HW 1

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(i) Given $X_i, \ldots, X_n$ are i.i.d. Poisson(λ), and the family of estimators for λ is $\hat{\lambda}_{\alpha,\beta} = \frac{\sum_{i=1}^n X_i + \alpha}{n+\beta}$, where $\alpha, \beta \geq 0$, the expectation $E[\hat{\lambda}_{\alpha,\beta}]$ can be computed as follows:

$$\begin{aligned}
E[\hat{\lambda}_{\alpha,\beta}] &= E\left[\frac{\sum_{i=1}^n X_i + \alpha}{n+\beta}\right] \\
&= \frac{1}{n+\beta} E\left[\sum_{i=1}^n X_i + \alpha\right] \\
&= \frac{1}{n+\beta}\left(E\left[\sum_{i=1}^n X_i\right] + E[\alpha]\right) \\
&= \frac{1}{n+\beta}\left(n\lambda + \alpha\right) \\
&= \frac{n\lambda + \alpha}{n+\beta}
\end{aligned}$$

The variance $Var(\hat{\lambda}_{\alpha,\beta})$ is computed as:

$$\begin{aligned}
Var(\hat{\lambda}_{\alpha,\beta}) &= Var\left(\frac{\sum_{i=1}^n X_i + \alpha}{n+\beta}\right) \\
&= \frac{1}{(n+\beta)^2} Var\left(\sum_{i=1}^n X_i\right) \\
&= \frac{1}{(n+\beta)^2}\left(nVar(X_i)\right) \\
&= \frac{1}{(n+\beta)^2}(n\lambda) \\
&= \frac{n\lambda}{(n+\beta)^2}
\end{aligned}$$

(ii) The estimator $\hat{\lambda}_{\alpha,\beta}$ is biased if $E[\hat{\lambda}_{\alpha,\beta}] \neq \lambda$. Setting $E[\hat{\lambda}_{\alpha,\beta}] = \lambda$, we have:

$$\begin{aligned}
\frac{n\lambda + \alpha}{n + \beta} &= \lambda \\
n\lambda + \alpha &= \lambda(n + \beta) \\
n\lambda + \alpha &= n\lambda + \lambda\beta \\
\alpha &= \lambda\beta
\end{aligned}$$

Thus, the estimator is biased when $\alpha \neq \lambda\beta$.

(iii) The MSE of the estimator can be calculated as follows:

$$\begin{aligned}
MSE(\hat{\lambda}_{\alpha,\beta}) &= E[(\hat{\lambda}_{\alpha,\beta} - \lambda)^2] \\
&= Var(\hat{\lambda}_{\alpha,\beta}) + (E[\hat{\lambda}_{\alpha,\beta}] - \lambda)^2 \\
&= \frac{n\lambda}{(n + \beta)^2} + \left(\frac{n\lambda + \alpha}{n + \beta} - \lambda\right)^2 \\
&= \frac{n\lambda}{(n + \beta)^2} + \left(\frac{\alpha - \lambda\beta}{n + \beta}\right)^2 \\
&= \frac{n\lambda + (\alpha - \lambda\beta)^2}{(n + \beta)^2}
\end{aligned}$$

(iv) You would prefer using $\hat{\lambda}_{\alpha,\beta}$ over $\hat{\lambda}$ when the MSE of $\hat{\lambda}_{\alpha,\beta}$ is less than that of $\hat{\lambda}$. Otherwise, you would prefer $\hat{\lambda}$.

LLM Usage: All work was done in VSCode with GitHub Copilot integration. The integration "provides code suggestions, explanations, and automated implementations based on natural language prompts and existing code context," and also offers autonomous coding and an in-IDE chat interface that is able to interact with the current codebase. Only the Copilot provided automatic inline suggestions for both LaTex and Python in `.tex` and `.ipynb` Jupyter notebook files respectively were taken into account / used.