AAE 538: Air-Breathing Propulsion

Lecture 3: Fundamentals of Compressible Flow (continued)

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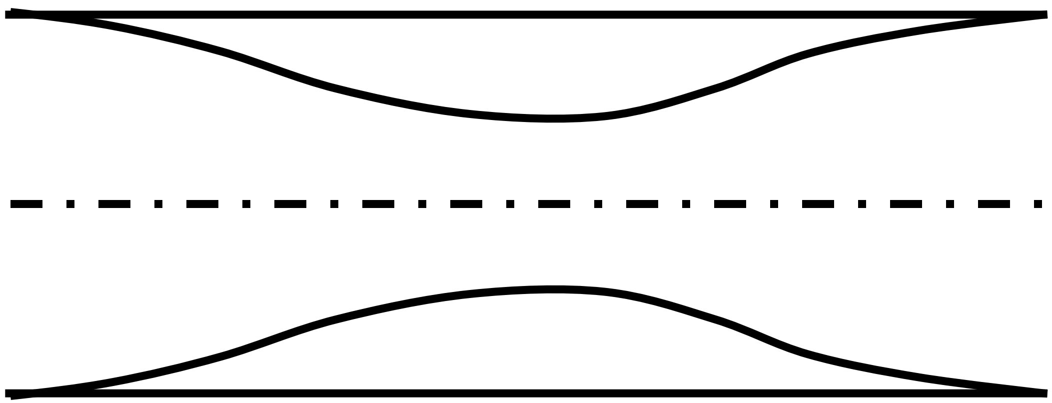
Mass Flow Function

- Consider the mass flow through a variable area duct.



The continuity equation states that the mass flow at any axial station is constant:

- By introducing the equation of state for an ideal gas, the definition of the stagnation pressure, and the definition of the stagnation temperature, we can express the mass flow rate in terms of ____________________________.
  - The resulting relation is known as the mass flow function.

Mass Flow Function

- To begin, incorporating the ideal gas relation into the continuity equation yields:

Multiplying and dividing through by the stagnation pressure, the square root of the stagnation temperature, and the ratio of the specific heats, we find the following expression for the mass flow rate at plane 1.

Rearranging to find stagnation-static ratios and other known relations.

$$\dot{m} = \frac{p_1}{RT_1} u_1 A_1 =$$

Mass Flow Function

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Simplifying, the mass flow rate at any axial location can now be written in terms of the stagnation properties and area as:

$$\dot{m} = \rho_1 u_1 A_1 = \left(\sqrt{\frac{\gamma}{R}} \frac{p_{o,1} A_1}{\sqrt{T_{o,1}}} \right) M_1 \frac{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{1/2}}{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\frac{\gamma}{\gamma - 1}}}$$

Combining the similar Mach number expressions, we find:

Mass Flow Function

It is convenient to define Mach number variation on the mass flow rate through the dimensionless function

As a result, we can write the continuity equation as:

$$\dot{m} = \left(\frac{p_{o,1} A_1}{\sqrt{T_{o,1}}}\right)\sqrt{\frac{\gamma}{R}}\, D_1 = \left(\frac{p_{o,2} A_2}{\sqrt{T_{o,2}}}\right)\sqrt{\frac{\gamma}{R}}\, D_2$$

- Studying the behavior of the function $D(M)$:
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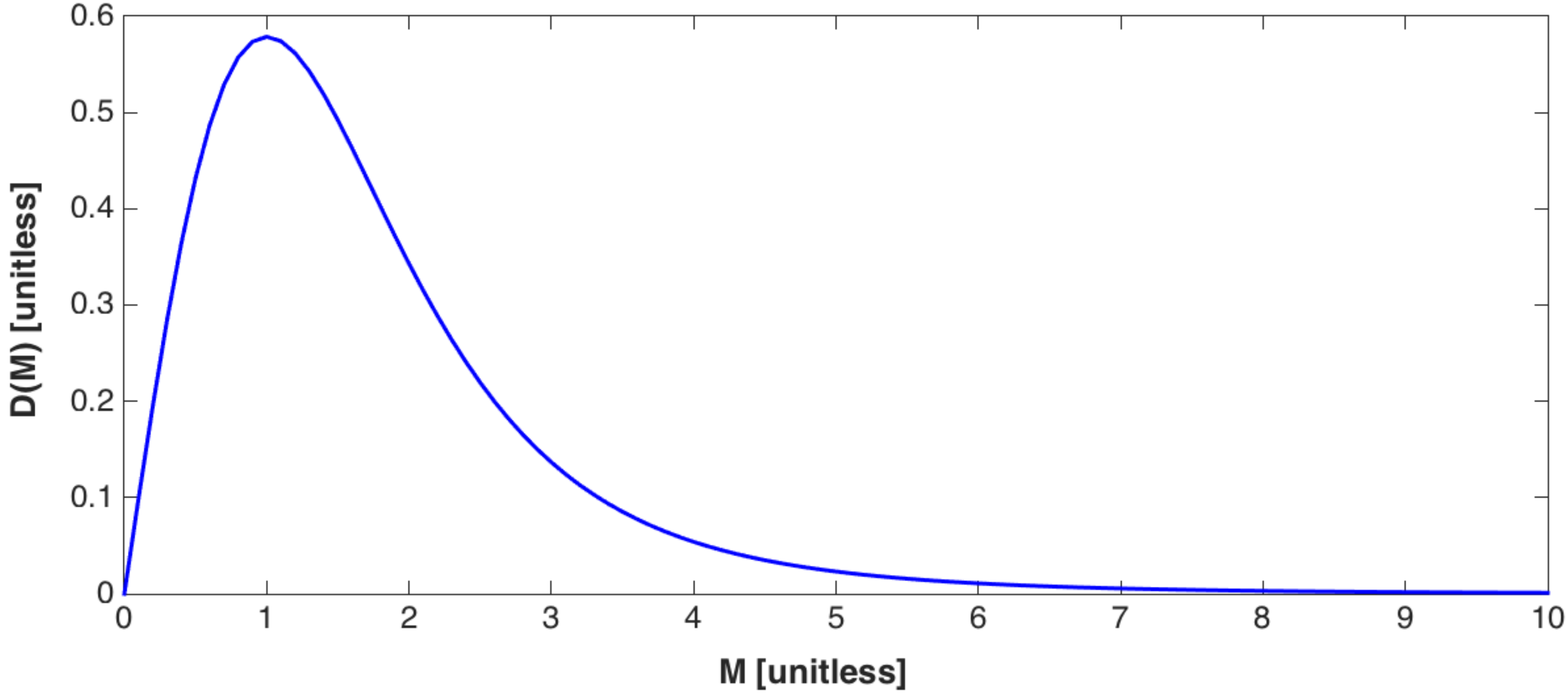
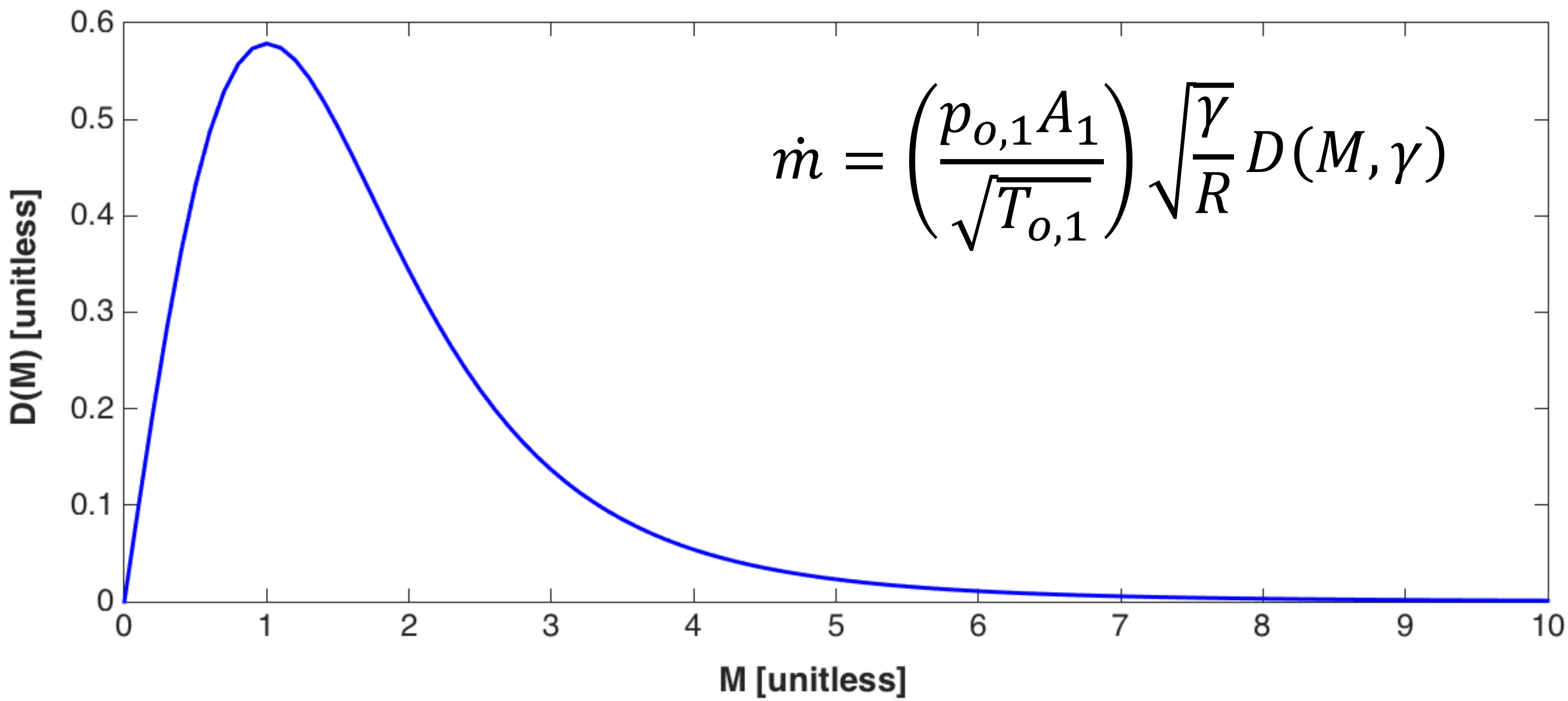
Mass Flow Function



Mass Flow Function



- Inspection of the function reveals that this maximum occurs at $M = 1$.
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- For a given Mach number, the mass flux is proportional to the upstream stagnation pressure and inversely proportional to the stagnation temperature.

Mass Flow Function

- Returning to our flow through a variable area duct, let's assume:
  - No heat addition and no work

- If we further take the assumption that our irreversibilities are small,
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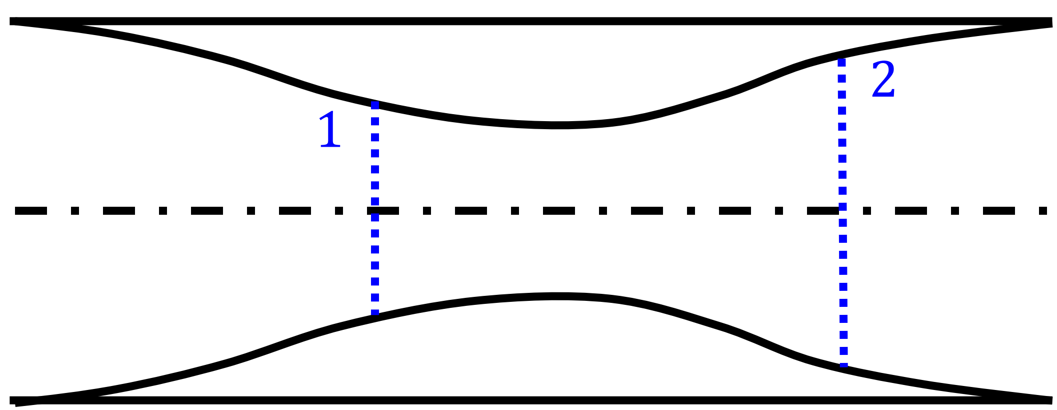
- Finally, assuming ideal gas behavior with constant properties, the continuity equation between planes 1 and 2 reduces to:



Mass Flow Function

- With these developments, it becomes very easy to compute the Mach number for any given area.

- Knowing the Mach number throughout the length of the duct, we can the readily compute the static temperature and pressure
  - From our known ______________________________ quantities.

- An important conclusion to draw, here, is that the conservation of mass can be expressed in terms of either static quantities:

$$\dot{m} = \rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

Or in terms of stagnation quantities:

$$\dot{m} = \left(\frac{p_{o,1} A_1}{\sqrt{T_{o,1}}}\right)\sqrt{\frac{\gamma}{R}}D_1 = \left(\frac{p_{o,2} A_2}{\sqrt{T_{o,2}}}\right)\sqrt{\frac{\gamma}{R}}D_2$$

Example

Consider an isentropic, one-dimensional flow in a variable area duct in which no heat is added to the fluid and no work is done by it. The pressure, temperature, velocity, and area at location 1 are: p_1, T_1, u_1, and A_1. Find the conditions are location 2 where the area is A_2. Assume no work is done, no heat is added, and that the process can be approximated as isentropic. Assume the fluid is air ($\gamma = 1.4$)

Example

Example

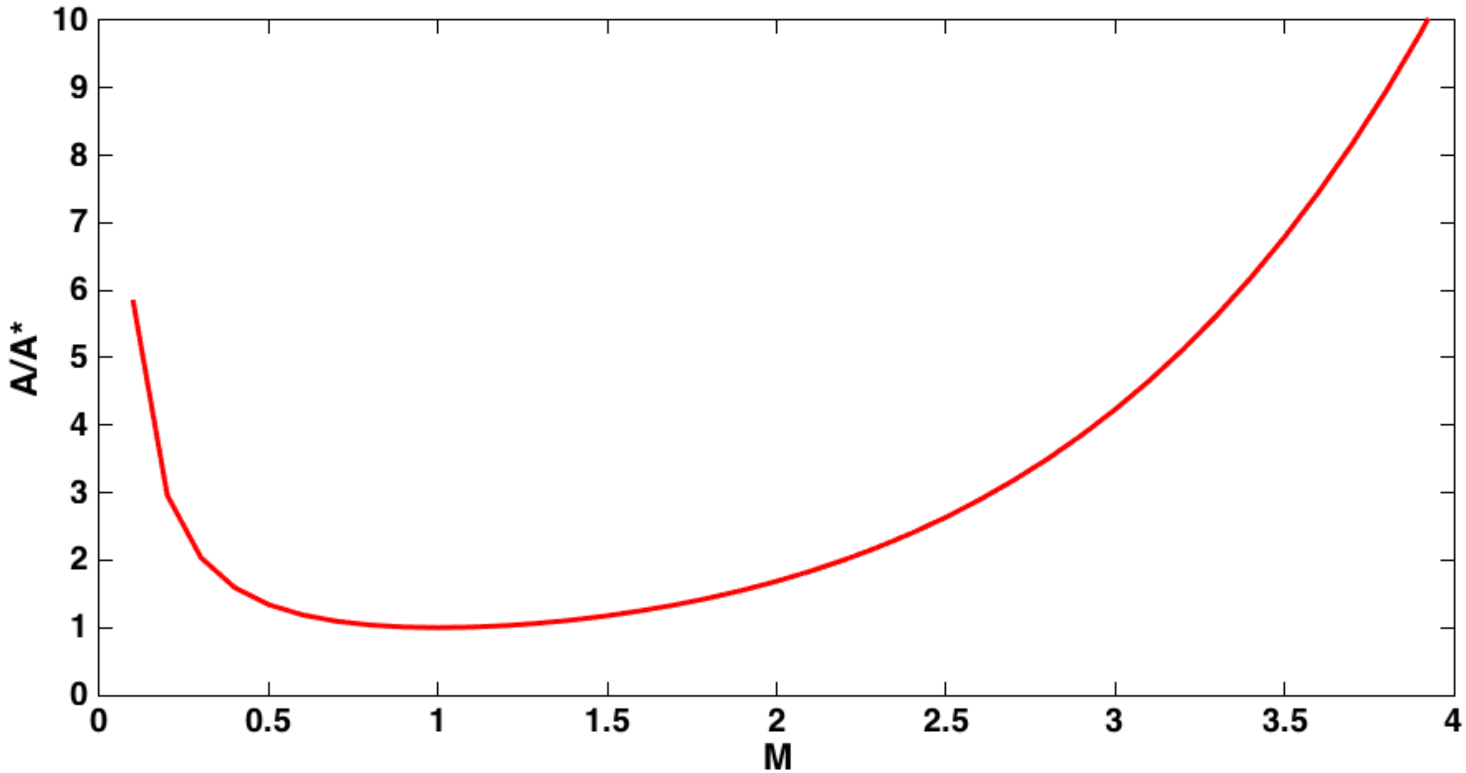
Other characteristics of D(M)

- The maximum value of $D(M)$ (always at $M = 1$), depends on the ration of specific heats. For air with $\gamma = 1.4$, $D(1) =$
  - At ambient conditions ($298K$ and $0.1\ MPa$), the maximum mass flux is
- For steady, isentropic flow in a channel, the stagnation quantities and mass flow rate are constant. Hence, we can relate the area ratio to $D(M)$ through a reference condition, A^*, where location 2 is ________________________________.

$$\frac{\dot{m}\sqrt{T_o}}{p_o}\sqrt{\frac{R}{\gamma}} = A_1 D_1 = A_2 D_2 \quad \Rightarrow$$

- Suppose the stagnation temperature or pressure changes between locations 1 and 2. We can use A^* as a references state – the minimum possible area to pass a given mass flow rate at the known stagnation conditions. From continuity, we can show:

Other characteristics of D(M)



- The quantity A/A* can be a useful measure of how much area a design needs to pass a certain flow.
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A brief statement on units…

- British engineering units are used throughout the U.S. and are, therefore, the most intuitive for many of us.
- SI units provide the most consistency for analysis and result in the fewest errors.
  - It is suggest that you convert from British to SI, perform your analysis, then convert back from SI to British at the end, if necessary.
- The units of temperature are Kelvin (K)
  - If a change in temperature is needed, degrees C may be used.
  - In formulas with property changes (e.g. an equation of state) or temperature ratios, absolute units must be used.
- The units of pressure are N/m^2 or Pa.
  - Similarly, for a change in pressure the gauge pressure can be used.
  - In formulas with an equation of state or pressure ratios, the absolute pressure (gauge + atmospheric) must be used.

Iteration Procedures

- In the example, above, it was necessary to obtain the solution to a nonlinear equation for Mach number.
  - In particular, we must solve for M_2 given the following relation:

$$D_2 = \frac{A_1 D_1}{A_2} = \frac{M_2}{\left(1 + \frac{\gamma - 1}{2} M_2^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}$$

where $\frac{A_1 D_1}{A_2}$ is known.

- One convenient method of computing this solution, numerically, is by means of Newton's method. The basic steps are as follows:
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Iteration Procedures

- For the present equation, we expand the function $D(M)$ in a Taylor Series to obtain the following:

  where the superscript k refers to the k^{th} iteration value and $M_2^{(k)}$ refers to the guess at the k^{th} iteration.
- Solving for the new value of the guess, $M_2^{(k+1)}$, we find the following:

$$M_2^{(k+1)} = M_2^{(k)} + \frac{\frac{A_1 D_1}{A_2} - D\left(M_2^{(k)}\right)}{\left(\frac{dD}{dM}\right)_{2,k}}$$

- If the slope is constant, the value, $M_2^{(k+1)}$, obtained from this solution will be
- In the more general case where the slope is variable (as it is for the D function), the value for $M_2^{(k+1)}$ will only be an ______________________________.
  - Substitute. And repeat… until the error has reached an acceptable level.

Other Notes on Newton's Method

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- The initial guess needs to be reasonable or the solution will diverge.
  - For this example, M_1 would be a good starting guess in the example.
- Convergence criteria can be established several ways
  - A common implementation is to stop the iteration once the change in the guessed value between the new guess and the previous guess is very small.
- The slope is always evaluated at the variable-value of the previous guess (k).
  - In general, the derivative can be approximated numerically.
  - For analytical functions like the mass flow function, the derivative can be found analytically. Using the standard differentiation methods from calculus, we find:

$$\frac{dD}{dM} = \frac{D}{M}\frac{1-M^2}{1+\frac{\gamma-1}{2}M^2}$$

Thrust Flow Function

- Returning to the momentum equation, we also seek to express the force-momentum balance in terms of the stagnation conditions and the Mach number.

  Beginning with the expression we developed for 1D, steady flow with no body forces.

$$\rho_i u_i^2 A_i + p_i A_i + F_x = \rho_e u_e^2 A_e + p_e A_e$$

  We first simplify by substituting in the mass flow rate

  Next, we introduce the perfect gas relation and factor out the pressure-force term

Thrust Flow Function

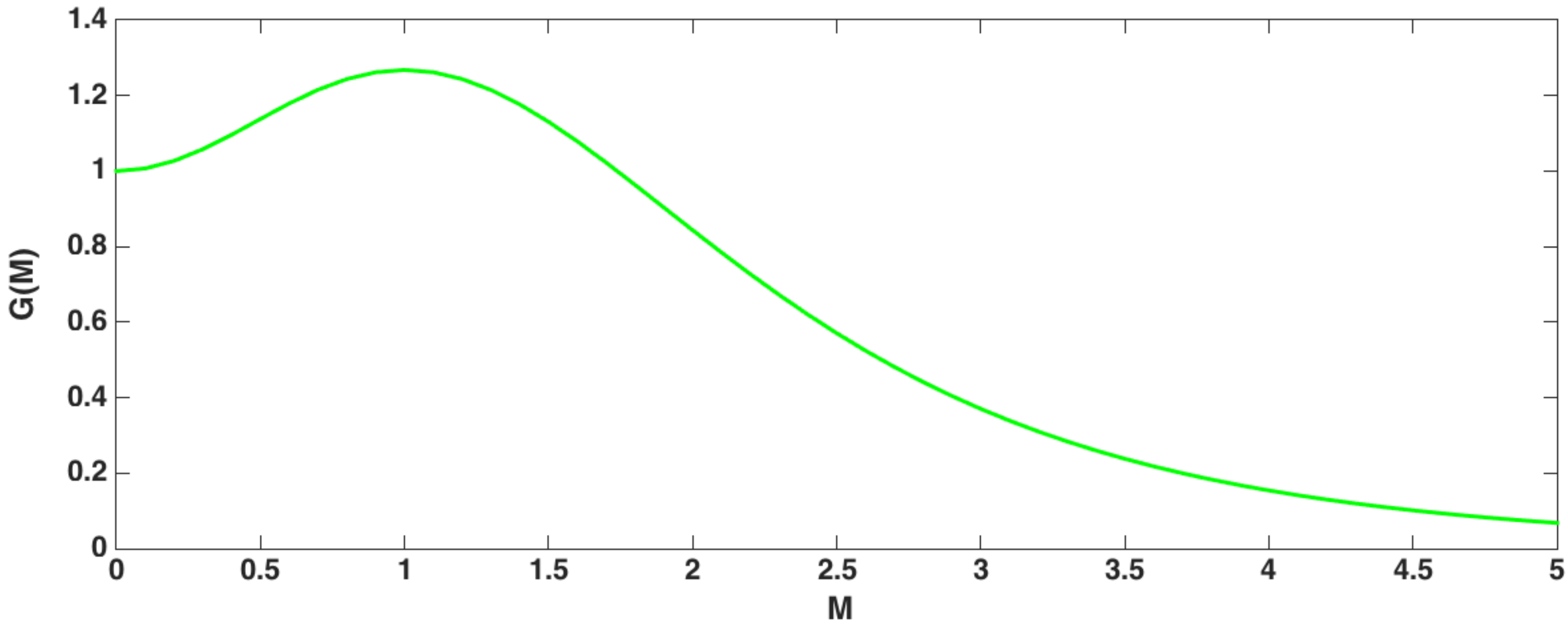
Multiplying and dividing the last term inside the parentheses by the ratio of specific heats (γ) yields:

Plugging in, then multiplying and dividing both sides by the stagnation pressure gives a formulation of the conservation of momentum for the steady, 1D flow of an ideal gas in terms of stagnation pressure and Mach number:

Defining the second Mach number function as the Thrust (momentum) Flow Function

Thrust Flow Function

- For a given area and stagnation pressure, the maximum stream thrust occurs at $M = 1$.
  - Where the stream thrust is the force on a duct to counteract a change in momentum within the duct.



What we really want is to be able to compute the thrust for a given mass flow, total temperature, and Mach number

Thrust Flow Function

- We can also express the thrust flow function in terms of the mass flow. Here we compute the ratio of $D(M)$ and $G(M)$ to find: