Chapter 3
Accuracy of Numerical Methods

In the previous chapter we discussed global error, the error $|v^n - u^n|$ at some time step n between the exact solution u^n and our numerical approximation v^n. In this chapter we will be concerned with local error. Local error is the error incurred in one step of the numerical method. Although our goal is to obtain small global errors, it is the local errors over which we have direct control.

10 Self-Assessment

Before reading this chapter, you may wish to review...

- telescoping series [Wikipedia: Telescoping series]
- Taylor series expansions [18.01 Lecture 38: Video]

After reading this chapter you should be able to...

- describe the difference between local and global errors
- evaluate the local truncation error for a given numerical method
- describe the relationship between local truncation error and local order of accuracy
- implement the midpoint method

11 Local error

Let's first understand the connection between local and global errors. If local errors are the errors incurred in a single step of the method and global errors are the total errors incurred up to a time step $N = T/\Delta t$, it stands to reason that that the number of steps $T/\Delta t$ should play a critical role in relating the two types of errors.

Let e^n be the global error at time step n, then

$$u^N - v^N = e^N = (e^1 - e^0) + (e^2 - e^1) + \cdots + (e^N - e^{N-1}) = \sum_{n=1}^{N} \Delta e^n \tag{12}$$

where $e^0 = u^0 - v^0 = 0$ and $\Delta e^n = e^n - e^{n-1}$ is the change in error from step $n-1$ to n (aka, the local error). Suppose that our numerical method has global order of accuracy (see 2) of p, then we can expect that $e^N = \mathscr{O}(\Delta t^p)$. Suppose that the local errors $\Delta e^n = \mathscr{O}(\Delta t^q)$ for all time steps n, then

$$\mathscr{O}(\Delta t^p) = e^N = \sum_{n=1}^{N} \Delta e^n = N\mathscr{O}(\Delta t^q). \tag{13}$$

Recall from above that $N = T/\Delta t$, and therefore,

$$\mathscr{O}(\Delta t^p) = \frac{T}{\Delta t}\mathscr{O}(\Delta t^q) = \mathscr{O}(\Delta t^{q-1}), \tag{14}$$

giving the relation $p = q-1$, or equivalently, $q = p+1$. This means that if the global order of accuracy is p, we can expect that the local error behaves as $\mathscr{O}(\Delta t^{p+1})$. We must note that errors need not sum this way, particularly if the numerical method is not stable. In Chapter 5 we will establish conditions to guarantee convergence.

12 Local truncation error

Any multi-step numerical method can be written in the abstract form

$$v^{n+1} = g(v^{n+1}, v^n, v^{n-1}, \ldots, t^{n+1}, t^n, t^{n-1}, \ldots, \Delta t); \tag{15}$$

that is, there is a function g that takes as input the numerical approximations from previous time steps (and in the case of implicit methods, also time step $n+1$ itself, see Chapter 9) and gives as output the numerical approximation v^{n+1} at time step $n+1$.

The local error is the error incurred in one time step of the numerical method. In order to analyze this error, we think of determining v^{n+1} using data from the exact solution $u(t)$. That is, we let $v^{n+1} = g(u^{n+1}, u^n, u^{n-1}, \ldots, t^{n+1}, t^n, t^{n-1}, \ldots, \Delta t)$ and calculate the error $v^{n+1} - u^{n+1}$. This error is called the *local truncation error*.

Definition 1 (Local truncation error). The local truncation error τ is the error incurred by one step of a numerical method given exact data, i.e.,

$$\tau \equiv g(u^{n+1}, u^n, u^{n-1}, \ldots, t^{n+1}, t^n, t^{n-1}, \ldots, \Delta t) - u^{n+1} \tag{16}$$

In the following example, we determine the local truncation error for the forward Euler method.

Example 1. Recall that the forward Euler method (see (7)) is given by the equation

$$v^{n+1} = v^n + \Delta t\, f(v^n, t^n) \tag{17}$$

and therefore in this case

$$g(v^n, t^n, \Delta t) = v^n + \Delta t\, f(v^n, t^n). \tag{18}$$

The local truncation error is

$$\tau = g(u^n, t^n, \Delta t) - u^{n+1} = u^n + \Delta t\, f(u^n, t^n) - u^{n+1}. \tag{19}$$

We are solving the ODE $u_t = f(u,t)$ and therefore $f(u^n, t^n) = u_t^n$; thus,

$$\tau = u^n + \Delta t u_t^n - u^{n+1}. \tag{20}$$

Substituting the Taylor series expansion

$$u^{n+1} = u^n + \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n + \mathscr{O}(\Delta t^3) \tag{21}$$

into (20), we find

$$\tau = u^n + \Delta t u_t^n - (u^n + \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n + \mathscr{O}(\Delta t^3)) = -\frac{1}{2}\Delta t^2 u_{tt}^n + \mathscr{O}(\Delta t^3). \tag{22}$$

The leading term of the local truncation error for forward Euler is $-\frac{1}{2}\Delta t^2 u_{tt}^n = \mathscr{O}(\Delta t^2)$.

13 Local order of accuracy

While the local truncation error is explicitly the error in one step of the numerical method given exact data, the local order of accuracy describes the behavior of that error as it relates to the time step Δt.

Definition 2 (Local order of accuracy). Let τ be the local truncation error associated to a given multi-step numerical method. The method has local order of accuracy p if

$$\tau = \mathscr{O}(\Delta t^{p+1}) \quad \text{as} \quad \Delta t \to 0. \tag{23}$$

Note that the local order of accuracy is defined to be one less than the order of the leading term in the local truncation error. Therefore, the local and global orders of accuracy will be the same for each method, and we can discuss the order of accuracy of a method without confusion.

Exercise 1. What is the local order of accuracy for the forward Euler method?

(a) $p = 0$
(b) $p = 1$
(c) $p = 2$
(d) $p = 3$

In the next class, we will be exercising the convergence and accuracy analysis tools on the *midpoint method*, which we now introduce.

14 Midpoint Method

We will derive the midpoint method using a definition of the derivative. For a differentiable function $u(t)$, the derivative is given by

$$u_t(t) = \lim_{\Delta t \to 0} \frac{u(t+\Delta t) - u(t-\Delta t)}{2\Delta t}. \tag{24}$$

Now suppose $t = t^n$, then in our notation, we have

$$u_t^n = \lim_{\Delta t \to 0} \frac{u^{n+1} - u^{n-1}}{2\Delta t}. \tag{25}$$

We will now make this an approximation by eliminating the limiting process; i.e., we write

$$u_t^n \approx \frac{u^{n+1} - u^{n-1}}{2\Delta t}. \tag{26}$$

Recall that $u_t^n = f(u^n, t^n)$ is the governing equation and rearrange to find

$$u^{n+1} \approx u^{n-1} + 2\Delta t\, f(u^n, t^n) \tag{27}$$

Substituting our numerical approximation to the solution, we arrive at the midpoint method

$$v^{n+1} = v^{n-1} + 2\Delta t\, f(v^n, t^n), \qquad n \geq 1. \tag{28}$$

It is important to note that the midpoint rule is a two-step method: v^{n+1} is obtained using information from steps $n-1$ and n. This means that the midpoint method cannot be applied for the first time step ($n = 0$); instead one must use a one-step method like forward Euler for the first step.

Thought Experiment Assume that v^0 is given and that v^1 has been obtained by a one-step method. Make a sketch indicating how the midpoint method leads to v^2.

Exercise 2. Let $\Delta t = 1$, $v^0 = u(0)$, and $v^1 = u(1)$. For which of the following functions $u(t)$ will the midpoint method exactly predict $v^2 = u(2)$.

(a) $u(t) = 2 - t$
(b) $u(t) = 4 - t^2$
(c) $u(t) = 3t - t^3$
(d) (a) and (b)