HW 0

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(a) Let $x_1, \ldots, x_n$ be real values. Then for the quadratic function $f(\theta) = \sum_{i=1}^{n} w_i(x_i - \theta)^2$ where $w_i > 0$ for all i, the optimal solution θ^* denoted by $\theta^* = \arg\min_\theta f(\theta)$ can be calculated as follows:

$$\begin{aligned}
\frac{d}{d\theta} f(\theta) &= \frac{d}{d\theta} \sum_{i=1}^{n} w_i(x_i - \theta)^2 \\
&= \sum_{i=1}^{n} w_i \cdot 2(x_i - \theta) \cdot (-1) \\
&= -2 \sum_{i=1}^{n} w_i(x_i - \theta) \\
&= -2 \left(\sum_{i=1}^{n} w_i x_i - \theta \sum_{i=1}^{n} w_i \right)
\end{aligned}$$

Setting the derivative to zero to find the minimum:

$$\begin{aligned}
-2 \left(\sum_{i=1}^{n} w_i x_i - \theta \sum_{i=1}^{n} w_i \right) &= 0 \\
\sum_{i=1}^{n} w_i x_i - \theta \sum_{i=1}^{n} w_i &= 0 \\
\theta \sum_{i=1}^{n} w_i &= \sum_{i=1}^{n} w_i x_i \\
\theta^* &= \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}
\end{aligned}$$

Thus, the optimal solution is the weighted average of the x_i's. If some weights are negative, the function may not be convex, and the solution may not correspond to a minimum.

(b) (i) Given $2n$ kids are randomly divied into two equal subgroups, the probability that the two tallest kids end up in the the same subgroup can be calculated as follows:

$$\begin{aligned}
P(\text{tallest in same group}) &= P(\text{both in group 1}) + P(\text{both in group 2}) \\
&= \frac{\binom{2n-2}{n-2}}{\binom{2n}{n}} + \frac{\binom{2n-2}{n-2}}{\binom{2n}{n}} \\
&= 2 \cdot \frac{\binom{2n-2}{n-2}}{\binom{2n}{n}} \\
&= 2 \cdot \frac{\frac{(2n-2)!}{(n-2)!(n)!}}{\frac{(2n)!}{(n)!(n)!}} \\
&= 2 \cdot \frac{(2n-2)!n!}{(n-2)!(2n)!} \\
&= 2 \cdot \frac{n(n-1)}{(2n)(2n-1)} \\
&= \frac{n(n-1)}{(2n-1)(n)} \\
&= \frac{n-1}{2(2n-1)}
\end{aligned}$$

(ii) The probabilty that the two tallest kids end up in different subgroups is:

$$\begin{aligned}
P(\text{tallest in different groups}) &= 1 - P(\text{both tallest in same group}) \\
&= 1 - \frac{n-1}{2(2n-1)} \\
&= \frac{2(2n-1) - (n-1)}{2(2n-1)} \\
&= \frac{4n - 2 - n + 1}{2(2n-1)} \\
&= \frac{3n-1}{2(2n-1)}
\end{aligned}$$

(c) We know $P(\text{knows answer}) = p$, and $P(\text{doesn't konw answer}) = 1 - p$. Also, $P(\text{correct}|\text{knows answer}) = 0.99$ and $P(\text{correct}|\text{doesn't know answer}) = 1/k$. Then $P(\text{knew answer}|\text{correct})$ can be calculated using Bayes' Theorem:

$$
\begin{aligned}
P(\text{knew answer}|\text{correct}) &= \frac{P(\text{correct}|\text{knew answer}) \cdot P(\text{knew answer})}{P(\text{correct})} \\
&= \frac{P(\text{correct}|\text{knew answer}) \cdot P(\text{knew answer})}{P(\text{correct}|\text{knew answer}) \cdot P(\text{knew answer}) + P(\text{correct}|\text{didn't know answ}} \\
&= \frac{0.99 \cdot p}{0.99 \cdot p + \frac{1}{k} \cdot (1-p)}
\end{aligned}
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(a) a