mathbb{1} * (-)$$

and

$$q_*q^*(-) \simeq L_{\mathcal{V}}\mathbb{1} * (-)$$

Proposition 3.30 (Proposition 8.5 of [GS13])**.** *There is an action $*_U$ of $\mathcal{K}(U)$ on $\mathcal{A}(U)$ defined by commutativity diagram*

$$\begin{tikzcd}
\mathcal{K} \times \mathcal{A} \arrow[r, "p^* \times q^*"] \arrow[d, "*"'] & \mathcal{K}(U) \times \mathcal{A}(U) \arrow[d, "*_U"] \\
\mathcal{A} \arrow[r, "q^*"'] & \mathcal{A}(U)
\end{tikzcd}$$

We then prove the analog of Proposition 3.29 on $\mathcal{A}$.

Proposition 3.31 (Proposition 8.6 of [GS13])**.** *For any $\mathcal{P} \in U$, there is an equivalence between $\Gamma_{\mathcal{P}}\mathcal{A}$ and $\Gamma_{\mathcal{P}}\mathcal{A}(U)$ induced by q^*, q_*.*

Proof. It is not hard to see $\Gamma_{\mathcal{P}}\mathcal{A}$ is contained by $q_*(\mathcal{A}(U))$, so p^* is fully faithful when restrict to $\Gamma_{\mathcal{A}}$. so it remained to show the image of p^* restrict to $\Gamma_{\mathcal{P}}\mathcal{A}$ is $\mathcal{A}(U)$. We note for each $a \in \mathcal{A}$, we have

$$q^*(\Gamma_{\mathcal{P}}\mathbb{1} * a) = p^*\Gamma_{\mathcal{P}}\mathbb{1} *_U q^*a \simeq \Gamma_{\mathcal{P}}\mathbb{1}_U *_U q^*a$$

so this implies $q^*\Gamma_{\mathcal{P}}\mathcal{A} = \Gamma_{\mathcal{P}}\mathcal{A}(U)$. □

Definition 3.32. a localizing $\mathcal{K}-$module $\mathcal{L}$ is *minimal* if it has no proper trivial localizing $\mathcal{K}-$submodules.

The minimality is also equivalent to say for any non-zero $a \in \mathcal{L}$, we have

$$\mathrm{Loc}^*(a) = \mathcal{L}$$

Theorem 3.33. *Suppose $\mathcal{K}$ has a model and that there exist an open cover $\mathrm{Spc}(\mathcal{K}^c) = \bigcup_i^n U_i$ such that the action $\mathcal{K}(U_i)$ on $\mathcal{A}(U_i)$ yields bijectiion*

$$\{\textit{subset of } \sigma\mathcal{A}(U_i)\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \mathrm{LOC}^*(\mathcal{A}(U_i))$$

then σ and τ gives a bijection

$$\{\textit{subset of } \sigma\mathcal{A}\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \mathrm{LOC}^*(\mathcal{A})$$

The proof will based on the following key observation

Lemma 3.34. *If $\mathcal{A}$ is a module over $\mathcal{K}$ with an action that satisfies the local-to-global principle. Then $\tau\sigma = \mathrm{id}$ if and only if $\varGamma_{\mathcal{P}}\mathcal{A}$ are minimal submodules.*

Proof. ($\Leftarrow$)Let $\mathcal{L}$ be a submodule of $\mathcal{A}$. We recall

$$\sigma\mathcal{L} = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c) \mid \varGamma_{\mathcal{P}}\mathcal{L} \neq 0\}$$

so for each $x \in \mathcal{L}$, if $\mathcal{P} \in \mathrm{supp}(a)$, we must have $\mathcal{P} \in \sigma\mathcal{A}$. From this, conclude $\mathcal{L} \subset \tau\sigma\mathcal{L} = \{a \in \mathcal{A} \mid \mathrm{supp}(a) \subset \sigma\mathcal{L}\}$.

Conversely, if $a \in \tau\sigma\mathcal{L}$, then $\mathrm{supp}(a) \subset \sigma\mathcal{L}$ and by local-to-global prinicple

$$a \in \mathrm{Loc}^*(a) = \mathrm{Loc}^*(\varGamma_{\mathcal{P}}\mathbb{1} * a \mid \mathcal{P} \in \sigma\mathcal{L})$$

If $\mathcal{P} \in \sigma\mathcal{L}$, then $\varGamma_{\mathcal{P}}\mathcal{L} \neq 0$, so it is a non-zero localizing submodule of $\varGamma_{\mathcal{P}}\mathcal{A}$, and by minimally, we have $\varGamma_{\mathcal{P}}\mathcal{A} = \varGamma_{\mathcal{P}}\mathcal{L} \subset \mathcal{L}$. This implies $a \in \mathcal{L}$ and so $\tau\sigma\mathcal{L} = \mathcal{L}$.

($\Rightarrow$) This direction is straightforward once we realize the following fact: given a non-zero object $\varGamma_{\mathcal{P}}\mathbb{1} * a \in \varGamma_{\mathcal{P}}\mathcal{A}$, then

$$\mathrm{supp}(\varGamma_{\mathcal{P}}\mathbb{1} * a) = \mathrm{supp}(\varGamma_{\mathrm{V}(\mathcal{P})}\mathbb{1} * (L_{\mathrm{Z}(\mathcal{P})}\mathbb{1} * a))$$

because of of Proposition 3.14 (4), this is equal to

$$\begin{aligned} &= \mathrm{V}(\mathcal{P}) \cap \mathrm{supp}(L_{\mathrm{Z}(\mathcal{P})}\mathbb{1} * a) \\ &= \mathrm{V}(\mathcal{P}) \cap (\mathrm{Spc}(\mathcal{K}^c) - \mathrm{Z}(\mathcal{P})) \cap \mathrm{supp}(a) \end{aligned}$$

Recall that $\mathrm{Z}(\mathcal{P}) = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}^c) \mid \mathcal{P} \not\subset \mathcal{Q}\}$ so $\mathrm{Spc}(\mathcal{K}^c - \mathrm{Z}(\mathcal{P})) = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}^c) \mid \mathcal{P} \subset \mathcal{Q}\} = \mathrm{U}(\mathcal{P})$. Clearly, $\mathcal{P} \in \mathrm{supp}(x)$ and $\mathrm{V}(\mathcal{P}) \cap \mathrm{U}(\mathcal{P}) = \{\mathcal{P}\}$, so

$$\mathrm{supp}(\varGamma_{\mathcal{P}}\mathbb{1} * a) = \mathrm{V}(\mathcal{P}) \cap (\mathrm{Spc}(\mathcal{K}^c) - \mathrm{Z}(\mathcal{P})) \cap \mathrm{supp}(a) = \{\mathcal{P}\}$$

This implies $\sigma\varGamma_{\mathcal{P}}\mathcal{A} = \{\mathcal{P}\}$. Because now the σ and τ are bijective, so there is no non-trivial localizing submodule of $\varGamma_{\mathcal{P}}\mathcal{A}$. $\square$

Proof of Theorem 3.33. In Proposition 3.18, we have seen that σ is left inverse to τ. So, by previous lemma, it is enough to show $\Gamma_{\mathcal{P}}\mathcal{A}$ is minimal for each $\mathcal{P} \in \sigma\mathcal{A}$. There is an open subset U_i containing $\mathcal{P}$, and by Proposition 3.31, $\Gamma_{\mathcal{P}}\mathcal{A} \simeq \Gamma_{\mathcal{P}}\mathcal{A}(U_i)$. The latter category is minimal $\mathcal{A}(U_i)$ localizing $\mathcal{K}(U_i)-$submodule by hypothesis and according to the diagram in Proposition 3.30, it is also minimal with respect to the action of $\mathcal{K}$. □

3.5 Example: Noetherian Commutative Ring

Now, let's get our hands dirty and use the machinery we construct to compute some specific examples. The first example is $\mathscr{D}(R)$ for R is noetherian. Our target in this section is to give an alternative proof for the following classification theorem.

Theorem 3.35 (Theorem 2.8 of [AM92])**.** *There is a one-to-one correspondence between the collection*

$$\{\textit{subset of}\, \mathrm{Spec}(R)\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \mathrm{LOC}(\mathscr{D}(R))$$

Here are some important facts we will use

1. $\mathscr{D}(R)$ is a rigidly-compactly generated tensor triangulated category that has a model with $\mathrm{Spc}(\mathscr{D}^{\mathrm{perf}}(R))$ noetherian. Therefore, the Theorem 3.24 guarantee the the action $\mathscr{D}(R) \times \mathscr{D}(R) \xrightarrow{\otimes} \mathscr{D}(R)$ has local-to-global principle.

2. Because $\{R\}$ is a set of compact generator, that is $\mathrm{Loc}(R) = \mathscr{D}(R)$, so $\mathrm{LOC}(\mathscr{D}(R)) = \mathrm{LOC}^{\otimes}(\mathscr{D}(R))$.(Lemma 2.38 and Lemma 2.37).

3. The reason we put $\mathrm{Spec}(R)$ rather than $\mathrm{Spc}(\mathscr{D}^{\mathrm{perf}}(R))$ is that we have claimed that there is a natural isomorphism $\mathrm{Spc}(\mathscr{D}(R)) \simeq \mathrm{Spec}(R)$ in Example 1.39, so we will identify this two spaces via this isomorphism. We also remind the reader that

   $$\sigma(\mathcal{L}) = \{\mathfrak{p} \in \mathrm{Spc}(R) \mid \Gamma_{\mathfrak{p}}\mathcal{L} \neq 0\}$$

   for localizing subcataegory $\mathcal{L}$ and

   $$\tau W = \{X \in \mathscr{D}(R) \mid \mathrm{Supp}(x) \subset W\}$$

Next, we need to describe the Rickard idempotent.

Definition 3.36. Let $x \in R$, then the *stable Koszul complex* is a chain complex concentrated on the degree 0 and 1

$$K_\infty(x) = \cdots \to R \to R_f \to 0 \to \dots$$

where the only non-zero morphism is the canonical map to localization. Given a sequence of element $\mathbf{f} = \{f_1, \dots, f_n\}$, we then define

$$K_\infty(\mathbf{f}) = K_\infty(f_1) \otimes \cdots \otimes K_\infty(f_n)$$

If $I = (f_1, \dots, f_n) \subset R$ is an ideal, we will define $K_\infty(I) = K_\infty(f_1) \otimes \cdots \otimes K_\infty(f_n)$.

Remark 3.37. Of course, the reason why we call $K_\infty(\mathbf{f})$ a stable Koszul complex is because it is a "stabilization" of the (unstable) Koszul chain complex, which is defined as

$$K_d(\mathbf{f}) = (R \xrightarrow{f_1^d} R) \otimes \cdots \otimes (R \xrightarrow{f_i^d} R)$$

Indeed, if we realize the localization R_f could be constructed as

$$\varinjlim_{i \in \mathbb{N}} (\cdots \to R \xrightarrow{\times f} R \xrightarrow{\times f} R \xrightarrow{\times f} R \to \dots)$$

we then see the degree one stable Koszul chain complex $K_\infty(f_i) = (R \to R_{f_i})$ is the colimit of degree one (unstable) Koszul chain complex $K_d(f_i) = (R \xrightarrow{f_i^d} R)$. Because the colimit commutes with the tensor product, we get $K_\infty(\mathbf{f})$ as the colimit of $K_d(\mathbf{f})$.

There is a natural map $K_\infty(\mathbf{f}) \xrightarrow{\chi} R$, and the cone object (or fibration) of this map is called *Cech complex* $\check{C}(\mathbf{f})$. More explicitly, $\check{C}(I)$ is exactly $\Sigma(\ker(\chi))$, so

$$\check{C}(\mathbf{f})^i = K_\infty^{i+1}(I), \forall i \geqslant 0$$

For example, if $I = (f_1, f_2)$, then

$$\check{C}(I) = \cdots \to R_{f_1} \oplus R_{f_2} \to R_{f_1 f_2} \to \dots$$

On the other hand, both $K_\infty(\mathbf{f})$ and $\check{C}(\mathbf{f})$ are bounded chain complex of $R-$flat module, which means there $K-$*flat*: the functor $K_\infty(\mathbf{f}) \otimes -$ and $\check{C}(\mathbf{f} \otimes -)$ over $K(R)$ preserve quasi-isomorphism.

Proposition 3.38. *For any ideal $I = (f_1, \dots, f_n) \subset R$, and $\mathfrak{p} \in \mathrm{Spc}(R)$we have*

(i) $\Gamma_{\mathrm{V}(I)} R \simeq K_\infty(I)$.

(ii) $L_{\mathrm{V}(I)} R \simeq \check{C}(I)$.

(iii) $L_{\mathrm{Z}(\mathfrak{p})} R \simeq R_{\mathfrak{p}}$.

Proof. The proof is relatively tedious, so we only sketch the idea. Note that the (i) and (ii) exactly restate of [Gre01], Lemma 5.8.

Lemma 3.39 (Proposition 5.6 of [Gre01]). *$\tau(\mathrm{V}(I))$ is generated as a localizing tensor ideal by any one of the following three modules: R/I, $K_\infty(I)$ and $K_1(\mathbf{f})$.*

Because we have a triangle

$$K_\infty(I) \to R \to \check{C}(I)$$

and $K_\infty(I) \in \tau(\mathrm{V}(I))$, so if we can show $\check{C}(I) \in \mathscr{D}(R)/\mathrm{Loc}(\tau(\mathrm{V}(I)))$, then the uniqueness of the idempotent triangle in Proposition 2.33 (3) will automatically conclude the result. Recall from the construction of the localizing sequence that this is the same show

$$\mathrm{Hom}(\mathrm{Loc}(\tau(\mathrm{V}(I)), \check{C}(I)) = 0$$

Applying the previous Lemma, it is enough to check $\mathrm{Hom}(-, \check{C}(I)) = 0$ for "$-$" $= R/I$ or "$-$" $= K_\infty(I)$ or "$-$" $= K_1(I)$.

Lemma 3.40. $\mathrm{Hom}(K_1(I), (\check{C})(I)) = 0$

Proof. We prove this by proceeding with an induction. When $n = 1$, we have

$$K_1(I) = \cdots \to 0 \to R \xrightarrow{\times f} R \to 0 \to \ldots$$

$$\check{C}(I) \cdots \to R_f \to 0 \to \ldots$$

Then

$$\begin{aligned}\mathrm{Hom}(K_1(I), \check{C}(I)) &= \bigoplus \mathrm{Hom}(R, \check{C}(I)) \\ &= \check{C}(I) \oplus \check{C}(I) \\ &= \cdots \to R_f \xrightarrow{f} R_f \to \ldots\end{aligned}$$

because $\times f$ is an isomorphism between R_f, so $\mathrm{Hom}(K_1(I), \check{C}(I)) = (\cdots \to R_f \xrightarrow{f} R_f \to \ldots) \simeq 0$. Now, we do the induction part $n-1 \Rightarrow n$. By tensor-hom adjunction

$$\mathrm{Hom}(K_1(f_1) \otimes \cdots \otimes K_1(f_n), \check{C}(I)) \simeq \mathrm{Hom}(K_1(f_1) \otimes \cdots \otimes K_1(f_{n-1}), \mathrm{Hom}(K_1(f_n), \check{C}(I))$$

Lemma 3.41. $\mathrm{Hom}(K_1(f_n), \check{C}(I)) \simeq \mathrm{Hom}(K_1(f_n), \check{C}(f_1, \ldots, f_{n-1}))$

Proof. Rewrite $\check{C}(I)$ as

$$\bigoplus_i (\bigoplus_t R_{f_{i_1} \ldots f_{i_t}})$$

Because finite direct sum commutes with Hom, we then have

$$\begin{aligned}\mathrm{Hom}(K_1(f_n), \check{C}(I)) &\simeq \mathrm{Hom}(K_1(f_n), \check{C}(f_1, \ldots, f_n)) \\ &= \bigoplus_i (\bigoplus_t \mathrm{Hom}(K_1(f_n), R_{f_{i_1} \ldots f_{i_t}}) \oplus \bigoplus_s \mathrm{Hom}(K_1(f_n), R_{f_{i_1} \ldots f_{i_s} f_n}))\end{aligned}$$

where $f_n \neq f_{i_t}, \forall i, t$. Clearly

$$\bigoplus_i \bigoplus_t \mathrm{Hom}(K_1(f_n), R_{f_{i_1} \ldots f_{i_t}}) \simeq \mathrm{Hom}(K_1(f_n), \check{C}(f_1, \ldots, f_{n-1}))$$

On the other hand

$$\mathrm{Hom}(K_1(f_n), R_{f_{i_1} \ldots f_{i_s} f_n}) \simeq \cdots \to 0 \to R_{f_{i_1} \ldots f_{i_s} f_n} \xrightarrow{\times f_n} R_{f_{i_1} \ldots f_{i_s} f_n} \to 0 \to \ldots$$

$''\times f_n''$ is an isomorphism between $R_{f_{i_1} \ldots f_{i_s} f_n}$, so the chain complex above is 0, which implies

$$\bigoplus_s \mathrm{Hom}(K_1(f_n), R_{f_{i_1} \ldots f_{i_s} f_n}) = 0$$

which will then imply the result.

□

Applying the tensor-hom adjunction and previous lemma, we then have

$$
\begin{aligned}
&= \mathrm{Hom}(K_1(f_1) \otimes \cdots \otimes K_1(f_{n-1}), \mathrm{Hom}(K_1(f_n), \check{C}(I)) \\
&= \mathrm{Hom}(K_1(f_1) \otimes \cdots \otimes K_1(f_{n-1}), \mathrm{Hom}(K_1(f_n), \check{C}(f_1, \ldots, f_{n-1})) \\
&= \mathrm{Hom}(K_1(f_1) \otimes \cdots \otimes K_1(f_{n-1}) \otimes K_1(f_n), \check{C}(f_1, \ldots, f_{n-1})) \\
&= \mathrm{Hom}(K_1(f_1) \otimes \cdots \otimes K(f_n), \mathrm{Hom}(K_1(f_{n-1}), \check{C}(f_1, \ldots, f_{n-1}))
\end{aligned}
$$

Using the previous lemma again, we reduce to

$$
\begin{aligned}
&= \mathrm{Hom}(K_1(f_1) \otimes \cdots \otimes K(f_n), \mathrm{Hom}(K_1(f_{n-1}), \check{C}(f_1, \ldots, f_{n-2})) \\
&= \mathrm{Hom}(K_1(f_1) \otimes \cdots \otimes K_1(f_{n-1}) \otimes K_1(f_n), \check{C}(f_1, \ldots, f_{n-2}))
\end{aligned}
$$

Continue this process, we will eventually have

$$
= \mathrm{Hom}(K_1(f_2) \otimes \cdots \otimes K_1(f_{n-1}) \otimes K_1(f_n), \mathrm{Hom}(K_1(f_1), \check{C}(f_1))
$$

whereby the basic case, is equal to

$$
\mathrm{Hom}(K_1(f_2) \otimes \cdots \otimes K_1(f_{n-1}) \otimes K_1(f_n), 0) = 0
$$

□

We next show the (iii), which exactly comes from the fact that the full subcategory

$$
\tau \mathrm{U}(\mathfrak{p}) = \{X \in \mathscr{D}(R) \mid \mathrm{supp}(X) \subset \mathrm{U}(\mathfrak{p})\}
$$

is the essential image of $\mathscr{D}(R_{\mathfrak{p}})$, and combined with the fact $\mathscr{D}^{\mathrm{perf}}(R)/\tau \mathrm{Z}(\mathfrak{p}) \simeq \mathscr{D}^{\mathrm{perf}}(R_{\mathfrak{p}})$. □

The previous proposition gives $\varGamma_{\mathfrak{p}} R = K_\infty(\mathfrak{p}) \otimes R_{\mathfrak{p}} \simeq K_\infty(\mathfrak{p})_{\mathfrak{p}}$. We already collected all the necessary ingredients to prove the Theorem 3.35.

Proof of Theorem 3.35. As Lemma 3.34 suggests, it is enough to show the $\varGamma_{\mathfrak{p}}\mathscr{D}(R)$ is minimal for each $\mathfrak{p} \in \mathrm{Spc}(R)$. For a prime $\mathfrak{p}$, we will denote $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ as the associated residue field.

Lemma 3.42. *There are the following localizing subcategories are equal*

$$
\varGamma_{\mathfrak{p}}\mathscr{D}(R) = \mathrm{Loc}(\varGamma_{\mathfrak{p}} R) = \mathrm{Loc}(k(\mathfrak{p}))
$$

Proof. For the first equality, because every localizing subcategory tensor is ideal, and it is clearly $\mathrm{Loc}(\varGamma_{\mathfrak{p}} R) \supset \varGamma_{\mathfrak{p}}\mathscr{D}(R)$. Hence we conclude the result.

For the second equality, we first note that

$$
\varGamma_{\mathfrak{p}} R \otimes k(\mathfrak{p}) \simeq k(\mathfrak{p})
$$

so $k(\mathfrak{p}) \in \varGamma_{\mathfrak{p}}\mathscr{D}(R)$. For converse inclusion, we claim that $K(\mathfrak{p}^i)_{\mathfrak{p}}$ are in the $\mathrm{Loc}(k(\mathfrak{p}))$ for each i. To do this, we shall first recall that a module M over R is said to have *finite length* if it admits a filtration

$$
0 = M_0 \subset M_1 \subset \cdots \subset M_n = M
$$

such that M_{i+1}/M_i is a simple module over R.

Lemma 3.43. *For a local noetherian ring $(R, \mathfrak{m}, k)$, any simple module N is isomorphic to k.*

Proof. Take a non-zero $m \in M$, then we can define a natural map

$$f : A \to N$$
$$a \mapsto a * m$$

This is a non-zero map as $1 * m = m$, so $\mathrm{Im}(f)$ is a submodule of M. As M is simple, $\mathrm{Im}(f) = M$. Hence $A/\ker(f) \simeq M$, but $\ker(f)$ must be $\mathfrak{m}$, because, by the third isomorphism theorem, the submodule of M is one-to-one corresponds to module contains $ker(f)$. Thus, we get $k \simeq M$. □

Lemma 3.44. *Any finite length module M is in $\langle k \rangle$.*

Proof. Let's proceed with an induction on i. The basic case $i = 0$ is easy, now suppose this is correct for $i - 1$, then from the short exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} = k \to 0$$

we see because $M_{i-1} \in \langle k \rangle$, and $k \in \langle k \rangle$, we have $M_i \in \langle k \rangle$.

□

Lemma 3.45. *For a local noetherian ring $(R, \mathfrak{m}, k)$, any bounded chain complex X over R with finite length homology group is in the $\langle k \rangle \subset \mathscr{D}(R)$.*

Proof. Again we can do the induction on the length n of the chain complex. The basic case $n = 1$ is just the Lemma 3.44. Now suppose this is correct for $n - 1$, then because for any bounded chain complex, we have a short exact sequence

$$\tau_{n-1\leqslant} X \to X \to X/\tau_{n-1\leqslant} X$$

where $\tau_{n-1\leqslant} X$ is the $n - 1$ truncation chain complex of X, i.e $\tau_{n-1\leqslant} X_i = X_i$ if $i \leqslant n - 1$, and $\tau_{n-1\leqslant} X_i = X_i = 0$ if $i > n - 1$. If X has length n, then $X/\tau_{n-1\leqslant} X$ is just a finite length module. Therefore, we have $X \in \langle k \rangle$ by induction.

□

Because the homology group of the Koszul complex has a finite length, so $K(\mathfrak{p}^i)_{\mathfrak{p}}$ is a bounded chain complex with a finite length homology group. Hence, $K(\mathfrak{p}^i)_{\mathfrak{p}} \in \langle (k(\mathfrak{p})) \rangle \subset \mathscr{D}(R_{\mathfrak{p}})$, as $\mathscr{D}(R_{\mathfrak{p}})$ is fully faithful subcategory of $\mathscr{D}(R)$, we then get

$$K(\mathfrak{p}^i)_{\mathfrak{p}} \in \langle k(\mathfrak{p}) \rangle \subset \mathrm{Loc}(k(\mathfrak{p}))$$

In the meanwhile, the localizing subcategory $\mathrm{Loc}(k(\mathfrak{p}))$ is closed under the direct limit. This is a corollary from the fact that $\mathbf{Ch}(R)$ is a Grothendieck abelian category, which implies for a sequence of objects $\{X_i\}_{i \in I}$, the homotopy limit $\mathrm{hocolim} X_i$ is quasi-isomorphic to the direct limit $\varinjlim X_i$(see [TM15], Lemma 2* or [AM93], Remark 2.2 for a proof). Hence, we finish the proof by the fact that $K_\infty(\mathfrak{p})_{\mathfrak{p}}$ is directed limit of $K(\mathfrak{p}^i)_{\mathfrak{p}}$.

□

Lemma 3.46. *The localizing subcategories $\Gamma_{\mathfrak{p}}(\mathscr{D}(R))$ are minimal.*

Let $X \in \mathscr{D}(R)$, then because $\mathrm{Loc}(k(\mathfrak{p}))$ is tensor ideal, we have

$$X \otimes k(\mathfrak{p}) \simeq \bigoplus_{i\in\mathbb{Z}} \Sigma^i k(\mathfrak{p})^{(\Lambda_i)}$$

where $k(\mathfrak{p})^{(\Lambda_i)}$ denote coproduct of λ_i copies of $k(\mathfrak{p})$. Hence, for any $X \neq 0$ in $\Gamma_{\mathfrak{p}}\mathscr{D}(R)$, we must have $k(\mathfrak{p}) \otimes X \neq 0 \Rightarrow k(\mathfrak{p}) \in \mathrm{Loc}(X)$, and this give

$$\mathrm{Loc}(k(\mathfrak{p})) \subset \mathrm{Loc}(X) \subset \Gamma_{\mathfrak{p}}\mathscr{D}(R)$$

Thus, we conclude that $\Gamma_{\mathfrak{p}}\mathscr{D}(R)$ is minimal. □

3.6 Example: Representation of Categories over Commutative Noetherian Ring

The purpose of this is to show you that $\mathscr{D}(R)$ is not the only example that will satisfy the local-to-global principle. Indeed, Greg Stevenson also illustrates another example of the singularity category of hypersurface in [GS16], section 3 and we recommend that interested readers see this section for details on this example. We will mainly focus on ideas and avoid being involved in too many technical discussions. The result discussed here basically comes from [AG15], and the machinery introduced in this section could used to, for instance, classify the localizing subcategory of derived category of $R-$linear path algebra of Dykin quiver.

To begin with, we first introduce our object. We will let R be a noetherian commutative ring and $\mathcal{C}$ be arbitrary small category. This could be, for example, a category of quiver representation, a groupoid, or a poset.

Definition 3.47. We define the *category of (right) $R-$module* $\mathrm{Mod}_R\mathcal{C}$ as the contravariant functors from $\mathcal{C}$ to $R-$modules $\mathrm{Mod}_R\mathcal{C} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}R]$

Advertisement 3.48 (*"Infinity Category"*)**.** People generally use the convention $[-,-]$ rather than $\mathrm{Fun}(-,-)$ to tell the reader that they're working with "Infinity-category". There are many reasons to work with "infinity category" rather than ordinary category. For example, recall that the presheaf over the category $\mathcal{C}$ is a function $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. The Yoneda embedding then tells us the $\mathcal{C}$ could be embedded as a full subcategory of the presheaf category via representable functor. Therefore we can identify each object X with its representable functor $\mathrm{Hom}(-, X)$. This identification is powerful, but not good enough. For instance, if we consider the $\mathcal{C}$ as the category of topological space, scheme, manifold... then we also want to talk about the topological information (Homology/homotopy)in the presheaf under the identification, but this is hard because presheaf is not a "Space" under this definition. Hence, we want to update the $\mathrm{Hom}(-, X)$, as a "Mapping Space" $[-, X] : \mathcal{C}^{\mathrm{op}} \to$ "Space". In the meanwhile, we also want to have a Yoneda embedding so we can now identify the "Space" $[-, X]$ with X and discuss the topological invariant on presheaf directly.

But how do we do this? Or more precisely, what is the "Space"? This philosophical question is relevant to *Grothendieck Homotopy Hypnothesis*, where Grothendieck believes the "Space" is "Infinity Groupoid". As the name suggests, it means that it is a category with 1-morphism and 2-morphism...and all morphisms are invertible. This might sound crazy to you, but the good news is that we can really do it! Through the serious work by Jacob Lurie, and Andre Joyal... we now have a model for the "infinity-category": use the simplicial set, the "infinity category" is exactly the simplicial that has filler for all inner horns. In this model, the "infinity groupoid" is Kan complex. (This is not the only model we can construct an infinity category. Depending on your faith, you can choose to accept Siegel's complete space as an infinity category or, moreover, some people choose to work with the "model-free" infinity category. However, the simplicial set is the most widely accepted model.)

We have a functor $\mathrm{Sing} : \mathrm{Top} \to \mathrm{Sset}$ This functor will take all CW-complex to the Kan complex(infinity groupoid) as we want because most of the topological space we like is CW-complexes. In additionally, the map between two objects could be identified as a Kan complex, so they're "Space". Another good news is that every ordinary category, which we called 1-Cat could be fully faithful embedded into the category of $\infty-$categories, so the ordinary category is just a special "infinity category". Finally, we can have the category of "Space": it is the $\infty-$category of the category of Kan complex(this is a very big category! and the way you construct it is to use a functor called *homotopy coherent never*, which send simplicial enriched category to infinity category). Now we can do everything we want. This is the very beginning of the story of infinity categories, and if you like this idea, DON'T HESITATE TO CLICK [Kerdon] or [HTT09] and [HA15] TO START!!!

Lemma 3.49 (Lemma 2.2 of [AG15])**.** $\mathrm{Mod}_R\mathcal{C}$ *is a Grothendieck abelian category with enough projective objects*

Proof. Recall that the Grothendieck abelian category is an abelian category that satisfies (1): the existence and exactness of filtered colimit, and (2): possesses a generator, i.e. there is an object C such that $\mathrm{Hom}(-, C)$ is a fully faithful functor to the category of set. Clearly, $\mathrm{Mod}_R\mathcal{C}$ abelian because it is a functor category from small category to abelian category. We also note that the direct sum of the set of representable functors is indeed a generator (Yoneda Lemma). Because the filtered colimits are computed pointwise, so $\mathrm{Mod}_R\mathcal{C}$ satisfy (1) because $\mathrm{Mod}(R)$ satisfy it. Finally, the projective objects are summands of the direct sum of representable objects, together with Yoneda Lemma, we can show it has enough projective objects. □

Definition 3.50. The $R-$*linearzation* of $\mathcal{C}$, denoted by $R\mathcal{C}$ is a category with same object with $\mathcal{C}$ and whose Hom-set are R-free modules

$$\mathrm{Hom}_{R\mathcal{C}}(c, c') = \bigoplus_{f \in \mathrm{Hom}_{\mathcal{C}}(c,c')} Rf$$

with obvious composition.

For a free $R-$linear category $R\mathcal{C}$, there is an action

$$\mathrm{Mod}(R) \times \mathrm{Mod}_R\mathcal{C} \xrightarrow{\otimes_R} \mathrm{Mod}_R\mathcal{C}$$

which is defined by pointwise tensor product, i.e. for $R-$moduel M and a $\mathcal{C}$-module F, for each $C \in \mathcal{C}$

$$(M \otimes_R F) = M \otimes_R F(C)$$

Lemma 3.51 (Lemma 2.7 of [AG15]). *The natural functor $F : \mathrm{Mod}_R R\mathcal{C} \to \mathrm{Mod}_R\mathcal{C}$ is a equivalence for any small category $\mathcal{C}$. Moreover, this equivalence is compatible with the action defined above.*

Given a homomorphism of ring $\phi : R \to S$, then we have a base change functor and restriction functor which gives an adjunction

$$\mathrm{Mod}_R\mathcal{C} \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathrm{Mod}_S\mathcal{C}$$

where $\phi^* : F \mapsto S \otimes_R F$.

Because $\mathrm{Mod}_R(R\mathcal{C})$ is an abelian category, we can consider its unbounded derived category $\mathscr{D}(R\mathcal{C})$. The action defined above will then generalize to the action of the unbounded derived category

$$\mathscr{D}(R) \times \mathscr{D}(R\mathcal{C}) \to \mathscr{D}(R\mathcal{C})$$

Note that the projective objects of $\mathrm{Mod}_R\mathcal{C}$ are pointwise $R-$flat module (Hom-set are free $R-$module), so the functor $\phi^*(-) = S \otimes_R (-)$ could be generalized to a well-defined total derived tensor product $\mathbf{L}\phi^*(-) = S \otimes_R^{\mathbb{L}} -$. Hence, we have an adjunction

$$\mathscr{D}(R\mathcal{C}) \underset{\mathbf{L}\phi_*}{\overset{\mathbf{L}\phi^*}{\rightleftarrows}} \mathscr{D}(S\mathcal{C})$$

The construction here is very useful to understand the localization categories of $\mathscr{D}(R\mathcal{C})$: Let $\mathfrak{p} \in \mathrm{Spec}(R)$, and $k(\mathfrak{p})$ as the associated residue field of $\mathfrak{p}$, we define

$$\mathcal{F} = \coprod_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathrm{LOC}(\mathscr{D}(k(\mathfrak{p})\mathcal{C}))$$

then there is an "obvious map

$$\mathcal{F} \xrightarrow{s} \mathrm{Spec}(R)$$

such that $s^{-1}(\mathfrak{p}) = \mathrm{LOC}(\mathscr{D}(k(\mathfrak{p}))\mathcal{C})$. Also, for the morphism $\phi : R \to k(\mathfrak{p})$, we have the adjunction

$$\mathscr{D}(R\mathcal{C}) \overset{k(\mathfrak{p})\otimes_R^{\mathbb{L}}}{\rightleftarrows} \mathscr{D}(k(\mathfrak{p})\mathcal{C})$$

together we can define a morphism

$$\mathrm{LOC}\mathscr{D}(R\mathcal{C}) \underset{g}{\overset{f}{\rightleftarrows}} \{\text{section } l \text{ of } \mathcal{F} \xrightarrow{s} \mathrm{Spec}(R)\}$$

defined on a localizing subcategory $\mathcal{L}$ and section l by

$$f(\mathcal{L}) : \mathfrak{p} \mapsto \mathrm{Loc}(k(\mathfrak{p}) \otimes L) \subset \mathscr{D}(k(\mathfrak{p})\mathcal{C})$$

and

$$g(l) = \{X \in \mathscr{D}(R\mathcal{C}) \mid k(\mathfrak{p}) \otimes_R^{\mathbb{L}} X \in l(\mathfrak{p})\}$$

Theorem 3.52 (Corollary 4.3 of [AG15])**.** *The map f and g are inverse to each other*

The proof is technical, we will give the rough idea.

Proposition 3.53 (Theorem 3.5 of [AG15])**.** *Given $X \in \mathscr{D}(R\mathcal{C})$, there is an equality*

$$\mathrm{Loc}(X) = \mathrm{Loc}(k(\mathfrak{p}) \otimes_R^{\mathbb{L}} X \mid \mathfrak{p} \in \mathrm{Spec}(R))$$

In particular, $k(\mathfrak{p}) \otimes X \simeq 0$ for all $\mathfrak{p}$ if and only if $X \simeq 0$

Proposition 3.54 (Proposition 4.15 of [GS16])**.** *Given a section l of $\mathcal{F} \xrightarrow{s} \mathrm{Spec}(R)$, there is an equality*

$$g(l) = \{X \mid k(\mathfrak{p}) \otimes_R^{\mathbb{L}} X \in l(\mathfrak{p}), \forall \mathfrak{p} \in \mathrm{Spec}(R)\} = \mathrm{Loc}(l(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R))$$

where $l(\mathfrak{p})$ is viewed as a subcategory of $\mathscr{D}(R\mathcal{C})$ through the restricting functor alone $R \to k(\mathfrak{p})$.

Proof of Theorem of 3.52. Fixed a localizing subcategory $\mathcal{L}$ of $\mathscr{D}(R\mathcal{C})$ and a section l of s, we can now compute directly

$$\begin{aligned} gf(\mathcal{L}) &= \mathrm{Loc}(f(\mathcal{L})(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)) \\ &= \mathrm{Loc}(k(\mathfrak{p}) \otimes_R^{\mathbb{L}} \mathcal{L} \mid \mathfrak{p} \in \mathrm{Spec}(R)) \\ &= \mathcal{L} \end{aligned}$$

and finally

$$\begin{aligned} fg(\mathcal{L})(\mathfrak{p}) &= \mathrm{Loc}(k(\mathfrak{p}) \otimes_R^{\mathbb{L}} g(l)) \\ &= \mathrm{Loc}(k(\mathfrak{p}) \otimes_R^{\mathbb{L}} l(\mathfrak{p})) \\ &= l(\mathfrak{p}) \end{aligned}$$

$\square$

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