

# HW 1

Kevin Lin

10/9/2025

## 1

Let event  $A$  be the event that in a group of  $n$  people, no one shares a birthday. Let event  $B$  be the complement of event  $A$ , such that in a group of  $n$  people, at least 2 people share a birthday. Thus,  $P(B) = 1 - P(A)$ . We can calculate  $P(A)$  by starting with the first person, who can have any birthday. The next person must have a different birthday, thus have a  $\frac{364}{365}$  chance of not sharing a birthday with the first person. This goes on until the  $n$ th person, who has a  $\frac{365-n+1}{365}$  chance of not sharing a birthday with any of the previous  $n-1$  people. Thus, we have:

$$P(A) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-n+1}{365} = \frac{1}{365^n} \cdot \frac{365 \cdot 364 \cdots (365-n+1)}{1}$$

Recognize that  $\frac{365 \cdot 364 \cdots (365-n+1)}{1} = \frac{365!}{(365-n)!} \therefore$

$$P(A) = \frac{365!}{365^n(365-n)!}$$

We want to find a minimum  $n$  such that  $P(B) \geq 0.5 \therefore P(A) < 0.5$ . Solving  $P(A) < 0.5$  for  $n$ , we find that the minimum  $n$  is 23, where  $P(A) \approx 0.493$  and  $P(B) \approx 0.507$ .

Thus, in a group of  $n$  unrelated individuals, the probability that at least 2 people share a birthday is  $P(B) = 1 - \frac{365!}{365^n(365-n)!}$ . This probability exceeds 0.5 when  $n \geq 23$ .

## 2

### 2.1

Each bag has a probability of  $\frac{1}{3}$  of being chosen. The probability of drawing a blue marble at random is thus the sum of the probabilities of drawing a

blue marble from each bag multiplied by the probability of choosing that bag. Thus, we have:

$$P(\text{blue marble}) = \frac{1}{3} \cdot \frac{25}{100} + \frac{1}{3} \cdot \frac{40}{100} + \frac{1}{3} \cdot \frac{55}{100} = \frac{2}{5} = 0.4$$

## 2.2

If the first bag has the probability of 0.5 of being chosen, the other two bags then have a probability of 0.25 of being chosen. Thus, we now have:

$$P(\text{blue marble}) = 0.5 \cdot \frac{25}{100} + 0.25 \cdot \frac{40}{100} + 0.25 \cdot \frac{55}{100} = \frac{29}{80} = 0.3625$$

## 3

For the gambler's current fortune of  $k$  dollars, they can either win or lose on the next gamble. Thus by the law of total probability, we have:

$$\begin{aligned} q_k &= \frac{1}{2} \cdot q_{k+1} + \frac{1}{2} \cdot q_{k-1} \\ 2q_k &= q_{k+1} + q_{k-1} \\ q_{k+1} - q_k &= q_k - q_{k-1} \end{aligned}$$

This shows that the difference between consecutive  $q$  values is constant, thus  $q_k$  is linear and can express  $q_k$  as  $q_k = A + Bk$  for some constants  $A, B$ . We solve for  $A, B$  using the boundary conditions where  $q_0 = 1$  and  $q_N = 0$ :

$$\begin{aligned} q_0 &= A + B \cdot 0 \therefore A = 1 \\ q_N &= 1 + BN = 0 \therefore B = -\frac{1}{N} \\ \therefore q_k &= 1 - \frac{k}{N} \end{aligned}$$

## 4

For a given element  $x$  in  $\Omega$ , the probability of  $x$  being in either set  $A$  or  $B$  is equally likely because  $A$  and  $B$  are independently selected subsets of  $\Omega$ . Thus, for any element  $x$  to satisfy  $A \subseteq B$ , element  $x$  must either be:

- $x \notin A$  &  $x \notin B$
- $x \notin A$  &  $x \in B$
- $x \in A$  &  $x \in B$

The only case that does not satisfy  $A \subseteq B$  is if  $x \in A$  &  $x \notin B$ . Since each of the 4 cases are equally likely, the probability of  $x$  satisfying  $A \subseteq B$  is  $\frac{3}{4}$ . Since this must be true for all  $n$  elements in  $\Omega$ , the probability of  $A \subseteq B$  is consequently:

$$P(A \subseteq B) = \left(\frac{3}{4}\right)^n$$

## 5

### 5.1

Each coin is equally likely to be chosen with a probability of  $\frac{1}{3}$ . The probability of getting tails when a coin is flipped is thus the sum of the probabilities of getting tails with each coin multiplied by the probability of choosing that coin. Thus, we have:

$$P(T) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 = \frac{1}{3}$$

### 5.2

$$\begin{aligned} P(\text{fake coin}|\text{H}) &= \frac{P(\text{H}|\text{fake coin}) \cdot P(\text{fake coin})}{P(\text{H})} \\ &= \frac{1 \cdot \frac{1}{3}}{\frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1} = \frac{1}{2} \end{aligned}$$

## 6

Any joint distribution function must satisfy:

1.  $F(-\infty, x_2) = F(x_1, -\infty) = 0$
2.  $F(\infty, \infty) = 1$
3.  $F(x_1, \infty) = F(x_1)$  &  $F(\infty, x_2) = F(x_2)$
4. monotonicity:  $F(a_1, a_2) \leq F(b_1, b_2)$  if  $a_1 \leq b_1$  &  $a_2 \leq b_2$
5.  $P(a_1 < X \leq b_1, a_2 < Y \leq b_2) = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2) \geq 0$

### 6.1

For  $F(x_1, x_2) = 1 - e^{-x_1 - x_2}$  if  $x_1, x_2 \geq 0$ :

1.  $F(-\infty, x_2) = F(x_1, -\infty) = 0$  is satisfied as  $x_1, x_2 \geq 0$

2.  $F(\infty, \infty) = 1 - e^{-\infty - \infty} = 1 - 0 = 1$  is satisfied
3.  $F(x_1, \infty) = 1 - e^{-x_1 - \infty} = 1 - 0 = 1$  and  $F(\infty, x_2) = 1 - e^{-\infty - x_2} = 1 - 0 = 1$  is satisfied
4. monotonicity is satisfied as  $e^{-x}$  is a decreasing function
5.  $P(a_1 < X \leq b_1, a_2 < Y \leq b_2) = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$   
 $= (1 - e^{-b_1 - b_2}) - (1 - e^{-a_1 - b_2}) - (1 - e^{-b_1 - a_2}) + (1 - e^{-a_1 - a_2})$   
Let  $a_1 = 0, a_2 = 0, b_1 = t_1 > 0, b_2 = t_2 > 0$ , then:  
 $P(0 < X \leq t_1, 0 < Y \leq t_2) = (1 - e^{-t_1 - t_2}) - (1 - e^{0 - t_2}) - (1 - e^{-t_1 + 0}) + (1 - e^{0 + 0})$   
 $= 1 - e^{-t_1 - t_2} - 1 + e^{-t_2} - 1 + e^{-t_1} = e^{-t_2} + e^{-t_1} - e^{-t_1 - t_2} - 1$   
 $= (e^{-t_1} - 1)(1 - e^{-t_2})$   
 $(e^{-t_1} - 1) < 0$  and  $(1 - e^{-t_2}) > 0$ , therefore  $P(0 < X \leq t_1, 0 < Y \leq t_2) < 0$   
which violates condition 5.

Thus,  $F(x_1, x_2) = 1 - e^{-x_1 - x_2}$  if  $x_1, x_2 \geq 0$  is not a valid joint distribution function.

## 6.2

For  $F(x_1, x_2) = 1 - e^{-\min(x_1, x_2)} - \min(x_1, x_2)e^{-\min(x_1, x_2)}$  if  $x_1, x_2 \geq 0$ :

1.  $F(-\infty, x_2) = F(x_1, -\infty) = 0$  is satisfied as  $x_1, x_2 \geq 0$
2.  $F(\infty, \infty) = 1 - e^{-\infty} - \infty \cdot e^{-\infty} = 1 - 0 - 0 = 1$  is satisfied
3.  $F(x_1, \infty) = 1 - e^{-x_1} - x_1 e^{-x_1} = F(x_1)$  and  $F(\infty, x_2) = 1 - e^{-x_2} - x_2 e^{-x_2} = F(x_2)$  is satisfied
4. monotonicity is satisfied as  $e^{-x}$  is a decreasing function and  $\min(x_1, x_2)$  is increasing
5.  $P(a_1 < X \leq b_1, a_2 < Y \leq b_2)$  Again, let  $a_1 = 0, a_2 = 0, b_1 = t, b_2 = t$  where  $t > 0$ , then:  
 $P(0 < X \leq t, 0 < Y \leq t) = F(t, t) - F(0, t) - F(t, 0) + F(0, 0)$   
 $= (1 - e^{-t} - te^{-t}) - (1 - e^0 - 0 \cdot e^0) - (1 - e^0 - 0 \cdot e^0) + (1 - e^0 - 0 \cdot e^0)$   
 $= 1 - e^{-t} - te^{-t}$ , which has a minimum of 0 when  $t = 0$  and is positive for all  $t > 0$  as shown from condition 4.

Thus,  $F(x_1, x_2) = 1 - e^{-\min(x_1, x_2)} - \min(x_1, x_2)e^{-\min(x_1, x_2)}$  if  $x_1, x_2 \geq 0$  is a valid joint distribution function. Marginal distribution functions  $F_1(x_1)$  and  $F_2(x_2)$  are determined in condition 3.

## 7

For  $\max(X, Y)$ , both  $X$  and  $Y$  must be less than or equal to some value  $t$  for  $\max(X, Y)$  to be possible. Thus:

$$P(\max(X, Y) \leq t) = P(X \leq t, Y \leq t)$$

$$\text{By independence, } P(X \leq t, Y \leq t) = P(X \leq t)P(Y \leq t)$$

$$\therefore F_{\max}(t) = F(t)G(t)$$

For  $\min(X, Y)$ , either  $X$  or  $Y$  must be greater than to some value  $t$  for  $\min(X, Y)$  to be possible. Thus:

$$P(\min(X, Y) > t) = P(X > t, Y > t)$$

$$\text{By independence, } P(X > t, Y > t) = P(X > t)P(Y > t)$$

$$\text{By law of total probability, } P(X > t) = 1 - P(X \leq t) = 1 - F(t)$$

$$\text{Likewise, } P(Y > t) = 1 - G(t)$$

$$\therefore P(\min(X, Y) > t) = (1 - F(t))(1 - G(t))$$

$$\text{By law of total probability, } P(\min(X, Y) \leq t) = 1 - P(\min(X, Y) > t)$$

$$\therefore F_{\min}(t) = 1 - (1 - F(t))(1 - G(t)) = F(t) + G(t) - F(t)G(t)$$