HW 1

Kevin Lin

1/16/2026

1

(i) Given $X_i, \ldots, X_n$ are i.i.d. Poisson(λ), and the family of estimators for λ is $\hat{\lambda}_{\alpha,\beta} = \frac{\sum_{i=1}^n X_i + \alpha}{n+\beta}$, where $\alpha, \beta \geq 0$, the expectation $E[\hat{\lambda}_{\alpha,\beta}]$ can be computed as follows:

$$\begin{aligned} E[\hat{\lambda}_{\alpha,\beta}] &= E\left[\frac{\sum_{i=1}^n X_i + \alpha}{n+\beta}\right] \\ &= \frac{1}{n+\beta} E\left[\sum_{i=1}^n X_i + \alpha\right] \\ &= \frac{1}{n+\beta}\left(E\left[\sum_{i=1}^n X_i\right] + E[\alpha]\right) \\ &= \frac{1}{n+\beta}(n\lambda + \alpha) \\ &= \frac{n\lambda + \alpha}{n+\beta} \end{aligned}$$

The variance $Var(\hat{\lambda}_{\alpha,\beta})$ is computed as:

$$\begin{aligned} Var(\hat{\lambda}_{\alpha,\beta}) &= Var\left(\frac{\sum_{i=1}^n X_i + \alpha}{n+\beta}\right) \\ &= \frac{1}{(n+\beta)^2} Var\left(\sum_{i=1}^n X_i\right) \\ &= \frac{1}{(n+\beta)^2}(nVar(X_i)) \\ &= \frac{1}{(n+\beta)^2}(n\lambda) \\ &= \frac{n\lambda}{(n+\beta)^2} \end{aligned}$$

(ii) The estimator $\hat{\lambda}_{\alpha,\beta}$ is biased if $E[\hat{\lambda}_{\alpha,\beta}] \neq \lambda$. Setting $E[\hat{\lambda}_{\alpha,\beta}] = \lambda$, we have:

$$\begin{aligned} \frac{n\lambda + \alpha}{n + \beta} &= \lambda \\ n\lambda + \alpha &= \lambda(n + \beta) \\ n\lambda + \alpha &= n\lambda + \lambda\beta \\ \alpha &= \lambda\beta \end{aligned}$$

Thus, the estimator is biased when $\alpha \neq \lambda\beta$.

(iii) The MSE of the estimator can be calculated as follows:

$$\begin{aligned} MSE(\hat{\lambda}_{\alpha,\beta}) &= E[(\hat{\lambda}_{\alpha,\beta} - \lambda)^2] \\ &= Var(\hat{\lambda}_{\alpha,\beta}) + (E[\hat{\lambda}_{\alpha,\beta}] - \lambda)^2 \\ &= \frac{n\lambda}{(n+\beta)^2} + \left(\frac{n\lambda + \alpha}{n + \beta} - \lambda\right)^2 \\ &= \frac{n\lambda}{(n+\beta)^2} + \left(\frac{\alpha - \lambda\beta}{n + \beta}\right)^2 \\ &= \frac{n\lambda + (\alpha - \lambda\beta)^2}{(n+\beta)^2} \end{aligned}$$

(iv) You would prefer using $\hat{\lambda}_{\alpha,\beta}$ over $\hat{\lambda}$ when the MSE of $\hat{\lambda}_{\alpha,\beta}$ is less than that of $\hat{\lambda}$. Otherwise, you would prefer $\hat{\lambda}$.