Learning Dissipative Chaotic Dynamics with Boundedness Guarantees

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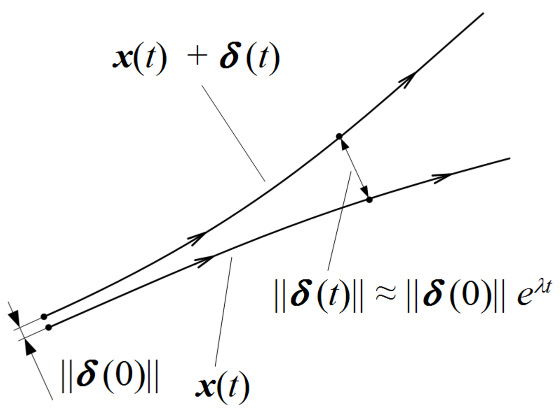
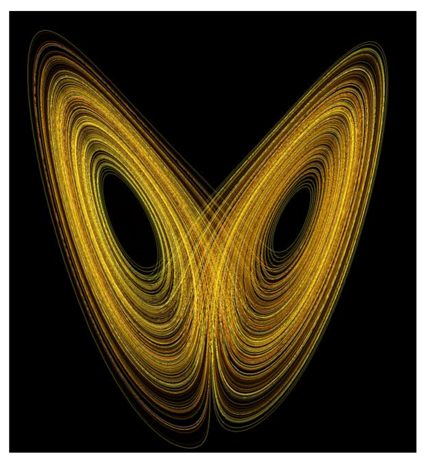
Learning Dissipative Chaos

Chaos: *exponential separation* after infinitesimal perturbation

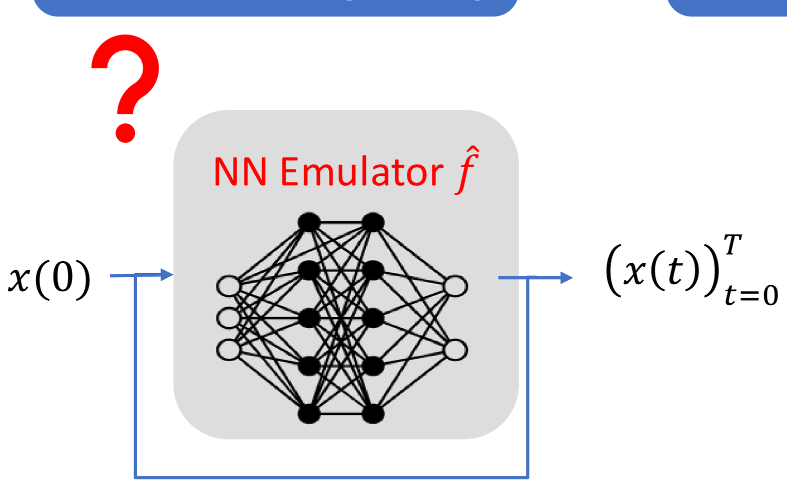
Dissipative chaos: system is *ergodic on the attractor*





Accurate trajectory prediction ✗ **Invariant Statistics** ✓

Model Stability Bounded Trajectory → Long Trajectory Rollout → Invariant Statistics



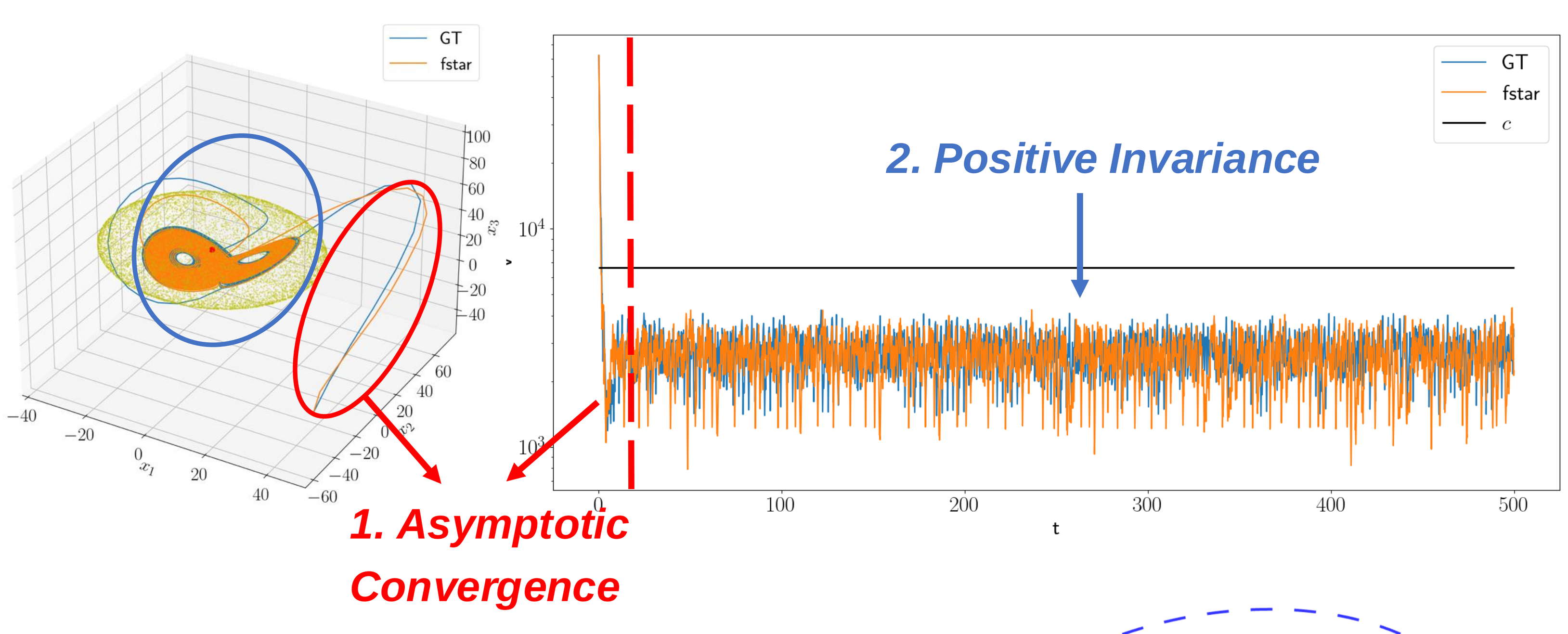
Model stability unknown

Challenges [1]:
Data-driven methods can experience finite-time blow up!

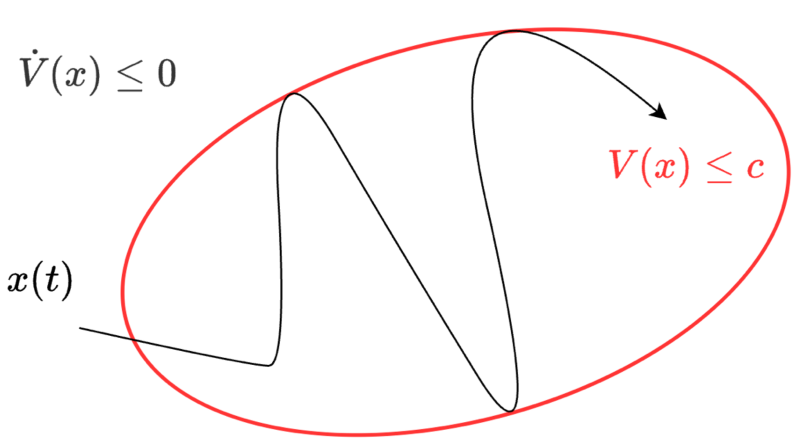
Key Contributions

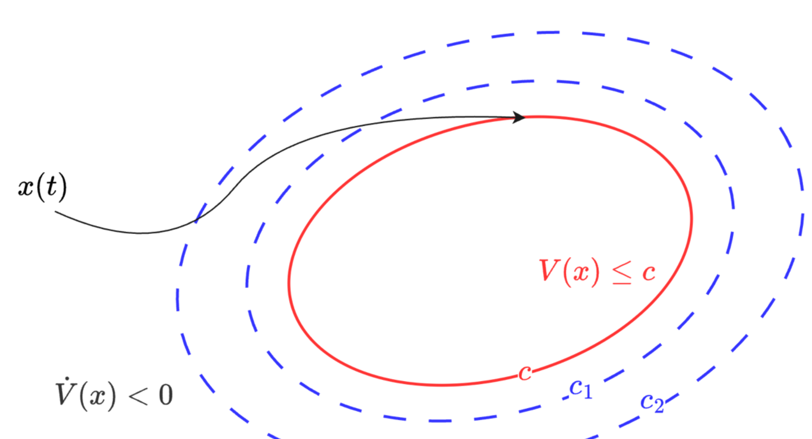
- Formal ***Trajectory boundedness Guarantees*** (No finite-time blowup)
- Simultaneous learning of ***dynamics and energy representation***
- Preserves invariant statistics and ***identify an attractor outer-estimate***

Control-theoretic Perspective: Lyapunov - Energy



1. Asymptotic Convergence:
Losing energy outside level set





2. Positive Invariance:
Similar to Lasalle's Invariance
No energy gain outside

Main Theoretical Results: Algebraic Condition for Dissipativity

Existence of a Lyapunov function V: lower-bounded radially unbounded C^1 function
Given a level set $M(c) = \{x \in \mathbb{R}^n : V(x) \le c\}$

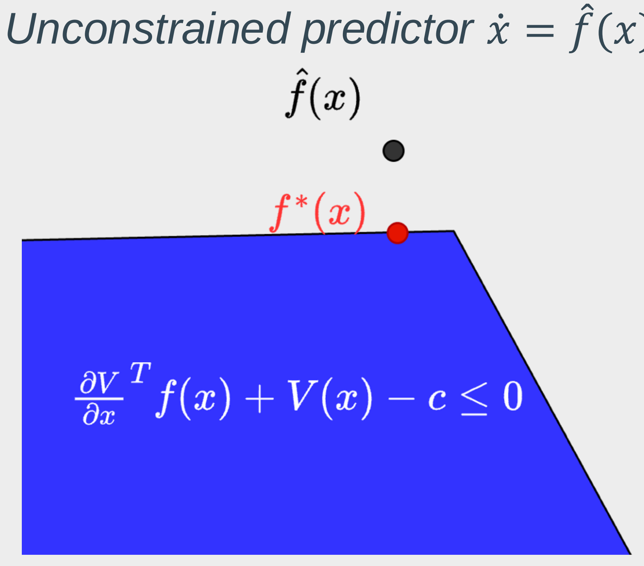
Exterior condition $\forall x \in \{x \in \mathbb{R}^n : V(x) > c\}, \dot{V}(x) < 0$ *Difficult to enforce*

Global condition $\forall x \in \mathbb{R}^n, \dot{V}(x) + V(x) - c \le 0$ *can verify pointwise*

(inspired by s-procedure [3])

Constrain dynamics $f(x)$ to be dissipative

Unconstrained predictor $\dot{x} = \hat{f}(x)$ *Minimum distance projection*



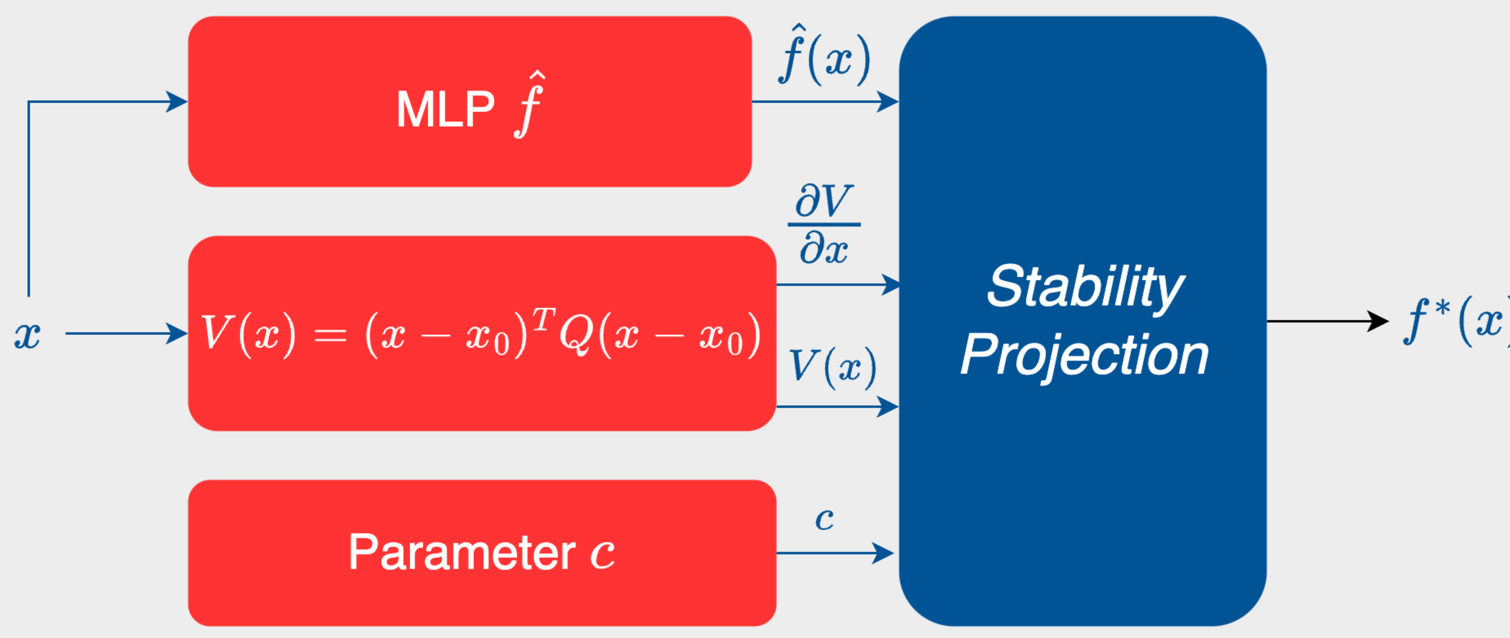
$$f^*(x) = Proj(c, \hat{f}(x), \frac{\partial V}{\partial x}, V(x))$$

$$= argmin_{f(x)} \|f(x) - \hat{f}(x)\|^2$$

$$\text{subject to } \frac{\partial V}{\partial x}^T f(x) + V(x) - c \le 0$$

Closed-form solution
General constrained Neural Network results [4]

Intrinsically Dissipative Architecture



- Dynamics emulator $\hat{f}$
- Energy function representation $V(x)$
- Invariant level set: $M(c) = \{x : V(x) \le c\}$

Simultaneously learned to produce dissipative dynamics predictor $f^*(x)$

Stability Projection: unconstrained $\hat{f} \to f^*$

It's just a special ReLU layer!

$$f^*(x) = \hat{f}(x) - \frac{\partial V}{\partial x}^T \frac{ReLU\left(\frac{\partial V}{\partial x}\hat{f}(x) + V(x) - c\right)}{\|\frac{\partial V}{\partial x}\|^2}$$

Volume Regularization

Loss Function: Prediction loss over sampled trajectories of N steps

$$L = \sum_i \sum_{k=1}^{N} \left\| x^i(k\Delta t) - \hat{x}^i(k\Delta t) \right\|^2 + L_{reg}$$

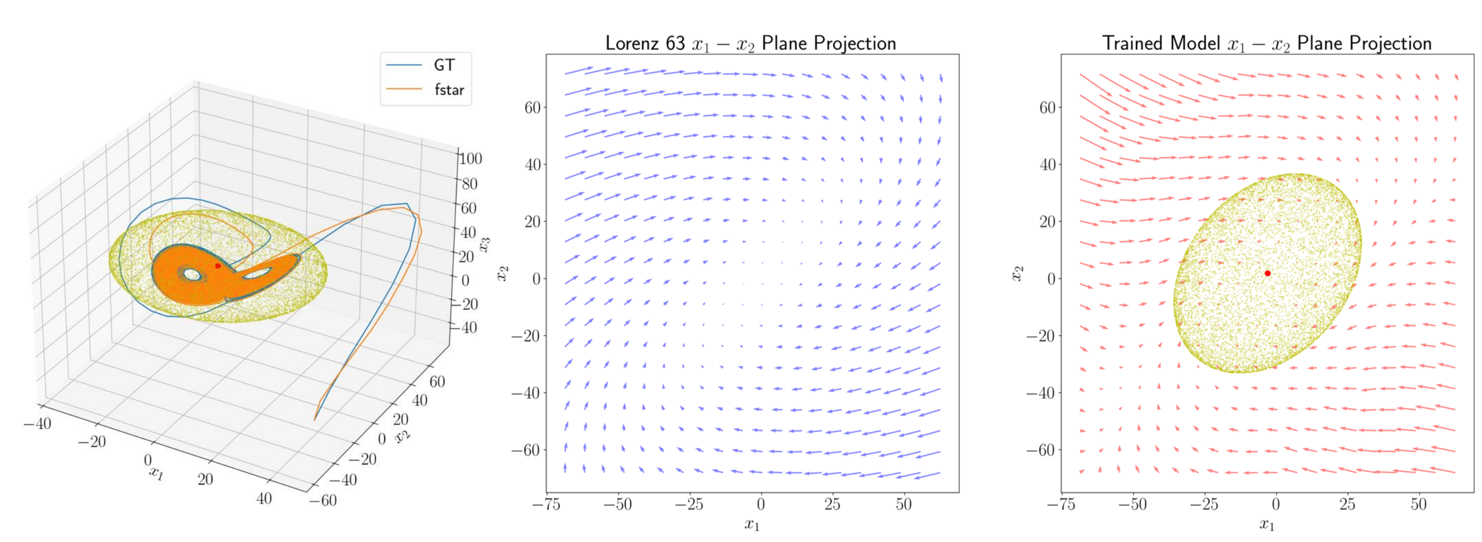
Ground truth Predicted

Observation: Multiple V, c pairs could work for this optimization

Regularization: Pick the tightest level set

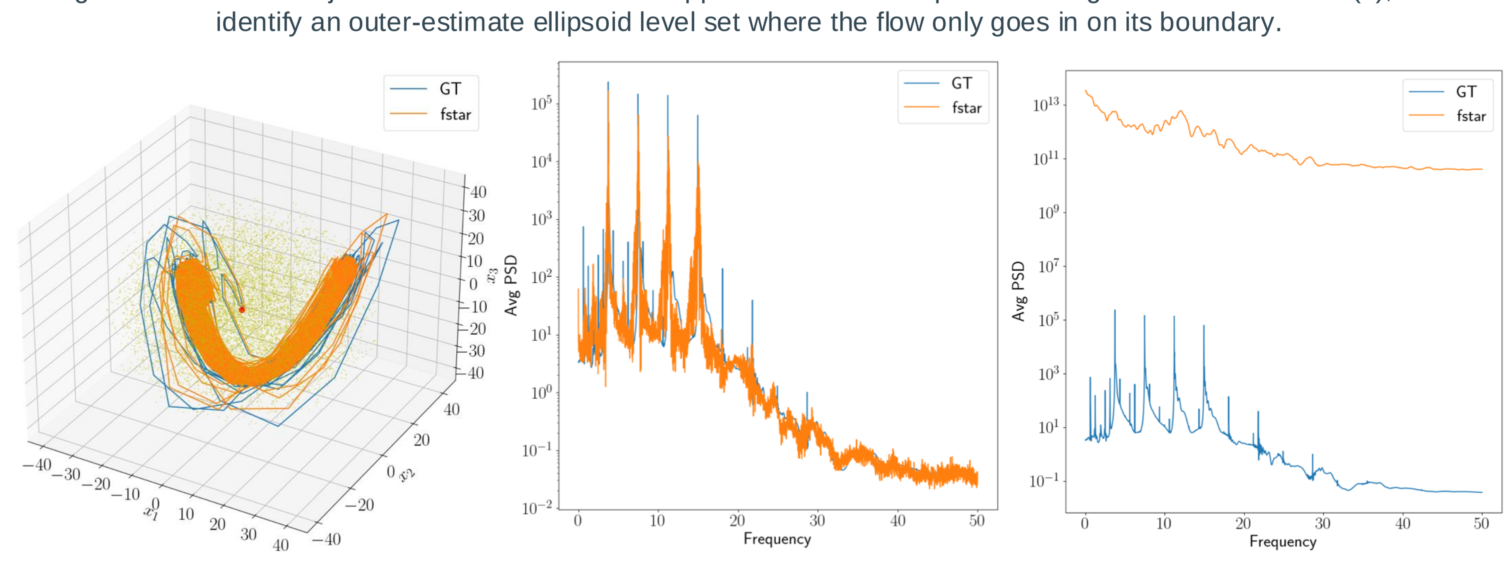
$L_{reg} = \lambda\, vol(\{x : V(x) \le c\})$ *volume regularizer*

Numerical Simulations



(a) Trajectory rollout (with projection) (b) The outer-estimate of the attractor and dissipative flow on its boundary

Figure 1. Simulated trajectories for Lorenz 63. Our approach is able to reproduce the ground truth attractor (a), and identify an outer-estimate ellipsoid level set where the flow only goes in on its boundary.



(a) Trajectory rollout (with projection) (b) Fourier energy spectrum (with projection) (c) spectrum (without projection)

Figure 2. Simulated trajectories for a truncated KS ODE. Our approach is able to reproduce the ground truth energy spectrum (b) and the attractor (a), while the vanilla MLP without projection experiences finite-time blow-up (c).

	MLP spectrum error %	MLP #Unbounded Trajectories	**Proposed Model spectrum error%**
Lorenz 63	6.37×10^{-4}	0/25	$\mathbf{7.40 \times 10^{-4}}$
Lorenz 96	N/A	24/25	$\mathbf{1.96 \times 10^{-3}}$
Truncated KS	N/A	6/25	$\mathbf{3.37 \times 10^{-3}}$

Table 1. Numerical simulations show that the proposed approach provides accurate energy spectrum prediction, while the MLP without the stability projection layer experiences finite-time blow-up very often especially for more complex systems.

References

[1] Lu, Zhixin, Brian R. Hunt, and Edward Ott. "Attractor reconstruction by machine learning." *Chaos: An Interdisciplinary Journal of Nonlinear Science* 28.6 (2018).

[2] Strogatz, Steven H. *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*. CRC press, 2018.

[3] Parrilo, Pablo A. "Semidefinite programming relaxations for semialgebraic problems." *Mathematical programming* 96 (2003): 293-320.

[4] Min, Youngjae, Anoopkumar Sonar, and Navid Azizan. "Hard-Constrained Neural Networks with Universal Approximation Guarantees." *arXiv preprint arXiv:2410.10807* (2024).

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