Representation of Quiver and its Application

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Notation

Throughout the note, we will assume an algebraically closed field k (e.g $k = \mathbb{C}$). A will always be an finite dimensional algebra over k.

We will use $\mathrm{mod}A$, $\mathrm{Mod}A$ for the category of left(right) finitely generated module over A and all left(right) modules over A respectively. $D(\mathcal{C})$ will be derived category of an abelian category $\mathcal{C}$ and $D^b(\mathcal{C})$ will be bounded derived category of $\mathcal{C}$

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1 What is a Quiver?

1.1 Quiver

A Quiver $Q = (Q_0, Q_1, s, t)$ is a quadruple with the data:
1.A set Q_0 where the elements are called **vertices or point**
2. A set Q_1 where the elements are called **arrow**

3. Two maps $s, t : Q_1 \to Q_0$ associate for each arrow $\alpha \in Q_1$, where $s(\alpha)$ is the **source** of α and $t(\alpha)$ is the **target** of α.

So a Quiver is really just a directed graph where Q_1, Q_0 describe the underlying graph of a Quiver Q, and we will follow all terminology in the graph to describe the Quiver(for example, connected Quiver means the underlying graph is connected...)

To associate Quiver with the representation of algebra, we shall introduce the idea of the representation of Quiver.

Definition 1.1. A representation of Quiver $(M_a, \phi_\alpha)_{a \in Q_0, \alpha \in Q_1}$ consist of the following data:
(1) For each point $a \in Q_0$, we associate a finte dimension $k-$vector space M_a.
(2) for each arrow $\alpha : a \to b$ in Q_1, we associate a $k-$linear map $\phi_\alpha : M_a \to M_b$.
We will use M_a in this note for simplicity when the context is clear.

Example 1.2. The **loop quiver** is a quiver with a unique point and an arrow

$$\begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd}$$

So a representation of loop quiver is just a vector space V and an endomorphism ϕ_α

Example 1.3. A **Kronecker Quiver** is a quiver as follow

$$\bullet \rightrightarrows \bullet$$

Category of Representation of Quiver

The representation of a Quiver Q will form a category, which we denote it by $Rep(Q)$, where the morphism is defined as follow.

Definition 1.4. Given two representation $(M_a, \phi_\alpha), (N_a, \rho_\alpha)$, a morphism $f : M_a \to N_a$ is a collection $f = (f_a)_{a \in Q_0}$, where $f_a : M_a \to N_a$ a $k-$linear map and for each arrow $\alpha : a \to b$, we have

$$\rho_\alpha f_a = f_b \phi_\alpha$$

for the associated map ϕ_α, ρ_α.

For example, if we have representation $(V, f), (W, g)$ of loop Quiver, then the morphism between these two representations will be all $h \in \mathrm{Hom}(V, W)$ such that $hf = gh$.
Remark:

In fact, as you may expect, the $Rep(Q)$ is an Abelian Category, and we can easily see:
(1) For two representation $(M_a, \phi_\alpha), (N_a, \rho_\alpha)$ the $\mathrm{Hom}((M_\alpha), (N_\alpha))$ is a vector space.
(2) Finite direct sum exists in this category: given two representations $(M_a, \phi_\alpha), (N_a, \rho_\alpha)$, we define $M_a \oplus N_a$ by $(M_a \oplus N_a, (\phi_\alpha, \rho_\alpha))$.
(3) Let $f : M_a \to N_a$ be a morphism between the representation of Quiver, we can construct its kernel (L_a, δ_α) by: L_a is $\ker(f_a)$, δ_α is the restriction of ϕ_α to L_a. Similarly, we can construct the cokernel, image, and coimage of f.
(4) Finally, it is easy to see the first isomorphism theorem holds in this category.

Remark 1.5. Indeed, $Rep(Q)$ is not just an Abelian category but equivalent to a category of the module of "some" algebra. We will see what is this algebra in the next section and show this equivalence in section 1.3

1.2 Path algebra

The path algebra kQ of a Quiver Q is an $k-$algebra where all paths with lengths bigger than or equal to 0 form a basis of the underlying k vector space and with the multiplication given by the concatenation of a path. If two paths can not be concatenated, then their product is 0.

Example 1.6. For path algebra of loop Quiver

$$\begin{tikzcd} a \arrow[loop above, "\alpha"] \end{tikzcd}$$

We can notice the unique point a is the unit of path algebra, and the basis of the algebra is $\{a, \alpha, \alpha^2, \dots\}$, which is an infinite dimension vector space. The multiplication rule is given by

$$a\alpha^i = \alpha^i a = \alpha^i$$

$$\alpha^i \alpha^j = \alpha^{i+j}$$

You can notice the behavior of this basis is very similar to the basis of polynomial algebra $k[x]$ and in fact

$$x \mapsto \alpha, 1 \mapsto a$$

will induce an $k-$ linear isomorphism between kQ and $k[x]$

Example 1.7. Let's use Kroneckner Quiver as an example

$$\begin{tikzcd} a \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & b \end{tikzcd}$$

The corresponding path algebra kQ is a 4 dimensional k vector space with base $\{a, b, \alpha, \beta\}$. The multiplication is $a * \alpha = \alpha, a * \beta = \beta$, and $a * b = 0, \alpha * \beta = 0, \alpha * b = \alpha$. Notice $b * \alpha = 0$ as the $t(b) = b$ but $s(\alpha) = a$, so we can't concatenate b and α in this way. Also notice $a * a = a$ and $b * b = b$, so every point are idempotent in path algebra.

The basic property of path algebra kQ is determined by the underlying graph of Q.

Lemma 1.8. *Let Q be an Quiver, kQ be its path algebra*
(1) kQ is associative algebra
(2) kQ is finite dimensional if and only if Q is finite, acyclic Quiver.
(3) kQ has a unit if and only if Q is finite Quiver.

Proof. (1) This is straightforward from definition.
(2) ($\Rightarrow$) If kQ is finite dimension, and suppose Q contains a cycle α, then $\alpha, \alpha^2\alpha^3 \dots$ will be distinct infinite path. By definition of path algebra, the base of kQ will consist of infinitely many elements. Similarly, if Q_0 contains infinitely many points, the base of kQ will consist infinity many elements
($\Leftarrow$) This direction is easy, as the assumption finite and acyclic implies it contains finite many paths and points.
(3)($\Rightarrow$) If Q_0 is finite, then as each point is idempotent, and each path has a unique source, so $\sum_{a \in Q_0} a$ is unit.
($\Leftarrow$) If $|Q_0|$ is infinity. Suppose by contrary that it has a unit $1 = \sum_{i=1}^{m} \lambda_i w_i$ where λ_i is a scalar and w_i is a path, then we notice the set Q_0' which contains all source of w_i is a finite set, so if we take $a \in Q_0/Q_0'$, then $a * 1 = 0$, which is an contradiction.

$\square$

The path algebra is an important "bridge" to connect the representation of finite dimensional algebra and Quiver representation. Especially, There is a correspondence between the property of the underlying graph and the property of path algebra where you have seen some simple examples from above. To see further connection, we need the idea of bounded path algebra and basic algebra.

Definition 1.9. Given a Quiver$Q$, let $A = kQ$ be its path algebra. The arrow Ideal R is a two-sided ideal of A that is generated by all the arrows of Q.

Remark: It is straightforward to see from the definition that the arrow ideal is naturally graded (as $k-$vector space) by the length of the path, which means, we have decomposition

$$R = kQ_1 \oplus kQ_2 \oplus \dots$$

where kQ_i is the subspace of kQ that generated by all path with length i.

Definition 1.10. For the path algebra $A = kQ$, an Ideal $\mathbb{I}$ is called **admissible** if there is some $n \geqslant 2$ such that

$$R^n \subset \mathbb{I} \subset R^2$$

If $\mathbb{I}$ is an admissible ideal of kQ, then the pair $(Q, \mathbb{I})$ is said to be **bounded quiver**, the algebra $kQ/\mathbb{I}$ is called **bound quiver algebra**

Remark 1.11. Notice when Q is acyclic, then there is a $n > 0$ such that $R^n = 0$, so this means we can let $\mathbb{I}$ be 0.

Example 1.12. For Kronecker Quiver, $R = \langle \alpha, \beta \rangle$, and it doesn't have any path with a length bigger or equal to 2, so the only possible admissible ideal is 0.

Example 1.13. Let Q be the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\lambda"] \arrow[dl, "\omega"] \\ & 4 \arrow[ul, "\sigma"] & \end{tikzcd}$$

Then we notice $\mathbb{I} = \langle \alpha\beta, \omega\sigma \rangle$ is an admissible ideal but $\mathbb{I}' = \langle \alpha\beta - \lambda \rangle$ is not because $\lambda \notin R^2$, so $\alpha\beta - \lambda$ is not an element in R^2

We should notice the bound quiver algebra $kQ/\mathbb{I}$ is also finite dimension $k-$vector space because kQ/R^n is finite dimension

Remark 1.14. We also notice when Q is a finite quiver, then an admissible ideal $\mathbb{I}$ is finitely generated. This is more or less just from the short exact sequence

$$0 \to R^n \to \mathbb{I} \to \mathbb{I}/R^n \to 0$$

We notice it will be sufficient to show both R^n and $\mathbb{I}/R^n$ are finitely generated. The case for R^n is straightforward if we realize it is just generated by all the paths with length n.
On the other hand, $\mathbb{I}/R^n$ is an ideal of $kQ/\mathbb{I}$ which is finite dimensional, so it is finitely generated as kQ module.

Basic Algebra

For an algebra A, an idempotent e is called **central** if it is in the center of A. Two idempotents e_1, e_2 are called orthogonal if $e_1e_2 = e_2e_1 = 0$. The idempotent e is called primitive if it can not be written as the sum of two orthogonal idempotent.
Every algebra will have two idempotents, namely 1 and 0. The idempotents that are different from these two are called non-trivial idempotents. If A has a non-trivial central idempotent e, then $1 - e$ is also another nontrivial idempotent and e and $1 - e$ are orthogonal. In this case, we have $1 = e + (1 - e)$, so this implies the decomposition $A = eA \oplus (1-e)A$. if e is not primitive, then it means eA is decomposable (e.g if $e = e_1 + e_2$ then $eA = M_1 \oplus M_2$ for $M_i = e_i eA$), otherwise, it is indecomposable, so the existence of central idempotent of A implies we can decompose A into two algebra, if we can't decompose A into two algebra, then A is called **connected** (this terminology will make sense once soon).

Because A is finite-dimensional over k, an important result ([2] I.4.10, we will use this result later) is it will always admit a decomposition

$$A = P_1 \oplus \cdots \oplus P_n$$

Where each P_i are indecomposable right ideals of A, this means $1 = p_1 + \cdots + p_n$ and each p_i is primitive idempotent and pairwise orthogonal. In this case, we have $P_i = p_i A$. This decomposition is called **indecomopsable decomposition** of A and The set $\{p_1, \ldots, p_n\}$ is called **complete set of primitve orthogonal idempotents** of A.

Remark 1.15. Moreover, P_i are all the indecomposable projective modules of A (up to isomorphism). This is another important fact that we will use later, see I.1.4 and I.1.4 in [2] for more details).

Definition 1.16. A $k-$algebra A with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotent is called **basic** if $e_i A \not\cong e_j A$ for all $i \neq j$

For each algebra A, we can associate a basic algebra, which we denote by A^b as follows:
First, let $e_A = e_{j_1} + e_{j_2} + \cdots + e_{j_m}$ where e_{j_i} are chosen from $\{e_1, \ldots, e_n\}$ such that $e_{j_i} A \not\cong e_{j_s} A$ if $j_i \neq j_s$ and for each $e_s \in \{e_1, \ldots, e_n\}$, $e_s A \cong e_{j_s} A$ for one of the module in $e_{j_1} A, e_{j_2} A, \ldots, e_{j_m} A$.
Secondly, consider

$$A^b = e_A A e_A$$

This is an $k-$algebra where the unit is e_A, notice each $e_{j_i} = e_A e_{j_i} e_A \in e_A A e_A$, and $\{e_{j_1}, \ldots, e_{j_n}\}$ is a complete set of primitive orthogonal idempotent of A^b. Moreover, A^b is basic, but this fact is relatively hard to prove. See [2] I.6.10 for a complete proof.

The A^b carry a lot of important information of A, for example

Theorem 1.17. *[2]* $\mathrm{mod} A$ *is equivalent to* $\mathrm{mod} A^b$

Let's back to path algebra. From the example of Krockner Quiver, we can easily see all the points will form a complete set of primitive orthogonal idempotent of kQ. This is in fact correct more generally for any finite Quiver.

Proposition 1.18. *Let Q be a finite Quiver, then the set $\{a, a \in Q_0\}$ will form a complete set of primitive orthogonal idempotent for kQ*

Proof. By Lemma 1.7, we have $1 = \sum_{a \in Q_0} a$, and also, it is quite clearly $ab = 0$ for two different points, so it only remains to show each point is primitive. Here we use the I.4.8 from [2] that if $\mathrm{End}(akQ) = akQa$ is local, then a is primitive. This is the same as showing the only idempotent in algebra $akQa$ is a and 0. Suppose we have an nontrivial central idempotent $e = \lambda a + w$ where $\lambda \in k$ and w is the linear combination of cycle through a with length $\geqslant 1$, then

$$0 = e^2 - e = (\lambda^2 - \lambda)a + (2\lambda - 1)w + w^2$$

implies $w = 0$ and $\lambda^2 = \lambda$. Hence $\lambda = 1$ or $\lambda = 0$, in the previous case, we have $e = a$, the latter case we have $e = 0$. $\square$

Unfortunately, not every path algebra of finite quiver Q is basic. For example, consider

$$\begin{tikzcd} a \arrow[r, bend left, "\alpha"] & b \arrow[l, bend left, "\beta"] \end{tikzcd}$$

The corresponding path algebra kQ has two indecomposable direct summand akQ and bkQ, where the first one has $\{a, \alpha, \alpha\beta, (\alpha\beta)^2, \dots\}$ and the second has $\{b, \beta, \beta\alpha, (\beta\alpha)^2, \dots\}$ as basis, and by the map

$$a \mapsto b$$

$$\alpha \mapsto \beta$$

$$\alpha\beta \mapsto \beta\alpha$$

we can get an isomorphism $akQ \cong bkQ$. To avoid this problem, we should require the quiver to be acyclic. This example also shows that the path algebra is connected if the underlying graph is connected (so connected is a reasonable name for this kind of algebra). The converse is also true, and we put it in the following summarization which concludes the parallel between the property of path algebra and the underlying graph.

Theorem 1.19. *Let Q be finite, connected, acyclic, and connected quiver, then the path algebra is an associative finite dimensional algebra with complete set of orthogonal primitive set $\{a, a \in Q_0\}$ which has unit and is connected.*

Conversely, given a basic algebra A, and $\{e_1, \dots, e_n\}$ be a complete set of orthogonal primitives idempotent, we can Construct a Quiver, denoted by Q_A, by
(1): We think of each decomposition e_iA as points, the cardinal $|Q_0|$ will be same as the n.
(2): For any two "points" e_iA and e_jA, we think $\dim_k(e_i(\mathrm{rad}A/\mathrm{rad}^2A)e_j)$, the dimension as $k-$vector space, as the number of arrow from i to j.

One of the advantages of this Q_A is that it doesn't depend on the choice of the complete set of primitive orthogonal idempotent (see [2] II.3.2). So this Quiver is "canonically" for each basic algebra A. Furthermore, the following theorem shows this basic algebra could also be recovered from its canonical quiver.

Theorem 1.20. *For a basic, connected algebra A, Q_A is a connected graph, and there exists an admissible ideal $\mathbb{I}$ of kQ_A such that $A \cong kQ_A/\mathbb{I}$*

From the above discussion, we see for any finite dimensional algebra A, we have

$$\mathrm{mod}A \cong \mathrm{mod}A^b \cong \mathrm{mod}kQ_{A^b}/\mathbb{I}$$

, and this tells us why we want to understand the quiver.

1.3 Representation of Quiver is Representation of Path algebra

As mentioned above, the representation of Quiver is the same as the representation of "some" algebra. In this section, we will show this algebra is just the path algebra of Q. To state our main theorem, we need to know how to define the representation for bound quiver

Definition 1.21. Let Q be a finite Quiver, $M = (M_a, \rho_\alpha)$ be a representation of Q. For any nontrivial path $\mu = \alpha_1\alpha_2\dots\alpha_n$ from a to b, we define the **evaluation** on the μ to be the $k-$linear map from M_a to M_b by

$$\rho_\mu = \rho_{\alpha_1}\rho_{\alpha_2}\cdots\rho_{\alpha_n}$$

We can also extend this idea to linear combination of paths with the same source and target; namely, if we have

$$\sigma = \sum_{i=1}^{n} \lambda_i\mu_i$$

be such a linear combination, then

$$\rho_\mu = \sum_{i=1}^{m} \lambda_i\rho_{\mu_i}$$

If every path of the linear combination $\sigma = \sum_{i=1}^{n} \lambda_i\mu_i$ has a length at least 2, then we call σ a **relation** in Q. By remark 1.14, we know the admissible ideal is generated by a finite set of relations.
Now, if we have a finite quiver Q and $\mathbb{I}$ be admissible ideal, then a representation (M_a, ρ_α) of Q is called **bound by** $\mathbb{I}$ if we have

$$\rho_\mu = 0$$

for all relation $\mu \in \mathbb{I}$, and as mention, this happen if and only if $\rho_{\mu_i} = 0$ for all i if $\mathbb{I} = \langle\mu_1, \mu_2, \dots, \mu_n\rangle$. We denote the category of representation bound by $\mathbb{I}$ by $Rep(Q, \mathbb{I})$

Example 1.22. Let Q be the quiver

$$\begin{tikzcd}
 & & 6 & & \\
1 & 2 \arrow[l, "\alpha"'] & 3 \arrow[l, "\beta"'] \arrow[u, "\lambda"] \arrow[r, "\gamma"'] & 4 \arrow[r, "\sigma"] & 5
\end{tikzcd}$$

bound by the relation $\gamma\sigma$, then

, then the following representation is bound by $\gamma\sigma$

$$\begin{tikzcd}
 & & k & & \\
k & k \arrow[l, "1"'] & k \arrow[l, "1"'] \arrow[u, "1"] \arrow[r, "0"] & 0 \arrow[r, "0"] & 0
\end{tikzcd}$$

Now we can state our main theorem

Theorem 1.23. *For a finite connected Quiver Q, and $A = kQ/\mathbb{I}$ where $\mathbb{I}$ is admissible ideal, we have*

$$\mathrm{Mod}A \cong Rep(Q, \mathbb{I})$$

So, when Q is acyclic, we can choose $\mathbb{I}$ to be 0, and this will give us

Theorem 1.24. *For a finite connected acyclic Quiver Q, and $A = kQ$, we have*

$$\mathrm{Mod}A \cong Rep(Q)$$

Proof. To prove this theorem, we will construct two explicit functors $\mathcal{F} : \mathrm{Mod}A \to Rep(Q, \mathbb{I}), \mathcal{G} : Rep(Q, \mathbb{I}) \to \mathrm{Mod}A$ and show they're mutually invertible.

Construction of functor $\mathcal{F}$: Let M be an $A = kQ/\mathbb{I}$ module, we define a representation $\mathcal{F}(M) = (M_a, \phi_\alpha)_{a \in Q_0, \alpha \in Q_1}$ as follow:
Let $a \in Q_0$, $a' = a + \mathbb{I}$ be the corresponding primitive idempotent in A. Similarly, if $\alpha \in Q_1$, then we let α' be its corresponding residue class in A. Now we let $M_a = Ma'$, and if there is an arrow $\alpha : a \to b$, then we define the $\rho_\alpha : M_a \to M_b$ by $\rho_\alpha(ma') = ma'\alpha' b' = ma'\alpha'$ for $x = ma \in M_a$.

Now, we check this construction defines a functor. First, clearly, the (M_a, ρ_α) is a representation of quiver, and it is bound because: if $\sigma = \sum_{i=1}^{m} \lambda_i \mu_i$ is a relation in $\mathbb{I}$ from a to b where $\mu_i = \alpha_{i,1} \dots \alpha_{i,j}$, then

$$\rho_\sigma(x) = \sum_{i=1}^{m} \lambda_i \rho_{\mu_i}(x)$$

$$= \sum_{i=1}^{m} \lambda_i \rho_{\alpha_{i,1} \dots \rho_{i,j}}(x)$$

$$= \sum_{i=1}^{m} \lambda_i (x\alpha'_{i,1} \dots \alpha_{i,j})'$$

$$= x * \sigma' = x * 0 = 0$$

So, the image is correct. For a morphism $f : M \to N$, and $(M_a, \rho_\alpha), (N_a, \phi_\alpha)$ be the corresponding representation then we define $\mathcal{F}(f)$ as: for $ma \in Ma$, $f_a(ma) = f(ma) = f(ma^2) = f(ma)a \in Na = N'_a$, so f_a define a $k-$linear $M_a \to N_a$. If $\alpha : a \to b$, then we can easily verify that $\phi_\alpha f_a(x) = \phi_\alpha(f(x)a) = f(x)\alpha' = f(x\alpha') = f_b(x\alpha') = f_b\rho_\alpha(x)$ as we desire. Also, it is easy to see the composition and identity preserved by $\mathcal{F}$, so it is a functor.

Construction of functor $\mathcal{G}$: Let (M_a, ρ_α) be a bound representation, then we define an associated $kQ/\mathbb{I} = A-$module $G(M)$ be: $G(M) = \oplus_{a \in Q_0} M_a$. Let $x = (x_a)_{a \in Q_0}$, to define the kQ structure on $G(M)$, it is sufficient to define multiplication μx for each path μ. If μ is a point a, then we set

$$x\mu = xa = x_a$$

If$\mu = \alpha_1\alpha_2 \dots \alpha_n$ a nontrivial path from a to b, then let $\rho_{\alpha_1}\rho_{\alpha_2}\dots\rho_{\alpha_n} : M_a \to M_b$ be associated map, then we put

$$(x_a)_{a\in Q_0}\mu = (\rho_\mu(x_a))_{a\in Q_0}$$

so, the the only nonzero term is the $b-th$ term. By the definition, if we have any relation in $\sigma \in \mathbb{I}$, then $(x_a)\sigma = (\rho_\sigma(x_a)) = 0$, so $G(M)$ is a module of $kQ/\mathbb{I}$.

Secondly, if $f = (f_a)$ is a morphism between two representation $(M_a, \rho_\alpha), (N_a, \phi_\alpha)$. We notice, as $k-$vector space, we naturally have a $k-$linear map $f = \oplus f_a : \oplus M_a \to N_a$, and because of specific construction, we notice for a path $w : a \to b$

$$f(xw) = f_b(x_a w)(\text{recall } b\text{-th coordinate is the only non zero term }) = f_b(\rho_\alpha(x_b))$$

$$= \phi_\alpha f_a(x_b) = f(x)w$$

So this map is also left $A-$module map.
Again, it is easy to see identity and composition are preserved $\mathcal{G}$, so we finish our construction.

Thanks to the specific construction of $\mathcal{F}$ and $\mathcal{G}$, it is very straightforward to see they're mutually invertible, and this finishes our prove.

□

1.4 Gabriel Theorem

In representation theory, one of the most important questions is

Given an algebra A, how to classify all the simple and indecomposable representations?

The Gabriel theorem is to answer this question for all the path algebra of finite, acyclic, connected Quiver.

Definition 1.25. For a finite dimension algebra A, if it has a finite number of isomorphism classes of indecomposable finite dimensional right A-module, then we call it **Representation-finite**, or of finite type. Otherwise, this algebra is called **Representation-infinite**

Because of the result from sections 1.2 and 1.3, we can extend this notation to Quiver, and call a Quiver is representation finite or in finite type if the corresponding path algebra is representation finite. The Gaberiel theorem is to classify all the finite, acyclic, connected quiver which is representation-finite. In this section, our Q is always finite, connected, acyclic Quiver

Theorem 1.26. *(Gabriel) Let Q be a finite, connected, acyclic Quiver, $A = kQ$, then A is representation finite if and only if the underlying undirected graph is one of the following*

$$
\begin{tikzcd}
\mathbb{A}_m & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \cdots \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet & m \geqslant 1
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \bullet \arrow[dr, dash] & & & & & & \\
\mathbb{D}_n & & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \cdots \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet & n \geqslant 4 \\
 & \bullet \arrow[ur, dash] & & & & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & \bullet \arrow[d, dash] & & \\
\mathbb{E}_6 & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & \bullet \arrow[d, dash] & & & \\
\mathbb{E}_7 & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & & \bullet \arrow[d, dash] & & & \\
\mathbb{E}_8 & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet
\end{tikzcd}
$$

Remark 1.27. These graphs are also called Dykin diagrams. They play a very important role in the Lie algebra for both finite dimension (Classification of semisimple Lie algebra) and infinite dimension (Classification of "type" of Kac-Moody algebra: affine, finite, and indefinite).

Remark 1.28. You should be aware how strong of this theorem is: the theorem is for underlying **undirected** graph, which means, for example, both quiver

$$
\begin{tikzcd}
 & & \bullet & & \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l] \arrow[r] \arrow[u] & \bullet \arrow[r] & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & \bullet \arrow[d] & & \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l]
\end{tikzcd}
$$

are representation-finite as they're underlying undirected graph are $\mathbb{E}_6$

The proof of the Gaberial theorem is hard and technical, so we will not present complete proof here(see chapter VII in [2] if you're interested). Instead, we will illustrate the main idea of the proof. The theorem has two parts: the "if" part, and the "only if" part. The proof of each part exactly corresponds to two important technique, root system, and reflective functor, which is widely used in various areas of modern mathematics.

Root system

In the following section, the set $\{e_1, \ldots, e_n\}$ be canonical basis of free abelian group $\mathbb{Z}^n$

Definition 1.29. A quadratic form $q = q(x_1, \ldots, x_n)$ on $\mathbb{Z}^n$ with n indeterminate is called **integral quadratic form** if it is on the form

$$q(x_1, \ldots, x_n) = \sum_{i}^{n} x_i^2 + \sum_{i<j} a_{ij} x_i x_j$$

for $a_{ij} \in \mathbb{Z}, \forall i, j$

The evaluation of integral quadratic form q on a vector $\mathbf{x} = [x_1, \ldots, x_n]$ will give us a map from $\mathbb{Z}^n$ to $\mathbb{Z}$, and depending on its value, we call a vector $\mathbf{x}$ is **positive** if $\mathbf{x} \neq 0$ and each $x_i \geqslant 0$. An integral quadratic form is called **weakly positive** if $q(\mathbf{x}) > 0$ for all $\mathbf{x} > 0$; it is called **positive semi-definite** if $q(\mathbf{x}) \geqslant 0, \forall \mathbf{x} \in \mathbb{Z}$, and **positive definite** if $q(\mathbf{x}) > 0, \forall \mathbf{x} \neq 0$,**indefinite** if there exists a nonzero vector **x**such that $q(\mathbf{x}) < 0$. If we have a positive semi-definite form q. then we define the **radical** of q by

$$\mathrm{rad} q = \{\mathbf{x} \in \mathbb{Z}, q(\mathbf{x}) = 0\}$$

the element of radical called **radical vector**. Also, we notice the radical element of a form q will form a subgroup of $\mathbb{Z}^n$. Finally, a vector $\mathbf{x} \in \mathbb{Z}^n$ is called the root of q is $q(\mathbf{x}) = 1$, and it is called **positive root** if **x**is positive.

We can associate an integral quadratic form with the Quiver

Definition 1.30. Let Q be a finite connected acyclic quiver, an **quadratic form** of Q is defined as

$$q_Q(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 + \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}$$

for $\mathbf{x} = [x_1, \ldots, x_n] \in \mathbb{Z}^n$

Now let's see how to use the root system to prove the "if" part. First, an important result is

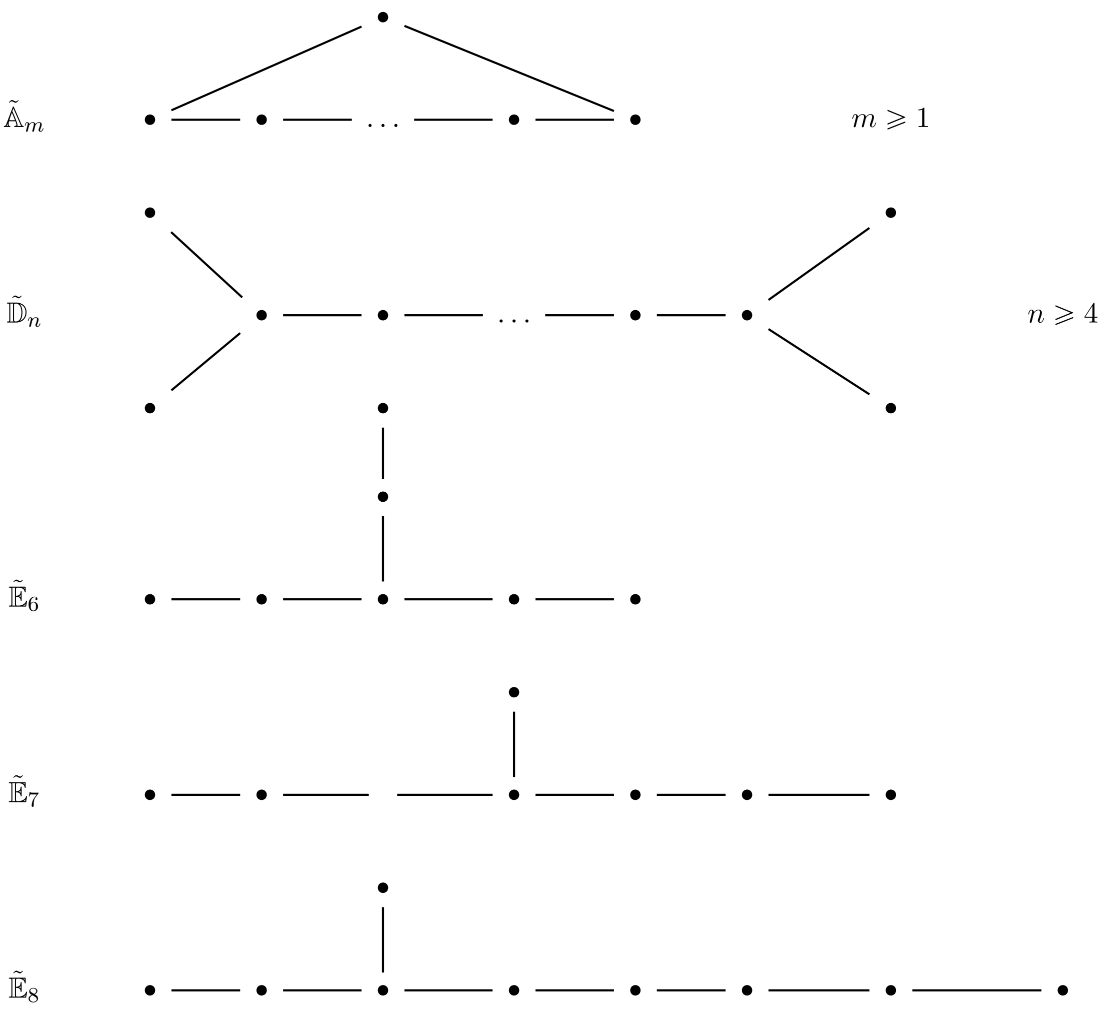
Proposition 1.31. *A weakly positive integral quadratic form q has only finitely many positive roots*

By some nontrivial observation ([2] VII.3 and VII.4), we will notice that every quadratic form q_Q associated with a quiver Q is weakly positive. Then the conclusion will be directly from the following theorem

Theorem 1.32. *If the underlying graph of Q is a Dykin diagram, then $M \mapsto \dim M$ will induce an isomorphism between isomorphism classes of indecomposable $A = kQ$ module and the subset $\{\boldsymbol{x} \in \mathbb{N}^n; q_Q(\boldsymbol{x}) = 1\}$ of all positive roots of q_Q*

Reflective Functor

In this part, we will show how to prove the "only if" part. This part requires our graph not Dykin, so we want to find some "common property" for the graph in this case. To do this, we need to introduce the "exotic" version of Dykin graph, which is sometimes called **Euclidean Graph**



Proposition 1.33. *(combinatoric fact) For a connected acyclic finite graph, if it is not one of the Dykin graph, then it should contain one of the Euclidean graphs as a subgraph.*

We should also notice if Q is a Quiver, Q' is a subquiver, then we can always embed the representation (M'_a, ρ'_α) of Q' to Q through a fully faithful functor $F : Rep(Q') \to Rep(Q)$ by setting all "additional" points and arrow be 0. Thanks to fully faithful, if M_a corresponds to an indecomposable module in kQ', then it will also be an indecomposable module in kQ. By the above discussion, we realize that if we can show every Euclidean graph is representation-infinite, then we can finish the "only if" part. This question is hard because we need to show for any quiver that the underlying undirected graph is

Euclidean are representation-infinite. The case for $\tilde{\mathbb{A}}_m$ is relatively easy by the following observation.

Lemma 1.34. *Any Quiver Q that the underlying graph is $\tilde{\mathbb{A}}_m$ is representation-infinite*

Proof. Let Q such quiver, we can suppose our quiver is numbered by $1, 2, \ldots, m, m+1$ and there is one arrow $\alpha : 1 \to 2$. Now given arbitrary scalar $\lambda \in k$, we construct an representation $M(\lambda) = (M_i^{(\lambda)}, \rho_\beta^{(\lambda)})$as follow:

(1) We assign k to every point $M^{(}\lambda)_i$
(2) For ρ_β^λ, if $\beta = \alpha$, then $\rho_\beta^{(\lambda)}(x) = \lambda x$. Otherwise, it is an identity map between k

We first show there are infinitely such representations (up to isomorphism). Choose another scalar μ, by commutativity relation, then we notice any non-zero morphism $f = (f_i)_{i \in \{1,2,\ldots,m+1\}}$ from $M(\lambda)$ to $M(\mu)$ will be: $f_1 = f_2 = \cdots = f_{m+1}$, and hence each f_i should be an isomprhism. Also, from the commutativity in $\alpha : 1 \to 2$, we have

$$\mu f(1) = \rho_\alpha^{(\mu)} f_1(1) = f_2 \rho_\alpha^{(\lambda)}(1) = f_2(\lambda) = \lambda f_2(1)$$

because $f_1 = f_2$, so we get $\lambda = \mu$, so for any two distinct scalar in k, they will define different $M(\lambda)$. Because k has infinitely many elements, there are infinitely such representations.

Secondly, we observe that $f \in \mathrm{End}(M(\lambda))$ is completely decided by f_1, by the above discussion, we know it is exactly a scalar multiplication (as any such map will define an isomorphism and satisfy the commutativity relation), so $\mathrm{End}(M(\lambda)) = k$ which implies it is indecomposable. $\square$

For the remaining 4 cases, we need to use the reflective functor, to motivate the reader why we need it, we start with some combinatoric results.

Let Q be an quiver, $a \in Q_0$, then we can define an "action" of a to Q to create a new quiver $aQ = (Q_1', Q_0', s', t')$: (1) For any arrow with a source or target is a, reverse the direction. (2) Except (1), we don't change anything.

Example 1.35. Let Q be a quiver

$$\begin{tikzcd} a \arrow[r] & b \arrow[dr] & \\ & & c \end{tikzcd}$$

Then aQ will be the quiver

$$\begin{tikzcd} a & b \arrow[l] \arrow[dr] & \\ & & c \end{tikzcd}$$

Now, we need to mention two special points of a quiver. One is called **sink**, which means it is not a source for any path or arrow. For example, for Q in the above example, c is a sink. Another one is called **sources**, which means it is not the target of any arrow or path. The a is a source of Q from the above example.

Lemma 1.36. *Let Q and Q' be two trees with the same underlying undirected graph, then there exists a sequence $i_1, \dots, i_n$ of point of Q such that*
(1): for each $1 \leqslant s \leqslant n$, i_s is a sink in $i_{s-1} \dots i_1 Q$
(2)$i_n \dots i_1 Q = Q'$

Because all the graphs of Euclidean graphs are trees, this lemma implies that we don't need to show all the quivers that underlying undirected graphs in Euclidean are representation-infinity. Instead, we can only show one of them, let's call it $Q_{\tilde{\mathbb{D}}_m}$(rep.$Q_{\tilde{\mathbb{E}}_6}, Q_{\tilde{\mathbb{E}}_7}, Q_{\tilde{\mathbb{E}}_8}$) is representation-infinity, then if we construct a functor S_a for quiver Q which we want it to (1): send a representation of Q to aQ (2) S_a preserve the indecomposable module, that Q is representation-infinity if and only if aQ is of representation-infinity. If we can construct such a functor, then the lemma 1.36 shows that any quiver Q with the underlying undirected graph is Euclidean representation infinity if and only if $Q_{\tilde{\mathbb{D}}_m}$(or.$Q_{\tilde{\mathbb{E}}_6}, Q_{\tilde{\mathbb{E}}_7}, Q_{\tilde{\mathbb{E}}_8}$) is representation-infinity by the composition of functor $S_{i_s} \dots S_{i_1}$. The first target is true by the following lemma, and the functor we want is called the reflective functor.

Lemma 1.37. *The following Quivers are representation-infinite*

$$\begin{tikzcd}
\bullet \arrow[dr] & & & & & & \bullet \arrow[dl] & \\
& \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[r] & \bullet & & n \geqslant 4 \\
\bullet \arrow[ur] & & & & & & \bullet \arrow[ul] &
\end{tikzcd}$$

$$\begin{tikzcd}
& & \bullet \arrow[d] & & \\
& & \bullet \arrow[d] & & \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
& & & \bullet \arrow[d] & & & \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
& & \bullet \arrow[d] & & & & & \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}$$

Now, we can conclude the "only if" part by constructing the reflective functor.

Definition 1.38. Let Q be an Quiver, $a \in Q$, $M = (M_a, \rho_\alpha$ be a representation, we define

$$S_a : Rep(Q) \to Rep(aQ)$$

by:
(a) $S_a(M) = (M'_a, \rho'_\alpha)$ where

(1)$M'_i = M_i$ if $i \neq a$. If $i = a$, then

$$M'_a = \ker(\oplus_{\alpha:s(\alpha)\to a} M_{s(\alpha)\to M_a})$$

The kernel of direct sum of all the arrow α in Q with target in a.

(2) $\rho'_\alpha = \rho_\alpha$ for all arrow $\alpha : i \to j$ with $j \neq a$. If $j = a$, then $\rho'_\alpha : M_a \to M_i$ is composition of all inclusion of M'_a to $\oplus_{\beta:s(\beta)\to a} M_{s(\beta)}$ with the projection onto the each direct summand M_i.

(b) for a morphism $f = (f_i)_{i\in Q_0} : M \to N$, for $M = (M_a, \rho_\alpha), N = (N_a, \phi_\alpha)$, we define $f' = S_a(f)$ be: if $i \neq a$, then $f'_i = f_i$. In $i = a$, then f'_a is the unique map to make the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & S_a(M)_a \arrow[r] \arrow[d, dashed, "f'_a"] & \oplus_{s(\alpha)\to a} M_{s(\alpha)} \arrow[r, "(\rho_\alpha)_\alpha"] \arrow[d, "\oplus_\alpha f_{s(\alpha)}"] & M_a \arrow[d, "f_a"] \\
0 \arrow[r] & S_a(N)_a \arrow[r] & \oplus_{s(\alpha)\to a} N_{s(\alpha)} \arrow[r, "(\phi_\alpha)_\alpha"] & N_a
\end{tikzcd}$$

By theorem 1.24, we can expect to construct a similar functor S'_a between the module category of path algebra kQ and kQ_a, unfortunately, the construction will rely on the existence of a special kind of module in path algebra called ”APR-tilting module”, which will be a bit out of our scope. (We will know what is tilting module is later. The APR-tilting module is one of the examples of titling modules, but the construction will rely on some properties of quiver. See [2] VI.2 if you're interested in the specific construction). The fact that reflective functor preserves the isomorphism classes of the indecomposable module will also require the construction of functor S'_a. Again, interested readers can refer [2] for more details.

2 Thrall Conjecture

The Thrall Conjecture is a Conjecture about the representation of finite dimensional algebra over k. It claims:

Conjecture: A finite-dimensional $k-$algebra is either representation finite or there exist indecomposable modules for any dimension.

This conjecture has been solved (even in the case k is an arbitrary field) by a series of works. The purpose of this section is to introduce the Auslander-Reiten Translation and use it to give a simple proof of this conjecture.

2.1 Almost Split Short Exact Sequence

The almost split short exact sequence will be the main object we deal with in the Auslander-Reiten theory. The theory confirms the existence of such a short exact sequence in the $\mathrm{mod}A$. Contrary to split short exact sequence, the almost short sequence is designed for indecomposable modules. We will introduce what is almost a short exact sequence and also another class of morphism called "irreducible" morphism in this section.

Definition 2.1. Let L, M, N module in $\mathrm{mod}A$, then
(1) a morphism $f : L \to M$ is called **left minimal** if for every $h \in \mathrm{End}(M)$ such that $hf = f$ is an auomotphism
(2) a morphism $g : M \to N$ is called **right minimal** if every $s \in \mathrm{End}(M)$ such that $gs = g$ is an automorphism.
(3)a morphism $f : L \to M$ is called **left almost split** if
(i) f is not section
(ii) for every morphism $u : L \to U$ that is not section, there is $u' : M \to U$ such that $u'f = u$
(4)a morphism $g : M \to N$ is called **right almost split** if
(i) g is not a retraction.
(ii) for every $v : V \to N$ that is not retraction, there is a $v' : V \to M$ such that $gv' = v$.
Finally we have **left minimal almost split=left minimal+left almost split**, similarly for **right minimal almost split**

The almost split morphism encodes the property "indecomposable" of module.

Proposition 2.2. *(1) If $f : L \to M$ is a left almost split morphism in $\mathrm{mod}A$, then L is indecomposable*
(2)If $g : M \to N$ is a right almost split morphism in $\mathrm{mod}A$, then the module is indecomposable

Proof. We only prove (1) as (2) is dually. Suppose by contradiction that $L = L_1 \oplus L_2$, let p_i be the projectition of L to L_i, then there is a $u_i : M \to L_i$ such that $u_i f = p_i$, but this will imply $uf = 1$ which means f is a section, then we get contradiction. □

Definition 2.3. A short exact sequence in $\mathrm{mod}A$

$$0 \xrightarrow{f} L \xrightarrow{g} M \to N \to 0$$

is called **almost split sequence** if f is left minimal almost split and g is right almost split.

Similar to the split short exact sequence, almost split sequence also has its own "splitting lemma". Before introducing this lemma, we shall invite another class of morphism.

Definition 2.4. A morphis $f : M \to N$ in $\mathrm{mod}A$ is called irreducible if
(1) f is neither section not retraction
(2) if $f = f_1 f_2$, then either f_1 is retraction or f_2 is section

An irreducible morphism between indecomposable module X, Y in $\mathrm{mod}A$ is completed determined by a $k-$vector space $\mathrm{rad}(X, Y)/\mathrm{rad}^2(X, Y)$. Here $\mathrm{rad}(X, Y)$ means all the non-invertible morphism of $\mathrm{Hom}(X, Y)$. The $\mathrm{rad}^2(X, Y)$ is all morphism in the form fg, where $f \in \mathrm{rad}(X, Z), g \in \mathrm{rad}(Z, Y)$ for some object (not necessary indecomposable!) in $\mathrm{mod}A$.

Indeed, when X, Y are two indecomposable module, then $f : X \to Y$ is irreducible if and only if $f \in \mathrm{rad}(X, Y)/\mathrm{rad}^2(X, Y)$([2]IV). Because of this reason, we will call $\mathrm{rad}(X, Y)/\mathrm{rad}^2(X$ **space of irreducible morphism** and write it as $\mathrm{Irr}(X, Y)$

Theorem 2.5. *("splitting lemma" of almost short exact sequence)*
Let

$$0 \xrightarrow{f} L \xrightarrow{g} M \to N \to 0$$

be a short exact sequence in $\mathrm{mod}A$*, then the following statement are equvialent*
(1) The given sequence is almost split
(2) L is indecomposable, and g is right almost split
(3) N is indecomposable, and f is left almost split
(4) f is left minimal almost split
(5) g is right minimal almost split
(6) L, N are indecomopesable, and f, g are irreduicble.

2.2 Auslander-Reiten Translation

In this section, we will introduce the Auslander-Reiten theory, which shows the existence of an almost split short exact sequence in $\mathrm{mod}A$. The theorem relies on a functor called Auslander-Reiten Translation.

Definition 2.6. Given module M in $\mathrm{mod}A$ with a minimal projective resolution

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

then we call the $\mathrm{coker}(f_1)^t$ in

$$0 \to M^t \to P_0^t \xleftarrow{f_q^t} \mathrm{coker}(f_1)^t \to 0$$

be the **transpose** of M, and write it as TrM. Here M^t means the $k-$dual of module M.

Definition 2.7. The **Auslander-Reiten Translation** is the functor

$$\tau = D\mathrm{Tr}, \tau^{-1} = \mathrm{Tr}D$$

where D is the $A-$ dual of module.

With the Auslander-Reiten Translation, we can now state our main theorem

Theorem 2.8. *Let M be an indecomposable nonprojective module in $\mathrm{mod}A$, then there is some module E, F such that*

$$0 \to \tau M \to E \to M \to 0$$
$$0 \to M \to F \to \tau^{-1} M$$

are two almost split short exact sequences.

Remark 2.9. (Not an important remark) The Auslander-Reiten theorem could be generated categorically. More precisely, we can generate it to any **mutilocular abelian category** $\mathcal{C}$ where $\mathcal{C}$ is a $k-$linear category and each object has an indecomposable decomposition, and the endomorphism ring of the indecomposable object is local.

With the Auslander-Reiten translation, we can consider another combinatoric model, called Auslander-Reiten quiver, and this time we will "graph" out the category.

Construction of Auslander-Reiten Quiver: The quiver will be constructed over $\mathrm{mod}A$, and we will denote this quiver by $\Gamma(\mathrm{mod}A)$. The point is isomorphism classes of indecomposable module. Let $[L], [M]$ be two points, then there is an arrow from $[L]$ to $[M]$ if and only if $\mathrm{Irr}(L, M) \neq 0$, which means there is an irreducible morphism in $\mathrm{Hom}(L, M)$. An important property is for any points $[M]$, the number of its successor will be same as the number of predecessors of $[\tau M]$, and this is not hard to check by Auslander-Reiten theory. On the other hand, if $[M]$ is projective, then $[L]$ should be a direct summand of $\mathrm{rad}M$, so the predecessor of the projective module is always finite. Finally any point $[M]$ will have finite many successors and predecessors

Now, we already have all the things to prove the Thrall conjecture, and let's start to prove it.

2.3 The Proof of Thrall Conjecture

Let A be an algebra, a sequence of irreducible morphism in $\mathrm{mod}A$ in the form

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} M_n$$

with M_i indecomposable is called a **chain of irreducible morphism** from M_0 to M_n of length n. The following two lemmas will be the key to the proof of Thrall conjecture. We only show how to prove the first one, the second one can be found in [2] IV.5.

Lemma 2.10. *Let $n \in \mathbb{N}$ and let M and N be indecomposable right $A-$module where $\mathrm{Hom}_A(M, N) \neq 0$. Suppose there exists no chain of irreducible morphism from M to N of length $< n$. Then*
(1) There exists a chain of irreducible morphism

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} M_n$$

and a morphism $g : M_n \to N$ with $gf_n \dots f_2 f_1 \neq 0$
(2) There exists a chain of irreducible morphisms

$$N_n \xrightarrow{g_n} N_{n-1} \xrightarrow{g_{n-2}} \dots \xrightarrow{g_1} N_0 = N$$

and a morphism $f : M \to N_n$ with $g_1 \dots g_n f \neq 0$

Proof. These two statements are parallel, so we only prove (1). The proof of (2) will be similar. Our strategy is to prove by induction on n.

Case $n = 0$: then the statement is the same as say there is a chain of irreducible morphism in length $\geqslant 1$ between M and N, so there is nothing to show in this case.

Case $n \Rightarrow n+1$ Suppose now we have M, N such that there is no chain of irreducible morphism of length $< n+1$. By induction, we have a chain of irreducible morphism

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} M_n$$

and a morphism $g : M_n \to N$ such that $gf_n \dots f_1 \neq 0$. The hypothesis implies that g is not an isomorphism. Because both M_n and N are indecomposable, g is not a section. By Auslander-Reiten theory, we can consider a left minimal almost split morphism starting from M_n

$$h = (h_1, \dots, h_s) M_n \to \bigoplus_{i=i}^{s} L_s$$

Because g is not a section, so the definition of left minimal almost split morphism implies there is a $u = (u_i)_{i=1}^s \bigoplus_{i=i}^s L_s \to N$ such that $g = uh = \sum_{j=1}^s u_j h_j$. Hence, we have $0 \neq gf_n \dots f_1 = \sum_{j=1}^s u_j h_j f_n \dots f_1 \neq 0$, that means there is a $1 \geqslant j \geqslant s$ such that $u_j h_j f_n \dots f_1 \neq 0$. Then because h_j is irreducible, by setting $M_{n+1} = L_u, f_{n+1} = h_j, g' = u_j$, we complete the induction.

□

Lemma 2.11. *For a natural number $b \in \mathbb{N}$, let*

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots M_{2^b-1} \xrightarrow{f_{2^b-1}} M_{2^b}$$

be a chain of nonzero nonisomprhisms in $\mathrm{mod} A$ *where all M_i are indecomposable of length $\leqslant b$, then $f_{2^b-1} \dots f_1 = 0$*

Remark 2.12. The statement " M_i are indecomposable of length $\leqslant n$" means it will have a composition series with length less or equal to n (see [2] I.3 if you don't know what is composition series.)

Theorem 2.13. *If A is a basic connected finite-dimensional $k-$algebra, if $\Gamma(\mathrm{mod} A)$ has a connected component $\mathcal{C}$ whose module is of bounded length, then $\mathcal{C}$ is finite and $\mathcal{C} = \Gamma(\mathrm{mod} A)$. In particular, A is representation-finite.*

Proof. Suppose b is the bound of the length for the indecomposable module X in $\mathcal{C}$ for $[X]$ is a point in $\mathcal{C}$. Let M, N be two indecomposable modules where such that $\mathrm{Hom}(M, N)$. If $[M]$ is a point in $\mathcal{C}$, then we claim $[N]$ will also be a point in $\mathcal{C}$. Indeed, we notice that there is a chain of irreducible morphism between M and N of length smaller than $n = 2^b - 1$, because if this is not the case, then lemma 2.1 implies a chain of irreducible morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} M_n$$

with a $g : M_n \to N$ and $gf_n \dots f_1 \neq 0$, but by lemma 2.2, we should have $f_n \dots f_1 = 0$ which is a contradiction.

Now let $[M]$ be an arbitrary point in $\mathcal{C}$, then there should be an indecomposable module P such that $\mathrm{Hom}(P, M) \neq 0$ (because we have $\mathrm{Hom}(A, M) \neq 0$, so remark I.1.15 implies this fact), so we have $[P]$ is a point in $\mathcal{C}$. Now, recall that from theorem I.1.20 that Q_A is a connected graph, and $A \cong kQ_A/\mathbb{I}$. If $a, b \in (Q_A)_0$, then $akQb$ correspond to all path between a and b, so the connectedness implies what given an $a \in (Q_A)_0$, for any other points b, there exist a sequence of points $a = b_0, \dots, b_n = b$ such that $\overline{b_i}kQ/\mathbb{I}\overline{b_{i+1}} \neq 0$ (or $\overline{b_{i+1}}kQ/\mathbb{I}\overline{b_i} \neq 0. \neq 0$), where $\overline{b_i}$ is the residue class in of b_i in $kQ/\mathbb{I}$. Again, by remark I.1.15, the above discussion will be translated to for any other indecomposable projective module P', there is a sequence of indecomposable projective module $P = P_0, \dots, P_s = P'$, such that $\mathrm{Hom}(P_i, P_{i+1}) \neq 0$ or $\mathrm{Hom}(P_{i+1}, P_i) \neq 0$ because $\mathrm{Hom}(P_i, P_{i+1}) = \mathrm{Hom}(e_iA, e_jA) = e_jAe_i$. So, we have $[P']$ is a point in $\mathbb{C}$. Hence, we can deduce that $\mathcal{C} = \Gamma(\mathrm{mod}A)$ because for any M, we have an indecomposable projective module P such that $\mathrm{Hom}(P, M) \neq 0$.

To show the finiteness, we notice for any indecomposable module M, we have an indecomposable projective module such that $\mathrm{Hom}(P, M) \neq 0$, so there is an irreducible morphism from P to M, that means every module would be a successor of the projective module. Because in the Auslander-Reiten quiver, the number of successors is finite, and there are only finitely many nonisomorphic indecomposable projective. Hence, we conclude that there are only finite many points in $\mathcal{C}$

□

Then we notice that theorem 2.4 confirms the Thrall conjecture because not representation finite will imply the length of the module has no bound, so we have a module in any given dimension.

3 Coherent sheaves of $\mathbb{P}^1_k$

3.1 Tilting theorem

The Tilting theory is to compare the module between two algebras. The very original version is given by Brenner and Butler in [3] for the Torsion class and the Torision-free class. Later, Rickard generalizes it in the derived category in[4][5]. A more categorical version is developed by [6] and [7]. We will give a brief introduction to all three versions in this section.
As we mentioned, the tilting was designed for comparing the module between two algebras, and these two algebras are A, and $\mathrm{End}(T)$ for a "tilting module", which is the main object we will study.

Definition 3.1. A pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}A$ is called **torsion** pair if the following three conditions are satisfied
(1) $\mathrm{Hom}(M, N) = 0, \forall M \in \mathcal{T}, N \in \mathcal{F}$

(2) $\mathrm{Hom}(M,-)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$
(3) $\mathrm{Hom}(-,N)|_{\mathcal{T}} = 0$ implies $M \in \mathcal{F}$

One of the ways to find the Torsion is to consider $Gen(T)$ and $Cogen(\tau T)$ for a module T. In most of cases, they won't become a torsion pair, but when T is very nice, for example when T is a "Titlting module", which is defined by the following.

Definition 3.2. An $A-$module T is called tilting module if
(1) $\mathrm{pd}T_A \leqslant 1$
(2) $\mathrm{Ext}^1(T,T) = 0$
(3) There is a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$
where T', T'' is direct summand of direct sum of T, or this is same to say they're in the $\mathrm{add}T$.

Now let $\mathcal{T}(T) = \{M_A|\mathrm{Ext}^1_A(T,M) = 0\}$ and $\mathcal{F}(T) = \{M_A|\mathrm{Hom}(T,M) = 0\}$, we will in the next theorem that when T is a tilting module, then $Gen(T)$ and $Cogen(\tau T)$ will be a torsion pair, and they're exactly $\mathcal{T}(T)$ and $\mathcal{F}(T)$

Theorem 3.3. *The following conditions are equivalent*
(1) T is a tilting $A-$module
(2) $Gen(T) = \mathcal{T}(T)$ (3) $Cogen\tau T = \mathcal{F}(T)$
(4) $(\mathcal{T}(T), \mathcal{F}(T)$ is a torsion pair

And this will induce the first version of the tilting theorem by Brenner and Butler. To begin with, we first notice that T is also a $B = \mathrm{End}(T)-$module, and in fact, it is a tilting $B-$dmoule, so we shall introduce the corresponding Torsion pair $(\mathcal{X}(T), \mathcal{Y}(T)$ for T are B module. Indeed, they'll be exactly

$$\mathcal{X}(T) = \{X_B|\mathrm{Hom}_B(X, DT) = 0\} = \{X_B|X \otimes T = 0\}$$

$$\mathcal{Y}(T) = \{Y_B|\mathrm{Ext}^1_B(Y, DT) = 0\} = \{Y_B|\mathrm{Tor}^B_1(Y,T) = 0\}$$

Theorem 3.4. *Let A be an algebra, T be tilting module, $B = \mathrm{End}(T_A)$, then we have*
(1) there is a $k-$algebra isomorphism $A \to B^{op}$
(2) The functor $\mathrm{Hom}_A(T,)$ and $- \otimes_B T$ are mutually inverse between $\mathcal{T}(T)$ and $\mathcal{Y}(A)$
(3) The functor $\mathrm{Ext}^1(T,-)$ and $\mathrm{Tor}^1(-,T)$ are mutually invrse between $\mathcal{F}(T)$ and $\mathcal{X}(T)$

Rickard generates the idea of a tilting module in the derived category of the module. More precisely, he shows that

Theorem 3.5. *Let A, B be two associative ring (not necessary finite dimensional $k-$algebra), then the following are equvialent*
(1) $D^b(A)$ is triangulated equivalent to $D^b(B)$
(2) B is isomorphic to $\mathrm{End}(T)$, where T is an object in $K^b(P_A)$, where P_A is category of all finitely generated projective module, such that
(1) $\mathrm{Hom}(T, T[i] = 0), \forall i \neq 0$
(2) $\mathrm{add}(T)$ generates $K^b(P_A)$ as triangulated category

So this theorem shows that whenever we have a derived equivalence between two algebras, there is a "Tilting theory" corresponding to it.

3.2 Kronecker Quiver and $\mathcal{D}^b(\mathbb{P}^1_k)$

Finally, we generate the tilting module to tilt objects in the category and show applications to the coherent sheaf of $\mathbb{P}^1$. Let $\mathcal{C}$ be an $k-$linear Grothendieck Abelian category, then an object T is called **Tilting object** if $\mathrm{RHom(T,-)} : D(\mathcal{C}) \to D(\mathrm{ModEnd}(T))$ is an equivalence. The following condition is given by [6] and [7].

Proposition 3.6. *Suppose $\mathcal{C}$ is locally Noetherian of finite homological dimension, and let $T \in \mathcal{C}$ with the following property:*
(1) T is an noethrian object
(2) $\mathrm{Ext}^i(T,T) = 0, \forall i > 0$
(3) The closure of $\mathrm{add}(T)$ contains a set of generators of $\mathcal{C}$.
Then T is an tilting object, if moreover, $\mathrm{End}(T)$ is Noetherian, then there $\mathrm{RHom(T,)}$ is a equivalence between bounded derived category

To end this section, we will show that $T = \mathscr{O}(-1) \oplus \mathscr{O}$ is a tiliting object in *QCoh*($\mathbb{P}^{\Bbbk}$). It satisfies (1) because $\mathscr{O}$ and $\mathscr{O}(-)$ are the only subobject of T. (2) follows from the fact $\mathrm{Ext}^1(\mathscr{O}(m), \mathscr{O}(n)) = \mathrm{Ext}^1(\mathscr{O}, \mathscr{O}(n-m)) = H^1(\mathscr{O}(n-m)) = 0$. Also, by the fact that $\mathscr{O}(-n), n \in \mathbb{N}$ form system of generator of $\mathcal{C}$ ([8]), so this result follow from the short exact sequence

$$0 \to \mathscr{O}(-n-1) \to \mathscr{O}(-n) \oplus \mathscr{O}(-n) \to \mathscr{O}(-n+1)$$

So we get T is a tilting object. Moreover, we notice that $\mathrm{End}(T)$ is isomorphic to the path algebra of Kronecker quiver: by decomposition, we found that T is four-dimensional vector space with basis $\{a', b', c, d\}$ from $a' \in \mathrm{End}(\mathscr{O}(-1)), b' \in \mathrm{End}(\mathscr{O}) = \mathscr{O}$ and $c, d \in \mathrm{Hom}(\mathscr{O}(-1), \mathscr{O})$, and we also notice $a'b' = 0$, $a'c = c, a'd = d$ and $cb' = c, db' = d$, so the map

$$a' \mapsto a, b' \mapsto b, c \mapsto \alpha, d \mapsto \beta$$

will give an isomorphism of $k-$algebra. One way to think it is

$$\mathscr{O}(-1) \underset{d}{\overset{c}{\rightrightarrows}} \mathscr{O}$$

, and hence we get $D^b(\mathbb{P}^1) \cong D^b(\bullet \rightrightarrows \bullet)$.

References

[1] **A. A. Beilinson** , *Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra*, Funct Anal Its Appl, 1978.

[2] **Ibrahim Assem**, **Daniel Simson,**, **Andrzej Skowronski**, *Elements of the Representation Theory of Associative Algebras*, Mathematics (general), Mathematics, Algebra Series: London Mathematical Society Student Texts (65), Cambridge University Press, January 2010

[3] **S.Brenner**, **M.C.R.Butler**, *The equivalence of certain functors occurring in the representation theory and species*, J.London Math. Soc, 14(1976) 183-87

[4] **Jeremy Rickard**, *Morita Theory for Derived Categories*, J.London Math. Soc, Volume s2-39, Issue 3, June 1989

[5] **Jeremy Rickard**, *Derived Equivalences As Derived Functors*, J.London Math. Soc, Volumes 2-43, Feb 1991

[6] **A.I.Bondal** *Representation of associative algebras and coherent sheaves* , Mathematics of the USSR-Izvestiya, 1990, Volume 34, Issue 1, Pages 23–42

[7] **D.Baer** *Tilting sheaves in representation theory of algebras*, Manuscripta Math 60, 323–347 (1988)

[8] **B.Keller**, *derived categories and tilting*, https://webusers.imj-prg.fr/ bernhard.keller/ictp2006/lecturenotes/keller.pdf