HW 2

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1

Define deviation $D = E[(X-c)^2]$ for some constant c. We want to find the value of c that minimizes D. We can expand D as follows:

$$D = E[(X-c)^2] = E[X^2 - 2cX + c^2] = E[X^2] - 2cE[X] + c^2$$

To find the minimum value of D, we can take the derivative of D with respect to c and set it equal to 0:

$$\frac{dD}{dc} = -2E[X] + 2c = 0 \therefore c = E[X]$$

Thus, the D is minimized when $c = E[X]$.

2

Note that each jump is independent and identically distributed. For the current particle's current position k, it can either hop left or right with probability p and $1-p$ respectively. Thus, we can express the expected position $E[X_n]$ after n jumps as follows:

$$E[X_n] = \sum_{k=1}^{n} E[X_k] = nE[X_k]$$
$$E[X_k] = (-1)p + (1)(1-p) = 1 - 2p$$
$$\therefore E[X_n] = n(1-2p)$$

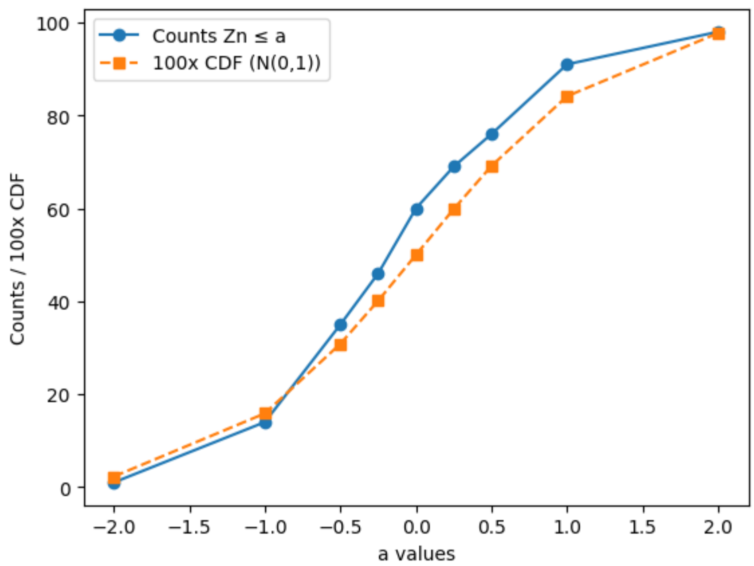
Likewise, we can calculate the variance $Var(X_n)$ as:

$$Var(X_n) = \sum_{k=1}^{n} Var(X_k) = nVar(X_k)$$
$$Var(X_k) = E[X_k^2] - (E[X_k])^2$$
$$E[X_k^2] = (-1)^2 p + (1)^2(1-p) = 1$$
$$Var(X_k) = 1 - (1-2p)^2 = 4p(1-p)$$
$$\therefore Var(X_n) = 4np(1-p)$$

3

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

np.random.seed(0)

X_bar_n = np.mean(np.random.uniform(0, 1, 100))
mu_X = 0.5
var_X_bar = (1/12)/100
std_X_bar = np.sqrt(var_X_bar)
print("Mean:", mu_X)
print("Var:", var_X_bar)

a_vals = [-2, -1, -0.5, -0.25, 0, 0.25, 0.5, 1, 2]
z_vals = []
for _ in range(100):
    sample_mean = np.mean(np.random.uniform(0, 1, 100))
    z_vals.append((sample_mean - mu_X) / np.sqrt(var_X / 100))

counts = [int((np.array(z_vals) <= a).sum()) for a in a_vals]
cdf = 100 * norm.cdf(a_vals)

plt.plot(a_vals, counts, "o-", label="Counts Zn <= a")
plt.plot(a_vals, cdf, "s--", label="100x CDF (N(0,1))")
plt.legend()
plt.xlabel("a values")
plt.ylabel("Counts / 100x CDF")
plt.show()
```

```
Mean: 0.5
Var: 0.0008333333333333333
```

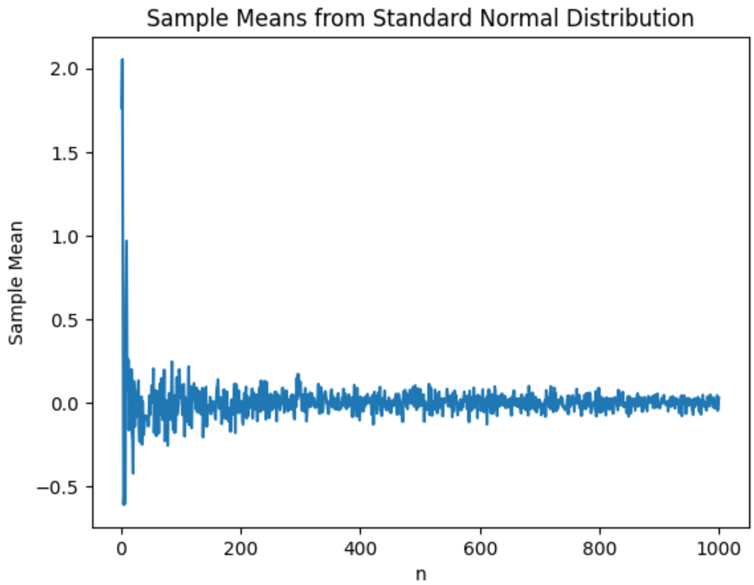


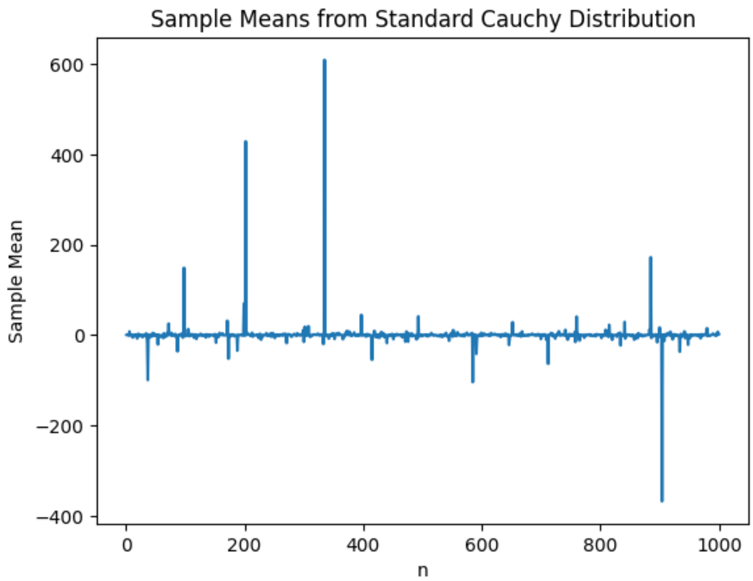
Because each X is IID uniform(0, 1), we have $E[X_n] = \mu_X = 0.5$. The variance of the sample mean is likewise $Var(X_n) = \sigma^2/n = (1/12)/100$. The empirical counts of $Z_n \leq a$ closely follow the CDF of the standard normal distribution, which aligns with the Central Limit Theorem. As n increases, the distribution of sample means approaches a normal distribution, regardless of the original distribution of the data (in this case, uniform).

4

```
import numpy as np
import matplotlib.pyplot as plt

np.random.seed(0)

Xbars_normal = []
Xbars_cauchy = []
for i in range(1000):
    Xbars_normal.append(np.mean(np.random.standard_normal(size=i + 1)))
    Xbars_cauchy.append(np.mean(np.random.standard_cauchy(size=i + 1)))

def plot(Xbars, title):
    plt.plot(range(1, 1001), Xbars)
    plt.xlabel("n")
    plt.ylabel("Sample Mean")
    plt.title(title)
    plt.show()

```

```
plot(Xbars_normal, "Sample Means from Standard Normal
    Distribution")
plot(Xbars_cauchy, "Sample Means from Standard Cauchy
    Distribution")
```





The Cauchy distribution of sample means differs from the normal distribution as the Cauchy distribution has no finite mean (expectation is undefined). Note:

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx \\
&\approx \int_{-\infty}^{\infty} \frac{1}{x} dx = \text{undefined}
\end{aligned}
$$