DSC 210 Numerical Linear Algebra, Fall 2025

Homework problems for Topic 1: *Linear Algebra Basics*

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Write your solutions to the following problems by typing them in LaTeX. Unless otherwise noted by the problem's instructions, show your work and provide justification for your answer. Homework is due via Gradescope at **23rd October 2025, 11:59 PM**.
Late Policy: If you submit your homework after the deadline we will apply a late penalty of 10% per day.

Guidelines for Homework Related Questions:

(a) As a general rule, we can help you understand the homework problems and explain the material from the corresponding lectures, but we cannot give you the entire solution.

(b) Regarding debugging programming questions: We ask you to do some debugging on your own first, including printing out intermediate values in your algorithms, trying a simpler version of the problem, etc.

(c) We will not be pre-grading the homework, i.e. we won't confirm if the answer you have is correct.

AI Usage Policy:

(a) Code: You may use LLMs to debug your code; however, you may not use LLMs to generate your entire code, and code must be reviewed and tested.

(b) Writing: You may use LLMs to correct grammar, style and latex issues; however, you may not use LLMs to generate entire solutions, sentences or paragraphs. All writing must be in your own voice.

Academic Integrity Policy:

The UC San Diego Academic Integrity Policy (formerly the Policy on Integrity of Scholarship) is effective as of September 25, 2023 and applies to any cases originating on or after September 25, 2023. The university expects both faculty and students to honor the policy. For students, this means that all academic work will be done by the individual to whom it's assigned, without unauthorized aid of any kind. If violations of academic integrity occur, the same Sanctioning Guidelines apply regardless of which policy was effective for that case.

For more information on how the policy is implemented, refer to the most current procedures. Remember: When in doubt about what constitutes appropriate collaboration or resource use, please ask TAs before proceeding. It's always better to clarify expectations than to risk an academic integrity violation. Academic integrity violations can have serious consequences for your academic record, and you will get zero grades.

You can access the Homework Template using the following link: `https://www.overleaf.com/read/vfhcmsppvskp`

Question 1: Property of triangular matrices (20 points)

Given L_1 and L_2 are two lower triangular matrices of size $n \times n$, prove that $L_1 L_2$ is also a lower triangular matrix. Further, prove by induction that multiplication of $m\,(m > 2)$ lower triangular matrices $(L_1, L_2, ..., L_m)$ is also a lower triangular matrix.

Solution:

Let $A = L_1 L_2$, then:

$$A_{ij} = \sum_{k=1}^{n} (L_1)_{ik} (L_2)_{kj}$$

In order for A to be a lower triangular matrix, we need to show that $A_{ij} = 0$ for all $i < j$.
In L_1, $(L_1)_{ik} = 0$ for all $k > i$.
In L_2, $(L_2)_{kj} = 0$ for all $j > k$.
Thus, for A_{ij} to be non-zero, we need $k \leq i$ and $j \leq k$.
Combining these two inequalities, we get $j \leq k \leq i$.
However, for $i < j$, there is no k that satisfies this condition, thus $A_{ij} = 0$ for all $i < j$, and A is a lower triangular matrix.

We already know that $L_1 L_2$ is a lower triangular matrix (base case, product of 2 lower triangular matrices will result in a lower triangular matrix). $L_1 L_2 L_3 = (L_1 L_2) L_3 = A L_3$, which circles back to our base case of multiplying two lower triangular matrices. Any further products will also result multiplying two lower triangular matrices, thus, by induction, we can conclude that the product of m lower triangular matrices will ultimately result in a lower triangular matrix.

Question 2: Matrix operations (20 points)

Let $\mathbf{B}$ be a 4×4 matrix to which we apply the following 7 operations sequentially and get a final matrix $\mathbf{D}$:

(i) double column 1,

(ii) halve row 3,

(iii) add row 1 to row 4,

(iv) interchange columns 2 and 3,

(v) subtract row 2 from each of the other rows,

(vi) replace column 4 by column 1,

(vii) delete column 2 (so that the column dimension is reduced by 1).

(a) Express each operation (i) to (vii) as a matrix and the final matrix $\mathbf{D}$ as a product of 8 matrices. (10 points)

(b) Write the final result again as a product of $\mathbf{ABC}$, i.e. write matrix $\mathbf{D} = \mathbf{ABC}$ and find $\mathbf{A}, \mathbf{C}$. (5 points)

(c) Write Python code to verify your answers in parts a, and b. Show the answers and code. (5 points) Let

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

Hint: You can use NumPy for matrix operations.

Solution:

(a) We can write each operation as a matrix E_n where n is the operation number. We know that in order to modify $\mathbf{B}$, we can either left multiply or right multiply by E_n depending on whether we are modifying rows or columns respectively. Any operation begins with I. Any operation on row i requires changing $E_n[i,:]$ and any operation on column j requires changing $E_n[:,j]$. Considering each row / column as a vector, we can modify other rows / columns with respect to the primary row / column by modifying the respective value at their intersection in the vector (i.e. applying an operation using row i onto row j would require modifying $E_n[i,j]$).

(i) Thus,

$$E_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\mathbf{D}_i \leftarrow \mathbf{B}E_1$

(ii)

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\mathbf{D}_{ii} \leftarrow E_2\mathbf{B}E_1$

(iii)

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

where $\mathbf{D}_{iii} \leftarrow E_3 E_2 \mathbf{B} E_1$

(iv)

$$E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\mathbf{D}_{iv} \leftarrow E_3 E_2 \mathbf{B} E_1 E_4$

(v)

$$E_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

where $\mathbf{D}_{v} \leftarrow E_5 E_3 E_2 \mathbf{B} E_1 E_4$

(vi)

$$E_6 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\mathbf{D}_{vi} \leftarrow E_5 E_3 E_2 \mathbf{B} E_1 E_4 E_6$

(vii)

$$E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\mathbf{D}_{vii} \leftarrow E_5 E_3 E_2 \mathbf{B} E_1 E_4 E_6 E_7$.

Thus, we can express $\mathbf{D}$ as:

$$\mathbf{D} = E_5 E_3 E_2 \mathbf{B} E_1 E_4 E_6 E_7$$

(b) We know matrix multiplication is associative, thus we can group the matrices as follows:

$$\mathbf{D} = (E_5 E_3 E_2)\mathbf{B}(E_1 E_4 E_6 E_7)$$

Thus, we can express $\mathbf{A}$ and $\mathbf{C}$ as:

$$\mathbf{A} = E_5 E_3 E_2, \quad \mathbf{C} = E_1 E_4 E_6 E_7$$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) Code:

```python
import numpy as np

B = np.array([[1, 2, 3, 4],
              [5, 6, 7, 8],
              [9, 10, 11, 12],
              [13, 14, 15, 16]])

# double column 1
E1 = np.array([[2, 0, 0, 0],
               [0, 1, 0, 0],
               [0, 0, 1, 0],
               [0, 0, 0, 1]])
B @ E1
```

```
array([[ 2,  2,  3,  4],
       [10,  6,  7,  8],
       [18, 10, 11, 12],
       [26, 14, 15, 16]])
```

```python
# halve row 3
E2 = np.array([[1, 0, 0, 0],
               [0, 1, 0, 0],
               [0, 0, 0.5, 0],
               [0, 0, 0, 1]])
E2 @ B @ E1
```

```
array([[ 2. ,  2. ,  3. ,  4. ],
       [10. ,  6. ,  7. ,  8. ],
       [ 9. ,  5. ,  5.5,  6. ],
       [26. , 14. , 15. , 16. ]])
```

```python
# add row 1 to row 4
E3 = np.array([[1, 0, 0, 0],
               [0, 1, 0, 0],
               [0, 0, 1, 0],
               [1, 0, 0, 1]])
E3 @ E2 @ B @ E1
```

```
array([[ 2. ,  2. ,  3. ,  4. ],
       [10. ,  6. ,  7. ,  8. ],
       [ 9. ,  5. ,  5.5,  6. ],
       [28. , 16. , 18. , 20. ]])
```

```python
# interchange columns 2 and 3
E4 = np.array([[1, 0, 0, 0],
               [0, 0, 1, 0],
               [0, 1, 0, 0],
               [0, 0, 0, 1]])
E3 @ E2 @ B @ E1 @ E4
```

```
array([[ 2. ,  3. ,  2. ,  4. ],
       [10. ,  7. ,  6. ,  8. ],
       [ 9. ,  5.5,  5. ,  6. ],
       [28. , 18. , 16. , 20. ]])
```

```python
# subtract row 2 from each of the other rows
E5 = np.array([[1, -1, 0, 0],
               [0, 1, 0, 0],
               [0, -1, 1, 0],
               [0, -1, 0, 1]])
E5 @ E3 @ E2 @ B @ E1 @ E4
```

```
array([[-8. , -4. , -4. , -4. ],
       [10. ,  7. ,  6. ,  8. ],
       [-1. , -1.5, -1. , -2. ],
       [18. , 11. , 10. , 12. ]])
```

```python
# replace column 4 by column 1
E6 = np.array([[1, 0, 0, 1],
               [0, 1, 0, 0],
               [0, 0, 1, 0],
               [0, 0, 0, 0]])
E5 @ E3 @ E2 @ B @ E1 @ E4 @ E6
```

```
array([[-8. , -4. , -4. , -8. ],
       [10. ,  7. ,  6. , 10. ],
       [-1. , -1.5, -1. , -1. ],
       [18. , 11. , 10. , 18. ]])
```

```python
# delete column 2
E7 = np.array([[1, 0, 0],
               [0, 0, 0],
               [0, 1, 0],
               [0, 0, 1]])
E5 @ E3 @ E2 @ B @ E1 @ E4 @ E6 @ E7
```

```
array([[-8., -4., -8.],
       [10.,  6., 10.],
       [-1., -1., -1.],
       [18., 10., 18.]])
```

```python
# Verify A
A = E5 @ E3 @ E2
A
```

```
array([[ 1. , -1. ,  0. ,  0. ],
       [ 0. ,  1. ,  0. ,  0. ],
       [ 0. , -1. ,  0.5,  0. ],
       [ 1. , -1. ,  0. ,  1. ]])
```

```python
# Verify C
C = E1 @ E4 @ E6 @ E7
C
```

```
array([[2, 0, 2],
       [0, 1, 0],
       [0, 0, 0],
       [0, 0, 0]])
```

```python
# Verify D = ABC
A @ B @ C
```

```
array([[-8., -4., -8.],
       [10.,  6., 10.],
       [-1., -1., -1.],
       [18., 10., 18.]])
```

Question 3: Matrix properties (20 points)

Prove that if a matrix $\mathbf{A}$ is triangular (upper or lower) then $\mathbf{A}^{-1}$ is also triangular. Further, use the result to show that if $\mathbf{A}$ is both triangular and orthogonal, then it is diagonal.

Solution:

Let $\mathbf{A}$ be $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, then $\mathbf{A}^{-1} = \frac{1}{ad}\begin{bmatrix} d & -b \\ 0 & a \end{bmatrix}$, which is also upper triangular. The case for lower triangular matrices is similar.

Now, let $\mathbf{A}$ be an $(n+1) \times (n+1)$ upper triangular matrix:

$$\mathbf{A} = \begin{bmatrix} A_1 & a_2 \\ \mathbf{0} & x \end{bmatrix}$$

Where A_1 is an $n \times n$ upper triangular matrix, a_2 is an $n \times 1$ vector, $\mathbf{0}$ is a $1 \times n$ 0 vector, and x is a scalar. Let $\mathbf{A}^{-1}$ then be in a similar format:

$$\mathbf{A}^{-1} = \begin{bmatrix} B_1 & b_2 \\ b_3 & y \end{bmatrix}$$

Where B_1 is an $n \times n$ matrix, b_2 is an $n \times 1$ vector, b_3 is a $1 \times n$ vector, and y is a scalar.

We know that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_{n+1}$:

$$\begin{bmatrix} A_1 & a_2 \\ \mathbf{0} & x \end{bmatrix} \begin{bmatrix} B_1 & b_2 \\ b_3 & y \end{bmatrix} = \begin{bmatrix} A_1B_1 + a_2b_3 & A_1b_2 + a_2y \\ xb_3 & xy \end{bmatrix} = \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

From the lower left multiplication, we can see that $b_3 = \mathbf{0}$ since x is non-zero (otherwise $\mathbf{A}$ would be singular and not invertible). Thus, we can simplifty the top left multiplication to $A_1B_1 = I_n$. Therefore, $B_1 = A_1^{-1}$, and by induction hypothesis, is also upper triangular. Thus:

$$\mathbf{A}^{-1} = \begin{bmatrix} A_1^{-1} & b_2 \\ \mathbf{0} & y \end{bmatrix}$$

which upper triangular. The case for lower triangular matrices is similar.

Now, let $\mathbf{A}$ be both triangular and orthogonal, then $\mathbf{A}^\top\mathbf{A} = \mathbf{I}$, so $\mathbf{A}^{-1} = \mathbf{A}^\top$. From our prior proof, if $\mathbf{A}$ is upper triangular, then $\mathbf{A}^{-1}$ is also upper triangular. However, if $\mathbf{A}$ is upper triangular, then $\mathbf{A}^\top$ is lower triangular. The only possible way for $\mathbf{A}^{-1}$ to be both upper and lower triangular is if it is diagonal, thus $\mathbf{A}$ is diagonal. The case for lower triangular matrices is similar.

Question 4: p-norm inequalities (20 points)

Let $\mathbf{x}$ be a real m-vector, the vector p-norms $\|\mathbf{x}\|_p$ are related by various inequalities, often involving the dimension of the vector, i.e. m. For each of the following, prove the inequality and give an example of a nonzero vector $\mathbf{x}$ for which *equality* is satisfied.

(a) $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2$. (7 points)

(b) $\|\mathbf{x}\|_2 \le \sqrt{m} \cdot \|\mathbf{x}\|_\infty$. (7 points)

(c) Plot a 2D contour of $\|\mathbf{x}\|_\infty = 1$, on the same chart also highlight regions where $\|\mathbf{x}\|_2 < 1$, $\|\mathbf{x}\|_2 = 1$ and $\|\mathbf{x}\|_2 > 1$. (6 points)

Solution:

(a) $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2$

By the definition of the p-norms, we have:

$$\|\mathbf{x}\|_\infty = \max_i |x_i|$$
$$\|\mathbf{x}\|_2 = \sqrt{\sum_i x_i^2}$$

Since $\|\mathbf{x}\|_\infty = x_k$ for some k, we can see that:

$$\|\mathbf{x}\|_2 = \sqrt{\sum_i x_i^2} \ge \sqrt{|x_k|^2} = |x_k| = \|\mathbf{x}\|_\infty$$

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, where $x_1 = 1$ and all other $x_i \ldots x_m = 0$, then $\|\mathbf{x}\|_\infty = 1$ and $\|\mathbf{x}\|_2 = 1$, satisfying equality.

(b) Use the definitions again, but recognize that each $|x_i| \le \|\mathbf{x}\|_\infty$. Then:

$$\sum_{i=1}^m x_i^2 \le \sum_{i=1}^m \|\mathbf{x}\|_\infty^2 = m \cdot \|\mathbf{x}\|_\infty^2$$
$$\sqrt{\sum_{i=1}^m x_i^2} \le \sqrt{m \cdot \|\mathbf{x}\|_\infty^2}$$
$$\|\mathbf{x}\|_2 \le \sqrt{m} \cdot \|\mathbf{x}\|_\infty$$

Let $\mathbf{x} = \mathbf{1}$, a vector of all ones, then $\|\mathbf{x}\|_\infty = 1$ and $\|\mathbf{x}\|_2 = \sqrt{m}$, satisfying equality.

(c) Code + figure below:

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.patches import Patch

x = np.linspace(-1.5, 1.5, 500)
y = np.linspace(-1.5, 1.5, 500)
X, Y = np.meshgrid(x, y)

plt.figure(figsize=(8, 8))
```

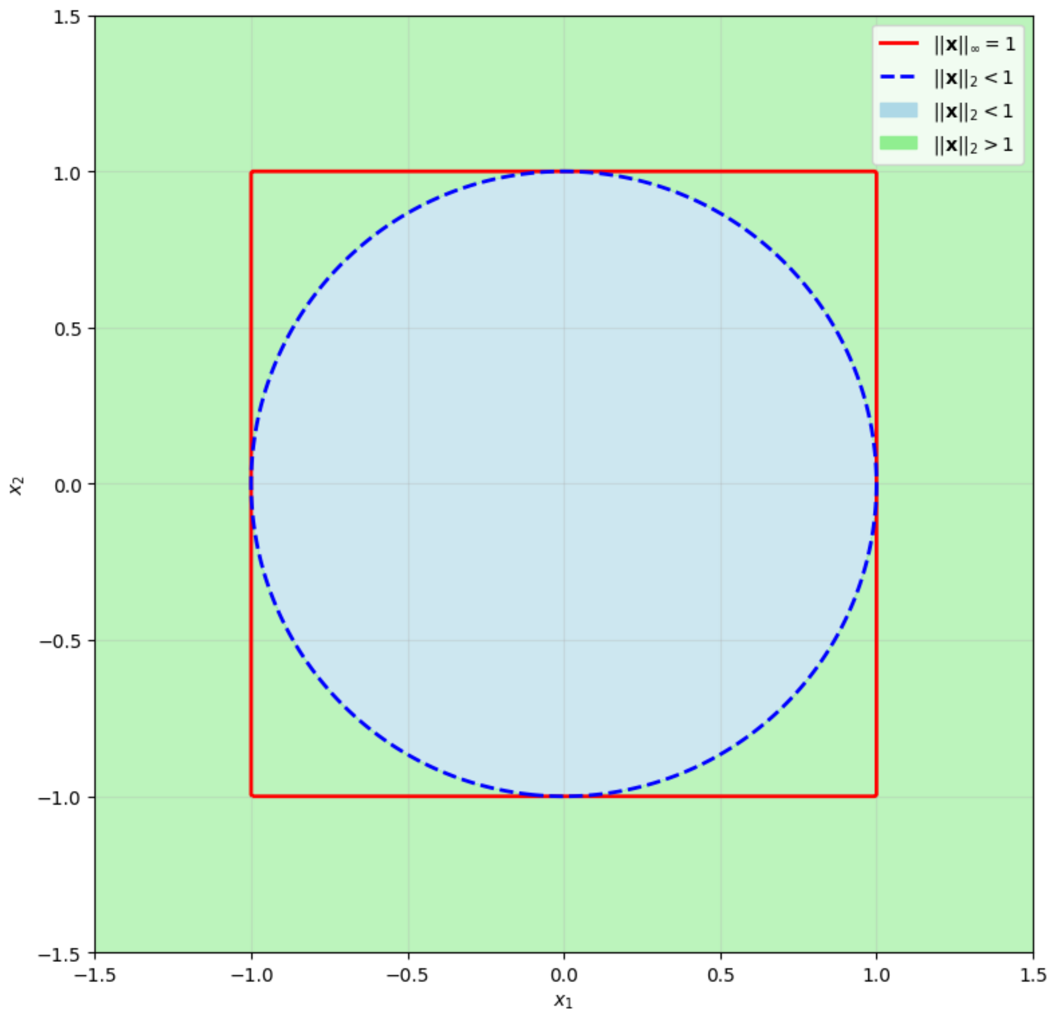
```python

norm_inf = np.maximum(np.abs(X), np.abs(Y))
norm_2 = np.sqrt(X**2 + Y**2)

plt.contour(X, Y, norm_inf, levels=[1], colors="red", linewidths=2)

plt.contourf(X, Y, norm_2, levels=[0, 1, 3], colors=["lightblue", "lightgreen"], alpha=0.6)
plt.contour(X, Y, norm_2, levels=[1], colors="blue", linewidths=2, linestyles="--")

plt.xlim(-1.5, 1.5)
plt.ylim(-1.5, 1.5)
plt.gca().set_aspect("equal")
plt.xlabel(r"$x_1$")
plt.ylabel(r"$x_2$")
plt.legend(handles=[
    plt.Line2D([0], [0], color="red", lw=2, label=r"$||\mathbf{x}||_\infty = 1$"),
    plt.Line2D([0], [0], color="blue", lw=2, linestyle="--", label=r"$||\mathbf{x}||_2 < 1$"),
    Patch(color="lightblue", label=r"$||\mathbf{x}||_2 < 1$"),
    Patch(color="lightgreen", label=r"$||\mathbf{x}||_2 > 1$")
], loc="upper right")
plt.tight_layout()
plt.grid(True, alpha=0.3)
plt.show()
```


$||\mathbf{x}||_\infty = 1$
$||\mathbf{x}||_2 < 1$
$||\mathbf{x}||_2 < 1$
$||\mathbf{x}||_2 > 1$
x_1
x_2
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

Question 5: Basic vector operations (20 points)

Given two 3-dimensional vectors $\mathbf{a}, \mathbf{b}$, and three matrices $A \in \mathbb{R}^{2\times3}, B \in \mathbb{R}^{3\times2}, C \in \mathbb{R}^{2\times3}$, scalars β_1, β_2 with the values below:

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\beta_{\mathbf{1}} = 4, \beta_{\mathbf{2}} = 5$$

(a) Compute the following operations by hand and show your work: (15 points)

(i) Vector operations: $\mathbf{a}+\mathbf{b}$, $\beta_1\mathbf{a}$, $\mathbf{a}\circ\mathbf{b}$, $\beta_1\mathbf{a}+\beta_2\mathbf{b}$ where $\circ$ denotes component-wise multiplication. (3 points)

(ii) Matrix operations: $\beta_1\mathbf{A}$, $\mathbf{A}+\mathbf{B}$, $\mathbf{A}+\mathbf{C}$. (3 points)

(iii) Transpose operations: $(\mathbf{AB})^\top$, $\mathbf{B}^\top\mathbf{A}^\top$, $(\mathbf{A}^\top)^\top$, $(\mathbf{A}+\mathbf{C})^\top$. (3 points)

(iv) Inner products and outer product: $\langle\mathbf{a},\mathbf{b}\rangle$, $\langle\mathbf{b},\mathbf{a}\rangle$, $\langle\mathbf{a},\mathbf{a}\rangle$, $\langle\mathbf{b},\mathbf{b}\rangle$, $\beta_1\langle\mathbf{a},\mathbf{b}\rangle$, $\langle\beta_1\mathbf{a},\mathbf{b}\rangle$, $\mathbf{b}\mathbf{a}^\top$. (3 points)

(v) Determinants: $\det(\mathbf{AB})$, $\det(\mathbf{BC})$. (3 points)

(b) Implement all the parts above using python (any programming language of your choice) and show the answers and code. (5 points)

Solution:

(a) (i) $\mathbf{a}+\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 3+4 \\ 5+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}$

$\beta_1\mathbf{a} = 4\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4\cdot1 \\ 4\cdot3 \\ 4\cdot5 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 20 \end{bmatrix}$

$\mathbf{a}\circ\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \circ \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1\cdot2 \\ 3\cdot4 \\ 5\cdot6 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 30 \end{bmatrix}$

$\beta_1\mathbf{a}+\beta_2\mathbf{b} = 4\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4\cdot1+5\cdot2 \\ 4\cdot3+5\cdot4 \\ 4\cdot5+5\cdot6 \end{bmatrix} = \begin{bmatrix} 4+10 \\ 12+20 \\ 20+30 \end{bmatrix} = \begin{bmatrix} 14 \\ 32 \\ 50 \end{bmatrix}$

(ii) $\beta_1\mathbf{A} = 4\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 4\cdot1 & 4\cdot2 & 4\cdot3 \\ 4\cdot2 & 4\cdot4 & 4\cdot6 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 8 & 16 & 24 \end{bmatrix}$

$\mathbf{A}+\mathbf{B}$ DNE bc dimensions are incompatible

$\mathbf{A}+\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+0 & 3+0 \\ 2+0 & 4+0 & 6+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix}$

(iii) $(\mathbf{AB})^\top = \left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}\right)^\top = \begin{bmatrix} 1\cdot7+2\cdot9+3\cdot11 & 1\cdot8+2\cdot10+3\cdot12 \\ 2\cdot7+4\cdot9+6\cdot11 & 2\cdot8+4\cdot10+6\cdot12 \end{bmatrix}^\top =$

$\begin{bmatrix} 58 & 64 \\ 116 & 128 \end{bmatrix}^\top = \begin{bmatrix} 58 & 116 \\ 64 & 128 \end{bmatrix}$

$\mathbf{B}^\top\mathbf{A}^\top = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}^\top \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}^\top = \begin{bmatrix} 7 & 9 & 11 \\ 8 & 10 & 12 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} =$

$$\begin{bmatrix} 7\cdot 1+9\cdot 2+11\cdot 3 & 7\cdot 2+9\cdot 4+11\cdot 6 \\ 8\cdot 1+10\cdot 2+12\cdot 3 & 8\cdot 2+10\cdot 4+12\cdot 6 \end{bmatrix} = \begin{bmatrix} 58 & 116 \\ 64 & 128 \end{bmatrix}$$

$$(\mathbf{A}^\top)^\top = \left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}^\top\right)^\top = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}^\top = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$$(\mathbf{A}+\mathbf{C})^\top = (\text{part (ii)})^\top = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix}^\top = \begin{bmatrix} 2 & 2 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}$$

(iv) $\langle \mathbf{a}, \mathbf{b}\rangle = \mathbf{a}^\top\mathbf{b} = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 1\cdot 2 + 3\cdot 4 + 5\cdot 6 = 44$

$$\mathbf{a}\otimes\mathbf{b} = \mathbf{a}\mathbf{b}^\top = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 & 1\cdot 4 & 1\cdot 6 \\ 3\cdot 2 & 3\cdot 4 & 3\cdot 6 \\ 5\cdot 2 & 5\cdot 4 & 5\cdot 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 12 & 18 \\ 10 & 20 & 30 \end{bmatrix}$$

$$\langle \mathbf{b}, \mathbf{a}\rangle = \mathbf{b}^\top\mathbf{a} = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = 2\cdot 1 + 4\cdot 3 + 6\cdot 5 = 44$$

$$\mathbf{b}\otimes\mathbf{a} = \mathbf{b}\mathbf{a}^\top = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2\cdot 3 & 2\cdot 5 \\ 4\cdot 1 & 4\cdot 3 & 4\cdot 5 \\ 6\cdot 1 & 6\cdot 3 & 6\cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 12 & 20 \\ 6 & 18 & 30 \end{bmatrix}$$

$$\langle \mathbf{a}, \mathbf{a}\rangle = \mathbf{a}^\top\mathbf{a} = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = 1\cdot 1 + 3\cdot 3 + 5\cdot 5 = 35$$

$$\mathbf{a}\otimes\mathbf{a} = \mathbf{a}\mathbf{a}^\top = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 & 1\cdot 3 & 1\cdot 5 \\ 3\cdot 1 & 3\cdot 3 & 3\cdot 5 \\ 5\cdot 1 & 5\cdot 3 & 5\cdot 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 9 & 15 \\ 5 & 15 & 25 \end{bmatrix}$$

$$\langle \mathbf{b}, \mathbf{b}\rangle = \mathbf{b}^\top\mathbf{b} = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 2\cdot 2 + 4\cdot 4 + 6\cdot 6 = 56$$

$$\mathbf{b}\otimes\mathbf{b} = \mathbf{b}\mathbf{b}^\top = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 & 2\cdot 4 & 2\cdot 6 \\ 4\cdot 2 & 4\cdot 4 & 4\cdot 6 \\ 6\cdot 2 & 6\cdot 4 & 6\cdot 6 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 8 & 16 & 24 \\ 12 & 24 & 36 \end{bmatrix}$$

$\beta_1\langle \mathbf{a}, \mathbf{b}\rangle = 4\cdot 44 = 176$

$$\beta_1(\mathbf{a}\otimes\mathbf{b}) = 4\begin{bmatrix} 2 & 4 & 6 \\ 6 & 12 & 18 \\ 10 & 20 & 30 \end{bmatrix} = \begin{bmatrix} 8 & 16 & 24 \\ 24 & 48 & 72 \\ 40 & 80 & 120 \end{bmatrix}$$

$$\langle \beta_1\mathbf{a}, \mathbf{b}\rangle = (4\mathbf{a})^\top\mathbf{b} = \left(4\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}\right)^\top \begin{bmatrix} 4 & 12 & 20 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 4\cdot 2 + 12\cdot 4 + 20\cdot 6 = 176$$

$$\beta_1\mathbf{a}\otimes\mathbf{b} = (4\mathbf{a})\mathbf{b}^\top = \left(4\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}\right) \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 20 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 4\cdot 2 & 4\cdot 4 & 4\cdot 6 \\ 12\cdot 2 & 12\cdot 4 & 12\cdot 6 \\ 20\cdot 2 & 20\cdot 4 & 20\cdot 6 \end{bmatrix} =$$

$$\begin{bmatrix} 8 & 16 & 24 \\ 24 & 48 & 72 \\ 40 & 80 & 120 \end{bmatrix}$$

$\langle \mathbf{b}, \mathbf{a}^\top\rangle = \mathbf{b}^\top\mathbf{a}^\top =$ DNE bc dimensions are incompatible

$\mathbf{b}\otimes\mathbf{a}^\top = \mathbf{b}(\mathbf{a}^\top)^\top = \mathbf{b}\mathbf{a} =$ DNE bc dimensions are incompatible

(v) $\det(\mathbf{AB}) = \det\left(\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{bmatrix}\begin{bmatrix}7 & 8\\ 9 & 10\\ 11 & 12\end{bmatrix}\right) = \det\left(\begin{bmatrix}58 & 64\\ 116 & 128\end{bmatrix}\right) = 58 \cdot 128 - 64 \cdot 116 = 7424 - 7424 = 0$

$\det(\mathbf{BC}) = \det\left(\begin{bmatrix}7 & 8\\ 9 & 10\\ 11 & 12\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = \det\left(\begin{bmatrix}7 & 8 & 0\\ 9 & 10 & 0\\ 11 & 12 & 0\end{bmatrix}\right) = 7\det\left(\begin{bmatrix}10 & 0\\ 12 & 0\end{bmatrix}\right) - 8\det\left(\begin{bmatrix}9 & 0\\ 11 & 0\end{bmatrix}\right) + 0 = 7(10 \cdot 0 - 0 \cdot 12) - 8(9 \cdot 0 - 0 \cdot 11) + 0 = 0 - 0 + 0 = 0$

(b) Code:

```
a = np.array([1, 3, 5])
b = np.array([2, 4, 6])
A = np.array([[1, 2, 3],
              [2, 4, 6]])
B = np.array([[7, 8],
              [9, 10],
              [11, 12]])
C = np.array([[1, 0, 0],
              [0, 0, 1]])
beta_1 = 4
beta_2 = 5
```

```
a + b
```

```
array([ 3,  7, 11])
```

```
beta_1 * a
```

```
array([ 4, 12, 20])
```

```
a * b
```

```
array([ 2, 12, 30])
```

```
beta_1 * a + beta_2 * b
```

```
array([14, 32, 50])
```

```
beta_1 * A
```

```
array([[ 4,  8, 12],
       [ 8, 16, 24]])
```

```
A + B
```

Successful indication of DNE:

```
ValueError: operands could not be broadcast together with shapes (2,3) (3,2)
```

```
A + C
```

```
array([[2, 2, 3],
       [2, 4, 7]])
```

```python
(A @ B).T
```

```
array([[ 58, 116],
       [ 64, 128]])
```

```python
B.T @ A.T
```

```
array([[ 58, 116],
       [ 64, 128]])
```

```python
(A.T).T
```

```
array([[1, 2, 3],
       [2, 4, 6]])
```

```python
(A + C).T
```

```
array([[2, 2],
       [2, 4],
       [3, 7]])
```

```python
def inner_outer(a, b, mult=1):
    print(mult * np.inner(a, b))
    print(mult * np.outer(a, b))

inner_outer(a, b)
inner_outer(b, a)
inner_outer(a, a)
inner_outer(b, b)
inner_outer(a, b, beta_1)
inner_outer(beta_1 * a, b)
```

```
44
[[ 2  4  6]
 [ 6 12 18]
 [10 20 30]]
44
[[ 2  6 10]
 [ 4 12 20]
 [ 6 18 30]]
35
[[ 1  3  5]
 [ 3  9 15]
 [ 5 15 25]]
56
[[ 4  8 12]
 [ 8 16 24]
 [12 24 36]]
176
[[  8  16  24]
 [ 24  48  72]
 [ 40  80 120]]
```

```
176
[[  8  16  24]
 [ 24  48  72]
 [ 40  80 120]]
```

We need to hard reshape the vectors for numpy to not throw an error as the internal implementation of inner and outer products expect 1D arrays for vectors. *np.outer* will always return a matrix, so we use @ (matmul) operator to replicate the true intended outer product calculation and get the expected error.

```
np.inner(b.reshape(3, 1), a.reshape(3, 1).T)
```

```
ValueError: shapes (3,1) and (3,1) not aligned: 1 (dim 1) != 3 (dim 0)
```

```
b.reshape(3, 1).T @ a.reshape(3, 1).T
```

```
ValueError: matmul: Input operand 1 has a mismatch in its core dimension 0,
with gufunc signature (n?,k),(k,m?)->(n?,m?) (size 1 is different from 3)
```

```
np.linalg.det(A @ B)
```

```
np.float64(0.0)
```

```
np.linalg.det(B @ C)
```

```
np.float64(0.0)
```