

## HW 3

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### 1

Let  $X_1, X_2, \dots, X_n$  be IID such that  $X_i$  is uniform in  $[0, \theta]$ .

#### 1.1

For  $\hat{\theta} = \max(X_1, X_2, \dots, X_n)$ :

$$\begin{aligned} F_{\hat{\theta}}(x) &= P(\hat{\theta} \leq x) = P(X_1 \leq x, X_2 \leq x, \dots, X_n \leq x) \\ &= P(X_1 \leq x)P(X_2 \leq x) \cdots P(X_n \leq x) = \left(\frac{x}{\theta}\right)^n \text{ for } 0 \leq x \leq \theta \\ \therefore f_{\hat{\theta}}(x) &= \frac{d}{dx} F_{\hat{\theta}}(x) = \frac{d}{dx} \left(\frac{x}{\theta}\right)^n = \frac{n}{\theta^n} x^{n-1} \end{aligned}$$

The bias is  $E[\hat{\theta}] - \theta$ . Then expectation is:

$$\begin{aligned} E[\hat{\theta}] &= \int_0^\theta x f_{\hat{\theta}}(x) dx \\ &= \int_0^\theta x \cdot \frac{n}{\theta^n} x^{n-1} dx \\ &= \frac{n}{\theta^n} \int_0^\theta x^n dx \\ &= \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} \\ &= \frac{n}{n+1} \theta \end{aligned}$$

Thus, the bias is:

$$\text{Bias}(\hat{\theta}) = E[\hat{\theta}] - \theta = \frac{n}{n+1} \theta - \theta = -\frac{\theta}{n+1}$$

The standard error is:

$$\begin{aligned} \text{se}(\hat{\theta}) &= \sqrt{\text{var}(\hat{\theta})} \\ &= \sqrt{E[\hat{\theta}^2] - (E[\hat{\theta}])^2} \end{aligned}$$

$E[\hat{\theta}^2]$  is:

$$\begin{aligned}
 E[\hat{\theta}^2] &= \int_0^\theta x^2 f_{\hat{\theta}}(x) dx \\
 &= \int_0^\theta x^2 \cdot \frac{n}{\theta^n} x^{n-1} dx \\
 &= \frac{n}{\theta^n} \int_0^\theta x^{n+1} dx \\
 &= \frac{n}{\theta^n} \cdot \frac{\theta^{n+2}}{n+2} \\
 &= \frac{n}{n+2} \theta^2
 \end{aligned}$$

Thus, the standard error is:

$$\begin{aligned}
 \text{se}(\hat{\theta}) &= \sqrt{\frac{n}{n+2} \theta^2 - \left( \frac{n}{n+1} \theta \right)^2} \\
 &= \theta \sqrt{\frac{n}{n+2} - \frac{n^2}{(n+1)^2}} \\
 &= \theta \sqrt{\frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}} \\
 &= \theta \sqrt{\frac{n((n^2 + 2n + 1) - (n^2 + 2n))}{(n+2)(n+1)^2}} \\
 &= \theta \sqrt{\frac{n}{(n+2)(n+1)^2}}
 \end{aligned}$$

The mean squared error is:

$$\begin{aligned}
\text{MSE}(\hat{\theta}) &= E[(\hat{\theta} - \theta)^2] \\
&= \text{var}(\hat{\theta}) + (\text{Bias}(\hat{\theta}))^2 \\
&= \left( \theta^2 \frac{n}{(n+2)(n+1)^2} \right) + \left( -\frac{\theta}{n+1} \right)^2 \\
&= \theta^2 \left( \frac{n}{(n+2)(n+1)^2} + \frac{1}{(n+1)^2} \right) \\
&= \theta^2 \left( \frac{n + (n+2)}{(n+2)(n+1)^2} \right) \\
&= \theta^2 \left( \frac{2n+2}{(n+2)(n+1)^2} \right) \\
&= \theta^2 \left( \frac{2(n+1)}{(n+2)(n+1)^2} \right) \\
&= \theta^2 \left( \frac{2}{(n+2)(n+1)} \right)
\end{aligned}$$

## 1.2

For  $\hat{\theta} = 2X_n = \frac{2}{n} \sum_{i=1}^n X_i$ :

The bias is  $E[\hat{\theta}] - \theta$ . Then expectation is:

$$\begin{aligned}
E[\hat{\theta}] &= E \left[ \frac{2}{n} \sum_{i=1}^n X_i \right] \\
&= \frac{2}{n} \sum_{i=1}^n E[X_i] \\
&= \frac{2}{n} \cdot n \cdot \frac{\theta}{2} \\
&= \theta
\end{aligned}$$

Thus, the bias is:

$$\text{Bias}(\hat{\theta}) = E[\hat{\theta}] - \theta = \theta - \theta = 0.$$

The standard error is:

$$\begin{aligned}
\text{se}(\hat{\theta}) &= \sqrt{\text{var}(\hat{\theta})} \\
&= \sqrt{\text{var}\left(\frac{2}{n} \sum_{i=1}^n X_i\right)} \\
&= \frac{2}{n} \sqrt{\text{var}\left(\sum_{i=1}^n X_i\right)} \\
&= \frac{2}{n} \sqrt{\sum_{i=1}^n \text{var}(X_i)} \\
&= \frac{2}{n} \sqrt{n \cdot \text{var}(X_i)} \\
&= \frac{2}{n} \sqrt{n \cdot \frac{\theta^2}{12}} \\
&= \frac{2}{n} \cdot \theta \cdot \sqrt{\frac{n}{12}} \\
&= \frac{\theta}{\sqrt{3n}}
\end{aligned}$$

Note that since the bias is 0, the MSE is just the variance:

$$\begin{aligned}
\text{MSE}(\hat{\theta}) &= \text{var}(\hat{\theta}) \\
&= \left(\frac{\theta}{\sqrt{3n}}\right)^2 \\
&= \frac{\theta^2}{3n}
\end{aligned}$$

## 2

Let  $X_1, \dots, X_n$  be IID with  $X_i$  being uniform in  $[a, b]$ .

### 2.1

The method of moments estimator for  $a$  is:

$$\begin{aligned}
E[X_i] &= \frac{a+b}{2} \\
\bar{X} &= \frac{a+b}{2} \\
\therefore \hat{a} &= 2\bar{X} - b
\end{aligned}$$

The method of moments estimator for  $b$  is:

$$\begin{aligned} E[X_i^2] &= \frac{a^2 + ab + b^2}{3} \\ \overline{X^2} &= \frac{a^2 + ab + b^2}{3} \\ \therefore 3\overline{X^2} &= a^2 + ab + b^2 \end{aligned}$$

Substituting in  $\hat{a}$ :

$$\begin{aligned} 3\overline{X^2} &= (2\bar{X} - b)^2 + (2\bar{X} - b)b + b^2 \\ &= 4\bar{X}^2 - 4\bar{X}b + b^2 + 2\bar{X}b - b^2 + b^2 \\ &= 4\bar{X}^2 - 2\bar{X}b + b^2 \end{aligned}$$

Thus, the method of moments estimator for  $b$  is:

$$\begin{aligned} b^2 - 2\bar{X}b + (4\bar{X}^2 - 3\overline{X^2}) &= 0 \\ \therefore \hat{b} &= \frac{2\bar{X} \pm \sqrt{(-2\bar{X})^2 - 4(4\bar{X}^2 - 3\overline{X^2})}}{2} \\ &= \bar{X} \pm \sqrt{3(\overline{X^2} - \bar{X}^2)} \end{aligned}$$

Arguably,  $a < b$ , so we can finalize the estimators for  $a$  and  $b$  respectively as the negative and positive roots:

$$\begin{aligned} \hat{a} &= 2\bar{X} - \left( \bar{X} + \sqrt{3(\overline{X^2} - \bar{X}^2)} \right) = \bar{X} - \sqrt{3(\overline{X^2} - \bar{X}^2)} \\ \hat{b} &= \bar{X} + \sqrt{3(\overline{X^2} - \bar{X}^2)} \end{aligned}$$

## 2.2

To find the MLE, we first calculate the likelihood function:

$$\begin{aligned} L(a, b) &= f(X_1, X_2, \dots, X_n; a, b) \\ &= \prod_{i=1}^n f(X_i; a, b) \\ &= \prod_{i=1}^n \frac{1}{b-a} \\ &= \left( \frac{1}{b-a} \right)^n \end{aligned}$$

for  $a \leq X_i \leq b$  for all  $i$ , and 0 otherwise. To maximize  $L(a, b)$ , we need to minimize  $b - a$ . Given the constraints, the smallest possible value for  $b - a$

occurs when  $a = \min(X_1, X_2, \dots, X_n)$  and  $b = \max(X_1, X_2, \dots, X_n)$ . Thus, the MLE for  $a$  and  $b$  are:

$$\begin{aligned}\hat{a} &= \min(X_1, X_2, \dots, X_n) \\ \hat{b} &= \max(X_1, X_2, \dots, X_n)\end{aligned}$$

### 3

Let  $X_1, \dots, X_n$  be IID  $\mathcal{N}(\mu, \sigma^2)$ . When estimating the variance, the biased estimator has a better MSE than the unbiased estimator. The unbiased estimator for variance is:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

The biased estimator for variance is:

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2$$

We know that  $E[S^2] = \sigma^2$  and  $E[\hat{\sigma}^2] = \frac{n-1}{n}\sigma^2$ . Thus, the bias for  $\hat{\sigma}^2$  is:

$$\text{Bias}(\hat{\sigma}^2) = E[\hat{\sigma}^2] - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = -\frac{\sigma^2}{n}$$

We can calculate the variance of  $S^2$  by using the property of the Chi Squared distribution:

$$\begin{aligned}\frac{(n-1)S^2}{\sigma^2} &\sim \chi_{n-1}^2 \\ \text{var}\left(\frac{(n-1)S^2}{\sigma^2}\right) &= \text{var}(\chi_{n-1}^2) = 2(n-1) \\ \frac{(n-1)^2}{\sigma^4} \text{var}(S^2) &= 2(n-1) \\ \text{var}(S^2) &= \frac{2\sigma^4}{n-1}\end{aligned}$$

Note that  $\hat{\sigma}^2 = \frac{n-1}{n}S^2$ . Thus, the variance of  $\hat{\sigma}^2$  is:

$$\begin{aligned}\text{var}(\hat{\sigma}^2) &= \text{var}\left(\frac{n-1}{n}S^2\right) \\ &= \left(\frac{n-1}{n}\right)^2 \text{var}(S^2) \\ &= \left(\frac{n-1}{n}\right)^2 \cdot \frac{2\sigma^4}{n-1} \\ &= \frac{2(n-1)}{n^2}\sigma^4\end{aligned}$$

Thus, the MSE of  $S^2$  is:

$$\begin{aligned}\text{MSE}(S^2) &= \text{var}(S^2) \\ &= \frac{2\sigma^4}{n-1}\end{aligned}$$

The MSE of  $\hat{\sigma}^2$  is:

$$\begin{aligned}\text{MSE}(\hat{\sigma}^2) &= \text{var}(\hat{\sigma}^2) + (\text{Bias}(\hat{\sigma}^2))^2 \\ &= \frac{2(n-1)}{n^2}\sigma^4 + \left(-\frac{\sigma^2}{n}\right)^2 \\ &= \frac{2(n-1)}{n^2}\sigma^4 + \frac{\sigma^4}{n^2} \\ &= \frac{2n-2+1}{n^2}\sigma^4 \\ &= \frac{2n-1}{n^2}\sigma^4\end{aligned}$$

Comparing the two MSEs:

$$\begin{aligned}\text{MSE}(S^2) &? \text{MSE}(\hat{\sigma}^2) \\ \frac{2\sigma^4}{n-1} &? \frac{2n-1}{n^2}\sigma^4 \\ \frac{2}{n-1} &? \frac{2n-1}{n^2} \\ 2n^2 &? (2n-1)(n-1) \\ 2n^2 &? 2n^2 - 3n + 1 \\ 0 &? -3n + 1 \\ 3n &? 1\end{aligned}$$

Since  $n$  is a positive integer and  $3n > 1$  for all  $n \geq 1$ , we can conclude that  $\text{MSE}(S^2) > \text{MSE}(\hat{\sigma}^2)$ . Thus, the biased estimator  $\hat{\sigma}^2$  has a better MSE than the unbiased estimator  $S^2$ .

## 4

Let  $P(X = 1) = P(X = -1) = 0.5$  and define:

$$X_n = \begin{cases} X & \text{with probability } 1 - \frac{1}{n} \\ e^n & \text{with probability } \frac{1}{n} \end{cases}$$

#### 4.1

To show that  $X_n$  converges in probability to  $X$ , we need to show that for any  $\epsilon > 0$ :

$$\lim_{n \rightarrow \infty} P(|X_n - X| \geq \epsilon) = 0$$

Note that:

$$\begin{aligned} P(|X_n - X| \geq \epsilon) &= P(X_n \neq X) \\ &= P(X_n = e^n) \\ &= \frac{1}{n} \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} P(|X_n - X| \geq \epsilon) = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

which shows that  $X_n$  converges in probability to  $X$ .

#### 4.2

As  $n \rightarrow \infty$ :

$$X_n \rightarrow \begin{cases} X & \text{with probability } 1 \\ \infty & \text{with probability } 0 \end{cases} = X \text{ with probability } 1$$

Thus,  $X_n$  converges to  $X$  in distribution.

#### 4.3

$E((X_n - X)^2)$  is:

$$E((X_n - X)^2) = E[(X_n - X)^2 | X_n = X] P(X_n = X) + E[(X_n - X)^2 | X_n = e^n] P(X_n = e^n)$$

$$\text{Note that } E[(X_n - X)^2 | X_n = X] = 0$$

$$= 0 \cdot \left(1 - \frac{1}{n}\right) + E[(e^n - X)^2] \cdot \frac{1}{n}$$

$$= E[e^{2n} - 2e^n X + X^2] \cdot \frac{1}{n}$$

$$= (e^{2n} - 2e^n E[X] + E[X^2]) \cdot \frac{1}{n}$$

$$E[X] = 1(0.5) + (-1)(0.5) = 0$$

$$E[X^2] = 1^2(0.5) + (-1)^2(0.5) = 1, \text{ so:}$$

$$= (e^{2n} - 0 + 1) \cdot \frac{1}{n}$$

$$= \frac{e^{2n} + 1}{n}$$



$$\lim_{n \rightarrow \infty} E((X_n - X)^2) = \lim_{n \rightarrow \infty} \frac{e^{2n} + 1}{n} = \infty$$

Thus,  $E((X_n - X)^2)$  does not converge to 0.