HW 1

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1

Let event A be the event that in a group of n people, no one shares a birthday. Let event B be the complement of event A, such that in a group of n people, at least 2 people share a birthday. Thus, $P(B) = 1 - P(A)$. We can calculate $P(A)$ by starting with the first person, who can have any birthday. The next person must have a different birthday, thus have a $\frac{364}{365}$ chance of not sharing a birthday with the first person. This goes on until the nth person, who has a $\frac{365-n+1}{365}$ chance of not sharing a birthday with any of the previous $n-1$ people. Thus, we have:

$$P(A) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-n+1}{365} = \frac{1}{365^n} \cdot \frac{365 \cdot 364 \cdots (365-n+1)}{1}$$

$$\text{Recognize that } \frac{365 \cdot 364 \cdots (365-n+1)}{1} = \frac{365!}{(365-n)!} \therefore$$

$$P(A) = \frac{365!}{365^n (365-n)!}$$

We want to find a minimum n such that $P(B) \geq 0.5 \therefore P(A) < 0.5$. Solving $P(A) < 0.5$ for n, we find that the minimum n is 23, where $P(A) \approx 0.493$ and $P(B) \approx 0.507$.

Thus, in a group of n unrelated individuals, the probability that at least 2 people share a birthday is $P(B) = 1 - \frac{365!}{365^n(365-n)!}$. This probability exceeds 0.5 when $n \geq 23$.

2

2.1

Each bag has a probability of $\frac{1}{3}$ of being chosen. The probability of drawing a blue marble at random is thus the sum of the probabilities of drawing a blue marble from each bag multiplied by the probability of choosing that bag. Thus, we have:

$$P(\text{blue marble}) = \frac{1}{3} \cdot \frac{25}{100} + \frac{1}{3} \cdot \frac{40}{100} + \frac{1}{3} \cdot \frac{55}{100} = \frac{2}{5} = 0.4$$

2.2

If the first bag has the probability of 0.5 of being chosen, the other two bags then have a probability of 0.25 of being chosen. Thus, we now have:

$$P(\text{blue marble}) = 0.5 \cdot \frac{25}{100} + 0.25 \cdot \frac{40}{100} + 0.25 \cdot \frac{55}{100} = \frac{29}{80} = 0.3625$$

3

For the gambler's current fortune of k dollars, they can either win or lose on the next gamble. Thus by the law of total probability, we have:

$$\begin{aligned} q_k &= \frac{1}{2} \cdot q_{k+1} + \frac{1}{2} \cdot q_{k-1} \\ 2q_k &= q_{k+1} + q_{k-1} \\ q_{k+1} - q_k &= q_k - q_{k-1} \end{aligned}$$

This shows that the difference between consecutive q values is constant, thus q_k is linear and can express q_k as $q_k = A + Bk$ for some constants A, B. We solve for A, B using the boundary conditions where $q_0 = 1$ and $q_N = 0$:

$$\begin{aligned} q_0 &= A + B \cdot 0 \therefore A = 1 \\ q_N &= 1 + BN = 0 \therefore B = -\frac{1}{N} \\ \therefore q_k &= 1 - \frac{k}{N} \end{aligned}$$

4

For a given element x in Ω, the probability of x being in either set A or B is equally likely because A and B are independently selected subsets of Ω. Thus, for any element x to satisfy $A \subseteq B$, element x must either be:

- $x \notin A$ & $x \notin B$
- $x \notin A$ & $x \in B$
- $x \in A$ & $x \in B$

The only case that does not satisfy $A \subseteq B$ is if $x \in A$ & $x \notin B$. Since each of the 4 cases are equally likely, the probability of x satisfying $A \subseteq B$ is $\frac{3}{4}$. Since this must be true for all n elements in Ω, the probability of $A \subseteq B$ is consequently:

$$P(A \subseteq B) = \left(\frac{3}{4}\right)^n$$

5

5.1

Each coin is equally likely to be chosen with a probability of $\frac{1}{3}$. The probability of getting tails when a coin is flipped is thus the sum of the probabilities of getting tails with each coin multiplied by the probability of choosing that coin. Thus, we have:

$$P(\mathrm{T}) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 = \frac{1}{3}$$

5.2

$$\begin{aligned} P(\text{fake coin}|\mathrm{H}) &= \frac{P(\mathrm{H}|\text{fake coin}) \cdot P(\text{fake coin})}{P(\mathrm{H})} \\ &= \frac{1 \cdot \frac{1}{3}}{\frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1} = \frac{1}{2} \end{aligned}$$

6

Any joint distribution function must satisfy:

1. $F(-\infty, x_2) = F(x_1, -\infty) = 0$
2. $F(\infty, \infty) = 0$
3. $F(x_1, \infty) = F(x_1)$ & $F(\infty, x_2) = F(x_2)$
4. monotonicity: $F(a_1, a_2) \leq F(b_1, b_2)$ if $a_1 \leq b_1$ & $a_2 \leq b_2$
5. $P(a_1 < X \leq b_1, a_2 < Y \leq b_2) = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2) \geq 0$

6.1

For $F(x_1, x_2) = 1 - e^{-x_1 - x_2}$ if $x_1, x_2 \geq 0$:

1. $F(-\infty, x_2) = F(x_1, -\infty) = 0$ is satisfied as $x_1, x_2 \geq 0$

2. $F(\infty, \infty) = 1 - e^{-\infty-\infty} = 1 - 0 = 1$ is satisfied

3. $F(x_1, \infty) = 1 - e^{-x_1-\infty} = 1 - 0 = 1$ and $F(\infty, x_2) = 1 - e^{-\infty-x_2} = 1 - 0 = 1$ is satisfied

4. monotonicity is satisfied as e^{-x} is a decreasing function

5. $P(a_1 < X \leq b_1, a_2 < Y \leq b_2) = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$
   $= (1 - e^{-b_1-b_2}) - (1 - e^{-a_1-b_2}) - (1 - e^{-b_1-a_2}) + (1 - e^{-a_1-a_2})$
   Let $a_1 = 0, a_2 = 0, b_1 = t_1 > 0, b_2 = t_2 > 0$, then:
   $P(0 < X \leq t_1, 0 < Y \leq t_2) = (1 - e^{-t_1-t_2}) - (1 - e^{0-t_2}) - (1 - e^{-t_1+0}) + (1 - e^{0+0})$
   $= 1 - e^{-t_1-t_2} - 1 + e^{-t_2} - 1 + e^{-t_1} = e^{-t_2} + e^{-t_1} - e^{-t_1-t_2} - 1$
   $= (e^{-t_1} - 1)(1 - e^{-t_2})$
   $(e^{-t_1} - 1) < 0$ and $(1 - e^{-t_2}) > 0$, therefore $P(0 < X \leq t_1, 0 < Y \leq t_2) < 0$ which violates condition 5.

Thus, $F(x_1, x_2) = 1 - e^{-x_1-x_2}$ if $x_1, x_2 \geq 0$ is not a valid joint distribution function.

6.2

For $F(x_1, x_2) = 1 - e^{-\min(x_1,x_2)} - \min(x_1, x_2)e^{-\min(x_1,x_2)}$ if $x_1, x_2 \geq 0$:

1. $F(-\infty, x_2) = F(x_1, -\infty) = 0$ is satisfied as $x_1, x_2 \geq 0$

2. $F(\infty, \infty) = 1 - e^{-\infty} - \infty \cdot e^{-\infty} = 1 - 0 - 0 = 1$ is satisfied

3. $F(x_1, \infty) = 1 - e^{-x_1} - x_1 e^{-x_1} = F(x_1)$ and $F(\infty, x_2) = 1 - e^{-x_2} - x_2 e^{-x_2} = F(x_2)$ is satisfied

4. monotonicity is satisfied as e^{-x} is a decreasing function and $\min(x_1, x_2)$ is increasing

5. $P(a_1 < X \leq b_1, a_2 < Y \leq b_2)$ Again, let $a_1 = 0, a_2 = 0, b_1 = t, b_2 = t$ where $t > 0$, then:
   $P(0 < X \leq t, 0 < Y \leq t) = F(t, t) - F(0, t) - F(t, 0) + F(0, 0)$
   $= (1 - e^{-t} - te^{-t}) - (1 - e^0 - 0 \cdot e^0) - (1 - e^0 - 0 \cdot e^0) + (1 - e^0 - 0 \cdot e^0)$
   $= 1 - e^{-t} - te^{-t}$, which has a minimum of 0 when $t = 0$ and is positive for all $t > 0$ as know from condition 4.

Thus, $F(x_1, x_2) = 1 - e^{-\min(x_1,x_2)} - \min(x_1, x_2)e^{-\min(x_1,x_2)}$ if $x_1, x_2 \geq 0$ is a valid joint distribution function. Marginal distribution functions $F_1(x_1)$ and $F_2(x_2)$ are determined in condition 3.

7

For $\max(X, Y)$, both X and Y must be less than or equal to some value t for $\max(X, Y)$ to be possible. Thus:

$$P(\max(X,Y) \leq t) = P(X \leq t, Y \leq t)$$
$$\text{By independence, } P(X \leq t, Y \leq t) = P(X \leq t)P(Y \leq t)$$
$$\therefore F_{\max}(t) = F(t)G(t)$$

For $\min(X, Y)$, either X or Y must be greater than to some value t for $\min(X, Y)$ to be possible. Thus:

$$P(\min(X,Y) > t) = P(X > t, Y > t)$$
$$\text{By independence, } P(X > t, Y > t) = P(X > t)P(Y > t)$$
$$\text{By law of total probability, } P(X > t) = 1 - P(X \leq t) = 1 - F(t)$$
$$\text{Likewise, } P(Y > t) = 1 - G(t)$$
$$\therefore P(\min(X,Y) > t) = (1 - F(t))(1 - G(t))$$
$$\text{By law of total probability, } P(\min(X,Y) \leq t) = 1 - P(\min(X,Y) > t)$$
$$\therefore F_{\min}(t) = 1 - (1 - F(t))(1 - G(t)) = F(t) + G(t) - F(t)G(t)$$