DSC 210 Numerical Linear Algebra, Fall 2025

Homework Problems for Topic 2: *Systems of Linear Equations*

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Write your solutions to the following problems by typing them in LaTeX. Unless otherwise noted by the problem's instructions, show your work and provide justification for your answer. Homework is due via Gradescope at **6th November 2025, 11:59 PM**.

Late Policy: If you submit your homework after the deadline we will apply a late penalty of 10% per day.

Guidelines for Homework Related Questions:

(a) As a general rule, we can help you understand the homework problems and explain the material from the corresponding lectures, but we cannot give you the entire solution.

(b) Regarding debugging programming questions: We ask you to do some debugging on your own first, including printing out intermediate values in your algorithms, trying a simpler version of the problem, etc.

(c) We will not be pre-grading the homework, i.e. we won't confirm if the answer you have is correct.

AI Usage Policy:

(a) Code: You may use LLMs to debug your code; however, you may not use LLMs to generate your entire code, and code must be reviewed and tested.

(b) Writing: You may use LLMs to correct grammar, style and latex issues; however, you may not use LLMs to generate entire solutions, sentences or paragraphs. All writing must be in your own voice.

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For more information on how the policy is implemented, refer to the most current procedures. Remember: When in doubt about what constitutes appropriate collaboration or resource use, please ask TAs before proceeding. It's always better to clarify expectations than to risk an academic integrity violation. Academic integrity violations can have serious consequences for your academic record, and you will get zero grades.

You can access the Homework Template using the following link: `https://www.overleaf.com/read/vfhcmsppvskp`

Question 1: Gauss elimination (20 points)

Use Gauss elimination to solve the following equations for $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$:

$$2x_1 + 6x_2 + 7x_3 = -11$$
$$-8x_1 + 10x_2 + 3x_3 = -15$$
$$9x_1 + 10x_2 + x_3 = 25$$

Solution:

We first convert the system of equations into an augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 6 & 7 & -11 \\ -8 & 10 & 3 & -15 \\ 9 & 10 & 1 & 25 \end{array} \right]$$

Next, we perform row operations to convert the matrix into an upper triangular form.
$R_1 = \frac{R_1}{2}$:

$$\left[\begin{array}{ccc|c} \frac{2}{2} & \frac{6}{2} & \frac{7}{2} & -\frac{11}{2} \\ -8 & 10 & 3 & -15 \\ 9 & 10 & 1 & 25 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 3 & \frac{7}{2} & -\frac{11}{2} \\ -8 & 10 & 3 & -15 \\ 9 & 10 & 1 & 25 \end{array} \right]$$

$R_2 = R_2 + 8R_1$:

$$\left[\begin{array}{ccc|c} 1 & 3 & \frac{7}{2} & -\frac{11}{2} \\ -8+8(1) & 10+8(3) & 3+8(\frac{7}{2}) & -15+8(-\frac{11}{2}) \\ 9 & 10 & 1 & 25 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 3 & \frac{7}{2} & -\frac{11}{2} \\ 0 & 34 & 31 & -59 \\ 9 & 10 & 1 & 25 \end{array} \right]$$

$R_3 = R_3 - 9R_1$:

$$\left[\begin{array}{ccc|c} 1 & 3 & \frac{7}{2} & -\frac{11}{2} \\ 0 & 34 & 31 & -59 \\ 9-9(1) & 10-9(3) & 1-9(\frac{7}{2}) & 25-9(-\frac{11}{2}) \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 3 & \frac{7}{2} & -\frac{11}{2} \\ 0 & 34 & 31 & -59 \\ 0 & -17 & -\frac{61}{2} & \frac{149}{2} \end{array} \right]$$

$R_3 = R_3 + \frac{R_2}{2}$:

$$\left[\begin{array}{ccc|c} 1 & 3 & \frac{7}{2} & -\frac{11}{2} \\ 0 & 34 & 31 & -59 \\ 0 & -17+\frac{34}{2} & -\frac{61}{2}+\frac{31}{2} & \frac{149}{2}+\frac{-59}{2} \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 3 & \frac{7}{2} & -\frac{11}{2} \\ 0 & 34 & 31 & -59 \\ 0 & 0 & -15 & 45 \end{array} \right]$$

Now we can solve for x_3, x_2, and x_1 using back substitution:
From the third row:

$$-15x_3 = 45 \therefore x_3 = -3$$

Plug in x_3 into the second row:

$$34x_2 + 31(-3) = -59 \therefore x_2 = 1$$

Plug in x_2 and x_3 into the first row:

$$x_1 + 3(1) + \frac{7}{2}(-3) = -\frac{11}{2} \therefore x_1 = 2$$

$$\therefore \mathbf{x} = \begin{bmatrix} 2 & 1 & -3 \end{bmatrix}^\top$$

Question 2: LU decomposition (20 points)

Perform LU decomposition for the matrix corresponding to the equations in **Question 1**. Using the matrices L and U, do forward and backward substitution to solve for **x**. Match your solution with that of **Question 1**.

Solution:

We can follow the steps of Gaussian elimination in **Question 1** to construct the L and U matrices. Remember from **HW 1** that any operation on row i requires changing $I[i, :]$ and applying it to the matrix on the left. However, for LU decomposition, you need to use the opposite sign of the row operation as L is the inverse of the sequence of operations used to reduce the matrix to U. The resulting U matrix is just the result of the row operation. Thus:

$R_1 = \frac{R_1}{2}$:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 3 & \frac{7}{2} \\ -8 & 10 & 3 \\ 9 & 10 & 1 \end{bmatrix}$$

$R_2 = R_2 + 8R_1$:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 3 & \frac{7}{2} \\ 0 & 34 & 31 \\ 9 & 10 & 1 \end{bmatrix}$$

$R_3 = R_3 - 9R_1$:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -8 & 1 & 0 \\ 9 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 3 & \frac{7}{2} \\ 0 & 34 & 31 \\ 0 & -17 & -\frac{61}{2} \end{bmatrix}$$

$R_3 = R_3 + \frac{R_2}{2}$:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -8 & 1 & 0 \\ 9 & -\frac{1}{2} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 3 & \frac{7}{2} \\ 0 & 34 & 31 \\ 0 & 0 & -15 \end{bmatrix}$$

Now we can solve for **x** using forward and backward substitution. For matrix L, we solve for **y** where $\mathbf{Ly = b}$:

From the first row:

$$2y_1 = -11 \therefore y_1 = -\frac{11}{2}$$

Plug in y_1 into the second row:

$$-8(-\frac{11}{2}) + y_2 = -15 \therefore y_2 = -59$$

Plug in y_1 and y_2 into the third row:

$$9(-\frac{11}{2}) - \frac{1}{2}(-59) + y_3 = 25 \therefore y_3 = 5$$

$$\therefore \mathbf{y} = \begin{bmatrix} -\frac{11}{2} & -59 & 5 \end{bmatrix}^\top$$

Now we can solve for **x** using backward substitution for matrix U where $\mathbf{Ux = y}$:

From the third row:

$$-15x_3 = 5 \therefore x_3 = -3$$

Plug in x_3 into the second row:

$$34x_2 + 31(-3) = -59 \therefore x_2 = 1$$

Plug in x_2 and x_3 into the first row:

$$x_1 + 3(1) + \frac{7}{2}(-3) = -\frac{11}{2} \therefore x_1 = 2$$

$$\therefore \mathbf{x} = \begin{bmatrix} 2 & 1 & -3 \end{bmatrix}^\top$$

which matches the solution from **Question 1**.

Question 3: QR decomposition (20 points)

Perform QR decomposition for the matrix corresponding to the equations in **Question 1** using the Gram-Schmidt algorithm. Using the decomposition, solve for **x**. Match your solution with that of **Question 1** and **Question 2**.

Solution:

We start with our matrix **A**:

$$\begin{bmatrix} 2 & 6 & 7 \\ -8 & 10 & 3 \\ 9 & 10 & 1 \end{bmatrix}$$

Thus:

$$\mathbf{u}_1 = (2, -8, 9)$$

$$\mathbf{q}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{(2, -8, 9)}{\sqrt{2^2 + (-8)^2 + 9^2}} = \frac{(2, -8, 9)}{\sqrt{149}} = \left(\frac{2}{\sqrt{149}}, \frac{-8}{\sqrt{149}}, \frac{9}{\sqrt{149}} \right)$$

$$\mathbf{u}_2 = (6, 10, 10) - \text{proj}_{\mathbf{q}_1}(6, 10, 10) = (6, 10, 10) - ((6, 10, 10) \cdot \mathbf{q}_1)\, \mathbf{q}_1$$

$$= (6, 10, 10) - \left(\frac{6 \cdot 2}{\sqrt{149}} + \frac{10 \cdot (-8)}{\sqrt{149}} + \frac{10 \cdot 9}{\sqrt{149}} \right) \frac{(2, -8, 9)}{\sqrt{149}} = (6, 10, 10) - \frac{22}{149}(2, -8, 9)$$

$$= \left(6 - \frac{44}{149}, 10 + \frac{176}{149}, 10 - \frac{198}{149} \right) = \left(\frac{850}{149}, \frac{1666}{149}, \frac{1292}{149} \right)$$

$$\mathbf{q}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{\left(\frac{850}{149}, \frac{1666}{149}, \frac{1292}{149} \right)}{\sqrt{\left(\frac{850}{149} \right)^2 + \left(\frac{1666}{149} \right)^2 + \left(\frac{1292}{149} \right)^2}} = \frac{(850, 1666, 1292)}{\sqrt{850^2 + 1666^2 + 1292^2}}$$

$$= \left(\frac{850}{34\sqrt{4470}}, \frac{1666}{34\sqrt{4470}}, \frac{1292}{34\sqrt{4470}} \right) = \left(\frac{25}{\sqrt{4470}}, \frac{49}{\sqrt{4470}}, \frac{38}{\sqrt{4470}} \right)$$

$$\mathbf{u}_3 = (7, 3, 1) - \text{proj}_{\mathbf{e}_1}(7, 3, 1) - \text{proj}_{\mathbf{e}_2}(7, 3, 1)$$

$$= (7, 3, 1) - \left(\frac{7 \cdot 2}{\sqrt{149}} + \frac{3 \cdot (-8)}{\sqrt{149}} + \frac{1 \cdot 9}{\sqrt{149}} \right) \frac{(2, -8, 9)}{\sqrt{149}} - \left(\frac{7 \cdot 25}{\sqrt{4470}} + \frac{3 \cdot 49}{\sqrt{4470}} + \frac{1 \cdot 38}{\sqrt{4470}} \right) \frac{(25, 49, 38)}{\sqrt{4470}}$$

$$= (7, 3, 1) + \frac{1}{149}(2, -8, 9) - \frac{360}{4470}(25, 49, 38)$$

$$= (7, 3, 1) + \frac{1}{149}(2, -8, 9) - \frac{12}{149}(25, 49, 38)$$

$$= \left(7 + \frac{2}{149} - \frac{300}{149}, 3 - \frac{8}{149} - \frac{588}{149}, 1 + \frac{9}{149} - \frac{456}{149} \right) = (5, -1, -2)$$

$$\mathbf{q}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{(5, -1, -2)}{\sqrt{5^2 + (-1)^2 + (-2)^2}} = \frac{(5, -1, -2)}{\sqrt{30}} = \left(\frac{5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}, \frac{-2}{\sqrt{30}} \right)$$

The QR decomposition matrices are then as follows:

$$QR = \begin{bmatrix} q_1 | q_2 | q_3 \end{bmatrix} \begin{bmatrix} a_1 \cdot q_1 & a_2 \cdot q_1 & a_3 \cdot q_1 \\ 0 & a_2 \cdot q_2 & a_3 \cdot q_2 \\ 0 & 0 & a_3 \cdot q_3 \end{bmatrix}$$

Thus:

$$Q = \begin{bmatrix} \frac{2}{\sqrt{149}} & \frac{25}{\sqrt{4470}} & \frac{5}{\sqrt{30}} \\ \frac{-8}{\sqrt{149}} & \frac{49}{\sqrt{4470}} & \frac{-1}{\sqrt{30}} \\ \frac{9}{\sqrt{149}} & \frac{38}{\sqrt{4470}} & \frac{-2}{\sqrt{30}} \end{bmatrix}$$

$$R = \begin{bmatrix} 2(\frac{2}{\sqrt{149}}) + (-8)(\frac{-8}{\sqrt{149}}) + 9(\frac{9}{\sqrt{149}}) & 6(\frac{2}{\sqrt{149}}) + 10(\frac{-8}{\sqrt{149}}) + 10(\frac{9}{\sqrt{149}}) & 7(\frac{2}{\sqrt{149}}) + 3(\frac{-8}{\sqrt{149}}) + 1(\frac{9}{\sqrt{149}}) \\ 0 & 6(\frac{25}{\sqrt{4470}}) + 10(\frac{49}{\sqrt{4470}}) + 10(\frac{38}{\sqrt{4470}}) & 7(\frac{25}{\sqrt{4470}}) + 3(\frac{49}{\sqrt{4470}}) + 1(\frac{38}{\sqrt{4470}}) \\ 0 & 0 & 7(\frac{5}{\sqrt{30}}) + 3(\frac{-1}{\sqrt{30}}) + 1(\frac{-2}{\sqrt{30}}) \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{149} & \frac{22}{\sqrt{149}} & -\frac{1}{\sqrt{149}} \\ 0 & \frac{1020}{\sqrt{4470}} & \frac{360}{\sqrt{4470}} \\ 0 & 0 & \sqrt{30} \end{bmatrix}$$

We can now solve for $\mathbf{x}$ as:

$$QR\mathbf{x} = \mathbf{b}$$

$$R\mathbf{x} = Q^\top \mathbf{b}$$

$$Q^\top \mathbf{b} = \begin{bmatrix} \frac{2}{\sqrt{149}} & \frac{-8}{\sqrt{149}} & \frac{9}{\sqrt{149}} \\ \frac{25}{\sqrt{4470}} & \frac{49}{\sqrt{4470}} & \frac{38}{\sqrt{4470}} \\ \frac{5}{\sqrt{30}} & \frac{-1}{\sqrt{30}} & \frac{-2}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} -11 \\ -15 \\ 25 \end{bmatrix} = \begin{bmatrix} \frac{2(-11)+(-8)(-15)+9(25)}{\sqrt{149}} \\ \frac{25(-11)+49(-15)+38(25)}{\sqrt{4470}} \\ \frac{5(-11)+(-1)(-15)+(-2)(25)}{\sqrt{30}} \end{bmatrix} = \begin{bmatrix} \frac{323}{\sqrt{149}} \\ \frac{-60}{\sqrt{4470}} \\ \frac{-90}{\sqrt{30}} \end{bmatrix}$$

Now we can solve for $\mathbf{x}$ using backward substitution: From the third row:

$$\sqrt{30}x_3 = \frac{-90}{\sqrt{30}} \therefore x_3 = -3$$

Plug in x_3 into the second row:

$$\frac{1020}{\sqrt{4470}}x_2 + \frac{360}{\sqrt{4470}}(-3) = \frac{-60}{\sqrt{4470}} \therefore x_2 = 1$$

Plug in x_2 and x_3 into the first row:

$$\sqrt{149}x_1 + \frac{22}{\sqrt{149}}(1) - \frac{1}{\sqrt{149}}(-3) = \frac{323}{\sqrt{149}} \therefore x_1 = 2$$

$$\therefore \mathbf{x} = \begin{bmatrix} 2 & 1 & -3 \end{bmatrix}^\top$$

which matches the solution from **Question 1** and **Question 2**.

Question 4: Gram-Schmidt process (20 points)

Show that the residual vector $\mathbf{a}_i^{\perp}$ is orthogonal to $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{i-1}$ in the Gram-Schmidt process.

Solution:

The residual vector $\mathbf{a}_i^{\perp}$ is given by:

$$\mathbf{a}_i^{\perp} = \mathbf{a}_i - \sum_{j=1}^{i-1} \mathrm{proj}_{\mathbf{q}_j}(\mathbf{a}_i)$$

The projection can be expressed as:

$$\mathrm{proj}_{\mathbf{q}_j}(\mathbf{a}_i) = \left(\frac{\mathbf{a}_i \cdot \mathbf{q}_j}{\mathbf{q}_j \cdot \mathbf{q}_j}\right) \mathbf{q}_j$$

To prove orthogonality, we need to show that the dot product of $\mathbf{a}_i^{\perp}$ with each $\mathbf{q}_k$ (for $k = 1, 2, \ldots, i-1$) is zero:

$$\mathbf{a}_i^{\perp} \cdot \mathbf{q}_k = \left(\mathbf{a}_i - \sum_{j=1}^{i-1} \mathrm{proj}_{\mathbf{q}_j}(\mathbf{a}_i)\right) \cdot \mathbf{q}_k$$

Expanding this, we have:

$$\mathbf{a}_i^{\perp} \cdot \mathbf{q}_k = \mathbf{a}_i \cdot \mathbf{q}_k - \sum_{j=1}^{i-1} \left(\mathrm{proj}_{\mathbf{q}_j}(\mathbf{a}_i) \cdot \mathbf{q}_k\right)$$

Now, we know that the projection $\mathrm{proj}_{\mathbf{q}_j}(\mathbf{a}_i)$ is a scalar multiple of $\mathbf{q}_j$. Therefore, when we take the dot product with $\mathbf{q}_k$, we get:

$$\mathrm{proj}_{\mathbf{q}_j}(\mathbf{a}_i) \cdot \mathbf{q}_k = \left(\frac{\mathbf{a}_i \cdot \mathbf{q}_j}{\mathbf{q}_j \cdot \mathbf{q}_j}\right)(\mathbf{q}_j \cdot \mathbf{q}_k)$$

Since $\mathbf{q}_j$ and $\mathbf{q}_k$ are orthogonal for $j \neq k$, we have:

$$\mathbf{q}_j \cdot \mathbf{q}_k = 0 \quad \text{for } j \neq k$$

Thus, the sum reduces to:

$$\sum_{j=1}^{i-1} \left(\mathrm{proj}_{\mathbf{q}_j}(\mathbf{a}_i) \cdot \mathbf{q}_k\right) = \mathrm{proj}_{\mathbf{q}_k}(\mathbf{a}_i) \cdot \mathbf{q}_k$$

Substituting this back into our expression for $\mathbf{a}_i^{\perp} \cdot \mathbf{q}_k$, we get:

$$\mathbf{a}_i^{\perp} \cdot \mathbf{q}_k = \mathbf{a}_i \cdot \mathbf{q}_k - \mathrm{proj}_{\mathbf{q}_k}(\mathbf{a}_i) \cdot \mathbf{q}_k$$

$$\mathbf{a}_i^{\perp} \cdot \mathbf{q}_k = \mathbf{a}_i \cdot \mathbf{q}_k - \left(\frac{\mathbf{a}_i \cdot \mathbf{q}_k}{\mathbf{q}_k \cdot \mathbf{q}_k}\right)(\mathbf{q}_k \cdot \mathbf{q}_k)$$

$$\mathbf{a}_i^{\perp} \cdot \mathbf{q}_k = \mathbf{a}_i \cdot \mathbf{q}_k - \mathbf{a}_i \cdot \mathbf{q}_k = 0$$

Since this holds for all $k = 1, 2, \ldots, i-1$, the residual vector $\mathbf{a}_i^{\perp}$ is orthogonal to $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{i-1}$.

Question 5: Rank deficient matrices (20 points)

Compute the QR decomposition of following two matrices using Gram-Schmidt process and properties of the matrix **Q**.

i. $\mathbf{A} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

ii. $\mathbf{B} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

Note: matrices **Q** and **R** must be square matrices.

Solution:

i. $\mathbf{A} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\mathbf{u}_1 = (-1, -1, 1)$$

$$\mathbf{q}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{(-1, -1, 1)}{\sqrt{(-1)^2 + (-1)^2 + 1^2}} = \frac{(-1, -1, 1)}{\sqrt{3}} = \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$\mathbf{u}_2 = (1, -1, 1) - \text{proj}_{\mathbf{q}_1}(1, -1, 1) = (1, -1, 1) - ((1, -1, 1) \cdot \mathbf{q}_1)\, \mathbf{q}_1$$

$$= (1, -1, 1) - \left(\frac{1 \cdot (-1)}{\sqrt{3}} + \frac{-1 \cdot (-1)}{\sqrt{3}} + \frac{1 \cdot 1}{\sqrt{3}} \right) \frac{(-1, -1, 1)}{\sqrt{3}} = (1, -1, 1) - \frac{1}{3}(-1, -1, 1)$$

$$= \left(1 + \frac{1}{3}, -1 + \frac{1}{3}, 1 - \frac{1}{3} \right) = \left(\frac{4}{3}, \frac{-2}{3}, \frac{2}{3} \right)$$

$$\mathbf{q}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{\left(\frac{4}{3}, \frac{-2}{3}, \frac{2}{3} \right)}{\sqrt{\left(\frac{4}{3} \right)^2 + \left(\frac{-2}{3} \right)^2 + \left(\frac{2}{3} \right)^2}} = \frac{(4, -2, 2)}{\sqrt{4^2 + (-2)^2 + 2^2}}$$

$$= \left(\frac{4}{\sqrt{24}}, \frac{-2}{\sqrt{24}}, \frac{2}{\sqrt{24}} \right) = \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$$

Note that $\mathbf{R}_3 = -1 \cdot \mathbf{R}_2$, making **A** rank deficient. Thus, $\mathbf{q}_3$ must be a vector orthogonal to both $\mathbf{q}_1$ and $\mathbf{q}_2$. We can use the cross product to find such a vector:

$$\mathbf{q}_3 = \mathbf{q}_1 \times \mathbf{q}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{vmatrix} = \mathbf{i} \left(\frac{-1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} \right) - \mathbf{j} \left(\frac{-1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \cdot \frac{2}{\sqrt{6}} \right)$$

$$+ \mathbf{k} \left(\frac{-1}{\sqrt{3}} \cdot \frac{-1}{\sqrt{6}} - \frac{-1}{\sqrt{3}} \cdot \frac{2}{\sqrt{6}} \right) = \left(0, \frac{3}{\sqrt{18}}, \frac{3}{\sqrt{18}} \right) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

Thus:

$$Q = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$R = \begin{bmatrix} (-1)(\frac{-1}{\sqrt{3}}) + (-1)(\frac{-1}{\sqrt{3}}) + 1(\frac{1}{\sqrt{3}}) & 1(\frac{-1}{\sqrt{3}}) + (-1)(\frac{-1}{\sqrt{3}}) + 1(\frac{1}{\sqrt{3}}) & 1(\frac{-1}{\sqrt{3}}) + 1(\frac{-1}{\sqrt{3}}) + (-1)(\frac{1}{\sqrt{3}}) \\ 0 & 1(\frac{2}{\sqrt{6}}) + (-1)(\frac{-1}{\sqrt{6}}) + 1(\frac{1}{\sqrt{6}}) & 1(\frac{2}{\sqrt{6}}) + 1(\frac{-1}{\sqrt{6}}) + (-1)(\frac{1}{\sqrt{6}}) \\ 0 & 0 & 1(0) + 1(\frac{1}{\sqrt{2}}) + (-1)(\frac{1}{\sqrt{2}}) \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{3}} \\ 0 & \frac{4}{\sqrt{6}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

ii. $\mathbf{B} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

$\mathbf{u}_1 = (-1, 1, -1)$

$$\mathbf{q}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{(-1, 1, -1)}{\sqrt{(-1)^2 + 1^2 + (-1)^2}} = \frac{(-1, 1, -1)}{\sqrt{3}} = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right)$$

Note that $\mathbf{R}_2 = -1 \cdot \mathbf{R}_1$ and $\mathbf{R}_3 = \mathbf{R}_1$, making $\mathbf{B}$ rank deficient. We need $\mathbf{q}_2$ to be orthogonal to $\mathbf{q}_1$. We can choose any non zero nor parallel vector to $\mathbf{q}_1$:

$\mathbf{v} = (1, 0, 0)$

$\mathbf{u}_2 = \mathbf{v} - \text{proj}_{\mathbf{q}_1}(\mathbf{v}) = (1, 0, 0) - ((1, 0, 0) \cdot \mathbf{q}_1)\, \mathbf{q}_1$

$$= (1, 0, 0) - \left(\frac{1 \cdot (-1)}{\sqrt{3}} + \frac{0 \cdot 1}{\sqrt{3}} + \frac{0 \cdot (-1)}{\sqrt{3}} \right) \frac{(-1, 1, -1)}{\sqrt{3}} = (1, 0, 0) - \frac{-1}{3}(-1, 1, -1)$$

$$= \left(1 - \frac{1}{3}, 0 + \frac{1}{3}, 0 - \frac{1}{3} \right) = \left(\frac{2}{3}, \frac{1}{3}, \frac{-1}{3} \right)$$

$$\mathbf{q}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{\left(\frac{2}{3}, \frac{1}{3}, \frac{-1}{3} \right)}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2}} = \frac{(2, 1, -1)}{\sqrt{2^2 + 1^2 + (-1)^2}}$$

$$= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right)$$

$$\mathbf{q}_3 = \mathbf{q}_1 \times \mathbf{q}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{vmatrix} = \mathbf{i}\left(\frac{1}{\sqrt{3}} \cdot \frac{-1}{\sqrt{6}} - \frac{-1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} \right) - \mathbf{j}\left(\frac{-1}{\sqrt{3}} \cdot \frac{-1}{\sqrt{6}} - \frac{-1}{\sqrt{3}} \cdot \frac{2}{\sqrt{6}} \right)$$

$$+ \mathbf{k}\left(\frac{-1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \cdot \frac{2}{\sqrt{6}} \right) = \left(0, \frac{-3}{\sqrt{18}}, \frac{-3}{\sqrt{18}} \right) = \left(0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right)$$

Thus:

$$Q = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$R = \begin{bmatrix} (-1)(\frac{-1}{\sqrt{3}}) + 1(\frac{1}{\sqrt{3}}) + (-1)(\frac{-1}{\sqrt{3}}) & (-1)(\frac{-1}{\sqrt{3}}) + 1(\frac{1}{\sqrt{3}}) + (-1)(\frac{-1}{\sqrt{3}}) & (-1)(\frac{-1}{\sqrt{3}}) + 1(\frac{1}{\sqrt{3}}) + (-1)(\frac{-1}{\sqrt{3}}) \\ 0 & (-1)(\frac{2}{\sqrt{6}}) + 1(\frac{1}{\sqrt{6}}) + (-1)(\frac{-1}{\sqrt{6}}) & (-1)(\frac{2}{\sqrt{6}}) + 1(\frac{1}{\sqrt{6}}) + (-1)(\frac{-1}{\sqrt{6}}) \\ 0 & 0 & (-1)(0) + 1(\frac{-1}{\sqrt{2}}) + (-1)(\frac{-1}{\sqrt{2}}) \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{3} & \sqrt{3} & \sqrt{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$