HW 4

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1

For Poisson(λ), we have $E[X] = \lambda$ and $Var(X) = \lambda$. Thus, the method of moments estimator is $\hat{\lambda}_{MM} = \bar{X}$. We can calculate the MLE as follows:

$$
\begin{aligned}
P(X = x) &= \frac{e^{-\lambda}\lambda^x}{x!} \\
L(\lambda) &= \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \\
\ell(\lambda) &= \log(L(\lambda)) = \sum_{i=1}^{n} (-\lambda + x_i \log(\lambda) - \log(x_i!)) \\
\ell(\lambda) &= -n\lambda + \log(\lambda) \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \log(x_i!) \\
\frac{d\ell}{d\lambda} &= -n + \frac{1}{\lambda} \sum_{i=1}^{n} x_i = 0 \\
\hat{\lambda}_{MLE} &= \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{X}
\end{aligned}
$$

We can calculate the Fisher information $J(\lambda)$ as follows:

$$
\begin{aligned}
\ell(\lambda) &= x \log(\lambda) - \lambda - \log(x!) \\
\frac{d\ell}{d\lambda} &= \frac{x}{\lambda} - 1 \\
\frac{d^2\ell}{d\lambda^2} &= -\frac{x}{\lambda^2} \\
J(\lambda) &= -E\left[\frac{d^2\ell}{d\lambda^2}\right] = -E\left[-\frac{X}{\lambda^2}\right] = \frac{E[X]}{\lambda^2} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}
\end{aligned}
$$

2

2.1

The Fisher information of $f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}}$ is calculated as follows:

$$\begin{aligned}
\ell(\theta) &= -\frac{1}{2}\log(2\pi\theta) - \frac{x^2}{2\theta} \\
\frac{d\ell}{d\theta} &= -\frac{1}{2\theta} + \frac{x^2}{2\theta^2} \\
\frac{d^2\ell}{d\theta^2} &= \frac{1}{2\theta^2} - \frac{x^2}{\theta^3} \\
J(\theta) &= -E\left[\frac{d^2\ell}{d\theta^2}\right] = -E\left[\frac{1}{2\theta^2} - \frac{X^2}{\theta^3}\right] = -\left[\frac{1}{2\theta^2} - \frac{\theta}{\theta^3}\right] = \frac{1}{2\theta^2}
\end{aligned}$$

The Cramer-Rao bound on the MSE is given by:

$$\text{MSE}(\hat{\theta}) \geq \frac{1}{nJ(\theta)} = \frac{1}{n \cdot \frac{1}{2\theta^2}} = \frac{2\theta^2}{n}$$

2.2

The fisher information of $f(x;\theta) = \theta e^{-\theta x}$ is calculated as follows:

$$\begin{aligned}
\ell(\theta) &= \log(\theta) - \theta x \\
\frac{d\ell}{d\theta} &= \frac{1}{\theta} - x \\
\frac{d^2\ell}{d\theta^2} &= -\frac{1}{\theta^2} \\
J(\theta) &= -E\left[\frac{d^2\ell}{d\theta^2}\right] = -E\left[-\frac{1}{\theta^2}\right] = \frac{1}{\theta^2}
\end{aligned}$$

The Cramer-Rao bound on the MSE is given by:

$$\text{MSE}(\hat{\theta}) \geq \frac{1}{nJ(\theta)} = \frac{1}{n \cdot \frac{1}{\theta^2}} = \frac{\theta^2}{n}$$

3

Given $X_1 \dots X_n$ iid from some distribution, the empirical CDF is:

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I\{X_i \leq x\}$$

Then for each distinct point x, y, we have:

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I\{X_i \leq x\}$$
$$\hat{F}_n(y) = \frac{1}{n}\sum_{i=1}^{n} I\{X_i \leq y\}$$

So we can calculate the covariance as follows:

$$\begin{aligned}
Cov(\hat{F}_n(x), \hat{F}_n(y)) &= E[\hat{F}_n(x)\hat{F}_n(y)] - E[\hat{F}_n(x)]E[\hat{F}_n(y)] \\
&= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} I\{X_i \leq x\}\right)\left(\frac{1}{n}\sum_{j=1}^{n} I\{X_j \leq y\}\right)\right] - F(x)F(y) \\
&= \frac{1}{n^2}E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} I\{X_i \leq x\}I\{X_j \leq y\}\right] - F(x)F(y) \\
&= \frac{1}{n^2}\left(\sum_{i=j} E[I\{X_i \leq x\}I\{X_i \leq y\}] + \sum_{i\neq j} E[I\{X_i \leq x\}I\{X_j \leq y\}]\right) - F(x)F(y) \\
&= \frac{1}{n^2}\left(nF(\min(x,y)) + n(n-1)F(x)F(y)\right) - F(x)F(y) \\
&= \frac{1}{n}F(\min(x,y)) + \frac{n-1}{n}F(x)F(y) - F(x)F(y) \\
&= \frac{1}{n}F(\min(x,y)) - \frac{1}{n}F(x)F(y) \\
&= \frac{1}{n}\left(F(\min(x,y)) - F(x)F(y)\right)
\end{aligned}$$

4

4.1

The observed X has a mixed density:

$$f_\theta(x) = \theta f_1(x) + (1-\theta)f_0(x)$$

The Fisher information $J(\theta)$ is calculated as follows:

$$\ell(\theta; x) = \log(f_\theta(x)) = \log[\theta f_1(x) + (1-\theta)f_0(x)]$$
$$S(\theta; x) = \frac{d\ell}{d\theta} = \frac{f_1(x) - f_0(x)}{\theta f_1(x) + (1-\theta)f_0(x)}$$
$$\begin{aligned}
J(\theta) = E[S(\theta; X)^2] &= \int \left(\frac{f_1(x) - f_0(x)}{\theta f_1(x) + (1-\theta)f_0(x)}\right)^2 (\theta f_1(x) + (1-\theta)f_0(x))dx \\
&= \int \frac{(f_1(x) - f_0(x))^2}{\theta f_1(x) + (1-\theta)f_0(x)}dx
\end{aligned}$$

4.2

The Cramer-Rao lower bound on the MSE of an unbiased estimate of θ is given by:

$$\text{MSE}(\hat{\theta}) \geq \frac{1}{nJ(\theta)} = \frac{1}{n \int \frac{(f_1(x)-f_0(x))^2}{\theta f_1(x)+(1-\theta)f_0(x)} dx}$$

4.3

We want an unbiased estimator of θ. We want to then find $h(X)$ such that $E_\theta[h(X)] = \theta$:

$$\theta \int h(x) f_1(x) dx + (1-\theta) \int h(x) f_0(x) dx = \theta$$

Let $a = \int h(x)f_1(x)dx$ and $b = \int h(x)f_0(x)dx$. Then we have:

$$\theta a + (1-\theta)b = \theta$$
$$b + \theta(a-b) = \theta$$
$$b = 0, a = 1$$

So then $h(X)$ must satisfy:

$$\int h(x) f_0(x) dx = 0$$
$$\int h(x) f_1(x) dx = 1$$

We know $h(X)$ must be a combination of f_0 and f_1. Thus, we can let:

$$h(X) = af_1(X) + bf_0(X)$$
$$\int (af_1(x) + bf_0(x)) f_0(x) dx = 0 \implies a \int f_1(x) f_0(x) dx + b \int f_0(x)^2 dx = 0$$
$$\int (af_1(x) + bf_0(x)) f_1(x) dx = 1 \implies a \int f_1(x)^2 dx + b \int f_1(x) f_0(x) dx = 1$$

Let $A = \int f_1(x)^2, B = \int f_1(x)f_0(x), C = \int f_0(x)^2$. Then we have the system of equations:

$$
\begin{gathered}
aB + bC = 0 \\
aA + bB = 1 \\
\therefore \\
a = \frac{C}{AC - B^2}, b = -\frac{B}{AC - B^2} \\
\therefore h(X) = \frac{Cf_1(X) - Bf_0(X)}{AC - B^2} \\
\hat{\theta}(X) = \frac{\left(\int f_0(x)^2 dx\right) f_1(X) - \left(\int f_1(x)f_0(x)dx\right) f_0(X)}{\left(\int f_1(x)^2 dx\right)\left(\int f_0(x)^2 dx\right) - \left(\int f_1(x)f_0(x)dx\right)^2}
\end{gathered}
$$