Quatumn Group and Coboundary Category

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Contents

1 Coalgebra, Bialgebra and Hopf Algebra

Through the section, any map f means a linear map unless other mention.

1.1 Coalgebra

Let k be a fixed field, a **Algebra** over k is defined as a vector space A(or more generally a module) with additional operation:

multiplication: $\mu : A \otimes A \to A, a \otimes b \mapsto ab$, and a ring map: $\eta : k \to A$

such that the multiplication should be associative, and the scalar mutiplication $(k, a) \mapsto \eta(k)a$ is contained in the center $C(A)$, e.g, $k(ab) = (ka)b = a(kb)$. This is equivalent to say a **Algebra** is a triple (A, μ, η) with the following diagram (we call them associate and unit axiom respectfully) commute

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd} \qquad \begin{tikzcd} k \otimes A \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

If the multiplication is commutative, e.g, ab=ba, then it is equivalent to say that we have a flip map $\tau_A : a \otimes b \to b \otimes a$ such that the following diagram commute:

$$\begin{tikzcd} & A & \\ A \otimes A \arrow[ur, "\mu"] \arrow[rr, "\tau_A"] & & A \otimes A \arrow[ul, "\mu"'] \end{tikzcd}$$

A map $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is a morphism of Algebra if

$$\mu' \circ (f \otimes f) = f \circ \mu, f \circ \eta = \eta'$$

The idea of Coalegbra is reversing all the diagram in Alegbra.

Definition: A **Coalgebra** (C, Δ, ϵ) is a triple with a vector space C and two maps $\Delta : C \to C\otimes C$ (comutipilcation), $\epsilon : C \to k$ such that the following two diagrams (coassociate, counit) commute

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\Delta\otimes 1"] \\ C \otimes C \arrow[r, "1\otimes\Delta"] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} k \otimes C & C \otimes C \arrow[l, "\epsilon\otimes 1"'] \arrow[r, "1\otimes\epsilon"] & C \otimes k \\ & C \arrow[ul, "\cong"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

and similarly, a Coalgebra is called cocommutative if

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\tau_A"'] & & C \otimes C \end{tikzcd}$$

commute and τ_A is also the flip map.

Example 1.1.1: Any field k has a natural coalgeberea structure defined by $\Delta(1) = 1 \otimes 1, \epsilon(1) = 1$.

Example 1.1.2: Let X be a set, then a vector space $C = k[X] = \bigoplus_{x \in X} kx$ with a base X has the Coalgebra structure defined through $\Delta(x) = x \otimes x, \epsilon(x) = 1$.

Example 1.1.3: As a dual notation of vector space, one may think if the dual space A^* of a alegbra (A, μ, η) is a coalegbra. This is correct when A is finite dimension, because in this case, we have $\lambda : (A \otimes A)^* \cong A^* \otimes A^*$, so we could construct a coalgebra structure in A^* by $\Delta : \lambda^{-1} \circ \mu^*, \epsilon = \eta^*$, where $*$ means the transpose of linear map. Then once check this is a coalgebra structure in A^*.

Remark: The converse of this is always true, that is any dual space A^* of a coalgebra A is a algebra since the map $\lambda : A^* \otimes A^* \to (A \otimes A)^*, (f \otimes g)(x \otimes y) \mapsto f(x) \otimes g(y)$ is always exist and well-defined. One can construct a algebra structure by transpose and this map over A^*

Example 1.1.4: We can define the tensor product $C \otimes C'$of two Coalgebra $(C, \Delta, \epsilon), (C', \Delta', \epsilon')$, and it is still Coelgebra where the $\Delta^{\otimes} = (id \circ \tau_{C,C'} \circ id) \circ (\Delta \otimes \Delta'), \epsilon^{\otimes} = (\epsilon \otimes \epsilon')$

Remark: The morphism $Hom(C, C')$ in Category of Coalgebra *Coalg* is also an object in Vec_k, the category of vector space over k because any morphism of coalgebra is also linear. Such phenomena is called *Category Enriched in Vec_k*. We will talk more about this later.

Definition: A map $f : C \to C'$ between two Coalgebra (C, Δ, ϵ) and $(C', \Delta, \epsilon')$is a morphism of Coalgebra if

$$(f \otimes f) \circ \Delta = \Delta' \circ f, \epsilon = \epsilon' \circ f$$

Example 1.1.5: The exmple 1.1.3 imply whenever we have a morphism of finite dimension algebra $f : A \to B$, then its transpose $f^* : B^* \to A^*$ is morphism of coalgebra

Example 1.1.6 Back to the example 1.1.2. It is clearly $k[X] \otimes k[Y] \cong k[X \times Y]$ is an isomorphism of vector space, and in fact it is also an isomorphism of Coalgebra

Proof. The isomorphism is given by the map $f : (x \otimes y) \mapsto (x, y)$, and we need to check f is also a morphism of coalegbra. Clearly, we can have (1):$(f \otimes f) \circ \Delta^{\otimes}(x \otimes y) = (f \otimes f)((x \otimes y) \otimes (x' \otimes y')) = (x, y) \otimes (x', y') = \Delta \circ f(x \otimes y)$, and (2): $\epsilon(x, y) = (1, 1) = \epsilon^{\otimes} \circ f$. Hence this complete the proof □

Sweedler Notation: If x is an element of coalgebra (C, Δ, ϵ), then $\Delta(x) = \sum_i x'_i \otimes x''_i$. To get rid of subscript , Sweedler suggest us to use the following notation to represent $\Delta(x)$

$$\Delta(x) = \sum_{(x)} x' \otimes x''$$

So, for example the unit axiom could be represent as

$$\sum_{(x)} \epsilon(x')x'' = x = \sum_{(x)} x'\epsilon(x'')$$

To end this section, we introduce the coideal in Coalgebra

Defintion: Let (C, Δ, ϵ) be a Coalgebra, then a sub vector space I is called **Coideal** of C if

$$\Delta(I) \subset I \otimes C + C \otimes I; \epsilon(I) = 0$$

. If I is a coideal, then a **Quotient Coalgebra** $\overline{C} = C/I$ is defined as $(\overline{C}, \overline{\Delta}, \overline{\epsilon})$, where $\overline{\Delta}(C/I) = (C \otimes C)/(I \otimes C + C \otimes I) = C/I \otimes C/I$, $\overline{\epsilon} : C/I \to k$ by using the similar way that we factor $\overline{\Delta}$.

Example 1.1.7 Let $f : A \to B$ be a morphism of Coalgbera. Then for the $ker(f)$, we have $(f \otimes f) \circ \Delta(kerf) = \Delta' \circ f(ker(f))$, and the RHS is equal to 0, so we have $\Delta(kerf) \subset ker(f \otimes f) = C \otimes ker(f) + ker(f) \otimes C$. Similarly, $\epsilon(ker(f)) = \epsilon' \circ f(kerf) = 0$. Hence $ker(f)$ is a coideal.

Remark: Notice the $im(f)$ is in fact not necessary a coideal.

Let $C' = C/I$ be a quotient algebra of C, then we have unique surjective linear map $\pi : C \to C'$. This linear map will be a morphism of coalgbera for a unique coalgebra structure over C'

Proposition 1.1: There is a unique coalgebra structure of C' such that π is a morphism of Coalgebra.

Proof. Since $I \subset ker(\pi)$, so we can define a unique linear map $\epsilon_{C'}$,e.g. $\epsilon_{C'}(I) = 0, \epsilon_{C'}(C/I) = \epsilon(C/I)$ such that the following diagram commute

$$\begin{tikzcd} C \arrow[r, "\epsilon"] \arrow[d, "\pi"'] & k \\ C' \arrow[ur, "\epsilon_{C'}"'] & \end{tikzcd} \quad (1)$$

Then, Let $g = \Delta \circ (\pi \otimes \pi) : C \to C \otimes C \to C' \otimes C'$. Since $ker(\pi) = I$, so $\Delta(I) \subset ker(\pi \otimes \pi) = I \otimes C + C \otimes I$. Hence $I \subset ker(g)$ and we can use the similar way to define a unique linear map $\Delta_{C'}$ such that the following diagram commute

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\pi"'] & C \otimes C \arrow[d, "\pi\otimes\pi"] \\ C' \arrow[r, "\Delta_{C'}"'] & C' \otimes C' \end{tikzcd} \quad (2)$$

Through the (2) and coassociateive axiom of C, we get the following diagram commute

$$\begin{tikzcd}
C' \arrow[r, "\Delta'_C"] \arrow[ddd, bend right, "\Delta_{C'}"'] & C' \otimes C' \arrow[d, "\pi\otimes\pi"] \arrow[ddd, bend left, "1\otimes\Delta_{c'}"] \\
C \arrow[u, "\pi"] \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "1\otimes\Delta"] \\
C \otimes C \arrow[r, "\Delta\otimes 1"'] & C \otimes C \otimes C \arrow[d, "\pi\otimes\pi\otimes\pi"] \\
C' \otimes C' \arrow[u, "\pi\otimes\pi"] \arrow[r, "\Delta_{C'}\otimes 1"] & C' \otimes C' \otimes C'
\end{tikzcd}$$

which imply $(1 \otimes \Delta_{C'}) \circ \Delta_{C'} \circ \pi(C) = (\Delta_{C'} \otimes 1) \circ \Delta_{C'} \circ \pi(C) \ldots (3)$. Now, since the π is surjective, so $C' = im(\pi)$, and this imply the $\Delta_{C'}$ is comtiplication and satisfy coassciative through (3). Similary, using (1) and (2), we get $\epsilon_{C'}$ is a counit. Therefore, we get $(C', \Delta_{C'}, \epsilon_{C'})$ is a coalgbra. Also (1) and (2) will imply π is a morphism of coalgebra in this case. Now since both $\Delta_{C'}, \epsilon_{C'}$ are the unique linear map by our construction, so such coalgebra structure is unique. $\square$

It may be no surprise that the first isomorphism theorem will hold base on the following lemma

Lemma 1.1.2: Let $f : C \to C'$ be morphism of coalgebra. If $I \subset ker(f)$, then there is a unique coalgebra morphism $\bar{f}$ such that the following diagram commute

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\pi"'] & C' \\
C/I \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

Proof. If we treat $C, C', C/I$ as vector space, then clearly, we have a unique linear map $\bar{f}$ satisfy the requirement. We will show $\bar{f}$ is morphism of coalgebra.

First we have

$$\Delta_{C'} \circ \bar{f} \circ \pi(C) = \Delta_{C'} \circ f(C) = (f \otimes f) \circ \Delta_C = ((\bar{f} \circ \pi) \otimes (\bar{f} \circ \pi)) \circ \Delta_C = (\bar{f} \otimes \bar{f}) \circ (\pi \otimes \pi) \circ \Delta_C = (\bar{f} \otimes \bar{f}) \circ \Delta_{C/I} \circ \pi$$

. Since π is surjective, so we have $\Delta_{C'} \circ \bar{f} = (\bar{f} \otimes \bar{f}) \circ \Delta_{C/I}$. Using the same way, we get

$$\epsilon_{C'} \circ \bar{f} \circ \pi = \epsilon_{C'} \circ f = \epsilon_{C/I} \circ \pi$$

$\square$

Using the Lemma and Proposition 1.1, we conclude the first isomorphism theorem

Theorem 1.1.3: For any morphism of Coalgebra $f : A \to B$, we have $A/ker(f) \cong im(f)$.

Proof. This will be the similar proof of the case in group, ring or module. $\square$

1.2 Bialgebra

Let A be an vector space over k, we want to know the compatibility condition that we can define coalgebra (A, Δ, ϵ) and algebra (A, μ, η) simutanously over A

Theorem 1.2.1: The following two conditions are equivalent
(1): The maps μ, η are morphism of coalgebra
(2): The maps Δ, ϵ are morphism of algebra

Proof. All we need to just write down all commutative diagram. If the μ is a morphism of coalgebra, then we have the following two diagram commute,

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\mu} & A \\ {\scriptstyle (Id\otimes\tau\otimes Id)\circ(\Delta\otimes\Delta)}\downarrow & & \downarrow{\scriptstyle \Delta} \\ (A\otimes A)\otimes(A\otimes A) & \xrightarrow[\mu\otimes\mu]{} & A\otimes A \end{array} \qquad \begin{array}{ccc} A\otimes A & \xrightarrow{\epsilon\otimes\epsilon} & k\otimes k \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle Id} \\ A & \xrightarrow[\epsilon]{} & k \end{array}$$

Similarly, the μ is morphism of coalgebra means the following two diagram commute

$$\begin{array}{ccc} k & \xrightarrow{\eta} & A \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \Delta} \\ k\otimes k & \xrightarrow[\eta\otimes\eta]{} & A \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\eta} & A \\ & {\scriptstyle id}\nwarrow & \downarrow{\scriptstyle \mu} \\ & & k \end{array}$$

Notice these four commutativity diagrams are just same as the commutativity diagram for Δ, μ are morphism of algebra, and this imply both condition are equivalent. $\square$

Base on this result, we can equip coalgebra and algebra simutanously in a vector space H, which induce the following notation.

Definition: A Bialgebra $(H, \mu, \eta, \Delta, \epsilon)$ is a quintuple where (H, μ, η) is a algebra, (H, Δ, ϵ) is a coalgebra and μ, η is a morphism of coalgebra, Δ, ϵ is a morphism of algebra.

Example 1.2.1: By the example 1.1.3 and the remark, we have any dual space H^* of a finite dimension biaglebra H is also bialgebra.

Example 1.2.2: Any field is a bialgebra with trivial comutiplication, counit,mutiplication and unit

Example 1.2.3: In example 1.1.2, we define a coalgebra structure over $k[X]$ for G a set. Now if X is monoid with unit e and the associative map $\mu : X \times X \to X$. Then an algebra structure arise in $k[X]$ with unit e and mutiplication μ. Also we have

$$\Delta(xy) = (x \otimes x)(y \otimes y) = \Delta(x)\Delta(y)$$

$$\epsilon(xy) = 1 = \epsilon(x)\epsilon(1)$$

so both Δ and μ are morphism of algebra. Hence, $k[X]$ is a bialgebra.

Example 1.2.4 This example will introduce an important object of biaglebra. Recall that a tensor algebra $T(V)$over a vector space V is defined as $\bigoplus T^n(V), T^n(V) = V^{\otimes n}$. It has a universal property $Hom(V,W) \cong Hom_{Alg}(T(V),W)$. Now we show it is a bialgebera

Proposition 1.2.2: The tensor algebra has a unique Bialgebra structure defined by $\Delta(x) = x\otimes 1 + 1\otimes x, \epsilon(v) = 0$. This bialgebra is cocommutative and in particular, for element $v_1,\dots,v_n \in V$, we have $\epsilon(v_1 \dots v_n) = 0$ and

$$\Delta(v_1 \dots v_n) = 1 \otimes v_1 \dots v_n + \sum_{p=1}^{n-1}\sum_{\sigma} v_{\sigma(1)} \dots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \dots v_{\sigma(n)} + v_1 \dots v_n \otimes 1$$

where σ run over the element in S_n such that

$$\sigma(1) < \dots < \sigma(p), \sigma(p+1) < \dots \sigma(n)$$

. We call such permutation σ $(p, n-p)$ *shuffle*

Proof. The uniqueness come from the universal property of tensor algebra since $\Delta : T(V) \to T(V) \otimes T(V)$ and $\epsilon : T(V) \to k$ will be the only algebra morphism that its restriction on V will same as the desciption in the theorem.

Now given a squence of element $v_1,\dots,v_n \in V$, the fact $\epsilon(v_1,\dots v_n) = 0$ is trivial to check, so we will focus on computation of $\Delta(v_1 \dots v_n)$. We proceed an indction on n.
$n = 1$: the equation hold just by the definition.
$n-1 \Rightarrow n$: we have $\Delta(v_1 \dots v_n) = \Delta(v_1 \dots v_n)$

$$= \Delta(v_1 \dots v_n)(1 \otimes v_n + v_n \otimes 1)$$

$$= (1 \otimes v_1 \dots v_{n-1} + \sum_{p=1}^{n-2}\sum_{\sigma} v_{\sigma(1)} \dots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \dots v_{\sigma(n-1)} + v_1 \dots v_{n-1} \otimes 1)(1 \otimes v_n + v_n \otimes 1)$$

$$= 1 \otimes v_1 \dots v_{n-1}v_n + \sum_{p=1}^{n-2}\sum_{\sigma} v_{\sigma(1)} \dots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \dots v_{\sigma(n-1)}v_n + v_1 \dots v_{n-1} \otimes v_n + v_n \otimes v_1 \dots v_{n-1}$$

$$+ \sum_{p=1}^{n-2}\sum_{\sigma} v_{\sigma(1)} \dots v_{\sigma(p-1)}v_n \otimes v_{\sigma(p)} \dots v_{\sigma(n-1)} + v_1 \dots v_n \otimes 1$$

Where σ run over all *(p,n-1-p)-shuffle* in S_{n-1}. Let's rewrite the last summation to the form

$$= 1 \otimes v_1 \dots v_{n-1}v_n + \sum_{p=1}^{n-2}\sum_{\rho} v_{\rho(1)} \dots v_{\rho(p)} \otimes v_{\rho(p+1)} \dots v_{\rho(n-1)}v_n + v_1 \dots v_{n-1} \otimes v_n + v_n \otimes v_1 \dots v_{n-1}$$

$$+ \sum_{p=1}^{n-2}\sum_{\tau} v_{\tau(1)} \dots v_{\tau(p)}v_n \otimes v_{\tau(p+1)} \dots v_{\tau(n-1)} + v_1 \dots v_n \otimes 1$$

where the ρ run over *(p,n-1-p)-shuffle* of $S_{n-1}$and τ run over *(p-1,n-1)-shuffle* of the permuting set $\{1,\dots,n\}/\{p\}$. Now if $\sigma \in S_n$, then either (1): $\sigma(n) = n$, so the restriction of σ on S_{n-1} is ρ which is a *(p,n-1-p)-shuffle*, or (2): $\sigma(n) = p$, so the action of σ on $\{1,\dots,n\}/\{p\}$ is same as τ which is a *(p-1,n-1)-shuffle*. So this complete the induction.

The counit and coassociative is easily to check. To see the cocommutative, one should notice the cocommuativity is the consequence of the fact that the permutation

$$\begin{pmatrix} 1 & \dots & p & p+1\dots & n \\ n & \dots & p+1 & p\dots & 1 \end{pmatrix}$$

switches *(p,n-p)-shuffle* and *(n-p,p)-shuffle* $\square$

We now turn our attention to the primitive element.

Defintion: an element x in a bialgebra H is called primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$. We denote the set of all primitive element by *Prim(H)*

Remark: Notice any primitive element will have $\epsilon(x) = 0$ since $x = \epsilon(x)1 + \epsilon(x) = x + \epsilon(x)1$ (for any coalgebra, we always have $(id \otimes \epsilon)\Delta = id$), so the only way to vanish 1 is $\epsilon(x) = 0$

Proposition 1.2.3: The *Prim(H)* is closed under the commutator $[x, y] = xy - yx$.

Proof. Let $x, y \in Prim(H)$, then $\Delta(xy) = (= \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1 = 1 \otimes xy + x \otimes y = y \otimes x + xy \otimes 1$. Then we have

$$\Delta([x, y]) = 1 \otimes [x, y] = [x, y] \otimes y$$

which implies $[x, y]$ is primitive.

$\square$

Remark: The Primitive element forms a subvector space in a biaglebra. Hence, with the commutator, the primitive element will be a Lie algebra. So a functor $\mathcal{P} : Bialg \to Liealg$ arise through the assignment $H \mapsto Prim(H)$. We will talk more about this functor in Hopf algebra.

1.3 Hopf Algebra

Let $Hom(C, A)$ be all the linear map between vector space A and C, we want to discover its property when A is algebra and C is coalgebra. Now, let $f, g \in Hom(C, A)$, we define the the **convolution** $f * g$ be the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A$$

.

Proposition 1.3.1: $(Hom(C, A), *, \eta \circ \epsilon)$ is an algebra.

Proof. It is clear that the $*$ will be a bilinear map and $Hom(C, A)$ is closed under this operation. It remains to check the associativity and unit.
For associativity, we have

$$((f * g) * h)(x) = \sum_{(x)} (f(x')g(x''))h(x''') = \sum_{(x)} f(x')g(x'')h(x''') = f * (g * h)$$

and for the unit we have

$$(\eta\epsilon * f) = \sum_{x} \epsilon(x')f(x'') = f(\epsilon(x')x'') = f(x)$$

(notice $\epsilon(x) \in k$). So $\eta\epsilon$ is right unit. Using a similar way, we can check this is a left unit too. $\square$

If we have a Bialgebra H, then we may have the case $A = H = C$, so the $End(H)$ may equip with an addition algebra structure through convolution. This will induce the following defintion

Defintion: Let $(H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. An morphism $S \in End(H)$ is called **antipode** if

$$id * S = S * id = \eta\epsilon$$

. A bialgebra with an antipode is called **Hopf Algebra**

Remark: (1): Not every bialgebra has an antipode, but if it exists, it should be unique since if S and S' are antipodes, then

$$S = S * (\eta\epsilon) = S * (id * S') = (S * id) * S' = (\eta\epsilon) * S' = S'$$

(2) Some authors will require the antipode also be invertible to be a Hopf algebra. As you will see soon, there is some difference between these two definitions.
(3)Using the Sweedler notation, the definition of antipode is same to say

$$\sum_{(x)} x'S(x'') = \eta\epsilon(x) = \epsilon(x)1 = \sum_{(x)} S(x')x''$$

(4) The morphism of hopf algebra is indeed just the underlying morphism of bialgbera since one can check any bialgebra morphism will preserve antipode

The following theorem is an important proposition of the antipode S, and it probably will give you a reason why S is called antipode.

Proposition 1.3.2: Let $H^{op}_{cop} = (H, \Delta^{op}, \epsilon, \mu^{op}, \eta)$, then S is a bialgebra morphism from H to H^{op}_{cop} ,e.g

$$S(xy) = S(y)S(x), S(1) = 1, \forall x, y \in H(1)$$

and,

$$(S \otimes S)\Delta = \Delta^{op} S, \epsilon S = \epsilon(2)$$

Proof. We first prove (1) To see $S(1) = 1$, apply $x = 1$ to the fact $(id * S) = \eta\epsilon(1) = 1$

Now let $\rho, \phi \in Hom(H \otimes H, H)$ such that $\rho(x \otimes y) = S(xy), \phi(x \otimes y) = S(y)S(x)$. We want to show $\rho = \phi$.
Notice $(Hom(H \otimes H, H), *', \eta\epsilon^{\otimes})$ will again be a algebra. Without any confusion, we still denote it as$(Hom(H \otimes H, H), *, \eta\epsilon)$. We have

$$\rho * \mu = \sum_{(x \otimes y)} \rho((x \otimes y)')\mu((x \otimes y)'')$$

$$= \sum_{(x)(y)} \rho(x' \otimes y')\mu(x'' \otimes y'')$$

$$= \sum_{(x)(y)} \rho(x' \otimes y')x''y''$$

$$= \sum_{(x)(y)} S(x'y')x''y''$$
$$= \eta\epsilon(xy) = \epsilon(xy)$$

.

On the other hand, we hav e

$$\mu * \phi(x \otimes y) = \sum_{(x)(y)} x'y'S(y'')S(x'')$$
$$= \sum_{(x)(y)} x'(y'S(y''))S(x'')$$
$$\epsilon(y)\sum_{x} x'S(x'')$$
$$\epsilon(y)\epsilon(x)$$
$$\epsilon(xy)$$

Hence, we get $\rho * \mu = \mu * \phi = \eta\epsilon$ which imply ρ and ϕ are inverse of μ, so they should be equal.

Now, we turn to show (2). The trick we will use is same as above, we let $\rho = \Delta \circ S, \phi = (S \otimes S) \circ \Delta^{op}$, then one can check $\phi * \Delta = \Delta * \phi$, and we conclude $\phi = \rho$ by the same reason we use in the proof of (1). Lastly, we have

$$\epsilon \circ S(x) = \epsilon(S(\sum_{(x)} \epsilon(x')x'') = \epsilon(\sum_{(x)} \epsilon(x')S(x'')) = \epsilon(\eta\epsilon * S(x)) = \epsilon(\eta\epsilon(x)) = \epsilon(x)$$

as we expect.

□

Immediately, we get the following corollary.

Corollary Let

$$H = (H, \Delta, \epsilon, \mu, \eta)$$
$$H^{op} = (H, \Delta, \epsilon, \mu^{op}, \eta)$$
$$H^{cop} = (H, \Delta^{op}, \epsilon, \mu, \eta)$$
$$H^{op}_{cop} = (H, \Delta^{op}, \epsilon, \mu^{op}, \eta)$$

. Then H^{op}_{cop} is another Hopf algebra. If in addition, S is invertible, then we have

$$H^{op} \cong H^{cop},$$

So, we can see if we define hopf algebra is a bialgebra with invertible antipode, then this isomorphism will always hold. The invertible of antipode also imply the following fact.

Defintion: A **grouplike** element x in a coalgebra (H, Δ, μ) is a element such that $\Delta(x) = x \otimes x$

Proposition 1.3.3 Let H be a bialgebra, then the set of all grouplike element $G(H)$ is a monoid. If H is a Hopf algebra with invertible antipode S, then $G(H)$ is a group

Proof. We notice the unit "1" is a group-like element, so the $G(H)$ is a monoid. Now let S be a invertible element over H. First we notice $\epsilon(x) = 1$ by the fact $(id \otimes \epsilon)\Delta = id$, the using the definition of antipode we see $S(x)$ is the inverse of x □

We can also construct a Hopf algebra through over bialgebra H a through a algebra morphism $S : H \to H^{op}$.

Lemma 1.3.4: Assume H is a bialgebra generated by a set Xconsist of the elements $\sum_{(x)} x'S(x'') = \epsilon(x)1 = \sum_{(x)} S(x')x''$ where $S : H \to H^{op}$ a algebra morphism. Then S is an antipode.

Proof. The only thing we need to check is if $x, y \in X$, then $xy \in X$, which we left to the reader. □

1.3.1: the tensro algebra $T(V)$ is a hopf algebra with antipode determined by $S(1) = 1$ and $S(v_1 \dots v_n) = (-1)^n v_n \dots v_1, \forall v_i \in V$

Exapmle 1.3.2: Let G be a monoid, then the example 1.2.3 tells us it is a bialgebra, and it is a Hopf algebra if and only if G is a group because if S is an antipode, then $id * S(x) = xS(x) = \epsilon(x)1 = 1$, so $S(x) = x^{-1}$. Conversely, $S(x) = x^{-1}$ will give an antipode over $k[G]$.

Remark: Using the similar way in the exmaple 1.3.1, we can show that for a monoid G, $Fun(G, k)$ consist of all function from G to a field k is a bialgebra and will be a hopf algebra if G is a group with the antipode given by $S(f)(x) = f(x^{-1})$. In particular, if G is a affine algebraic group.

Example 1.3.3 The bialgebra $T(V)$ is a hopf algebra with the antipode *Example 1.3.4* This will be our majority example to discuss in this section. Let $\mathfrak{g}$ be a Lie algebra. A universal Enveloping algebra $U(\mathfrak{g})$ is defined as $T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])$. It is the "smallest" algebra that "contains" $\mathfrak{g}$ such that the commutator coincident with Lie bracket with the universal property $Hom_{LieAlg}(\mathfrak{g}, A) \cong Hom_{Alg}(U(\mathfrak{g}), A)$ for any associative algebra A. As a quotient of tensor algebra, it does have hopf algebra that defined in the same way of Tensor algebra (However, we should check all map is well-defined).

Now, let's take our attention to primitive element of $U(\mathfrak{g})$. As we mention previously, we have a functor $\mathcal{P} : Hopfalg \to LieAlg, H \mapsto Prim(H)$, conversely, we have $\mathcal{H} : LieAlg \to Hopfalg, \mathfrak{g} \mapsto U(\mathfrak{g})$.

Proposition 1.3.5: $\mathcal{P}$ is left adjoint to $\mathcal{H}$.

Proof. We notice if we have a $f \in \mathbf{Hom}_{HopfAlg}(G(\mathfrak{g}), H)$, then $f([x, y]) = f(\Delta(x \otimes y - y \otimes x)) = f(\Delta(x \otimes y)) - f(\Delta(y \otimes x)) = \Delta'(f(x) \otimes f(y)) - \Delta'(f(y) \otimes f(x)) = [f(x), f(y)]$, so the assignment $f \mapsto f \circ \Delta$ gives a map from f to a morphism of Lie algebra between $\mathfrak{g}$ and $\mathrm{Prim}(H)$. On the other hand, every $g \in \mathbf{Hom}_{LieAlg}(\mathfrak{g},\mathrm{Prim}(H))$ could be extended to a unique morphism of $\mathbf{Hom}_{HopfAlg}(G(\mathfrak{g}), H)$ by the universal property of $U(\mathfrak{g})$. So $\mathcal{H}$ is left adjoint to the $\mathcal{P}$

□

If k is a field with characteristics 0, then

Proposition 1.3.6: $\mathfrak{g} \cong Prim(U(\mathfrak{g}))$

Proof. By our construction, any element in $U(\mathfrak{g})$ is primitive, so every element in $\mathfrak{g}$ is primitive. We want to show that every primitive element is in $\mathfrak{g}$. Consider the $f : S(\mathfrak{g}) \to U(\mathfrak{g})$ defined $x_1 \otimes \cdots \otimes x_n \mapsto \frac{1}{n!} \sum_{\tau \in S_n} x_{\tau(1)} \ldots x_{\tau(n)}$. This is a coalgebra isomorphism(using PBW theorem), and the S, a symmetric algebra is just a universal enveloping algebra for an abelian Lie algebra, so it will suffice to show the conclusion for an abelian Lie algebra. In this case, since char(K)=0, so the dual space of coalgebra is the ring of formal power series and the primitive element should vanish for all elements in degree at least 2 (that all the elements could be decomposed), so this implies all the primitive elements should have degree 1 and hence they're inside the Lie algebra. □

Corollary 1.3.7: The functor $\mathcal{H} : LieAlg \to HopfAlg$ is fully faithful.

Remark: One may curious about in what condition, the functor $\mathcal{H}$ is equivalent? That is, what kind of Hopf algebra will be an enveloping algebra of a Lie algebra $\mathfrak{g}$? This question will induce the Milnor-Moore theorem.

2 Quantum Group

This section will be an warm-up section for the section 2.2, we will introduce the q-deformation on enveloping algebra of $\mathfrak{sl}(2)$ so that we have a quantum group $U_q(\mathfrak{sl}(2))$. This noncommutative and non-cocommutative algebra was introduced by Drinfeld as a more rigidity object than $\mathfrak{sl}(2)$. In this section, we will fix a ground field $k = \mathbb{C}$, but the whole theorem could be generated to more general case as we will see in the next section.

2.1 The quantum group $U_q(\mathfrak{sl}(2))$

2.1.1 quantum group $U_q(\mathfrak{sl}(2))$

We assume $q \neq 1$ or -1 and is not the root of unity throughout the whole section.

In the next few sections, we will introduce the quantum group $U_q(\mathfrak{g})$ when $\mathfrak{g} = \mathfrak{sl}(2)$. We first introduce some combinatoric notation. For an integer n, a quantum integer ($q-$integer) $[n]$ is defined as

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}$$

$[n]$ is never 0 if n is not root of unity. If q is root of unity. Let d be the smallest integer such that $q^d = 1$, we call d the order of q. We could assume the order should bigger than 2 by our assumption, and define

$$e = \begin{cases} d, & \text{d is odd} \\ d/2, & \text{d is even} \end{cases} \tag{1}$$

. if q is not a root of unity, we must have $q = e = \infty$, so we have

$$[n] = 0 \Leftrightarrow n \equiv 0 \bmod e$$

if $0 < k < n$, set $[0]! = 1$

$$[k]! = [1][2] \ldots [k]$$

, if $k > 0$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

finally, we have $q-$binomial equation

$$(x+y)^n = \sum_{k=0}^{n} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}$$

Remark: (1) Notice $[n] \to n$ and $\begin{bmatrix} n \\ k \end{bmatrix} \to \binom{n}{k}$ when $q \to 1$

(2) The definition given above could be extended to the case that q is indeterminate. In this case we denote $q-$interger by $[n]_q$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and we notice they are element of $k(q)$, the rational function of variable q.

(3) Let q be a indeterminate, or some complex number that is not root of unity, then using the induction we have

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^n \begin{bmatrix} n \\ k \end{bmatrix} + q^{-m+n+1} \begin{bmatrix} n \\ m-1 \end{bmatrix}$$

. So these elements are actually in $\mathbb{Z}[q, q^{-1}]$.

Defintion: The **Quantum group** $U_q(\mathfrak{sl}(2))$ is an algebra generated by four variable E, F, K, K^{-1} such that

$$KK^{-1} = K^{-1}K = 1 \cdots (1.1)$$

$$KEK^{-1} = q^2E, KFK^{-1} = q^{-1}F \cdots (1.2)$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} \cdots (1.3)$$

For simplicity, we denote it as U_q.

We Expect to recover $U(\mathfrak{sl}(2))$ from U_q by set $q = 1$. However, this is impossible by this definition. Hence, we give another equivalent presentation of U_q.

Proposition 2.1.1.1: The algebra U'_q generated by five variable E, F, K, K^{-1}, L with the following relation is isomorphic to U_q.

$$KK^{-1} = K^{-1}K = 1, \cdots (2.1)$$

$$KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F, \cdots (2.2)$$

$$[E, F] = L, (q - q^{-1}L) = K - K^{-1}, \cdot (2.3)$$

$$[L, E] = q(EK + K^{-1}E), [L, F] = -q^{-1}(FK + K^{-1}F) \cdots (2.4)$$

Remark As you see one of the benefits of this presentation is we can conclude all values of parameter q especially when $q = 1$. Hence, this presentation is in fact much more powerful than the definition we give, but for our purpose, it will be sufficient to use the simpler one we give.

Proof. Let $\rho : U_q \to U'_q$ be

$$\rho(E) = E, \rho(K) = K, \rho(F) = F$$

and $\psi : U'_q \to U_q$

$$\psi(E) = E, \psi(K) = K, \psi(E) = E, \psi(L) = [E, F]$$

.It is clearly that ρ is a well-defined morphism of algebra. We check the ψ is also a well-defined morphism of algebra. It will be sufficient to check all relation (2.1) to (2.4) that the image under ψ still holds in U_q. (2.1),(2.2) are easy to check, and for (2.3) we have

$$(q - q^{-1})\psi(L) = (q - q^{-1})[E, F] = K - K^{-1}$$

and for (2.4), we have

$$[\psi(L), \psi(E)] = [[E, F], E] = \frac{1}{q - q^{-1}}[K - K^{-1}], E$$

$$= \frac{(q^2 - 1)EK + (q^2 - 1)K^{-1}E}{q - q^{-1}}$$

$$= q(EK + K^{-1}E)$$

and one can check for the $[\psi(L), \psi(F)]$ in a similar way. So we get ψ is morphism of algebra.

Now by our definition, ψ and ρ are mutually inverse for all the generator, so it remains to check all the relation, e.g $\psi\rho(KEK^{-1}) = KEK^{-1}$ and $\rho\psi(KEK^{-1} = KEK^{-1}$. The only difficult one is (2.4), so we check

$$\rho\psi([L, K]) = \rho([[E, F], K])$$

$$= \rho(q(EK + K^{-1}E)$$

$$= q(EK + K^{-1}E) = [L, K]$$

and one may prove another one in a similar way. Hence we conclude that ψ and ρ are mutually inverse and therefore $U_q \cong U'_q$

□

To find the relation between $U = U(\mathfrak{sl}(2))$ and U'_1, we first recall that U is generated through 3 variables $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with the relation

$$[X, Y] = H, [H, X] = 2X, [H, Y] = -2Y$$

Proposition 2.1.1.2: $U'_1 \cong U[K]/(K^2 - 1), U \cong U'_1/(K - 1))$

Proof. When $q = 1$, we have

$$KK^{-1} = K^{-1}K = 1, \cdots (3.1)$$

$$KEK^{-1} = E, KFK^{-1} = F, \cdots (3.2)$$

$$[E, F] = L, 0 = K - K^{-1}, \cdot (3.3)$$

$$[L, E] = EK + K^{-1}E, [L, F] = -1(FK + K^{-1}F) \cdots (3.4)$$

The relation 3.1 and 3.2 imply K is central element. Using the (3.3), we have $K^2 = 1$, and the (3.4) could be rewritten as

$$[L, E] = 2EK, [L, F] = -2FK$$

. So, the map $K \mapsto K, E \mapsto XK, F \mapsto FK, L \mapsto H$ will give us an isomorphism of U_1' and $U[K]/(K^2 - 1)$. The second isomorphism could be gained through the function $X \mapsto E, Y \mapsto F, H \mapsto L$. □

In particularly, we can have a projection from U_1' to U be through the map defined in second isomorphism with a additional assignment $K \mapsto 1$. So, we may view U as deformation of U_q.U_q has many property parallel to U, but the biggest difference is that it is niether cocommutative nor commutative.

Proposition 2.1.1.3: The Δ, ϵ, S defined as follow, together with algebra structure over U_q is a Hopf algebra

$$\Delta(E) = 1 \otimes E + E \otimes K, \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$

$$\Delta(K) = K \otimes K\ \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\epsilon(E) = \epsilon(K) = 0, \epsilon(K) = \epsilon(K^{-1}) = 1$$

$$S(E) = -EK^{-1}, S(F) = -KF, S(K) = K^{-1}, S(K^{-1}) = K$$

Proof. (a): It will be easy to see the μ is an algebra morphism, we will show Δ is an algebra morphism. It will be sufficient to check

$$\Delta(K)\Delta(K^{-1}) = \Delta(K^{-1})\Delta(K) = 1$$

$$\Delta(K)\Delta(E)\Delta(K^{-1}) = q^2\Delta(E)$$

$$\Delta(K)\Delta(F)\Delta(K^{-1}) = -q^2\Delta(F)$$

$$[\Delta(E), \Delta(F)] = \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}$$

The first three relations are easily to check, so we only check the last one

$$[\Delta(E), \Delta(F)] = (1 \otimes E + E \otimes K)(K^{-1} \otimes F + F \otimes 1) - (K^{-1} \otimes F + F \otimes 1)(1 \otimes E + E \otimes K)$$

$$= K^{-1} \otimes EF + F \otimes E + EK^{-1} \otimes KF + EF \otimes K$$

$$-K^{-1} \otimes FE - K^{-1} \otimes FK - F \otimes E - FE \otimes K$$

$$= K^{-1} \otimes [E, F] + [E, F] \otimes K$$

$$= \frac{K^{-1} \otimes (K - K^{-1} + (K - K^{-1}) \otimes K)}{q - q^{-1}}$$

$$= \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}$$

.

Then one can easily check the Δ, μ will satisfy the coassociative and counit axiom. It only remain to show that S is antipode. We apply the propostion 1.3.3 here. First we show S is a morphism from H to H^{op}, that is same to show

$$S(K)S(K^{-1}) = S(K^{-1})S(K) = 1$$

$$S(K^{-1})S(E)S(K) = q^2 S(E)$$

$$S(K^{-1})S(F)S(K) = -q^2 S(F)$$

$$[S(F), S(E)] = \frac{S(K^{-1}) - S(K)}{q - q^{-1}}$$

Again, this is not pretty hard to show, and we just show the last one

$$[S(F), S(E)] = -KFEK^{-1} - EK^{-1}KF = [F, E]$$

$$= \frac{S(K^{-1}) - S(K)}{q - q^{-1}}$$

To conclude the result, we only need to check for all variable, we have

$$\sum_x x' S(x'') = \sum_x S(x')x'' = \epsilon(x)$$

□

This is Hopf algebra is neither commutative nor cocommutative because of the following Lemma

Lemma 2.1.1.4: A Hopf algebra is commutative or cocommutative if and only if $S^2 = Id$

Proof. We show the case for commutative, the cocommutative case could be proved in a similar way.

If H is commutative, we have

$$\sum_x x' S(x'') = \sum_x S(x'')x' = \eta\epsilon$$

Now consider $S * S^2$, we have

$$S * S^2 = \sum_x S(x')S^2(x'')$$

$$= S(\sum_x S(x'')x')$$

$$= S(\epsilon(x)1) = \epsilon(x)S(1) = \epsilon(x)1$$

So this imply S^2 is right inverse of S, and by the uniqueness of inverse, we have $id = S^2$

Conversely, if $S^2 = id$, then

$$\sum_x S(x'')x' = S^2(\sum_x S(x'')x')$$

$$= S(\sum_x x'' S(x'))$$

$$= S(\sum_x)x' S(x'')$$

$$= \sum_x S(x')x''$$

$$= \sum_x x' S(x'')$$

For the case of cocommutative, we have $\sum_x x'S(x'') = \sum_x x''S(x') = \eta\epsilon$ and one can use the similary way to get our conclusion. □

Proposition 2.1.1.5: The Hopf algebra U_q is not commutative and cocommutativ. when $q \neq 1$

Proof. We should notice $S^2(E) = q^2 E, S^2(F) = q^{-2} F$ □

Remark: With additional setting $\Delta(L) = K^{-1} \otimes L + L \otimes K, \epsilon(L) = 0, S(L) = -L$, we can equip a Hopf algebra over U_q', then we can see the isomorphism in proposition 2.1.1 is a Hopf isomorphism. Also, the isomorphism $U_1' \cong U[K]/(K^2 - 1)$ is isomorphism of Hopf algebra.

Remark: In this section, we only introduce the quantum group U_q over a field $\mathbb{C}$ for q a non-root of unity complex number. However we can actually modify the definition of U_q by replacing q of a variable v. That is, we can use the definition for U_q to define $U_v^{rat}(\mathfrak{sl}(2))$ over the field $k(v)$, the field of rational function of v, and k a field . The $k[v, v^{-1}]-$ subalgebra of U_q^{rat} generated by K, K^{-1}, E, F is called De Concini-Kac quantum group $U_v^{DK}(\mathfrak{sl}(2))$. The $k[v, v^{-1}]-$subalgebra of U_q^{rat} generated by K, K^{-1} and $E^{(n)} = \frac{E^n}{[n]_v!}, F^{(n)} = \frac{F^n}{[n]_v!}$ is called Lusztig quantum group U_v^D. Both quantum groups are Hopf algebra, and we can also notice there is an inclusion $U_v^{DK} \to U_v^L$. Moreover, we have $U_v^{DK} \cong U_q/(q - v)$ for $q \in k^\times$

2.1.2 Representation of $U_q(\mathfrak{sl}(2))$

We fix a q that is not the root of unity.

Definition: Let V be a U_q-module, λ a scalar. The subspace V^λ is generated by all the vector v such that $Kv = \lambda v$. The λ is called the **weight** of V is $V^\lambda \neq 0$

Definition: Let V be a U_q-module, λ a scalar. A vector $v \in V$ is called the **highest weight** of weight λ if $Ev = 0$, $Kv = \lambda v$. A U_q module is called **highest weight module** of highest weight λ if it is generated by the highest weight vector of weight λ.

The following relation of weight space V^λ will be useful

Lemma 2.1.2.1: $EV^\lambda \subset V^{q^2\lambda}, FV^\lambda \subset V^{q^{-2}\lambda}$.

Proof. For any $v \in V^{\lambda}$, we have

$$KEv = q^2 EKv = q^2 \lambda v$$

and similarly

$$KFv = q^{-2} FKv = q^{-2} \lambda v$$

$\square$

The highest weight vector in fact exist in any finite dimension U_q−module

Proposition 2.1.2.2: Any non-zero finite dimension U_q−module V has a highest weight vector.

Proof. The action of K on V gives an endomorphism $f : v \mapsto Kv$. Since we discuss over an algebraically closed field $k = \mathbb{C}$ and the dimension is finite, so we should have a eigenvalue λ of f and its correspond eigenvector v such that $Kv = \lambda v$. If $Ev = 0$, then the proof is completed, otherwise, we suppose $Ev \neq 0$. Now, let's consider the sequence of element element $Ev, E^2 v \dots E^m v \dots$. The lemma 2.2.1 tells us $E^m v \subset V^{q^{2m}} \lambda$, e.g. $E^m v = q^{2m} \lambda v$, and the action of E provide another endomorphism $v \mapsto Ev$. Hence, $E^m v$ are distinct eigenvector for distinct eigenvalue $q^{2m}\lambda$, since the dimension of V is finite, so there is a $n \in \mathbb{Z}$ such that $E^n v = 0$. Then $E^n v$ is the highest weight vector.

$\square$

Remark: The only possible eigenvalue of endomorphism induced by the action of E and K is 0 for a finite dimension U_q−module, since if we have a non-zero λ and it has a non-zero eigenvector v, then $E^m v = q^{2m} \lambda v$ are eigenvector for distinct eigenvalue $q^{2m}\lambda$ for all $m \in \mathbf{Z}$, so we have infinite eigenvalue which is impossible. The same process implies the endomorphism induced by action of K has only 0 as eigenvalue.

Let V be a finite dimension U_q−module that is generated by the highest vector, we want to show this module is simple

Lemma 2.1.2.3: Let v be a higest vector of weight λ. Set $v_0 = v, v_p = \frac{1}{[p]!} F^p v$, for $p > 0$, then we have

$$Kv_p = \lambda q^{-2p} v_p, Ev_p = \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} v_{p-1}, Fv_{p-1} = [p] v_p$$

Proof. The last one is trivial because $[p]v_p = F \frac{1}{[p-1]!} F^{p-1} v = Fv_{p-1}$. The remained two relation could be get through the following lemma $\square$

Lemma 2.1.2.4: For the $U_q(\mathfrak{sl}(2))$, the generators has the following relation for $m \geqslant 0, n \in \mathbb{Z}$

$$E^m K^n = q^{-2mn} K^n E^m, F^m K^n = q^{mn} K^n F^m$$

$$\begin{aligned}
[E, F^m] &= [m] F^{m-1} \frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q - q^{-1}} \\
&= [m] \frac{q^{(m-1)}K - q^{-(m-1)}K^{-1}}{q - q^{-1}} F^m
\end{aligned}$$

$$[F, E^m] = [m]\frac{q^{(m-1)}K - q^{-(m-1)}K^{-1}}{q - q^{-1}}E^{m-1}$$
$$= [m]E^{m-1}\frac{q^{(m-1)}K - q^{-(m-1)}K^{-1}}{q - q^{-1}}$$

Proof. For the first one, we notice

$$(KEK^{-1})^m = KEK^{-1}KEK^{-1}\dots KEK^{-1}KEK^{-1}$$
$$= KE^mK = q^{2m}E \Rightarrow KE^m = q^{2m}E^mK$$

then since

$$K^2E^m = q^{2m}KE^mK = q^{4m}E^mK^2$$

we can use induction on n to conclude the final result. Using the same way, we get the second one.

The third and fourth just easily gained by the induction on m □

Theorem 2.1.2.4: Let V be a finite dimension U_q-module generated by a highest weight vector v of weight λ. Then
(a) The scalar λ is in the form $\lambda = \epsilon q^n$ for $\epsilon = \pm 1$ and n is a integer that $n = dim(V) + 1$
(b)We have $v_p = 0$ for $p > n$. Moreover, $\{v_0, v_1 \dots, v_n\}$ is a basis of V
(c)The endomorphism $v \mapsto Kv$ is diagonalizable with (n+1) eigenvalue $\{\epsilon q^n, \epsilon q^{n-2}, \dots \epsilon q^{-n+2}, \epsilon q^{-n}\}$
(d) Any other highest vector are highest vector of λ and is scalar multiplication of v.
(e) The module V is simple.

Proof. (a) and (b): By the fact $Kv_p = \lambda q^{-2p}v_p$, we have the sequence of element $v_0, v_1, \dots, v_p$ are eigenvector for distinct eigenvalue λq^{-2p}. Since the V is finite dimension, so there is a $n \in \mathbb{Z}$ such that $v_n \neq 0, v_{n+1} = 0$. Then the fact $Fv_{p-1} = [p]v_p$ implies for $m > n, v_m = 0$ and $v_m \neq 0, m \leqslant n$. Moreover, we have

$$0 = Ev_{n+1} = \frac{q^{-n}\lambda - q^n\lambda^{-1}}{q - q^{-1}}v_n$$

. So we must have $q^{-n}\lambda = q^n\lambda^{-1}$, that is same to say $\lambda = \pm q^n$.
The set $\{v_0, \dots, v_n\}$ should be a basis of V since it is composed by eigenvectors for distinct eigenvalue; In fact, since V is generated by v as U_q-module, so the Lemma 2.1.2.3 and the definition of v_p will imply any element in V will in be the linear combination of $\{v_i\}_i$. We finish the proof of (a) and (b).
(c): The endomorphism is diagonalizable since it has a basis composed by the eigenvector of distinct eigenvalue $\{\epsilon q^n, \epsilon q^{n-2}, \dots \epsilon q^{-n+2}, \epsilon q^{-n}\}$ (use the fact $Kv_p = \lambda q^{-2p}v_p$ and $\lambda = \epsilon q^n$).
(d): Let v' be another highest weight vector, then by definition, it should be the eigenvector of $v \mapsto Kv$, so it should be the scalar of some v_i. However, since we also should have $Ev' = 0$, and $Ev_i = \frac{q^{-(i-1)}\lambda - q^{i-1}\lambda^{-1}}{q-q^{-1}}$, so the only possible case is when $i = 0$, which mean v' is scalar of $v_0 = v$.
(e) Let V' be a non-trivial U_q-submodule of V, then it should consist a highest weight vector space v'. By (d), it should be scalar multiple of v, by this imply $v \in V'$. Since V is generated by v, we must have $V \subset V'$ which imply $V = V'$ □

Theorem 2.1.2.5 Any simple finite dimension U_q-module is generated by a highest weight vector. Two finite dimension U_q-module generated by highest weight vector space in same weight are isomorphic.

Proof. Let V be a finite dimension simple U_q-module, then it consists a highest weight vector v which will generate a non-trivial submodule V' of V, since V simple, we must have $V' = V$. If we have two finite dimension module generated by v, v', the highest weight vector of same weight, then the map $f : v_i \mapsto v_i'$ will be an isomorphism. □

Remark: The theorem 2.1.2.4 and 2.1.2.5 implies we have unique (up to isomorphism) (n+1) dimension simple U_q-module generated by a highest weight vector of weight ϵq^n. We denote it as $V_{\epsilon,n}$ and its corresponding morphism of algebra $\rho_n : U_q \to End(V_{\epsilon,n})$. For example, when $n = 0$, we have $V_{\epsilon,0} = k$,

$$\rho_{\epsilon,0}(K) = \epsilon, \rho_{\epsilon,0}(E) = \rho_{\epsilon,0}(F) = 0$$

.

As we shown above, for a weight λ with special value, there is a module V generated by a highest weight vector v of weight λ. Our next target it to show that there is a highest weight module of arbitrary highest weight.

Let $\lambda \neq 0$ be arbitrary scalar, we define a infinite dimension vector space $V(\lambda)$ with the basis $\{v_i\}_{i\in\mathbf{N}}$. for $p > 0$, set

$$Kv_p = \lambda q^{-2p} v_p, K^{-1} v_p = \lambda^{-1} q^{2p} v_p$$

$$Ev_{p+1} = \frac{q^{-p}\lambda - q^p\lambda^{-1}}{q - q^{-1}} v_p, Fv_p = [p+1]v_{p+1}$$

and $Ev_0 = 0$

Proposition 2.1.2.6: The relation given above define a U_q-module structure for $V(\lambda)$. The element v_0 generates $V(\lambda)$ as U_q-module and it is a highest weight vector of weight λ.

Proof. By computation, we have

$$KK^{-1}v_p = v_p = K^{-1}Kv_p$$

$$KEK^{-1} = q^2 Ev_p, KFK^{-1} = q^{-2}Fv_p$$

and

$$\begin{aligned}
[E,F]v_p &= ([p+1]\frac{q^{-p}\lambda - q^p\lambda^{-1}}{q - q^{-1}} - [p]\frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}})v_p \\
&= \frac{q^{2p}\lambda - q^{2p}\lambda^{-1}}{q - q^{-1}} v_p \\
&= \frac{K - K^{-1}}{q - q^{-1}} v_p
\end{aligned}$$

The above relations show $V(\lambda)$ is a U_q-module.
Now, it is not hard to see $Kv_0 = \lambda v_0, Ev_0 = 0$, so v_0 is the highest weight vector of weight λ. Also, using the induction on p and the relation $Fv_p = [p+1]v_{p+1}$, we can conclude $v_p = \frac{1}{[p]!}F^p v_o$, so the v_0 generates the $V(\lambda)$ □

Defintion: The module $V(\lambda)$ is called **Verma module**.

The Verma module has the following universal property.

Proposition 2.1.2.7: For any U_q-module of highest weight λ, there is a surjective map $f : V(\lambda) \to V$.

Proof. Let v be the highest weight vector generates V, we set $f : v_p \mapsto \frac{1}{[p]!}F^p v$, then the relation in lemma 2.2.3 implies this map is linear. In particularly, we have $f(v_0) = v$, which implies this map is surjective.

□

2.1.3 When q is root of unity

We will briefly discuss the module of U_q when $q \in k^\times$ is a root of unity. Let's assume its order is d, then, a difference from the case that q is not root of unity is

$$[n] = 0 \Leftrightarrow n \equiv 0 \bmod e$$

where $e = d$ if d is odd and $e = d/2$ if d is even.

Proposition 2.1.3.1: Any non-trivial finite dimension U_q simple module are in the form $V_{\epsilon,n}$ where $\epsilon = \pm 1$ and $0 \leqslant n < e - 1$.

Proof. Since $1, q, q^2, \dots q^{2n}$ are distinct scalar for $n < 0$, so the proof will same as the case in q is not root of unity by consider the highest weight module of highest weight q^{2n} □

Now, the biggest difference will appear when $n > e$. Before we show our main result, we need two lemmas.

Lemma 2.1.3.2: E^e, F^e, K^e are in the centre of U_q.

Proof. This is just because of the relation of generator and Lemma 2.1.2.4. For example, by 2.1.2.4, $E^e K = q^{2e} K E^e$, and we notice $q^{2e} = 1$ by our definition of e, so E^e commute with K, also we have

$$[F, E^m] = [m]\frac{q^{(m-1)}K - q^{-(m-1)}K^{-1}}{q - q^{-1}}E^{m-1}$$

Since $[e] = 0$, so E^e commute with F. Using the similar way we can proof K^e, F^e are in the centre. □

Lemma 2.1.3.3: The action of centre element on finite dimension U_q-module is multiplication of scalar, e.g $c \in C(U_q)$, $v \in V$ a finite dimension simple module, then $c * v = \lambda v, \lambda \in k$

Proof. action of c on V induce an endomorphism $f : v \mapsto cv$. Since V is a finite dimension and k is algebraically closed by our assumption, so we should have an eigenvalue λ, then consider $f - \lambda Id$. This is not an isomorphism, so by Shur's Lemma, it should be 0. Hence, we get $f = \lambda Id$ as we desire. □

Remark: This Lemma is in fact correct for arbitrary irreducible finite-dimensional representation $\rho : A \to End(V)$ of an algebra A over an algebraically closed field,e.g $\rho(v) = \lambda v$.

Proposition 2.1.3.4: The U_q doesn't have simple module with dimension $n > e$.

Proof. We show by contradiction. Assume we have a simple module in dimension bigger than e, then we claim that there is a non-zero eigenvector of action of K such that $Fv = 0$

Proof. Suppose by contradiction that there is no such eigenvector, then let v be a non-zero eigenvector of the action of K, then I claim $v, Fv, \dots, F^{e-1}v$ will generate a non trivial submodule V' of V.

To show this, it will enough to show that the action of K, F, E to V' is invariant, e.g $EV' \subset V'$. Clearly, V' should invariant under K since v eigenvector of the action of K.

To see it is invariant under action of F, we see $F(F^n v) \in V'$ if $n < e - 1$. If $n = e - 1$, then

$$F(F^{e-1}v) = F^e v = \lambda v$$

because of the lemma 2.1.3.2 and 2.1.3.3. and this $\lambda \neq 0$ because otherwise we will have $F^{e-1}v$ is a vector such that $F(F^n v) \neq 0$ and it is a eigenvector of action of K which will contradict to our assumption.

To see the action of E is invariant for V', we need an element $C_q = EF + \frac{q^{-1}K+qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2}$. One can check this is an central element: It easy to see K commute with C_q because of the fact $KEFK^{-1} = KEK^{-1}KFK^{-1} = EF \Rightarrow KEF = EFK$. Also we have

$$FC_q = EFE + \frac{q^{-1}FK + q^{-1}FK^{-1}}{(q-q^{-1})^2} = EFE + \frac{q^{-1}q^2KF + qq^{-2}K^{-1}F^2}{q-q^{-1}} = EFE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}F = FC_q$$

and one can use the same way to show E is also commute with C_q.
Now, for a $p > 0$, we have

$$\begin{aligned} E(F^p v) &= EF(F^{p-1}v) \\ &= (C_q - \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2})(F^{p-1}v) \\ &= \alpha F^{p-1}v - \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2}F^{p-1}v \end{aligned}$$

and this clearly imply the action of E is invariant for V'. When $p = 0$, we use the same argument by the fact $v = \lambda^{-1}F^e v$ □

Hence, we now must have a eigenvector v of action of K and $Fv = 0$. Then we claim $v, Ev, \dots E^{n-1}v$ is a nontrivial submodule of V. We also check it is invariant under the action of K, E, F. This is clearly correct for K since v is an eigenvector vector of action of K and F because $Fv = 0$ and the Lemma 2.1.2.4. To see the action of E is invariant, we have $E(E^{e-1}(v)) = E^e v = \lambda v$ because of lemma 2.1.3.3 and 2.1.3.2. The case for $E^n v, n < e - 1$ is trivial, so we show all three action is invariant for V', which is a contradiction. □

Remark: The element C_q is called **Quantum Casimir element**, it plays an important role in the center Z_q of U_q when q is not root of unity. In fact, the Z_q is a polynomial algebra generated by C_q.

We already discussed the finite dimensional simple module with dimension bigger or smaller than e. We eventually end this section by introduce the case for dimension equal to e.

First, we construct two different modules in dimension e as follow: (1)The module $V(\lambda, a, b)$: Let $\{v_0, \ldots, v_{e-1}\}$ be the basis. For $0 \leqslant p \leqslant e-1$, set

$$Kv_p = \lambda q^{-2p} v_p$$

$$Ev_{p+1} = (\frac{q^{-p}\lambda - q^p\lambda^{-1}}{q-q^{-1}}[p+1] + ab)v_p$$

$$Fv_p = v_{p+1}$$

for λ, a, b are scalar and $Ev_0 = av_{e-1}, Fv_{e-1} = bv_0, Kv_{e-1} = \lambda q^{-2(e-1)} v_{e-1}$.
(2) The module $V(\mu, c)$: Set $\{v_0, \ldots, v_{e-1}\}$ be the basis. For $0 \leqslant p \leqslant e-1$, set

$$Kv_p = \mu q^{2p} v_p$$

$$Fv_{p+1} = \frac{q^{-p}\mu - q^p\mu^{-1}}{q-q^{-1}}[p+1]v_p$$

$$Ev_p = v_{p+1}$$

where μ, c are scalar and $Fv_0 = 0, Ev_{e-1} = cv_0, Kv_{e-1}\mu q^{-2} v_{e-1}$.

Remark: It is not hard to prove that there is an unique automorphism $\sigma : U_q \to U_q.K \mapsto K^{-1}, F \mapsto E, E \mapsto F$, then we observe $V(\mu, c) \cong V(\mu^{-1}, 0, c)$ by $ua \mapsto \sigma(u)a$, where $u \in U_q$

Theorem 2.1.3.5: Any finite dimensional simple module of U_q in dimension e is isomorphic to one of the three module,
(1)$V(\lambda, a, b), b \neq 0$
(2)$V(\lambda, a, 0), \lambda \neq \pm q^{j-1}, 1 \leqslant j \leqslant e-1$
(3)$V(\pm q^{j-1}, c), c \neq 0, 0 \leqslant j \leqslant e-1$.

Proof. Since the module is finite dimension and we discuss over a algebraically closed field $k = \mathbb{C}$, so the the action of K has an eigenvector v such that $kv = \lambda v$. Then, we have two cases:
(1) $Fv \neq 0$: By lemma 2.1.3.2 and 2.1.3.3, we have $F^e v = bv$ and $E^e v = av$. Here we can separate into two sub-cases
(a) if $b \neq 0$. Consider $v_0 = v, v_1 = Fv, \ldots, v_{e-1} = F^{e-1}v$. Through the definition, we immediately have $Fv_{e-1} = bv_0$, and for $Ev_{e-1} = av_0$ for a scalar, we just use the same trick in proposition 2.1.3.4 by using C_q. By lemma 2.1.2.4, we have

$$Kv_p = KF^p v = \lambda q^{-2p} F^p v = \lambda q^{-2p} v_p$$

$$Ev_{p+1} = (FE + [E, F^{p+1}])v_p = (\frac{q^{-p}\lambda - q^p\lambda^{-1}}{q-q^{-1}}[p+1] + ab)v_p$$

$$Fv_p = v_{p+1}$$

Hence, we get v_i are eigenvector of distinct eigenvalue, which implies they form a basis. Hence, we conclude the result.
(b) $b = 0$. almost everything same as (a), we only show why $\lambda \neq \pm q^{j-1}, 1 \leqslant j \leqslant e-1$, and this just by observing

$$Ev_e = 0 = \frac{q^{-(e-1)}\lambda - q^{(e-1)}\lambda^{-1}}{q - q^{-1}}[p+1]v_p$$

which mean $q^{-(e-1)}\lambda - q^{(e-1)}\lambda^{-1} = 0$, which imply $\lambda = \pm q^{e-1}$, and hence we can't have $q^{j-1}, 1 \leqslant j \leqslant e-1$

Remark: Notice in the case (b), we should have $Ev_i \neq 0, \forall i$ since otherwise, $\{v_i, v_{i+1}, \ldots, v_{e-1}\}$ is a non-trivial submodule of V by using the same construction in the proposition 2.1.3.4(switch the role of F and E).

(2) $Fv = 0$, for this case, we automatically have $Ev \neq 0$ otherwise we will have a nontrivial one dimension submodule $\{v\}$. Using the same construction in proposition 2.1.3.4, we have $v, Ev, \ldots, E^{e-1}v$ is a nontrivial submodule and hence equal to V. Then one can check it is in fact isomorphic to $V(\pm q^{1-j}, 0, a)$ which is same as $V(\pm q^{j-1}, a)$ by the remark before the proof. $\square$

Remark: We can see that the simple module in case (2) and (3) can't be an highest weight module since $Ev \neq 0$. In case (1), v is highest weight vector if and only if $a = 0$, and we should have $\lambda \neq \pm q^{1-j}, 0 \leqslant j \leqslant e-1$ because otherwise it is isomorphic to case (3) by the remark before the proof.

Remark: When q is root of unity, then

2.2 Quantum group $U_q(\mathfrak{g})$

In this section, we generate the idea of the quantum group for arbitrary symmetriable Kac-Moody algebra.

2.2.1 Kac-Moody Algebra

Will only discuss the Kac-Moody algebra over $\mathbb{C}$ in this section.

Defintion 2.2.1.1: Let A be a complex $n \times n$ matrix $(a_{ij})_{i,j \in n}$ with rank l, it is called **Generalized Cartan Matrix** if the following condition is satisfied:
(1): $a_{ii} = 2, \forall i$
(2):a_{ij} are non-positive integer if $i \neq j$
(3):$a_{ij} = 0$ if and only if $a_{ji} = 0$.
The A is called **decomposable** if there is a partition $n = I_1 \sqcup I_2$ such that $A = A_1 \oplus A_2$ where $A_1 = (A_{ij})_{i,j \in I_1}$ and similar for A_2. We call A **indecomposable** if it is not decomposable. It is called **symmetriable** if there is a diagonal matrix D with positive diagonal such that DA is symmetric.

Example 2.2.1: Let Q be a finite undirected graph with no self-loop. Then its adjacency matrix A, e.g A_{ij} =number of the edge between node i and j, is a symmetric matrix with diagonal 0. So $2I - A$ will be a symmetric generalized Cartan matrix; Moreover, we can notice any symmetric generalized Cartan Matrix can be obtained this way. In addition, $2I - A$ is indecomposable if and only if Q is connected.

The example above gives us an approach to classify the symmetric generalized Cartan matrix through the undirected graph, and in fact, we can use a family of directed graph called **Dykin diagram** to classify all the "finite" and "affine" generalized Cartan matrix.

The u in the proposition is a real column vector
Proposition 2.2.1.2: Let A be a indecomposable generalized Cartan matrix. Then, only one of the following three conditions will hold for both A and:
(Fin): $det(A) \neq 0$ and there exist a $u > 0$ such that $Au > 0$. If $Av = \geqslant 0$ then $v > 0$ or $v = 0$.
(Aff): $corank(A) = 1$ and there exists $u > 0$ such that $Au = 0$. If $Av \geqslant 0$ then $Av = 0$
(Ind): There exists $u > 0$ such that $Au < 0$. If $Av \geqslant 0$ and $v \geqslant 0$, then $v = 0$.

A generalized Cartan matrix satisfy (Fin) is said to be **Finite type** (resp **Affine**, **indefinite**).

Now, let $S(A)$ be the associate directed graph of a generalized Cartan matrix, e.g a_{ij} means the number of edges from node i to j, then if A is indecomposable, we have

Theorem 2.1.1.3 (1): If A is finite or affine type, then any proper subdiagram $S(A)$ is the union of (connected) Dykin diagram of finite type.
(2): if A is finite type, then $S(A)$ has no cycle.
(3) if A is affine type, then $S(A)$ contains cycle.

Remark: We can thhink the condition Aff as kind of infinite condition because a graph (or quiver) contains cycle implie they're path algebra is infinite dimensional.

Definition 2.2.1.4: A **realization** of a matrix $A = (x_{ij})$ with rank l is a triple $(\mathfrak{h}, \Pi, \breve{\Pi})$ where $\mathfrak{h}$ is a complex vector space, $\Pi = \{a_1, \dots, a_n\} \subset \mathfrak{h}^*$ and $\{\breve{a_1}, \dots, \breve{a_n}\} \subset \mathfrak{h}$ are indexed subset of $\mathfrak{h}^*, \mathfrak{h}$ with the following data:
(1): Both sets Π and $\breve{\Pi}$ are linearly independent.
(2): $\langle \breve{a}_i, a_j \rangle = a_j(\breve{a}_i) = x_{ij}$
(3): $dim(\mathfrak{h}) - n = n - l \Rightarrow dim(\mathfrak{h} = 2n - l)$

two realization $(\mathfrak{h}, \Pi, \breve{\Pi})$, $(\mathfrak{h}', \Pi', \breve{\Pi}')$ are isomorphic if there is an isomorphism $\phi : \mathfrak{h} \to \mathfrak{h}'$ such that $\phi(\Pi) = \Pi', \phi(\breve{\Pi}) = \breve{\Pi}'$

Theorem 2.2.1.5: For every $n \times n$ matrix $A = (x_{ij})_{i,j \in n}$, and suppose it has $rank(A) = l$ there exist unique realization up to isomorphism.

Proof. Given a matrix A, we set
$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$
for A_1 a $l \times n$ matrix with $rank(A_1) = l$ and A_2 a $(n - l) \times n$ matrix. Then consider the following $n \times (2n - l)$ matrix
$$C = \begin{pmatrix} A_1 & 0 \\ A_2 & I_{n-l} \end{pmatrix}$$

Now take $\mathfrak{c} = \mathbb{C}^{2n-1}$, set $\check{\Pi} = \{\check{a_1}, \ldots \check{a_n}, \check{a}_i = i\text{-th row of } C\}$, which is linearly independent set since A_1 and I are full rank. $a_1, \ldots, a_n \in \mathfrak{h}^*$ be the linear coordinate e.g $a_i(x_1, \ldots, x_{2n-1}) = x_i$, which is also linearly independent for same reason. Since both sets are linearly independent (, then this implies $\mathfrak{c}$ is a realization of A.

Conversely, let $(\mathfrak{h}, \Pi, \check{\Pi})$ be a realization of A then we extend the set Π by adding $a_{n+1}, \ldots, a_{2n-l}$ to be a basis such that we have

$$(\langle \check{a}_i, a_j \rangle) = \begin{pmatrix} A_1 & B \\ A_2 & D \end{pmatrix}$$

where B is a $l \times (n-l)$ matrix and D is invertible $(n-l) \times (n-l)$ matrix.

Now since D is invertible, we can change its basis to make it I, e.g the standard basis. Because A_1 has full rank, its columns are linearly independent, so it is possible to add some suitable linear combinations of $a_1, \ldots, a_l$ to $a_{n+1}, \ldots, a_{2n-l}$ so that we have B is 0, so eventually we get

$$\begin{pmatrix} A_1 & 0 \\ A_2 & I \end{pmatrix}$$

and this show any realization of A will also be the realization of C, and hence they should all isormophic to $\mathfrak{c}$. □

In analogy to the finite-dimensional case, the Π is called **root basis**, and the $\check{\Pi}$ is called **coroot basis**. Elements in Π are called **simple root** (resp. **simple coroot**). Set

$$Q = \sum_{i=1}^{n} \mathbb{Z} a_i, Q_+ = \sum_{i=i}^{n} \mathbb{Z}_+ a_i (resp. Q_-)$$

be two free abelian group, we call Q the **root lattice**. For $a = \sum_i k_i a_i \in Q$, the **height** of a is $ht(a) = \sum k_i$. Choose $\gamma_i \in \mathfrak{h}^*, i \in n$ such that

$$\langle \gamma_j, \check{a}_i \rangle = \delta_{ij}$$

the γ_j is called **Fundamental weights**. The **weight lattice** P is free abelian group generated by all fundamental weights and simple root. Similarly, we can define $\check{P}$, the dual weight lattice.

Remark: For any $\lambda \in P$, we have $\langle \lambda, \check{a}_i \rangle \in \mathbb{Z}$ for any coroot $\check{a}_i$

We now define our main target. First, we define a Lie algebra $\mathfrak{g}'(A)$ through a $n \times n$ matrix $A = (x_{ij})$ over $\mathbb{C}$ and its realization $(h, \Pi, \check{\Pi})$

Defintion: The Lie algebra $\mathfrak{g}'(A)$ is generated by $e_i, f_i (i = 1, \ldots, n)$ and $\mathfrak{h}$ with the following relation:

$$[e_i, f_j] = \delta_{ij} \check{a}_i \ldots (2.1)$$

$$[h, h'] = 0, \forall h, h' \in \mathfrak{h} \ldots (2.1)$$

$$[h, e_i] = \langle a_i, h \rangle e_i, a_i \in \Pi \ldots (2.1)$$

$$[h, f_i] = -\langle a_i, h \rangle f_i, \forall i \in n, h \in \mathfrak{h}, \ldots (2.1)$$

Denote $\mathfrak{n}'_+$(resp. $\mathfrak{n}'_-$) be the subalgebra generated by $e_1, \dots, e_n$ (resp.$f_1, \dots, f_n$).

Theorem 2.1.1.6: (1): $\mathfrak{g}'(A) \cong \mathfrak{n}'_+ \oplus \mathfrak{h} \oplus \mathfrak{n}'_+$
(2): $\mathfrak{n}'_-$ (resp. $\mathfrak{n}'_+$) is freely generated by $e_1, \dots, e_n$ (resp. $f_1, \dots, f_n$)
(3): The map $e_i \mapsto -f_i, f_i \mapsto -e_i, h \mapsto -h(h \in \mathfrak{h})$ can be uniquely extended to an involution automorphism ϕ of the Lie algebra $\mathfrak{g}'(A)$
(4): We has root space decomposition:

$$\mathfrak{g}'(A) = (\bigoplus_{a \in Q_+, a \neq 0} \mathfrak{g}'_{-a}) \oplus \mathfrak{h} \oplus (\bigoplus_{a \in Q_+, a \neq 0} \mathfrak{g}'_a)$$

respect to $\mathfrak{h}$, where $\mathfrak{g}'_a = \{x \in \mathfrak{g}'(A) | [h, x] = a(h)x, \forall h \in \mathfrak{h}\}$. Moreover, we have $dim(\mathfrak{g}'_{-a}) < \infty, \mathfrak{g}'_{-a} \subset \mathfrak{n}'_\pm$, for all $\pm a \in Q_+, a \neq 0$.
(5) Among all the ideal in $\mathfrak{g}'(A)$ intersect with $\mathfrak{h}$ trivially, there is an maximal ideal $\mathfrak{t}$. Furthermore, we have

$$\mathfrak{t} = (\mathfrak{t} \cap \mathfrak{n}'_-) \oplus (\mathfrak{t} \cap \mathfrak{n}'_+)$$

Remark: Notice in (4), $\mathfrak{g}_0 = \mathfrak{h}$

Proof. Let V be an $n-$dimensional complex vector space, with a basis $\{v_1, \dots, v_n\}$, and let $\lambda \in \mathfrak{h}*$ we first define an action of the generator of $\mathfrak{g}'(A)$ on $T(V)$ by

$$f_i(u) = v_i \otimes u$$

$$h(1) = \langle \lambda, h \rangle 1$$

and then inductively on s we have

$$h(v_j \otimes u) = hf_j(u) = f_j h(u) + f_i(u) = \langle a_j, h \rangle v_j \otimes u + v_j \otimes h(u), \forall u \in T^{s-1}(V)$$

$$e_i(1) = 0$$

and inductively on s we have

$$e_i(v_j \otimes u) = e_i f_j(v_u) = \delta_{ij} \check{a}_i(u) + v_j \otimes e_i(u), \forall u \in T^{n-1}(V)$$

The action given above defines a representation of $\mathfrak{g}'(A)$ on $T(V)$ since we can all the relation of (2.1). For the first one

$$([e_i, f_j](u) = e_i(v_j \otimes u) - v_j \otimes e_i(a) = \delta_{ij} \check{(a)}_i(u) + v_j \otimes e_i(u) - v_j \otimes e_i(u) = \delta_{ij} \check{(a)}_i(u)$$

The second relation is trivial since we clearly have $hh'(1) = h'h(1)$, that is the action of $\mathfrak{h}$ is diagonal. The third relation could be shown through induction on s of T^s. When $s = 0$, the relation clearly holds. Now let $u \in T^s$, and we can write it as $u = v_k \otimes u_1, u_1 \in T^{s-1}$, then we have

$$[h, e_j](v_k \otimes u_1) = h(\delta_{ij} \check{a}_i(u_1)) + h(v_k \otimes e_j(u_1)) - e_j(-\langle a_k, h \rangle)(v_k \otimes u_1) + v_k \otimes h(u_1)$$

$$= \langle a_j, h \rangle \delta_{ij} \check{a}_j(u_1) + v_k \otimes (he_j - e_j h)(u_1)$$

Finally, the last relation

$$[h, f_i](u) = h(v_i \otimes a) + v_i \otimes h(a)$$

$$= -\langle \check{a}_j \rangle f_i(u)$$

Based on this representation, We prove all the theorem
(3): This is trivial by our definition for ϕ.
(2): Consider the map $f_i \mapsto v_i$. This is a Lie homomorphism $\mathfrak{n'}_- \to T(V)$, so $T(V)$ is an enveloping algebra of $\mathfrak{n'}_-$, since $T(V)$ is free, so conclude that it is a universal enveloping algebra $U(\mathfrak{n'}_-)$. Then by PBW-theorem, we conclude $\mathfrak{n'}_-$ is freely generated by $f_i, \ldots, f_j$. Now apply the map ϕ we have $\mathfrak{n'}_- \cong \mathfrak{n'}_+$. Hence, we conclude the case for $\mathfrak{n'}_+$
(1): Let $\mathfrak{a} = \mathfrak{n'}_+\mathfrak{h} + \mathfrak{n'}_+$, we can see it is an ideal of $\mathfrak{g}'(A)$ since it is invariant under ad_{e_i}, ad_{f_j}, ad_h, and because it contains all the generator of $\mathfrak{g}'(A)$, so it should equal to $\mathfrak{g}'(A)$. We only need to show this sum is direct. If $u = n_- + h + n_+ = 0$, then using the representation on $T(V)$, we have $0 = u(1) = n_-(1) + \langle \lambda, h \rangle + 0$, so we must have $\langle \lambda, h \rangle = 0$ for all λ, which mean $h = 0$. By (2), we have a Canonical embedding $n_- \mapsto n_-(1)$, so by the injectivity, $n_-(1) = 0$ implies $n_- = 0$. Eventually, we have $n_+ = 0$, so this shows that the sum is direct.
(4): By the last three relations of (2.1), we can see

$$\mathfrak{n}'_{\pm} = \bigoplus \mathfrak{g'}_{\pm a}$$

By (2) $\mathfrak{n}'_-$ is generated by $e_i, i = 1, \ldots, n$, so we must have

$$dim(\mathfrak{g'}_a) \leqslant n^{|ht(a)|}$$

so together with (1), we conclude the result.
(5): We need the following lemma

Lemma 2.1.1.7: Let $\mathfrak{h}$ be a commutative Lie algebra, and V be an diagonalizable $\mathfrak{h}$ module, that is

$$V = \bigoplus_{a \in \mathfrak{h}^*} V_a, V_a = \{v \in V | h(v) = \langle a, h \rangle v\}$$

Then for any submodule U of V, we have

$$U = \bigoplus U \cap V_a$$

Proof. Let $u \in U$, Then it can be decompose as $u = \sum_{i=1}^{m} u_{a_i}, u_{a_i} \in V_{a_i}$, we want to show each u_{a_i} are in U. We can find a $h \in \mathfrak{h}$ such that $a_i(h)$ are distinct for all $i \in m$, then

$$h^k(u) = \sum_{i=1}^{m} a_i(h)^k u_{a_i}$$

from $k = 1, \ldots, m$ will form a system of linear. The matrix here is invertible since it has a full rank (all row and column are linearly independent since $a_i h$ are distinct.), and $h^k(u) \in U$ for all k since U is a submodule, hence we must have u_{a_i} are in U. □

In our case, we clearly have $\mathfrak{h}$ is commutative Lie algebra, so let $\mathfrak{k}$ be an ideal of $\mathfrak{h}$, we have

$$\mathfrak{k} = \bigoplus_{a \in \mathfrak{h}^*} (\mathfrak{g}'_a \cap \mathfrak{k})$$

this imply the sum of ideas that intersect with $\mathfrak{h}$ trivially will intersect with $\mathfrak{h}$ itself. Hence, let $\mathfrak{t}$ be the summation of all ideals that intersect with $\mathfrak{h}$ trivial, then it will be a unique maximal element with this property. In particular we clearly have $[f_i, \mathfrak{t} \cap \mathfrak{n}'_+] \subset \mathfrak{n'}_+$. Hence, we have $[\mathfrak{g}'(A), \mathfrak{t} \cap \mathfrak{n}'_+] \subset \mathfrak{t} \cap \mathfrak{n}'_+$. Similarly, we have $[\mathfrak{g}'(A), \mathfrak{t} \cap \mathfrak{n}'_-] \subset \mathfrak{t} \cap \mathfrak{n}'_-$. □

We now define our major target. Let A be $n \times n$ matrix and $(\mathfrak{h}, \Pi, \check{\Pi})$ be its realization, and $\mathfrak{g}'(A)$ be associated Lie algebra generated by e_i, f_i, Let $\mathfrak{t}$ be the maximal ideal Intersec $\mathfrak{h}$ trivially. We set $\mathfrak{g}(A) = \mathfrak{g}'(A)/\mathfrak{t}$.

Defintion: The matrix A is called **Cartan matrix** of $\mathfrak{g}(A)$. When A is generalized Cartan matrix, the quadruple $(\mathfrak{g}(A), \mathfrak{h}, \Pi, \check{\Pi})$ is called **Kac-Moody Algebra**. The subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$ is called **Cartan subalgebra**. The element e_i, f_j are called **Chevalley generators**. Two Kac-Moody algebra $(\mathfrak{g}(A), \mathfrak{h}, \Pi, \check{\Pi}), (\mathfrak{g}(A)_1, \mathfrak{h}_1, \Pi_1, \check{\Pi}_1)$ if there is a isomorphism $\rho : \mathfrak{g}(A) \to \mathfrak{g}(\mathfrak{A})_1$ such that $\rho(\mathfrak{h}) = \mathfrak{h}_1, \rho^*(\Pi) = \Pi_1, \rho(\check{\Pi}) = \check{\Pi}_1$.

Remark: Another way to define a Kac-Moody algebra is to use the definition of $\mathfrak{g}'(A)$ with additional **Serre relation**:
$(ad_{e_i})^{1-x_{ij}} e_j = 0$ when $i \neq j$, $(ad_{f_j})^{1-x_{ij}} f_j = 0$, $x_{ij} \in A$

We keep the same notation in the case $\mathfrak{g}'(A)$: we let $\mathfrak{n}_+$ be the subalgebra generated by e_i (resp. $\mathfrak{n}_-$). Since the $\mathfrak{t}$ is the maximal ideal intersect $\mathfrak{h}$ trivially, so all the result (1)-(4) of theorem 2.1.1.6 holds for any Kac-Moody algebra $\mathfrak{g}(A)$. The decomposition $\mathfrak{g}(A) \cong \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is called **triangular decomposition**.

Without no ambiguity, we denote a Kac-Moody algebra $\mathfrak{g}(A)$ by $\mathfrak{g}$. We end this section by discussing the Universal Enveloping algebra of $U(mathfrakg)$ so the reader can see the parallel with the quantum version. First, we notice, by using induction on n, we have

$$(ad_x)^n(y) = \sum_{K=0}^{n} (-1)^k \binom{n}{k} x^{n-k} y x^k, \forall x, y \in U(\mathfrak{g}), n \in \mathbb{Z}_{\geqslant 0}$$

Proposition 2.1.1. The Universal Enveloping algebra $U(\mathfrak{g})$ is generated by $e_i, f_j, \mathfrak{h}$ with the relation: all relation of (2.1) and additionally

$$\sum_{k=0}^{1-x_{ij}} (-1)^k \binom{1 - x_{ij}}{k} e^{1-x_{ij}-k} e_j e_i^k = 0, \forall i \neq j$$

$$\sum_{k=0}^{1-x_{ij}} (-1)^k \binom{1 - x_{ij}}{k} e^{1-x_{ij}-k} f_j f_i^k = 0, \forall i \neq j$$

Same to the case of $\mathfrak{g}$, we denote U^+ be the subalgebra of $U(\mathfrak{g})$ generated by e_i(resp. U^0 generated by $\mathfrak{h}$, U^- generated by f_j). We also define the **root space**

$$U_\alpha = \{u \in U(\mathfrak{g}) | [h, u] = \alpha(h)u, \forall h \in \mathfrak{h}, \alpha \in Q\}$$
$$U_\alpha^\pm = \{u \in U(\mathfrak{g}) | [h, u] = \alpha(h)u, \forall h \in \mathfrak{h}, \alpha \in Q_\pm\}$$

Proposition 2.1.1.8: $(1) U(\mathfrak{g}) \cong U^+ \otimes U^0 \otimes U^-$
$(2) U(\mathfrak{g}) \cong \bigoplus_{\alpha \in Q} U_\alpha$
$(3) U^\pm \cong \bigoplus_{\alpha \in Q} U_\alpha^\pm$

Proof. (1) Recall that by PBW theorem, we have for two lie algebra L_1, L_2 over a field, then $U(L_1 \oplus L_2) \cong U(L_1) \otimes U(L_2)$, so by triangular decomposition of $\mathfrak{g}$, we conclude the result.
(2): Clearly, every element should inside a unique U_α since this is uniquely determined by its generator which is the monomial of e_i, f_i, h by PBW theorem. To see this summation is direct, we see $U_\alpha \cap U_\beta = \varnothing$ for any two distinct $\alpha, \beta \in Q$, so we conclude the result.
(3) Using the same way in (2) □

2.2.2 Quantum group $U_q(\mathfrak{g})$

In this section, we will fix a ground field k (not necessary $\mathbb{C}$). We let q be a variable, and let $[n]_q, \begin{bmatrix} n \\ k \end{bmatrix}_q$ be the notation introduced in section 2.1.1.

Let $(A, \mathfrak{h}, \Pi, \check{\Pi})$ be a realization of a symmetriable generalized Cartan matrix $A = (x_{ij})_{i,j \in n}$, which is symmetried by a diagonal matrix $D = (s_i), s_i \in \mathbb{Z}_{\geqslant 0}$.

Definition: The quantum group $U_q(\mathfrak{g}) = U_q$ associate to $(A, \mathfrak{h}, \Pi, \check{\Pi})$ is an associative algebra over $k(q)$ generated by $e_i, f_i, q^h (h \in \check{P})$ with the relation

$$q^0 = 1, q^h q^{h'} = q^{h+h'}, h, h \in \check{P} \ldots (2.2)$$

$$q^h e_i q^{-h} = q^{a_i(h)} e_i, a_i \in \Pi, h \in \check{P} \ldots (2.3)$$

$$q^h f_i q^{-h} = q^{-a_i(h)} f_i \ldots (2.4)$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \ldots 2.5$$

$$\sum_{k=0}^{1-x_{ij}} (-1)^k \begin{bmatrix} 1 - x_{ij} \\ k \end{bmatrix} \}_{q_i} e_i^{1-x_{ij}-k} e_j e_i^k = 0, \forall i \neq j \ldots (2.6)$$

$$\sum_{k=0}^{1-x_{ij}} (-1)^k \begin{bmatrix} 1 - x_{ij} \\ k \end{bmatrix} \}_{q_i} f_i^{1-x_{ij}-k} f_j f_i^k = 0, \forall i \neq j \ldots (2.7)$$

where $q_i = q^{s_i}$, $K_i = q^{s_i h_i}, h_i \in \check{\Pi}$. For $a = \sum_i n_i a_i \in Q$, we use the notation $K_a = \prod K_i^{n_i}$.

Remark:(1) The $U_{q^i}(\mathfrak{sl}(2))$ is isomorphic to the subalgebra $U_q(\mathfrak{g}_i)$ of $U_q(\mathfrak{g})$ generated by by $e_i, f_i, K_i^{\pm}$, this gives an approach to get $U_q(\mathfrak{sl}(2))$
(2)The relation (2.6) and (2.7) are called **quantum Serre relation**. We also have denote the **quantum adjoin operator** as

$$(ad_q x)(y) = xy - q^{(\alpha|\beta)} yx$$

where x has the property $q^h x q^{-h} = q^\alpha, \forall h \in \check{P}$ (resp. y, β), and $(\alpha|\beta)$ is a bilinear map such that $(h_i|h) = a_i(h)/s_i, (\lambda_i|\lambda_s) = 0$(the fundamental root). So by induction, we can have

$$(ad_q x)^n(y) = \sum_{K=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y x^k, \forall x, y \in U(\mathfrak{g}), n \in \mathbb{Z}_{\geqslant 0}$$

Hence, the quantaum Serre relation could be rewritten as

$$(ad_q e_i)^{1-x_{ij}}(e_j) = 0, (ad_q f_j)^{1-x_{ij}}(f_j) = 0$$

for all $i \neq j$. The U_q is also a Hopf algebra which is same as the case in $U_q(\mathfrak{sl}(2))$.

Proposition 2.2.2.1: The U_q has Hopf algebra structure given by

$$\Delta(q^h) = q^h \otimes q^h, \Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i$$

$$\epsilon(q^h) = 1, \epsilon(e_i) = \epsilon(f_i) = 0$$

$$S(q^h) = q^{-h}, S(e_i) = -e_i K_i, S(f_i) = -K_i^{-1} f_i$$

Proof. We first need to check all three maps are well-defined, that is all relation closed under the the map. We only check the Serre relation here. The case for ϵ trivial since they just all 0 by defintion. For the antipode, we notice

$$S(e^n e_j e_i^k) = (-1)^n q_i^{n(n+x_{ij}-1)} e_i^k e_j e_i K_i^n K_j$$

$$S(e^n e_j e_i^k) = (-1)^n q_i^{n(n+x_{ij}-1)} K_n K_j f_i^k f_j f_i$$

which imply the Serre relation is closed under the antipode once we let $n = 1 - x_{ij}$.

Remark: To commute the $e^n e_j e_i^k$ to $e_i^k e_j e^n$, we can use the fact $q^h e_i e j q^h = q^{a_i} q^{a_j} e_j e_i$.

Now check the Δ is closed under relation. Using induction on n, we have

$$\Delta((ad_q e_i)^n(e_j)) = (ad_q e_i)^n(e_j) \otimes K_i^{-1} K_j^{-1}$$

$$+ \sum_0^{n-1} \tau_k^n q_i^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} e_i^{n-k} \otimes K_i^{-n+k} (ad_q e_i)^k (e_j)$$

$$+1 \otimes (ad_q e_i)^n (e_j)$$

where $\tau_k^n = \prod_{t=k}^{n-1}(1 - q_i^{2(t+x_{ij})})$. Now if we let $n = 1 - x_{ij}$, then we notice the middle term will be 0, which will finish the proof. The cocommutative, counit and axiom of antipode is easily to show. $\square$

Remark: We again notice $S^2 \neq Id$ in this case, so this Hopf algebra is also neither commutative nor cocommutative.

Denote U_q^+ (resp. U_q^-, U_q^0) be subalgebra generated by e_i (resp. f_j, q^h). We also let $U_q^{\geqslant 0}$ (resp. $U_q^{\leqslant 0}$) be the subalgebra generated by q^h and e_i (resp. f_j, q^h). We show that the $U_q(\mathfrak{g})$ will also have a triangular decomposition $U_q(\mathfrak{g}) \cong U_q^+ \otimes U_q^0 \otimes U_q^-$. To do this, let's first define an involution automorphism ϕ by

$$q^h \mapsto q^{-h}, e_i \mapsto f_i, f_i \mapsto e_i$$

This map has the following propositions which are trivally to prove.

Proposition 2.2.2.2:
(1) ϕ is involution automorphism
(2) The restriction of ϕ to U_q^+ is an algebra isomorphisim to U_q^-

Lemma 2.2.2.3:
(1) $U_q^{\geqslant 0} \cong U_q^0 \otimes U_q^+$
(2) $U_q^{\leqslant 0} \cong U_q^- \otimes U_q^0$

Proof. Because of the involution ϕ, once we prove one of them, we immediately get another one. Here, we show the second one.

Let $(\{f_\sigma\})_{\sigma \in \Omega}$ be the basis of $U_q^{\geqslant 0}$ which consist of monomial of f_i, then considet the map

$$\rho : f_\sigma \otimes g^h = f_\sigma g^h$$

because $q^h f_\sigma = q^{-a(h)} f_\sigma q^h = \rho(q^{a(h) f_\sigma \otimes q^h})$, for some $a \in Q_+$, so this map is surjective. To show this map is injective, we show the set $\{f_\sigma q^h | \sigma \in \Omega, h \in \check{P}\}$. Let's set $deg(f_i) = -a_i, deg(q^h) = 0, deg(e_i) = a_i$, then since all relation defined in $U_q(\mathfrak{g})$ is homogeneous under this setting, so we can have decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{a \in Q} (U_q)_a$$

where $(U_q)_a = \{u \in U_q(\mathfrak{g}) | q^h u q^{-h} = q^{a(h)}\},]$, that is every element in $(U_q)_a$ has degree a. Now suppose we have

$$\sum_{\sigma \in \Omega, h \in \check{P}} A_{\sigma,h} f_\sigma q^h = 0$$

we may rewrite it as

$$\sum_{b \in Q_+} (\sum_{deg(f_\sigma) = -b} A_{\sigma,h} f_\sigma q^h)$$

because of the decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{a \in Q} (U_q)_a$$

we must have each

$$\sum_{deg(f_\sigma) = -b, h \in \check{P}} A_{\sigma,h} f_\sigma q^h = 0$$

since they're in different $(U_q)_a$. Now, we apply the Δ to $\sum_{deg(f_\sigma)=-b} A_{\sigma,h} f_\sigma q^h = 0$. Because f_σ is monomial of f_i, so it has degree $-b \in Q_-$. So

$$\Delta(f_\sigma) = f_\sigma \otimes 1 + \cdots + K_b \otimes 1$$

so we have

$$\sum_{deg(f_\sigma) = -b, h \in \check{P}} \Delta(A_{\sigma,h} f_\sigma q^h) = 0$$

$$= \sum_{deg(f_\sigma) = -b, h \in \check{P}} A_\sigma (f_\sigma q^h \otimes q^h + \ldots K_b q^h \otimes q^h)$$

Then we have

$$\sum_{deg(f_\sigma)} = A_{\sigma,h}(f_\sigma g^h \otimes g^h)$$

by property of the tensor product, we should have one of the summations is 0, but since $\{p^h\}_{h\in\check{P}}$ is a linearly independent set, so we must have $\sum_{deg(f_\sigma)=-b} A_\sigma f_\sigma g^h = 0$ Now multiple g^{-h} in both side and use the linear independent of $\{f_\sigma\}_{\sigma\in\Omega}$, we conclude all $A_{\sigma,h}$ are 0.

□

Theorem 2.2.2.3 $U_q(\mathfrak{g}) \cong U_q^+ \otimes U_q^0 \otimes U_q^+$

Proof. The proof is basically same as the proposition 2.2.2.2, we let $\{f_\sigma\}_{\sigma\in\Omega}, \{e_\eta\}_{\eta\in\Omega'}$ then we consider the map $f_\sigma \otimes q^h \otimes e_\eta$. The map is surjective in same reason prop 2.2.2.2, then we show the set $\{f_\sigma q^h e_\eta\}$ is linearly independent. The strategy is same, we first restrict our case to the homogeneous term in a degree b, then compute the $\Delta(e_\eta)$ and $\Delta(f_\eta)$ so we can apply in $\sum_{h\in\check{P},b} C_{\sigma,h,\eta} f_\sigma q^h e_\eta$ which we get

$$\sum_{h\in\check{P},b} C_{\sigma,h,\eta}((f_\sigma \otimes 1 + \dots) + (q^h \otimes q^h)(\dots + 1 \otimes e_\eta))$$

Here, we choose the $e = deg(f_\sigma), c = deg(e_\eta)$ which is maximal and minimal, and the term in the summation with degree (e, c) should be 0. Hence we have

$$\sum_{h\in\check{P}, deg(f_\sigma)=e, deg(\eta)=c} C_{\sigma,h,\eta} f_\sigma (q^h \otimes q^h) e_\eta = 0$$

then use the linearly independent of $\{f_\sigma q^h\}$, we conclude the result. □

2.2.3 Representation of $U_q(\mathfrak{g})$

Defintion: A $U_q(\mathfrak{g})$ module V^q is called **weight module** if it admits a **weight root decomposition**:

$$V^q = \bigoplus_{\mu\in P} V_\mu^q, V_\mu^q = \{v \in V, q^h v = q^{\mu(h)} v | \forall h \in \check{P}\}$$

A vector $v \in V_\lambda^q$ is called **weight vector** of weight λ. If $e_i v = 0, \forall i$, then it is called a **maximal vector**. If V_λ^q is non trivial, then λ is called **weight** of V and $V_\lambda^q$is called **weight space** of λ. It dimension $dim(V_\lambda^q)$ is called **weight multiplicity** of λ. We denote the set of all weight of V^q as $wt(V^q)$, if $dim(V_\lambda^q) < \infty$ for all λ, then we define the **character** of

$$chV^q = \sum_\mu dim(V_\lambda^q) e^\lambda$$

where e^λ are standard basis element of the group algebra $k[P]$ with multiplication $e^\lambda e^\mu = e^{\lambda+\mu}$

We define a partially order $\geqslant$ in $\mathfrak{h}^*$ such that $\lambda \geqslant \mu \Leftrightarrow \lambda - \mu \in Q_+$

Defintion: Let $D(\lambda) = \{\mu \in P | \mu \leqslant \lambda\}$A category $\mathcal{O}$ consist of the following data:
(1): The objects are all weight modules V^q of $U_q(\mathfrak{g})$ finite dimensional weight space and there exists a finite number of an element $\lambda_1, \ldots, \lambda_s$ such that

$$wt(V^q) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s)$$

(2) The morphism is the homomorphism between modules.

The category $\mathcal{O}$ is an abelian category because of the following property
Proposition 2.2.3.1: Every submodule of weight module $V^q = \bigoplus_{\lambda \in P} V_q^\lambda$ is a weight module.

Proof. Suppose we have a non-zero submodule W of V^q that is not a weight module. Given summation $v_1 + \cdots + v_n$ in W such that n is the smallest integer such that every summand is not in the W, so each $v_i \in V_q^{\lambda_i}$. Now given an $h \in P$ such that $\lambda(h)_1 \neq \lambda_k(h)$ for at least one λ, then we notice $q^h v - q^{\lambda_1}(h)v$ will be a summation in at most $n-1$ that each summand is not in W, which is a contradiction. □

A weight module V_q is called **highest weight module** of the **highest weight** if there is a nonzero $v_\lambda \in V^q$ such that

$$e_i v_\lambda = 0, \forall i$$
$$q^h v_\lambda = q^{\lambda(h)} v \forall h \in \check{P}$$
$$V_q = U_q(\mathfrak{g}) v_\lambda$$

the vector v_q, which is unique up to the multiple because of the second requirement, is called **highest weight vector**. By triangular decomposition, we can see the highest weight module $V^q = U_q^- v_\lambda$. Then, we can easily see $dim(V_q^\lambda) = 1$ where the basis is $\{v_\lambda\}$ and the second requirement gives $V^q = \bigoplus_{\mu \leqslant \lambda} V_\mu^q$, so these properties guarantee the highest weight module is an object in $\mathcal{O}$.

Remark: The existence of highest weight module in category $\mathcal{O}$ is because of the property $wt(V^q) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s)$ which tells us the weight could not be infinite increasing for the object in $\mathcal{O}$

Proposition 2.2.3.2 Let M be a highest weight module in weight λ generated by a highest weight vector v, then
(1) Every submodule of M is a weight module, a submodule generated by the highest weight v_μ where $\mu < \lambda$ is proper. In particular, when M is simple, all of the highest weight vectors will be multiple of v
(2) M has a unique maximal submodule and unique simple quotient.
(3) All simple highest-weight module of weight λ is unique up to isomorphism. If M is one of these, then $dim(End((M)) = 1$

Proof. (1): This follow from the fact $V^q = \bigoplus_{\mu \leqslant \lambda} V_\mu^q$ and the proposition 2.2.3.1
(2): Each proper submodule is a weight module, and can't contain a weight λ since otherwise, any element in λ-space will generate the whole space (notice M is 1-dimension). So the summation of any submodule is proper. Then the summation of every proper submodule will be the unique maximal submodule.
(3): Take M_1, M_2 be two simple module in generated by highest weight vector v_1, v_2, then take

$N = M_1 \oplus M_2$. It is clear (v_1, v_2) is the highest weight vector, so N is the highest weight module. Since M_1, M_2 are maximal submodules, they should be isomorphic.

Since M is simple, $f \in End(M)$ is either 0 or isomorphism by Shur lemma. If f is an isomorphism, then $f(v) = cv$ for some multiplication c, so $f = cId$ □

Fix a $\lambda \in P$, let $J(\lambda)$ be a two-sided ideal generated by e_i, $q^h - q^{\lambda(h)}$, then the **Verma module** is defined as $M(\lambda) = U_q(\mathfrak{g})/J_\lambda$.Set $v_\lambda = 1 + J_\lambda$, then clearly $(q^h - q^\lambda)v_\lambda, e_i v_\lambda$ will be vanished in $M(\lambda)$, so this imply $M(\lambda)$ is also a highest weight module.
The construction of $M(\lambda)$ also implies the universal property: J_λ will annihilate any highest vector in weight λ, so $J(\lambda)$ will annihilate the module generated by it. Hence, there is a well-defined natural subjective map from $M(\lambda)$ to highest weight module in weight λ by sending generator to generator. So sometimes we call $M(\lambda)$ **universal highest weight module** in weight λ

Proposition: 2.2.3.3: (1) $M(\lambda)$ is free of rank 1, generated by $v_\lambda = 1 + J_\lambda$ as U_q^-−module
(2) Let $L(\lambda) = M(\lambda)$ quotient of the unique maximal submodule, then every simple module in $\mathcal{O}$ is isomorphic to $L(\lambda)$ for a $\lambda \in P$. Hence, every simple module is uniquely determined by its highest weight.

Proof. (1): All we need to do is to show the basis is free. Now suppose we have $uv_\lambda = 0, u \in U_q^-$. Then, by triangular decomposition, we can write $U_q(\mathfrak{g}) = U_q^- \otimes U_q^{\geqslant 0}$. So, the generator of $J(\lambda)$ implies it should completely lie in the $U_q^{\geqslant 0}$, so $J(\lambda) \cap U_q^- = 0$. This imply $u = 0$
(2): Since every module consists a highest weight vector, then the conclusion is straightforward by our discussion and proposition 2.2.3.2 □

Remark

2.2.4 When $q \to 0$: introduction to crystal base

Originally, the crystal base is a combinatorial tool introduced by Kashiwara to study the category$\mathcal{O}_{\text{int}}$, which is a subcategory of $\mathcal{O}$, when $q \to 0$. However, the idea has been generalized to a much more abstract way which we will see in the next chapter that provides us another approach to investigate the representation of Lie algebra.

Defintion: an element x in $U_q(\mathfrak{g})$ is called **locally nilpotent** for module M if for any element $m \in M$, there is an integer n such that $x^n m = 0$

The category $\mathcal{O}_{\text{int}}$ is a subcategory (not necessary abelian) of $\mathcal{O}$ where its object has the following additional requirement: e_i and f_i are locally nilpotent for all the object.

Remark: If e_i is locally nilpotent at an element v (resp. f_i), then we can notice $U_q^+ v$ (resp. $U_q^- v$) is finite dimension.(reader can refer the case in $U_q(\mathfrak{sl}(2))$)

The object in the $\mathcal{O}_{\text{int}}$ is called **integrable module**. We also set the **divide power** $e^{(s)} = e_i^s/([s]_{q^i}!)$ and $f^{(s)} = f_i^s/([s]_{q^i}!)$ Now, we define the **Kshiwara operator**, it is inspired by the following proposition of integrable module

Proposition 2.2.4.1: Let $M = \bigoplus_{\lambda \in P} M_\lambda$ be an object in $\mathcal{O}_{\text{int}}$, then every weight vector $u \in M_\lambda$ could be written in the form

$$u = u_0 + f_i u_1 + \dots f_i^{(s)} u_s$$

for every f_i, where $u_s \in M_{\lambda + k a_i} \cap ker(e_i)$ for all $k = 0, \dots, s$. Each u_s is uniquely determined by u and u_s only if $\lambda(h_i + k) \geqslant 0$

Proof. The existence of the this summation is based on the following lemma

Lemma 2.2.4.2: The object V^q in $\mathcal{O}_{\text{int}}$ could be decomposed into direct sum of finite dimensional irreducible $U_q(\mathfrak{g}_{(i)})-$module.

We will not provide a proof here, but here is some ideas how to prove it: The decomposition comes from the fact P and $\check{P}$ are free abelian group, so we can decompose each V_λ as $\bigoplus V_{a_i}$ where $\lambda = \sum \mathbb{Z} a_i$. They are clearly finite dimensional. Since every finite dimesnional simple module should be Verma module, so it just remained to check it is a Verma module.

Now, we show this summation is uniqueness by processing an induction on s. When $s = 0$ there is nothing to prove. Suppose it is correct for $s - 1$, and suppose we have

$$0 = \sum_{k=0}^{s} f_i^{(k)} u_k = u_0 \dots + f_i^{(s)} u_s$$

$$0 = e_i(u_0 = \dots f_i^{(s)}) u_s$$

$$0 = e_i u_0 + \dots + e_i f_i^{(s_i)} u_i$$

$$0 = \frac{K^i - K^{-i}}{q - q^{-1}} u_1 + \dots + \frac{K^i q_i^{-s+1} - K^{-i} q_i^{s-1}}{q - q^{-1}} f^{(s-1)} u_s$$

$$[\lambda(h_i) - 2]_{q^i} u_1 + \dots [\lambda(h_i) + s + 1]_{q^i} u_s$$

now by induction, we must have

$$[\lambda(h_i) - 2]_{q^i} u_1 = 0, \dots [\lambda(h_i) + s + 1]_{q^i} u_s = 0$$

Since the lenght of the $i-$string through any nonzero u_k is $\lambda(h_i) + 2k$, we should have $\lambda(h_i) + sk \geqslant k$, so we must have $\lambda(h_i) + k + 1 \geqslant 0$. Since q is indetermine, so $[\lambda(h_i) + k + 1]_{q^i} \neq 0$, so we must have $u_k = 0$ for all $k = 1, \dots s$, and this imply $u_0 = 0$ which complete the poof. □

Defintion: The **Kashiwara operator** $\overline{e_i}, \overline{f_i}$ on M is

$$\overline{e_i} u = \sum_{k=1}^{s} f_i^{(k-1)} u_k, \overline{f_i} u = \sum_{k=1}^{s} f_i^{(k+1)} u_k$$

Let $K = k(q)$, V a $K-$vector space, then for a subring $A \subset K$, a $A-$**lattice** of V is a $A-$submodule L of V such that $V \cong K \otimes L$, that is V is generated by L as $K-$vector space.

Now, Let A be a subring of $k(q)$ which consist of all rational function regular at q, e.g. $A = \{f(q) \in F(q) | f = g/h, f, g \in F[q], h(q) \neq 0\}$. This ring A is a local ring since it is localization of $k[q]$ at the ideal (q). Moreover, it is the field of quotient of $F[q]$, and by the map $f(q) \to f(0)$, we have $A/qA \cong k$.

Defintion: Let V be a $K-$vector space. A **Local base** of V at $q = 0$ is a pair (L, B) with the data:
(1) L is a free $A-$module and it is an $A-$lattice of V
(2) B is the basis of $k-$vector space L/qL

Remark: (1): This definition not just work for $q = 0$, indeed, one can define a local basis for any point in $\mathbb{P}^1 = spec(k[x]) \cup spec(k[x^{-1}])$. In this notes, we only care about the case $q = 0$, so we will simply say local base of V.
(2): The passage from L to L/qL is referred to take an **classical limit**, we should denot the image of v in L/qL by $\hat{v}$

Eample 2.2.4.3 Let B be the basis of V as $K-$vector space, then we can define a local basis (L, B) of V by setting L be a $A-$module generated by B. We call it the local basie associate with B.

Let $\{V_i\}$ be a family of $K-$vector space, (L_i, B_i) be the local basis of V_i, then $K \otimes \bigoplus_i L_i = \bigoplus_i (K \otimes L_i) = \bigoplus_i V_i$, so we have $L = \bigoplus_i L_i$ is free $A-$module lattice of $\bigoplus_i V_i$, and we also have $B = \bigsqcup B_i$ is a $k-$base of $\bigoplus_i L_i$. So (L, B) is a local base of $\bigoplus_i V_i$, which we denote it by $\bigoplus_i (L_i, B_i)$. Similarly, let V_1, V_2 be two $K-$vector space, then $L = L_1 \otimes L_2$ is a free $A-$lattice of $V_1 \otimes V_2$ and $B = B_1 \otimes B_2$ is a $k-$vector space of $(L_1/qL_1) \otimes (L_2/qL_2) \cong (L/qL)$, so we have (L, B) is a local base of $V_1 \otimes V_2$, which we denote by $(L_1, B_1) \otimes (L_2, B_2)$.

Defintion: Let $M = \bigoplus_{\lambda \in P} M_\lambda$ be an object in $\mathcal{O}_{\text{int}}$. A **crystal base** of M is a local base (L, B) of $K-$vector space consist of the following data:
(1) there is a local base (L_λ, B_λ) of each M_λ such that $(L, B) = \bigoplus_{\lambda \in P} (L_\lambda, B_\lambda)$
(2)$\overline{e_i} B \subset B \bigsqcup \{0\}, \overline{f}_i B \subset B \bigsqcup \{0\}$ for all i.
(3) for any $b, b' \in B$, we have $\overline{f_i} b = b'$ if and only if $b = \overline{e}_i b'$

The last condition could be presented in more visible way. We define an **I-colored arrow** on B by

$$b \xrightarrow{i} b', \text{if } b' = \overline{f_i} b(\text{ or } b = \overline{e}_i) b'$$

We may treat B as set of lattice and together with $I-$colored arrow, we get an graph which we call **crystal graph.**.

For $b \in B$, we set

$$\epsilon_i(b) = \max\{n \geqslant o | \overline{f_i}^n b \neq 0\}$$

$$\phi_i(b) = \max\{n \geqslant o | \overline{e_i}^n b \neq 0\}$$

this two set could be visualized as

$$\circ \xrightarrow{i} \ldots \xrightarrow{i} \circ \xrightarrow{i} \circ = b \xrightarrow{i} \circ \xrightarrow{i} \ldots \xrightarrow{i} \circ$$

where the string before the $b - th$ circle is ϵ_i and the string after it is ϕ_i. If $b \in B_\lambda$, we set the weight of b by $\text{wt}(b) = \lambda$, and we notice we have

$$\phi_i(b) - \epsilon_i(b) = \lambda(h_i)$$

this notation will allow us to define crystal base in abstract way.

Example 2.2.4.3 Crystal base of $U_q(\mathfrak{sl}(2))$: Let's go back to the case of $U_q(\mathfrak{sl}(2))$. Let $V(m)$ be a finite dimensional irreducible module in weight λ in dimensional $m+1$. We know it has a basis $\{v_0 = v, \dots, v_m\}$, where $v_i = f^{(i)}v$. Based one this, we can define an local base of $V(m)$ by

$$L(m) = \bigoplus_{k=0}^{m} Av_i$$

$$B(m) = \{\hat{v}_0, \dots, \hat{v}_m\}$$

It is not hard to see B is a $\mathbb{C}-$basis of A by using the fact $A/qA = \mathbb{C}$. So we get (L, B) is a local base of V, and the Kashiwara operator implie $\overline{f}v_i = v_{i+1}$ so its crystal graph contains a unique path

$$B(m) = \hat{v}_0 \to \cdots \to \hat{v}_m$$

A nice property of object in $\mathcal{O}_{\text{int}}$, wchich we will not pove here, is they're all semisimple. Namely, they're direct sum of irreducible highest weight module. In particularly, if we take finite dimensional object, then by the construction above, we can have a local base for every finite dimensional object in $\mathcal{O}_{\text{int}}$.

The case in $U_q(\mathfrak{sl}(2))$ may give you some intuitions for the existence of crystal base of irreducible highest weight of weight $\lambda \in P_+$ module in $\mathcal{O}_{\text{int}}$: take a highest weight vector v_λ, and consider the set $\{\overline{f}_{i_1} \dots \overline{f}_{i_r} v_\lambda | r \geqslant 0, i \in I\}$, then let $L(\lambda)$ be a free $A-$module spanned by this set, and set $B =$ the crystal limit of the set. However, the proof of this fact is in fact tedious and technical which relies on the trick called grant-loop argument by Kashiwara. In fact, Kashiwara shows a much more stronger argument about the object in $\mathcal{O}_{\text{int}}$.

Defintion: Two crystal base $(L_1, B_1), (L_2, B_2)$ are called **isomorphic** if there is a $A-$linear isomorphism $\phi : L_1 \to L_2$ such that
(1): ϕ commute with all kashiwara operator.
(2): The induced $F-$isomorphism $L_1/qL_1 \to L_2/qL_2$ is a bijection between $B_1 \bigsqcup\{0\} \to B_2 \bigsqcup\{0\}$.

Theorem(Kashiwara): Each $V(\lambda)$ in $\mathcal{O}_{\text{int}}$ has a unique crystal base (up to isomorphism).

So, together with the semisimple property, every object in $\mathcal{O}_{\text{int}}$ has a unique crystal base.

3 Coboundary Monoidal Category

3.1 Monoidal Category

Let C be a set, it is a monoid if it has a binary operation $*$ such that (1) $(a*b)*c = a*(b*c), \forall a, b, c \in S$ and (2) there is an element e that $e * a = a * e = a, \forall a \in S$. The idea of the Monoidal Category is the "Categorification" of the monoid set, that we "translate" the language of the set (element, function) to the language of Category (object, functor).

Definition: A **Monoidal Category** is a Category $\mathcal{C}$, together with a bifunctor $\otimes : C \times C \to C$(we call this functor tensor product or tensor functor), a unit object $\mathbf{1}$ and the natural isomorphisms.

$$a_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C), \forall A, B, C \in \mathcal{C}$$
$$L_X : \mathbf{1} \otimes X \cong X$$
$$R_X : X \otimes \mathbf{1} \cong X$$

such that the following diagram commute

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes Z) \otimes W \arrow[dl, "a_{X\otimes Y,Z,W}"'] \arrow[dr, "a_{X,Y,Z}\otimes 1_W"] & \\
(X \otimes Y) \otimes (Z \otimes W) \arrow[d, "a_{X,Y,Z\otimes W}"] & & (X \otimes (Y \otimes Z)) \otimes W \arrow[d, "a_{X,Y\otimes Z,W}"] \\
X \otimes (Y \otimes (Z \otimes W)) & & X \otimes ((Y \otimes Z) \otimes W) \arrow[ll, "1_X\otimes a_{Y,Z,W}"]
\end{tikzcd}$$

(this diagram is called pentagon axiom)

and,

$$\begin{tikzcd}
 & X \otimes Y \arrow[dl, "R_X\otimes 1_Y"'] \arrow[dr, "1_X\otimes L_Y"] & \\
(X \otimes \mathbf{1}) \otimes Y & & X \otimes (\mathbf{1} \otimes Y) \arrow[ll, "a_{X,1,Y}"]
\end{tikzcd}$$

(This diagram is called triangle identity)

if the the natural isomorphism a, R, L of $\mathcal{C}$ are exactly identity e.g $X \otimes \mathbf{1} = X$, rather than just isomorphism, then $\mathcal{C}$ is called **Strict Monoidal Category**

In general, we denote $(\mathcal{C}, \otimes, \mathbf{1}, a, L, R)$ or $(\mathcal{C}, \otimes, \mathbf{1})$ as a Monoidal Category.

Example 3.1.1: Let $\mathcal{C}$ be a (small) category, the functor category $End(\mathcal{C})$ which conclude all the functor to itself is a Monoidal category where the tensor product is just the composition $\circ$ of functor and the unit object is $1d_{\mathcal{C}}$

Example 3.1.2 Let G be a group, then the $Rep(G)$ denoted by the all representation of G over field F is a monoidal category. The tensor product of two representation $\rho : G \to GL(V_1), \mu : G \to GL(V_2)$ given by $\rho \otimes \mu : G \to GL(V_1 \otimes V_2)$ is associative, so it could be the tensor functor in $Rep(G)$, and the unit object is the trivial representation $\mathbf{1} : G \to F^{\times}$. Similarly, let $\mathfrak{g}$ be a lie algebra over a field F, then its representation $Rep(\mathfrak{g})$, form a monoidal category in a similar way.

Example 3.1.3 Let R be a ring, then the category $R - bimod$, the category of $R-$bimodule is a monoidal category that the tensor functor is the tensor product and the unit object is R itself. More generally, every additive category is actually a monoidal category, where the tensor functor is the direct sum and the unit object is the zero object.

In the monoid, the identity e is unique, so we want to check the unit object in monoidal category is also unique.

Proposition 3.1.4: The unit object I is unique (up to isomorphism)
Proof: Let I, I' be two object, then we have the natural isomorphism $R_I : X \otimes I \cong I$ and $L_{I'} = I' \otimes X \cong I'$. Now plug I' into R_I, I in to $L_{I'}$, we get $I' \cong I' \otimes I = I' \otimes I \cong I$, so $I \cong I'$.

3.2 Monoidal Functor

We already successfully categorificate the monoid set itself, now we want to categorificate the function between monoid set as functor between monoidal category.

Defintion: Let $(\mathcal{C}, \otimes, I_{\mathcal{C}}, a, L, R), (\mathcal{D}, \otimes', I_{\mathcal{D}}, a', L', R')$ be two monoidal category. A **Monoidal Functor** $F = (F_o, \phi, \epsilon)$ consists of the following data
(a): An underlying functor $F_o : \mathcal{C} \to \mathcal{D}$, which we also denote it by F if no ambiguity.
(b): A natural transformation $\phi_{X,Y} : FX \otimes' FY \to F(X \otimes Y)$
(c) a morphism $\epsilon : I_{\mathcal{D}} \to F(I_{\mathcal{C}})$

,and the following diagram commute

$$\begin{array}{ccc}
(FX \otimes' FY) \otimes' FZ & \xrightarrow{a'_{FX,FY,FZ}} & FX \otimes' (FY \otimes' FY) \\
\downarrow{\scriptstyle \phi_{X,Y} \otimes' 1_{FZ}} & & \downarrow{\scriptstyle 1_{FX} \otimes' \phi_{Y,Z}} \\
F(X \otimes Y) \otimes' FY & & FX \otimes' F(Y \otimes Z) \\
\downarrow{\scriptstyle \phi_{X \otimes Y, Z}} & & \downarrow{\scriptstyle \phi_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X \otimes (Y \otimes Z))
\end{array} \qquad (1.2.1)$$

$$\begin{array}{ccc}
I_{\mathcal{D}} \otimes FX & \xrightarrow{\epsilon \otimes 1} & F(I_{\mathcal{C}}) \otimes FX \\
\downarrow{\scriptstyle L'_{FX}} & & \downarrow{\scriptstyle \phi} \\
FX & \xleftarrow[F(L_X)]{} & F(I_{\mathcal{C}} \otimes X)
\end{array} \qquad (1.2.2)$$

$$\begin{array}{ccc}
FX & \xrightarrow{R'_{FX}} & FX \otimes' I_{\mathcal{D}} \\
\downarrow{\scriptstyle F(R_X)} & & \downarrow{\scriptstyle 1 \otimes' \epsilon} \\
F(X \otimes I_{\mathcal{C}}) & \xleftarrow[\phi_{FX, F(I_{\mathcal{C}})}]{} & FX \otimes' F(I_{\mathcal{C}})
\end{array} \qquad (1.2.3)$$

This functor F is called **Strong** if ϕ, ϵ are isomorphism. It is called **Normal** if ϵ is isomorphism, and it is called **Strict** if ϕ, ϵ are idenities.

Example 3.2.1: The forgetful functor send **Grp**, **Top**, **Monoid** to the **Set** are all monoidal. Another interesting example of forgetful functor is $Rep(G) \to Vec$, where we send the representation of group G to the vector space.

The monoidal functor between two monoidal category will form a category, and its morphism (or natural transformation) is defined as follow

Defintion Let (F, ϕ, ϵ), (G, ϕ', ϵ') be two Monidal functor, a natural transformation $\rho : F \to G$ is called **Monoidal natural transformation** if the following diagram commute:

$$\begin{tikzcd} FX \otimes FY \arrow[r, "\phi_{X,Y}"] \arrow[d, "\rho_X \otimes \rho_Y"] & F(X \otimes Y) \arrow[d, "\rho_{X \otimes Y}"] \\ GX \otimes GY \arrow[r, "\phi'_{X,Y}"] & G(X \otimes Y) \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, "\epsilon"] \arrow[dr, "\epsilon'"'] & F(I) \arrow[d, "\rho_I"] \\ & G(I) \end{tikzcd}$$

3.3 Coherence Theorem

Given a sequence of element with fixed ordered $x_1, \dots, x_n$ in a monoid, one may naturally ask how many way we can parenthesize the product, that is, how many different way to insert the parentheses. Through the combinatoric, we know the number is actually the Catalan number $\frac{1}{n+1}\binom{2n}{n}$, but because of the "associativity" property of the binary operation, we can identify two parenthesizings through a chain of "associativity" operation. For example, we can see when $n = 4$, $(ab)cd$ and $a((bc)d)$ are same since $(ab)cd = a(b(cd)) = a((bc)d)$

Similarly, in monoidal category, for any two distinct parenthesized products of n object, we can use a chain of "associativity" isomorphism to identify them. For example, the pentagon axiom helps us to identify all the possible parenthesized product in 4 objects, and one benefit is when we do computation for 4 objects, we can ignore the parentheses since the diagram of pentagon axiom commute, so every parenthesizing in 4 objects are canonical isomorphic.

However, for the case $n > 5$, we don't have a commute diagram formed by associativity isomorphism for all parenthesizings, so we may have more than one chain of associativity isomorphism between two distinct parenthesizing. The **Coherence Theorem**, which is proved by Mac Lane will help us to solve this problem. Before we prove the theorem, we first show an important proposition of monoidal category.

Defintion: A monoidal functor is **monoidal equivalence** if it is strong and equivalence.

Proposition 3.3.1: Every Monoidal category is monoidally equivalent to a strict Monoidal category.

Proof. Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}}, a, L, R)$ be a monoidal category, we separate the proof in several steps.

Step1: We construct a strict monoidal category from $\mathcal{C}$.
Consider a category $\mathcal{C}'$ whose objects are pairs (F, τ), where $F : \mathcal{C} \to \mathcal{C}$ a functor and a natural isomorphism $\tau_{X,Y} : X \otimes F(Y) \cong F(X \otimes Y)$. The morphism between two objects (F, τ) and (G, σ) is a natural transformation α such that the diagram commute

$$\begin{tikzcd} X \otimes F(Y) \arrow[r, "\tau"] \arrow[d, "1 \otimes \alpha_Y"] & F(X \otimes Y) \arrow[d, "\alpha_{X \otimes Y}"] \\ X \otimes G(Y) \arrow[r, "\sigma"] & G(X \otimes Y) \end{tikzcd}$$

and we can define a tensor product $\otimes' : ((F, \tau), (G, \sigma)) \mapsto (GF, \tau \circ \sigma)$, and here the $(\tau \circ \sigma)_{X,Y} : X \otimes GF(Y) \xrightarrow{\sigma_{X,F(Y)}} G(X \otimes F(Y)) \xrightarrow{\tau_{X,Y}} GF(X \otimes Y)$.
Also, we can construct the unit object $(1_{\mathcal{C}}, 1)$through identity functor $1_{\mathcal{C}}$, with the isomorphism $1 : X \otimes 1_{\mathcal{C}}(Y) \cong X \otimes Y$, and one can check

$$a' : ((F, \tau) \otimes' (G, \sigma)) \otimes' (Z, \omega) = (Z(GF), (\tau \circ \sigma) \circ \omega = ((ZG)F), \tau \circ (\sigma \circ \omega) = (F, \tau) \otimes' ((G, \sigma) \otimes' (Z, \omega))$$

$$L'_{(F,\tau)} : (1_{\mathcal{C}}, 1) \otimes' (F, \tau) = (1_{\mathcal{C}} F, 1 \circ \tau) = (F, \tau) = (F, \tau) \otimes' (1_{\mathcal{C}}, 1) = R'_{(F,\tau)}$$

We can easily verify the triangle identity and pentagon axiom. Hence the category $\mathcal{C}'$ is a strict monoidal category.

Step2: We construct a strong monoidal functor η bewteen $\mathcal{C}$ and $\mathcal{C}'$.
Consider $\eta : \mathcal{C} \to \mathcal{C}'$ defined as $X \mapsto ((- \otimes X), n_X)$, $\eta(f) = (- \otimes f)$ where f is a morphism in $\mathcal{C}$ and

$$(n_x)_{X,Y} = Y \otimes \eta(X)(Z) = Y \otimes (Z \otimes X) = (Y \otimes Z) \otimes X = \eta(X)(Y \otimes Z)$$

One can easily check this is a functor, and to construct a monoidal functor, we still need ϵ and ϕ, which we define as follow:
$\phi_{X,Y} : \eta(X) \otimes' \eta(F) = \eta(X \otimes Y)$
$\epsilon : (1_{\mathcal{C}}, 1) \to \eta(1_{\mathcal{C}})$
and through the fact
$\phi_{X,Y}(W) = X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes W = \eta(X \otimes Y)(W)$
$\epsilon : 1_{\mathcal{C}}(W) \cong (W \otimes 1) = \eta(\mathcal{C})$
both map are isomorphism, and all three diagrams

$$\begin{tikzcd} & (\eta(X) \otimes' \eta(Y)) \otimes' \eta(Z) \arrow[dl, "1 \otimes' \phi"'] \arrow[dr, "\phi \otimes' 1"] & \\ \eta(X) \otimes' (\eta(Y) \otimes \eta(Z)) \arrow[d, "\phi"] & & \eta(X \otimes Y) \otimes' \eta(Z) \arrow[d, "\phi"] \\ \eta(X \otimes (Y \otimes Z)) & & \eta((X \otimes Y) \otimes Z) \arrow[ll, "\eta(a)"] \end{tikzcd}$$

$$\begin{tikzcd}
(1_{\mathcal{C}}, 1) \otimes' \eta(X) \arrow[r, "\epsilon \otimes' 1"] \arrow[d, "="] & \eta(1_{\mathcal{C}}) \otimes' \eta(X) \arrow[d, "\phi"] \\
\eta(X) & \eta(1_{\mathcal{C}} \otimes X) \arrow[l, "\eta(L_X)"]
\end{tikzcd}$$

$$\begin{tikzcd}
\eta(X) \arrow[r, "\eta(R_X)"] \arrow[d, "="] & \eta(X \otimes 1_{\mathcal{C}}) \\
\eta(X) \otimes' (1_{\mathcal{C}}, 1) \arrow[r, "1 \otimes' \epsilon"] & \eta(X) \otimes' \eta(1_{\mathcal{C}}) \arrow[u, "\phi"']
\end{tikzcd}$$

are commute since if we plug any object W in to first two digram, we will get the pentagon axiom and triangle identity in $\mathcal{C}'$. The last one commute because of the following lemma.

Lemma: In the strict monoidal category, the diagram

$$\begin{tikzcd}
X \otimes Y \arrow[r, "1 \otimes R_Y"] \arrow[dr, "R_{X \otimes Y}"'] & X \otimes (Y \otimes I) \\
& (X \otimes Y) \otimes I \arrow[u, "a"']
\end{tikzcd}$$

commute.

Proof. Since the L, R, a are identity, so it will be equal to show the following diagram commute

$$\begin{tikzcd}
(X \otimes Y) \otimes I \arrow[r, "(1 \otimes R_Y) \otimes 1"] \arrow[dr, "R_{X \otimes Y} \otimes 1"'] & ((X \otimes (Y \otimes I)) \otimes I \\
& ((X \otimes Y) \otimes I) \otimes I \arrow[u, "a"']
\end{tikzcd}$$

Now consider the following diagram

$$
\begin{tikzcd}
X \otimes (Y \otimes I) \arrow[r, "1 \otimes R_{Y \otimes I}"] \arrow[rrr, bend left, "1"] & X \otimes ((Y \otimes I) \otimes) I \arrow[r, "a"] & X \otimes (Y \otimes (I \otimes I)) \arrow[r, "a"] \arrow[dd, "a"] & X \otimes (Y \otimes I) \arrow[dd, "a"] \\
(X \otimes Y) \otimes I \arrow[u, "a"] \arrow[r, "(1 \otimes R_Y) \otimes 1"] \arrow[dr, "R_{X \otimes Y} \otimes 1"'] \arrow[drrr, bend right, "1"'] & (X \otimes (Y \otimes I)) \otimes I \arrow[u, "a"'] & & \\
& ((X \otimes Y) \otimes I) \otimes I \arrow[u, "a \otimes 1"'] \arrow[r] & (X \otimes Y) \otimes (I \otimes I) \arrow[r] & (X \otimes Y) \otimes I
\end{tikzcd}
$$

This diagram will commute except the triangle part since it is either triangle identity or pentagon axiom, so all remaining part commute implies the triangle part commute. □

With this Lemma, the last diagram is commute if we plug any object in $\mathcal{C}$, so we get a strong monoidal functor (η, ϕ, ϵ)

Step3: η is equivlence.
Essentially surjective: Notice for any object (F, τ) in $\mathcal{C}'$, we have $F(X) \cong F(X \otimes 1) \cong X \otimes F(1)$, so any object in $\mathcal{C}'$ isomorphic to $\eta(F(1))$.
Full: Given a morphism $\rho : \eta(X) \to \eta(Y)$, then we have the following diagram commute

$$
\begin{tikzcd}
W \otimes \eta(X)(Z) \arrow[r, "n_X"] \arrow[d, "1 \otimes \rho_Z"] & \eta(X)(W \otimes Z) \arrow[d, "\rho_{W \otimes Z}"] \\
W \otimes \eta(Y)(Z) \arrow[r, "n_Y"'] & \eta(Y)(W \otimes Z)
\end{tikzcd}
$$

and we notice a composition of morphism $f : X \xrightarrow{(L_X)^{-1}} 1_{\mathcal{C}} \otimes X \xrightarrow{\rho_{1_{\mathcal{C}}}} 1_{\mathcal{C}} \otimes Y \xrightarrow{L_Y} Y$ will let the diagram

$$
\begin{tikzcd}
W \otimes \eta(X)(Z) \arrow[r, "n_X"] \arrow[d, "1 \otimes \eta(f)_Z"] & \eta(X)(W \otimes Z) \arrow[d, "\eta(f))_{W \otimes Z}"] \\
W \otimes \eta(Y)(Z) \arrow[r, "n_Y"'] & \eta(Y)(W \otimes Z)
\end{tikzcd}
$$

commute(because L is isomorphism), so this imply the $\eta(f) = \rho$ and the functor is full.
Faithful: When we have two morphism $\rho = \eta(f), \gamma = \eta(g)$ in $\mathcal{C}'$, then if $\rho = \gamma$, we should have $\eta(f) = - \otimes f = - \otimes g = \eta(g)$, In particular when "$-$" is $1_{\mathcal{C}}$, we get $1_{\mathcal{C}} \otimes f = f = g = 1_{\mathcal{C}} \otimes g$ through the L, so the functor is faithful.

□

Corollary_(Mac Lane Coherence theorem)_: Let $X_1, \ldots, X_n$ be object in $\mathcal{C}$, and let S_1, S_2 be two distinct parenthesizings of $X_1, \ldots, X_n$ in fixed order with (pssible) arbitrary insertion of unit object I. Then any two isomorphism $f, g : S_1 \to S_2$, obtained through compositing the unit isomorphism (L and R), associativity isomorphism and may have a inverse which tensored with a identity map, are identity (e.g $f = g$).

Remark: The theorem actually tells us the paretheizings in monoidal category of n objects will behave similarly in monoid, that we can identity any two parenthesizings, even with abitrary insertion of unit object, through a unique chain of associativity isomorphism (may conclude a unit ismorphism). The diagram we use to prove the Lemma in *Step2* in the previous proposition is a visible example for this theorem.

Proof. Let $\mathcal{C}$ be a monoidal category, then it will be monoidal equivalent to a strict monoidal category through a monoidal functor F. Consider the diagram representing f and g, then we pass it to $\mathcal{C}'$ through the F. Each arrow f_i in the diagram of f and g is either associativity isomorphism or unit isomorphism, so we can build a rectangle in 1.2.1 through $F(f_i)$ if f_i is a associativity isomorphism, and a rectangle in 1.2.2 or 1.2.3 if f_i is a unit isomorphism. Now, we get a "prism" diagram, where one base consists of all identity map (the associativity and unit isomorphism in $\mathcal{C}'$) and one base is the diagram of $F(f)$ and $F(g)$, and all sides are commutative. Hence, all the face in this diagram is commutative, and this implies $f = g$ □

3.4 Coboundary Monoidal Category

Let n be a positive integer, and , $1 \leqslant p < q, \leqslant n$, then let

$$\hat{s}_{p,q} = \begin{pmatrix} 1 & \ldots & p-1 & p & \ldots & q & q+1 & \ldots & n \\ 1 & \ldots & p-1 & q & \ldots & p & q+1 & \ldots & n \end{pmatrix}$$

we also say $p < q$ and $k < l$ are **disjoint** if $q < k$ or $p > l$, we say $p < q$ **contains** $k < l$ if $p \leqslant k < l \leqslant q$

A $n-$fruit cactus group C_n is generated by the generator $s_{p,q}$ with the relation

$$(1) : s_{p,q} \circ s_{p,q} = 1$$

$$(2) : s_{p,q} \circ s_{k,l} = s_{k,l} \circ s_{p,q} \text{ if } p < q \text{ and } k < l \text{ are disjoint}$$

$$(3) : s_{p,q} \circ s_{k,l} = s_{r,t} \circ s_{p,q} \text{if} p < q \text{ contains } k < l, \text{ where} \hat{s}_{p,q}(l) = r, \hat{s}_k = t$$

We can notice the S_n is generated by $\hat{s}_{i,i+1}$ and we have each $\hat{s}_{p,q}$ will satisfy the relation of cactus group, so there is an natural surjective homomorphism $C_n \to S_n$ extended from the map $s_{p,q} \mapsto \hat{s}_{p,q}$.

The Coboundary category is generalized from the cactus group

Defintion: A **Counbary monoidal category** is a monoidal category $\mathcal{C}$ together with natural isomorphism

$$\sigma^c_{U,V} : U \otimes V \to V \otimes U$$

that satisfy the following condition:

(1): $\sigma^c_{V,U} \circ \sigma^c_{U,V} = Id_{U \otimes V}$

(2): we have cactus relation for all object in $U, V, W \in \mathcal{C}$, that is the following diagram commute:

$$\begin{array}{ccc} U\otimes V\otimes W & \xrightarrow{\sigma^c_{U,V}\otimes Id} & V\otimes U\otimes W \\ {\scriptstyle Id\otimes\sigma^c_{V,W}}\downarrow & & \downarrow{\scriptstyle \sigma^c_{V\otimes U,W}} \\ U\otimes W\otimes V & \xrightarrow{\sigma^c_{U,W\otimes V}} & W\otimes V\otimes U \end{array}$$

The collection of the map $\sigma_{U,V}$ is called **cactus commutator**.

Remark: We denote the natural commutator for the collection of the isomorphism $\sigma_{U,V} : U \otimes V \to V \otimes U$, and the cactus commutator be the collection σ^c where they satisfy the additional requirements given above.
We can define a "group action" of cactus group on the counbardy category. Namely given $U_1, \ldots, U_n \in Ob(\mathcal{C})$, then for $1 \leqslant p < q \leqslant n$, then there is a naturally isomorphism

$$\sigma^c_{p,q} = id_{U_1\otimes\cdots\otimes U_{p-1}} \otimes \sigma^c_{U_p,U_{p+1}\otimes\cdots\otimes U_q} \otimes Id_{U_{q+1}\otimes\cdots\otimes U_n} :$$

$$U_1\otimes\cdots\otimes U_n \to U_1\otimes\cdots\otimes U_{p-1}\otimes\cdots\otimes U_{p-1}\otimes U_{p+1}\otimes\cdots\otimes U_q\otimes U_p\otimes U_{q+1}\otimes\cdots\otimes U_n$$

Using this fact, we can associate σ^c with the $s_{p,q}$ in the following way: we define $s_{p,p+1} = \sigma^c_{p,p+1}$ and $s_{p,q} = \sigma^c_{p,q} \circ s_{p+1,q}$ for $q > p + 1$, we also set $s_{p,p} = Id$. Hence we can see $s_{p,q}$ is an natural isomorphism $U_1 \otimes \cdots \otimes U_n \to U_1 \otimes \cdots \otimes U_{p_1} \otimes U_q \otimes U_{p+1} \otimes \cdots \otimes U_{q-1} U_p U_{q+1} \otimes \cdots \otimes U_n$

Lemma 3.4.1: The $s_{p,q}$ defined above will satisfy

$$(1) : s_{p,q} \circ s_{p,q} = 1$$

$$(2) : s_{p,q} \circ s_{k,l} = s_{k,l} \circ s_{p,q} \text{ if } p < q \text{ and } k < l \text{ are disjoint}$$

$$(3) : s_{p,q} \circ s_{k,l} = s_{r,t} \circ s_{p,q} \text{ if } p < q \text{ contains } k < l, \text{ where} \hat{s}_{p,q}(l) = r, \hat{s}_k = t$$

Remark: There is a more precisely definition for group action on category: Let $Aut(\mathcal{C})$ be all autoequivlenet monoidal functor on $\mathcal{C}$, $Cat(G)$ be the category of G where the object are just the element of G. Then, teh action of G on $\mathcal{C}$ is monoidal functor $Cat(G) \to Aut(\mathcal{C})$. In our case, in just mean a functor $\rho \mapsto U_{\hat{\rho}(1)} \otimes \cdots \otimes U_{\hat{\rho}(n)}$, for $\rho \in C_n$. Reader the parallel to the case that S_n acts on a monoid set.

3.5 $\mathfrak{g}$-Crystal category

As we said, the crystal base could be defined in more abstract way, so we can use it as combinatorial model for representation of Lie algebra $\mathfrak{g}$. This section will focus on the Lie algebra rather than quantum group. We let $\mathfrak{g}$ be a Lie algebra and P be its weight lattice, $\{h_i\}$ be simple root and $\{a_i\}$ be its simple coroot.

Definition: A **crystal** of $\mathfrak{g}$ is a finite set B together with the map

$$wt : B \to P$$

$$\epsilon_i, \phi_i : B \to \mathbb{Z}$$

$$\overline{e}_i, \overline{f}_i : B \to B \bigsqcup \{0\}$$

and for each $i \in I$ such that

$$(1) : \forall b \in B, \phi_i(b) - \epsilon_i = \langle wt(b), a_i \rangle, a_i \in \Pi$$

$$(2) : \epsilon_i(b) = \max\{n \geqslant o | \overline{f}_i^n b \neq 0\}, \phi_i(b) = \max\{n \geqslant o | \overline{e}_i^n b \neq 0\}$$

(3) : if $b \in B$ and $\overline{e}_i b \neq 0,$ then $wt(\overline{e}_i)b = wt(b)+a_i$. Similarly, we have$wt(\overline{f}_i b) = wt(b) - a_i$ if $\overline{f}_i b \neq 0$

$$(4) : \text{for all b,b'} \in B, \overline{f}_i b = b' \Leftrightarrow \overline{e}_i b' = b$$

.

Remark: We can think B as "base" of representation of $\mathfrak{g}$ or quantum group where $\overline{e}_i, \overline{f}_i$, are the Chevalley generator.

Again, we can associate a **crystal graph** with each crystal by the last proposition. The crystal is called **connected** if the underlying graph is connected. We also define the **highest weight crystal** of weight $\lambda \in P_+$, if there exist is an **highest weight vector** $b_\lambda \in B_\lambda$ of weight λ such that $\overline{e}_i b_\lambda = 0$ and B is generated by $\overline{f}_i b$.

Remark: (1) It is correct that every highest weight crystal is connected, but the converse is not true, and unlike the highest weight module, two highest weight crystal in same weight is not necessary isomorphic.
(2) Every crystal admits $B = \bigoplus_\lambda B_\lambda$, and this is a decomposition of the underlying crystal graph of B into connected component.

Defintion: The category of crystal of a Lie algebra $\mathfrak{g}$ consist of the following data:
(1) The object are crystals of $\mathfrak{g}$
(2) The morphism $f : B \to B'$ is a map $\Phi : B \bigsqcup \{0\} \to B' \bigsqcup \{0\}$ that satisfy:
(a)$\Phi(0) = 0$
(b)$\forall b \in B, \Phi(\overline{e}_i b) = \overline{e}_i \Phi(b)$ (resp. $\overline{f}_i$) if $\overline{e}_i b \neq 0$(resp. $\overline{f}_i b \neq 0$)
(c)Φ commute with wt, ϵ_i, ϕ_i for $\Phi(b) \neq 0$.

We denote this category as $\mathfrak{g}-$crystal. Furthermore, this category is a monoidal category by equipped with the following tensor rule.

Tensor rule: Let B, B' be two crystals. The tensor product $B \otimes B'$ is a set $\{b \otimes b' | b \in B, b' \in B'\}$ satisfy

$$wt(b \otimes b') = wt(b) + wt(b')$$
$$\epsilon_i(b \otimes b') = max(\epsilon_i(b), \epsilon_i(b') - \langle wt(b), h_i \rangle)$$
$$\phi_i(b \otimes b') = max(\epsilon_i(b'), \epsilon_i(b) + \langle wt(b'), h_i \rangle)$$

$$\overline{e}_i(b \otimes b') = \begin{cases} \overline{e}_i b \otimes b' & \text{If } \epsilon_i(b) \geqslant \phi_i(b') \\ b \otimes \overline{e}_i b' & \text{if } \epsilon_i(b) < \phi_i(b') \end{cases} \tag{2}$$

$$\overline{f}_i(b \otimes b') = \begin{cases} \overline{f}_i b \otimes b' & \text{If } \epsilon_i(b) \geqslant \phi_i(b') \\ b \otimes \overline{f}_i b' & \text{if } \epsilon_i(b) \leqslant \phi_i(b') \end{cases} \tag{3}$$

Here we use $b \otimes b'$ for (b, b'). It is not hard to see this $B \otimes B'$ is still a crystal, so the assignment $(B, B') \mapsto B \otimes B'$ is bifunctor. Moreover, this bifunctor will satisfy the associative axiom (moreover, it is strict.) because the we can see the requirement in the tensor rule only rely on the "maxiaml" one, so the order doesn't matter. Hence, $\mathfrak{g}-$crystal is a monoidal category

3.6 Construction of cactus commutator

The $\mathfrak{g}-$crystal is a monoidal category as we shown, and furthurmore it could be a coboundary category: we can construct the cactus commutator through the Schutzenberger involution

Schutzenberger involution: Let $\mathfrak{g}$ be a reductive or simple complex Lie algebra, I the set of vertices of its Dykin diagram, and let $\theta : I \to I$ be an automorphism of Dykin diagram such that $a_{\theta(i)} = -w_{\mathrm{o}}(a_i)$, where w_{o} is the longest element of Weyl group of $\mathfrak{g}$. We define a crystal $\overline{B}_\lambda$ with the underlying set $\{\overline{b}|b \in B_\lambda\}$ and

$$\overline{e}_i\overline{b} = \overline{\overline{f}_{\theta(i)}b}, \overline{f}_i\overline{b} = \overline{\overline{e}_{\theta(i)}b}, wt(\overline{b}) = w_{\mathrm{o}} * wt(b)$$

By our construction, $\overline{B}_\lambda$ is also a highest weight crystal, and there exists a crystal isomorphism from $\overline{B}_\lambda$ to B_λ (reader can refer [8] to see how to construct this isomorphism). On the other hand, there is a natural map of set $B_\lambda \to \overline{B}_\lambda : b \mapsto \overline{b}$. Composing these two maps, we get **Schutzenberger involution** $\xi_\lambda : B_\lambda \to B_\lambda$. We can see $\xi_\lambda \circ \xi_\lambda$ is a morphism of crystal because

$$\overline{e}_i\xi_\lambda(b) = \xi_\lambda(e_{\theta(i)}b), \overline{f}_i\xi_\lambda(f) = \xi_\lambda(f_{\theta(i)}b), wt(\xi(b)) = w_{\mathrm{o}}wt(b)$$

Moreover, $\xi_\lambda \circ \xi_\lambda = Id$ (so it is an involution) because of the following Shur Lemma

Lemma(*Shur*): The $Hom(B(\lambda), B(\mu))$ only contains identity if $\lambda = \mu$, and it is 0 otherwise.

Proof. Let $f : B \to B'$ be the morphism of highest weight crystal, then if $b \in B$ vanished by $\overline{e}_i$, then $f(b)$ is also vanished by $\overline{e}_i$, so the only possible morphism is send highest weight vector to highest weight vector.

□

Because we have the decomposition $\bigoplus_\lambda B_\lambda$for B, we can define an involution.$\xi_B : B \to B$, using this we can construct a commutator

Proposition 3.6.1: Let B, B' be two crystal, then $\sigma_{B,B'} : B \otimes B' \to B' \otimes B; b \otimes b' \mapsto \xi_{B\otimes B'}(\xi'_B(b') \otimes \xi_B(b))$ is an isomorphism of crystal (The $\sigma_{A,B}$ is in fact equal to $\xi_{B'\otimes B} \circ (\xi_{B'} \otimes \xi_B) \circ$ flip).

Proof. The map $\sigma_{A,B}$ is an bijective map of set since flip map and all ξ above are bijective map. It only remained to show this is a morphism of crystal.

For simplicity, we will use ξ to represent all the $\xi_B, \xi'_B, \xi_{B'\otimes B}$, then

$$\overline{e}_i\sigma_{B,B'}(b \otimes b') = \overline{e}_i\xi(\xi(b') \otimes \xi(b)) = (\xi(\overline{f}_{\theta(i)}(\xi(b') \otimes \xi(b))) = \xi(\xi(b') \otimes \overline{f}_{\theta(i)}\xi(b)) = \xi(\xi \otimes \xi(\overline{e_i}b))$$

$$= \sigma_{B,B'}(\overline{e}_i b \otimes b') = \sigma(\overline{e}_i(b \otimes b'))$$

Similarly, we have $\overline{f}_i$ and wt commute with the commutator. □

Lemma 3.6.2 $\sigma_{A,B} = \sigma_{B,A}^{-1}$.

Proof. We notice $\sigma^{-1}_{B,A} = \text{flip} \circ (\xi_B \otimes \xi_A) \circ \xi_{A\otimes B}$. Now let $(a,b) \in A \otimes B$, then we want to show that $\sigma_{A,B}((a\otimes b)) = \sigma^{-1}_{B,A}(a\otimes b)$. Since both map are isomorphism, then by Shur Lemma, this is same to show their map lie in the same connected component $B(\lambda)$.

The construction of ξ preserve the component, so if we let $(c\otimes d) = \xi_{A\otimes B}(a\otimes b)$, then it will be sufficient to check $(\xi(c)\otimes\xi(b))$ and $(\xi(b)\otimes\xi(a))$ are in the same component. However, since (c,d) lies in the same component of (a,b), so there exist some $i_1, \dots i_r, j_1, \dots, j_s$ such that $\overline{e}_{i_1}\dots\overline{e}_{i_r}\overline{f}_{j_1}\dots\overline{f}_{j_s}(a\otimes b) = (c\otimes d)$, and then it will be easily to see $f_{\theta(i_1)}\dots f_{i_r}e_{j_s}\dots e_{j_1}(\xi(b)\otimes\xi(a)) = (\xi(c)\otimes\xi(b))$. □

Immediately, we conclude the commutator $\sigma_{A,B}$ will satisfy the first proposition of cactus commutator by using the proposition 3.6.2 and the fact ξ is involution.
Corollary: We have $\sigma_{B,A}\circ\sigma_{A,B} = 1$.

Proposition 3.6.3: We have the following commutative for $\sigma_{A,B}$

$$\begin{array}{ccc} A\otimes B\otimes C & \xrightarrow{1\otimes\sigma_{B,C}} & A\otimes C\otimes B \\ {\scriptstyle \sigma_{A,B}\otimes 1}\downarrow & & \downarrow{\scriptstyle \sigma_{A,C\otimes B}} \\ B\otimes A\otimes C & \xrightarrow[\sigma_{C,B\otimes A}]{} & C\otimes B\otimes A \end{array}$$

Proof. The right-down path will give us

$$\xi(a, (\xi(c), \xi(b))) = \xi(\xi(c), \xi(b), \xi(a))$$

and the down-right path will give us

$$\xi((\xi(b), \xi(a)), c) = \xi(\xi(b), \xi(a), \xi(c))$$

□

Hence, the commutator σ will be an cactus commutator, and this imply that the $\mathfrak{g}-$crystal is a coboundary category.

Remrak: The construction of the commutator σ is given by Henriques and Kamnitzer in the paper [8]. Besides the cactus commutator over $\mathfrak{g}-$crystal, they also use a similar way to construct a cactus commutator over module of quantum group. Namely, we define $\xi : E_i \mapsto F_{\theta(i)}, F_i \mapsto E_{\theta(i)}, q^h \mapsto q^{w_o * h}$, and then we define a commutator σ in the same way. Then Henriques and Kamnitzer show this map is also a cactus commutator in module of quantum group, which mean the module of quantum of group is a coboundary category.

4 Reference

[1]: C. Kassel, *Quantum Groups*, volumn 155 of *Graduate Textbook in Mathematics.* Springer New York, 1995

[2]: G. Lusztig. *Introduction to quantum groups*, volume 110 of Progress in Mathematics. Birkhauser Boston Inc., Boston, MA, 1993.

[3]: M. Kashiwara. *On crystal bases.* Representations of groups (Banff, AB, 1994), volume 16 of *CMS Conf. Proc.*, pages 155–197. Amer. Math. Soc., Providence, RI, 1995

[4]: J.E.Humphrey, *Representations of Semisimple Lie Algebra in the BGG Category $\mathcal{O}$*, volume 94 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2002.

[5] V.Kac, *Infinite dimensional Lie Algebras*, volume 44 of *Progess in Mathematics*, Birkhäuser Boston, MA, 1993.

[6] M.E. Sweedler, *Hopf Algebra*, volume 44 of *Mathematics lecture notes series*, W. A. Benjamin, 1969

[7]: J. Hong and S.-J. Kang. *Introduction to quantum groups and crystal bases*, volume 42 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2002.

[8]: A. Henriques and J. Kamnitzer. Crystals and coboundary categories. *Duke Math. J.*, 132(2):191–216, 2006

[9]: 1. Etingof, Shlomo Gelaki, Dmitri Nikshych, Victor Ostrik. Tensor Category. *Mathematical Surveys and Monographs (2015) Volume: 205*