HW 3

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1

Let $X_1, X_2, \ldots, X_n$ be IID such that X_i is uniform in $[0, \theta]$.

1.1

For $\hat{\theta} = \max(X_1, X_2, \ldots, X_n)$:

$$F_{\hat{\theta}}(x) = P(\hat{\theta} \leq x) = P(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x)$$
$$= P(X_1 \leq x)P(X_2 \leq x)\cdots P(X_n \leq x) = \left(\frac{x}{\theta}\right)^n \text{ for } 0 \leq x \leq \theta$$
$$\therefore f_{\hat{\theta}}(x) = \frac{d}{dx}F_{\hat{\theta}}(x) = \frac{d}{dx}\left(\frac{x}{\theta}\right)^n = \frac{n}{\theta^n}x^{n-1}$$

The bias is $E[\hat{\theta}] - \theta$. Then expectation is:

$$\begin{aligned}
E[\hat{\theta}] &= \int_0^\theta x f_{\hat{\theta}}(x)\, dx \\
&= \int_0^\theta x \cdot \frac{n}{\theta^n}x^{n-1}\, dx \\
&= \frac{n}{\theta^n}\int_0^\theta x^n\, dx \\
&= \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} \\
&= \frac{n}{n+1}\theta
\end{aligned}$$

Thus, the bias is:

$$\text{Bias}(\hat{\theta}) = E[\hat{\theta}] - \theta = \frac{n}{n+1}\theta - \theta = -\frac{\theta}{n+1}$$

The standard error is:

$$\begin{aligned}
\text{se}(\hat{\theta}) &= \sqrt{var(\hat{\theta})} \\
&= \sqrt{E[\hat{\theta}^2] - (E[\hat{\theta}])^2}
\end{aligned}$$

$E[\hat{\theta}^2]$ is:

$$\begin{aligned}
E[\hat{\theta}^2] &= \int_0^\theta x^2 f_{\hat{\theta}}(x)\,dx \\
&= \int_0^\theta x^2 \cdot \frac{n}{\theta^n} x^{n-1}\,dx \\
&= \frac{n}{\theta^n} \int_0^\theta x^{n+1}\,dx \\
&= \frac{n}{\theta^n} \cdot \frac{\theta^{n+2}}{n+2} \\
&= \frac{n}{n+2}\theta^2
\end{aligned}$$

Thus, the standard error is:

$$\begin{aligned}
\mathrm{se}(\hat{\theta}) &= \sqrt{\frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2} \\
&= \theta\sqrt{\frac{n}{n+2} - \frac{n^2}{(n+1)^2}} \\
&= \theta\sqrt{\frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}} \\
&= \theta\sqrt{\frac{n((n^2+2n+1) - (n^2+2n))}{(n+2)(n+1)^2}} \\
&= \theta\sqrt{\frac{n}{(n+2)(n+1)^2}}
\end{aligned}$$

The mean squared error is:

$$\begin{aligned}
\mathrm{MSE}(\hat{\theta}) &= E[(\hat{\theta}-\theta)^2] \\
&= var(\hat{\theta}) + (\mathrm{Bias}(\hat{\theta}))^2 \\
&= \left(\theta^2 \frac{n}{(n+2)(n+1)^2}\right) + \left(-\frac{\theta}{n+1}\right)^2 \\
&= \theta^2 \left(\frac{n}{(n+2)(n+1)^2} + \frac{1}{(n+1)^2}\right) \\
&= \theta^2 \left(\frac{n+(n+2)}{(n+2)(n+1)^2}\right) \\
&= \theta^2 \left(\frac{2n+2}{(n+2)(n+1)^2}\right) \\
&= \theta^2 \left(\frac{2(n+1)}{(n+2)(n+1)^2}\right) \\
&= \theta^2 \left(\frac{2}{(n+2)(n+1)}\right)
\end{aligned}$$

1.2

For $\hat{\theta} = 2X_n = \frac{2}{n}\sum_{i=1}^{n} X_i$:
The bias is $E[\hat{\theta}] - \theta$. Then expectation is:

$$\begin{aligned}
E[\hat{\theta}] &= E\left[\frac{2}{n}\sum_{i=1}^{n} X_i\right] \\
&= \frac{2}{n}\sum_{i=1}^{n} E[X_i] \\
&= \frac{2}{n} \cdot n \cdot \frac{\theta}{2} \\
&= \theta
\end{aligned}$$

Thus, the bias is:

$$\mathrm{Bias}(\hat{\theta}) = E[\hat{\theta}] - \theta = \theta - \theta = 0.$$

The standard error is:

$$\begin{aligned}
\text{se}(\hat{\theta}) &= \sqrt{var(\hat{\theta})} \\
&= \sqrt{var\left(\frac{2}{n}\sum_{i=1}^{n} X_i\right)} \\
&= \frac{2}{n}\sqrt{var\left(\sum_{i=1}^{n} X_i\right)} \\
&= \frac{2}{n}\sqrt{\sum_{i=1}^{n} var(X_i)} \\
&= \frac{2}{n}\sqrt{n \cdot var(X_i)} \\
&= \frac{2}{n}\sqrt{n \cdot \frac{\theta^2}{12}} \\
&= \frac{2}{n} \cdot \theta \cdot \sqrt{\frac{n}{12}} \\
&= \frac{\theta}{\sqrt{3n}}
\end{aligned}$$

Note that since the bias is 0, the MSE is just the variance:

$$\begin{aligned}
\text{MSE}(\hat{\theta}) &= var(\hat{\theta}) \\
&= \left(\frac{\theta}{\sqrt{3n}}\right)^2 \\
&= \frac{\theta^2}{3n}
\end{aligned}$$

2

Let $X_1, \ldots, X_n$ be IID with X_i being uniform in $[a, b]$.

2.1

The method of moments estimator for a is:

$$\begin{aligned}
E[X_i] &= \frac{a+b}{2} \\
\bar{X} &= \frac{a+b}{2} \\
\therefore \hat{a} &= 2\bar{X} - b
\end{aligned}$$

The method of moments estimator for b is:

$$\begin{aligned}
E[X_i^2] &= \frac{a^2+ab+b^2}{3} \\
\overline{X^2} &= \frac{a^2+ab+b^2}{3} \\
\therefore 3\overline{X^2} &= a^2+ab+b^2
\end{aligned}$$

Substituting in $\hat{a}$:

$$\begin{aligned}
3\overline{X^2} &= (2\bar{X}-b)^2 + (2\bar{X}-b)b + b^2 \\
&= 4\bar{X}^2 - 4\bar{X}b + b^2 + 2\bar{X}b - b^2 + b^2 \\
&= 4\bar{X}^2 - 2\bar{X}b + b^2
\end{aligned}$$

Thus, the method of moments estimator for b is:

$$\begin{aligned}
b^2 - 2\bar{X}b + (4\bar{X}^2 - 3\overline{X^2}) = 0 \\
\therefore \hat{b} &= \frac{2\bar{X} \pm \sqrt{(-2\bar{X})^2 - 4(4\bar{X}^2 - 3\overline{X^2})}}{2} \\
&= \bar{X} \pm \sqrt{3(\overline{X^2} - \bar{X}^2)}
\end{aligned}$$

Arguably, $a < b$, so we can finalize the estimators for a and b respectively as the negative and positive roots:

$$\begin{aligned}
\hat{a} &= 2\bar{X} - \left(\bar{X} + \sqrt{3(\overline{X^2} - \bar{X}^2)}\right) = \bar{X} - \sqrt{3(\overline{X^2} - \bar{X}^2)} \\
\hat{b} &= \bar{X} + \sqrt{3(\overline{X^2} - \bar{X}^2)}
\end{aligned}$$

2.2

To find the MLE, we first calculate the likelihood function:

$$\begin{aligned}
L(a,b) &= f(X_1, X_2, \ldots, X_n; a, b) \\
&= \prod_{i=1}^{n} f(X_i; a, b) \\
&= \prod_{i=1}^{n} \frac{1}{b-a} \\
&= \left(\frac{1}{b-a}\right)^n
\end{aligned}$$

for $a \leq X_i \leq b$ for all i, and 0 otherwise. To maximize $L(a,b)$, we need to minimize $b-a$. Given the constraints, the smallest possible value for $b-a$

occurs when $a = \min(X_1, X_2, \ldots, X_n)$ and $b = \max(X_1, X_2, \ldots, X_n)$. Thus, the MLE for a and b are:

$$\hat{a} = \min(X_1, X_2, \ldots, X_n)$$
$$\hat{b} = \max(X_1, X_2, \ldots, X_n)$$

3

Let $X_1, \ldots X_n$ be IID $\mathcal{N}(\mu, \sigma^2)$. When estimating the variance, the biased estimator has a better MSE than the unbiased estimator. The unbiased estimator for variance is:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

The biased estimator for variance is:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

We know that $E[S^2] = \sigma^2$ and $E[\hat{\sigma}^2] = \frac{n-1}{n}\sigma^2$. Thus, the bias for $\hat{\sigma}^2$ is:

$$\text{Bias}(\hat{\sigma}^2) = E[\hat{\sigma}^2] - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = -\frac{\sigma^2}{n}$$

We can calculate the variance of S^2 by using the property of the Chi Squared distribution:

$$\begin{aligned}
\frac{(n-1)S^2}{\sigma^2} &\sim \chi^2_{n-1} \\
var\left(\frac{(n-1)S^2}{\sigma^2}\right) &= var(\chi^2_{n-1}) = 2(n-1) \\
\frac{(n-1)^2}{\sigma^4}var(S^2) &= 2(n-1) \\
var(S^2) &= \frac{2\sigma^4}{n-1}
\end{aligned}$$

Note that $\hat{sigma}^2 = \frac{n-1}{n}S^2$. Thus, the variance of $\hat{\sigma}^2$ is:

$$\begin{aligned}
var(\hat{\sigma}^2) &= var\left(\frac{n-1}{n}S^2\right) \\
&= \left(\frac{n-1}{n}\right)^2 var(S^2) \\
&= \left(\frac{n-1}{n}\right)^2 \cdot \frac{2\sigma^4}{n-1} \\
&= \frac{2(n-1)}{n^2}\sigma^4
\end{aligned}$$

Thus, the MSE of S^2 is:

$$\begin{aligned}\text{MSE}(S^2) &= var(S^2)\\ &= \frac{2\sigma^4}{n-1}\end{aligned}$$

The MSE of $\hat{\sigma}^2$ is:

$$\begin{aligned}\text{MSE}(\hat{\sigma}^2) &= var(\hat{\sigma}^2) + (\text{Bias}(\hat{\sigma}^2))^2\\ &= \frac{2(n-1)}{n^2}\sigma^4 + \left(-\frac{\sigma^2}{n}\right)^2\\ &= \frac{2(n-1)}{n^2}\sigma^4 + \frac{\sigma^4}{n^2}\\ &= \frac{2n-2+1}{n^2}\sigma^4\\ &= \frac{2n-1}{n^2}\sigma^4\end{aligned}$$

Comparing the two MSEs:

$$\begin{aligned}\text{MSE}(S^2) &\ ?\ \text{MSE}(\hat{\sigma}^2)\\ \frac{2\sigma^4}{n-1} &\ ?\ \frac{2n-1}{n^2}\sigma^4\\ \frac{2}{n-1} &\ ?\ \frac{2n-1}{n^2}\\ 2n^2 &\ ?\ (2n-1)(n-1)\\ 2n^2 &\ ?\ 2n^2-3n+1\\ 0 &\ ?\ -3n+1\\ 3n &\ ?\ 1\end{aligned}$$

Since n is a positive integer and $3n > 1$ for all $n \geq 1$, we can conclude that $\text{MSE}(S^2) > \text{MSE}(\hat{\sigma}^2)$. Thus, the biased estimator $\hat{\sigma}^2$ has a better MSE than the unbiased estimator S^2.

4

Let $P(X = 1) = P(X = -1) = 0.5$ and define:

$$X_n = \begin{cases} X & \text{with probability } 1 - \frac{1}{n} \\ e^n & \text{with probability } \frac{1}{n} \end{cases}$$

4.1

To show that X_n converges in probability to X, we need to show that for any $\epsilon > 0$:

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$$

Note that:

$$\begin{aligned} P(|X_n - X| \geq \epsilon) &= P(X_n \neq X) \\ &= P\left(X_n = e^n\right) \\ &= \frac{1}{n} \end{aligned}$$

Thus,

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = \lim_{n\to\infty} \frac{1}{n} = 0$$

which shows that X_n converges in probability to X.

4.2

As $n \to \infty$:

$$X_n \to \begin{cases} X & \text{with probability 1} \\ \infty & \text{with probability 0} \end{cases} = X \text{with probability 1}$$

Thus, X_n converges to X in distribution.

4.3

$E((X_n - X)^2)$ is:

$$E((X_n - X)^2) = E\left[(X_n - X)^2 | X_n = X\right] P(X_n = X) + E\left[(X_n - X)^2 | X_n = e^n\right] P(X_n = e^n)$$

Note that $E\left[(X_n - X)^2 | X_n = X\right] = 0$

$$\begin{aligned} &= 0 \cdot \left(1 - \frac{1}{n}\right) + E\left[(e^n - X)^2\right] \cdot \frac{1}{n} \\ &= E[e^{2n} - 2e^n X + X^2] \cdot \frac{1}{n} \\ &= \left(e^{2n} - 2e^n E[X] + E[X^2]\right) \cdot \frac{1}{n} \\ E[X] &= 1(0.5) + (-1)(0.5) = 0 \\ E[X^2] &= 1^2(0.5) + (-1)^2(0.5) = 1, \text{ so:} \\ &= \left(e^{2n} - 0 + 1\right) \cdot \frac{1}{n} \\ &= \frac{e^{2n} + 1}{n} \end{aligned}$$

$$\lim_{n\to\infty} E((X_n - X)^2) = \lim_{n\to\infty} \frac{e^{2n}+1}{n} = \infty$$

Thus, $E((X_n - X)^2)$ does not converge to 0.