Notes

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9/30/2025

1

$$\forall A \in \mathbb{R}$$
$$\frac{\frac{1}{8}}{\frac{1}{2}+\frac{1}{8}}$$
$$\frac{1}{2}$$
$$\Omega$$
$$\omega$$

A. A

B. B

2 Extended Monty Hall Problem

Suppose you have n doors, where behind 1 is the car and behind $n-1$ are goats. Monty Hall will open $k([0, n-2])$ doors as he has to leave one door unopened for you to switch to, and the original door you picked. The chance that you picked the car originally is $\frac{1}{n}$, hence the chance you didn't pick the car is $\frac{n-1}{n}$. When Monty Hall opens k doors, the probability that you should switch to win is now determined by the probability you didn't pick the correct door the first times multiplied by the new probability that you pick the correct door when you switch, given by:

$$\frac{n-1}{n} \cdot \frac{1}{n-k-1} = \frac{1}{n} \cdot \frac{n-1}{n-k-1}$$

Suppose Monty Hall opens $n-2$ doors, then when you switch the probablity of winning becomes apparent:

$$\frac{n-1}{n} \cdot \frac{1}{n-(n-2)-1} = \frac{n-1}{n}$$

In the standard Monty Hall problem where $n = 3$ and $k = 1$, the probability of winning when you switch is:

$$\frac{3-1}{3} \cdot \frac{1}{3-1-1} = \frac{2}{3}$$

3 Bayes

In a channel, you can send either a 0 or 1. The probability of sending a 0 is $\frac{2}{3}$ and the probability of sending a 1 is $\frac{1}{3}$. The probability that a number is received correctly as is $\frac{3}{4}$ and incorrectly is is $\frac{1}{4}$. What is P(1 sent — 1 received)?

$$P(1 \text{ sent}|1 \text{ received}) = \frac{P(1 \text{ received}|1 \text{ sent}) \cdot P(1 \text{ sent})}{P(1 \text{ received})}$$
$$= \frac{\frac{3}{4} \cdot \frac{1}{3}}{\frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4}} = \frac{\frac{1}{4}}{\frac{1}{6} + \frac{1}{4}} = \frac{3}{5}$$

4 (Cumulative) Distribution Function

The probability that $X(\omega)$ resides in the interval $(-\infty, x]$ depends on only the right endpoint x, hence it is convenient to:

$$F : x \mapsto P(X \leq x)$$
$$F(x) = P(X \leq x) = P(\omega : X(\omega) \leq x) = P(X^{-1}(-\infty, x])$$

Properties of F:

1. Right continuous
2. Increase monotonically
3. Has limits $F(-\infty) = 0$ and $F(\infty) = 1$
4. $P(a < X \leq b) = F(b) - F(a)$
   $P((a, b]) = P((-\infty, b]) - P((-\infty, a])$

- Determines probabilities of all types of intervals

- Once the CDF is known, the original probability space is in the background
- From this point, we can just deal with F, as if it were a real-line sample space governed by the distribution F

When the number of times $F(X-) \neq F(x)$ is countable, we say X is a discrete random variable. Note that intervals $F(x) - F(x-)$ are all disjoint at each point of a jump, therefore must contain a rational number of events within.

4.1 Discrete Distribution

If we can enumerate the point of jumps for any distribution function, we can define the probability of each jump point and hence achieve a discrete distribution.
Let $u(x)$ be a unit step function, and $u(x) = 1$ if $x \geq 0$, else 0.
F is a *discrete distribution* if it can be represented as:

$$F(x) = \sum_j p_j u(x - x_j)$$

where x_j is a countable set of real numbers, $p_j > 0$ are such that $\sum_j p_j = 1$.

4.2 Heaviside Distribution

$F(x) = u(x - x_0)$ is a discrete distribution with a fixed value x_0 with probability 1. Hence $P(X \geq x_0) = 1$ and $P(X < x_0) = 0$. Standard value for x_0 is 0. Can use indicator function to define such.

5 Moments

Let X be a random variable given the following distribution:

$$X = \begin{cases} 0, & 1-p \\ 1, & p \end{cases}$$

$$E[X^2] = 0^2 \cdot (1-p) + 1^2 \cdot p = p$$

$$E[X] = p$$

$$Var(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Now let X be a random variable following a uniform distribution:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & else \end{cases}$$

$$E[X] = \int_a^b x f_X(x) dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{x^2}{2}\bigg|_a^b$$

$$= \frac{1}{b-a} \cdot \left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

$$E[X^2] = \int_a^b x^2 f_X(x) dx = \int_a^b x^2 \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{x^3}{3}\bigg|_a^b$$

$$= \frac{1}{b-a} \cdot \left(\frac{b^3}{3} - \frac{a^3}{3}\right) = \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{b^2 + ab + a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4} = \frac{4(b^2 + ab + a^2) - 3(b^2 + 2ab + a^2)}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$

5.1 Moment Generating Functions

$$\psi_X(t) = E[e^{tX}] = \int e^{tx} f_X(x) dx$$

$$\psi_X(t) = E[1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + ...]$$

$$= 1 + tE[X] + \frac{t^2 E[X^2]}{2!} + \frac{t^3 E[X^3]}{3!} + ...$$

$$= \sum_{n=0}^{\infty} \frac{t^n E[X^n]}{n!} \psi_X(0) = 1$$

$$\psi'_X(t) = \sum_{n=1}^{\infty} \frac{t^{n-1} E[X^n]}{(n-1)!}$$

$$\psi'_X(0) = E[X]$$

$$\psi''_X(t) = \sum_{n=2}^{\infty} \frac{t^{n-2} E[X^n]}{(n-2)!}$$

$$\psi''_X(0) = E[X^2] \therefore$$

$$\psi_X^{(k)}(0) = E[X^k]$$