HW 4

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1

(i) Given design matrix $X \in \mathbb{R}^{n \times p}$ and response vector $y \in \mathbb{R}^n$, and that the columns of X are linearly dependent, we can prove that the OLS coefficient vector $\hat{\beta}$ is not unique as follows:
Since the columns of X are linearly dependent, there exists a non-zero vector $c \in \mathbb{R}^p$ such that $Xc = 0$. Let $\hat{\beta}$ be an OLS solution, i.e.,

$$\hat{\beta} = \arg\min_{\beta} \|y - X\beta\|_2^2.$$

Now consider another vector $\hat{\beta}' = \hat{\beta} + c$. We have:

$$X\hat{\beta}' = X(\hat{\beta} + c) = X\hat{\beta} + Xc = X\hat{\beta} + 0 = X\hat{\beta}.$$

Therefore, the residuals for both $\hat{\beta}$ and $\hat{\beta}'$ are the same:

$$\|y - X\hat{\beta}'\|_2^2 = \|y - X\hat{\beta}\|_2^2.$$

This shows that there are infinitely many solutions to the OLS problem, proving that $\hat{\beta}$ is not unique.
The form of the solution set can be expressed as:

$$\{\hat{\beta} + c : c \in \text{Null}(X)\},$$

where $\text{Null}(X)$ is the null space of X.

(ii) If the columns of X are linearly independent, and the linear model $y = x^T\beta + \epsilon$ holds with $\epsilon \sim \mathcal{N}(0, \sigma^2)$, then the maximum variance over all unit-norm linear combinations of the fitted coefficients $\max_{\|a\|_2=1} \text{Var}[a^T\hat{\beta}]$ can be derived as follows given $X = UDV^T$ (the SVD of X):
The OLS estimator is given by:

$$\hat{\beta} = (X^TX)^{-1}X^Ty.$$

The variance of $\hat{\beta}$ is:

$$\text{Var}[\hat{\beta}] = \sigma^2(X^TX)^{-1}.$$

Using the SVD of X, we have:

$$X^T X = V D^2 V^T,$$

so:

$$(X^T X)^{-1} = V D^{-2} V^T.$$

Therefore:

$$\text{Var}[\hat{\beta}] = \sigma^2 V D^{-2} V^T.$$

For any unit-norm vector a, we have:

$$\text{Var}[a^T \hat{\beta}] = a^T \text{Var}[\hat{\beta}] a = \sigma^2 a^T V D^{-2} V^T a.$$

Letting $b = V^T a$, we note that $\|b\|_2 = \|a\|_2 = 1$. Thus:

$$\text{Var}[a^T \hat{\beta}] = \sigma^2 b^T D^{-2} b = \sigma^2 \sum_{i=1}^{p} \frac{b_i^2}{d_i^2},$$

where d_i are the singular values of X. To maximize this variance, we should allocate all the weight to the smallest singular value $d_{\min}$:

$$\max_{\|a\|_2=1} \text{Var}[a^T \hat{\beta}] = \frac{\sigma^2}{d_{\min}^2}.$$

(iii) If the first $p-1$ columns of X are linearly independent, but the last column is a linear combination of the first $p-1$ columns, we can show that the maximum variance from part (ii) grows without bound as $\|z\|_2 \to \infty$ as follows:
If $z = 0$, then x_p lies exactly in the span of the first $p-1$ columns, so X has rank $p-1$. Therefore, the smallest singular value $d_{\min} = 0$, leading to an infinite variance:

$$\max_{\|a\|_2=1} \text{Var}[a^T \hat{\beta}] = \frac{\sigma^2}{d_{\min}^2} = \infty.$$

When $z \neq 0$, still $\|z\|_2$ grows extremely small, and as $z \to 0$, likewise, the smallest singular value scales w/ the pertubation, which shows that the maximum variance grows without bound as $\|z\|_2 \to 0$.

2

See attached Jupyter notebook for code and plots.

In [3]:
```
import numpy as np
import matplotlib.pyplot as plt
from datetime import datetime, timedelta
import yfinance as yf
```

In [4]:
```
data = yf.download(["NVDA", "SPY"], start= datetime.today() - timedelta(days = 1825
NVDA = data["NVDA"]
SPY = data["SPY"]
```

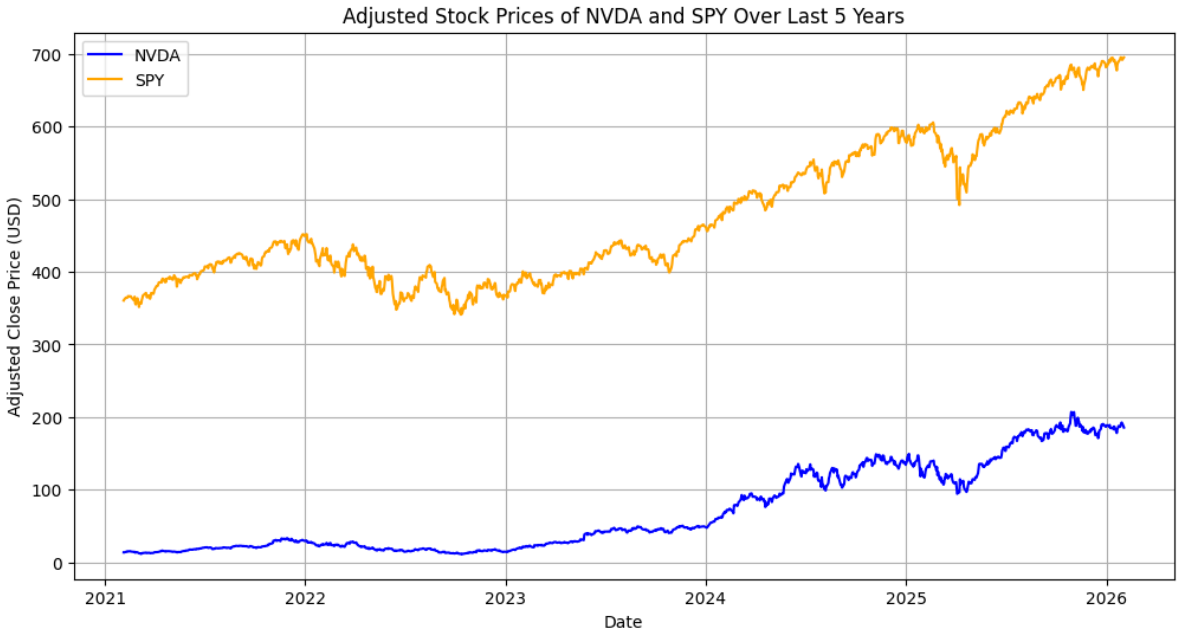
```
[**********************100%**********************]  2 of 2 completed
```

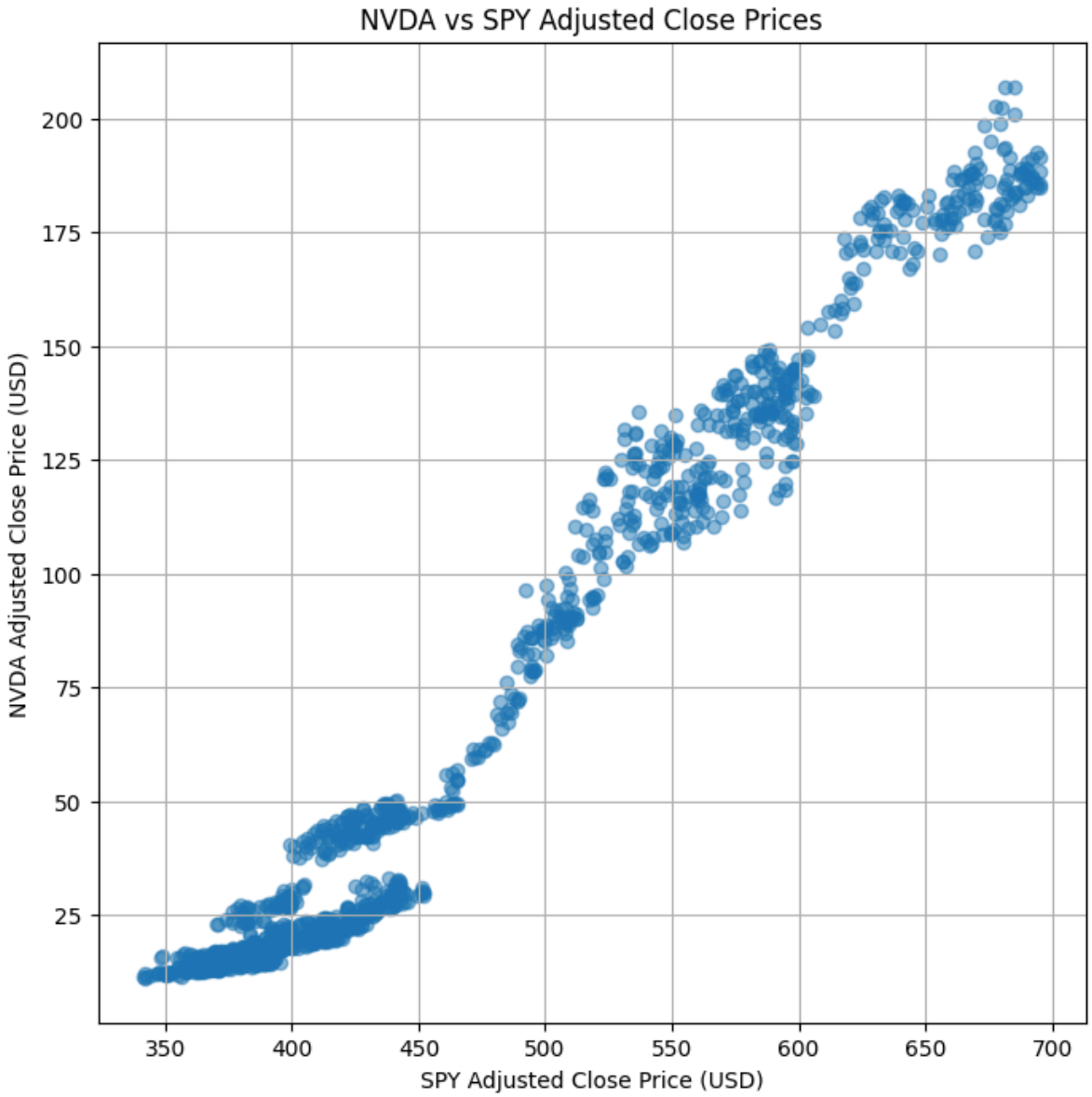
2

i

In [7]:
```
# plot adjusted stock prices as a function of time for both NVDA and SPY
plt.figure(figsize=(12, 6))
plt.plot(NVDA.index, NVDA.values, label='NVDA', color='blue')
plt.plot(SPY.index, SPY.values, label='SPY', color='orange')
plt.title('Adjusted Stock Prices of NVDA and SPY Over Last 5 Years')
plt.xlabel('Date')
plt.ylabel('Adjusted Close Price (USD)')
plt.legend()
plt.grid()
plt.show()

# scatter plot of NVDA versus SPY on the same trading day
plt.figure(figsize=(8, 8))
plt.scatter(SPY.values, NVDA.values, alpha=0.5)
plt.title('NVDA vs SPY Adjusted Close Prices')
plt.xlabel('SPY Adjusted Close Price (USD)')
plt.ylabel('NVDA Adjusted Close Price (USD)')
plt.grid()
plt.show()
```


Adjusted Stock Prices of NVDA and SPY Over Last 5 Years
NVDA
SPY
Adjusted Close Price (USD)
0
100
200
300
400
500
600
700
2021
2022
2023
2024
2025
2026
Date



The two stock trends seem to be extremely similar, rising and dropping at relatively the same times.

In [5]:
```
from statsmodels.api import OLS, add_constant
X = add_constant(SPY.values)
model = OLS(NVDA.values, X).fit()
print(model.summary())
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.970
Model:                            OLS   Adj. R-squared:                  0.970
Method:                 Least Squares   F-statistic:                 4.089e+04
Date:                Tue, 03 Feb 2026   Prob (F-statistic):               0.00
Time:                        17:10:47   Log-Likelihood:                -4693.6
No. Observations:                1255   AIC:                             9391.
Df Residuals:                    1253   BIC:                             9401.
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const       -211.7439      1.424   -148.724      0.000    -214.537    -208.951
x1             0.5926      0.003    202.224      0.000       0.587       0.598
==============================================================================
Omnibus:                       20.789   Durbin-Watson:                   0.063
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               14.635
Skew:                          -0.147   Prob(JB):                     0.000664
Kurtosis:                       2.561   Cond. No.                     2.40e+03
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly spe
cified.
[2] The condition number is large, 2.4e+03. This might indicate that there are
strong multicollinearity or other numerical problems.
```

Intercept: -211.8703, SE 1.425, p < 0.0001
This is the predicted value of NVDA when SPY is valued at $0. This has no meaning in it of itself, however given its extremely low p-value, is obviously extremely relevant to the model as it anchors the regression line. The SE is also extremely low, indicating a well estimated value.

Coefficient: 0.5929, SE 0.003, p < 0.0001
For every $1 increase in SPY, NVDA is projected to increase by 0.5929. With an extremely low SE and p-value, this is a positive and highly significant correlation.

R-Squared & Adjusted: 0.970 This is extremely high, meaning that 97% of the variation in NVDA prices are explained by SPY prices.

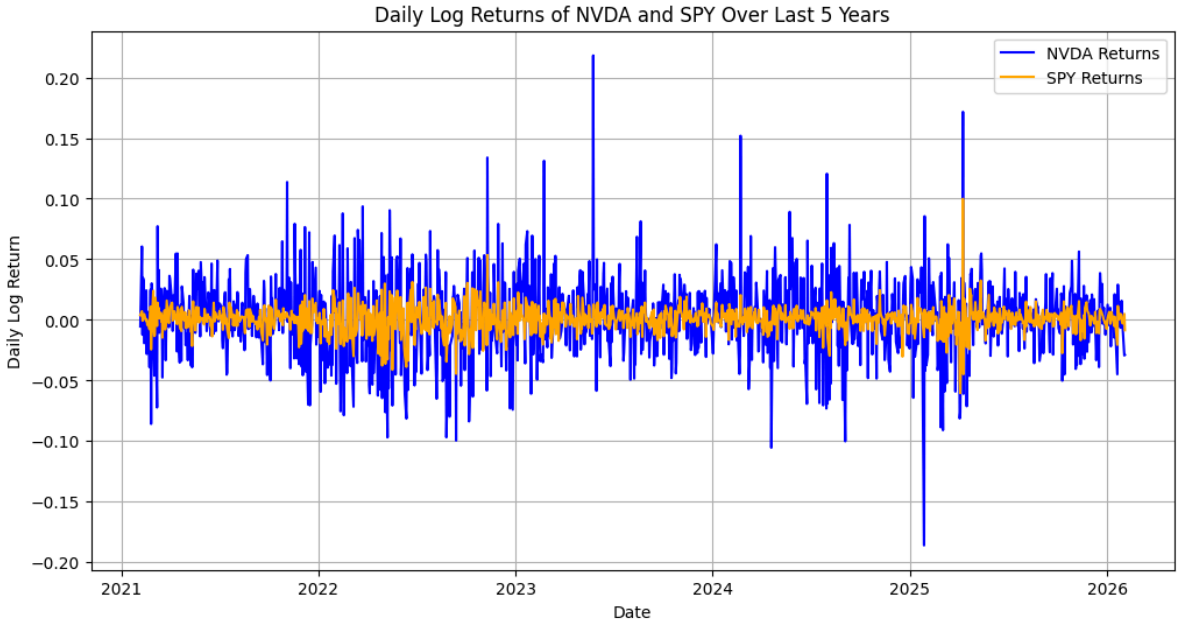
The very high R^2, t score, and confidence/p-value in the coefficients indicate a strong linear correlation between NVDA and SPY price. However, the Durbin-Watson score is 0.063. This is extremely low and a red flag in our model, as this indicates extreme positive autocorrelation, violating the independent error assumption, leading to an underestimation of the standard errors, which leads to inflated t-statistics and the false appearance of statistical significance. Thus, while the conditional mean relationship is well approximated by a linear function, the error structure is misspecified. Consequently, we can also see that the omnibus and JB statistics indicate non Guassian residuals.

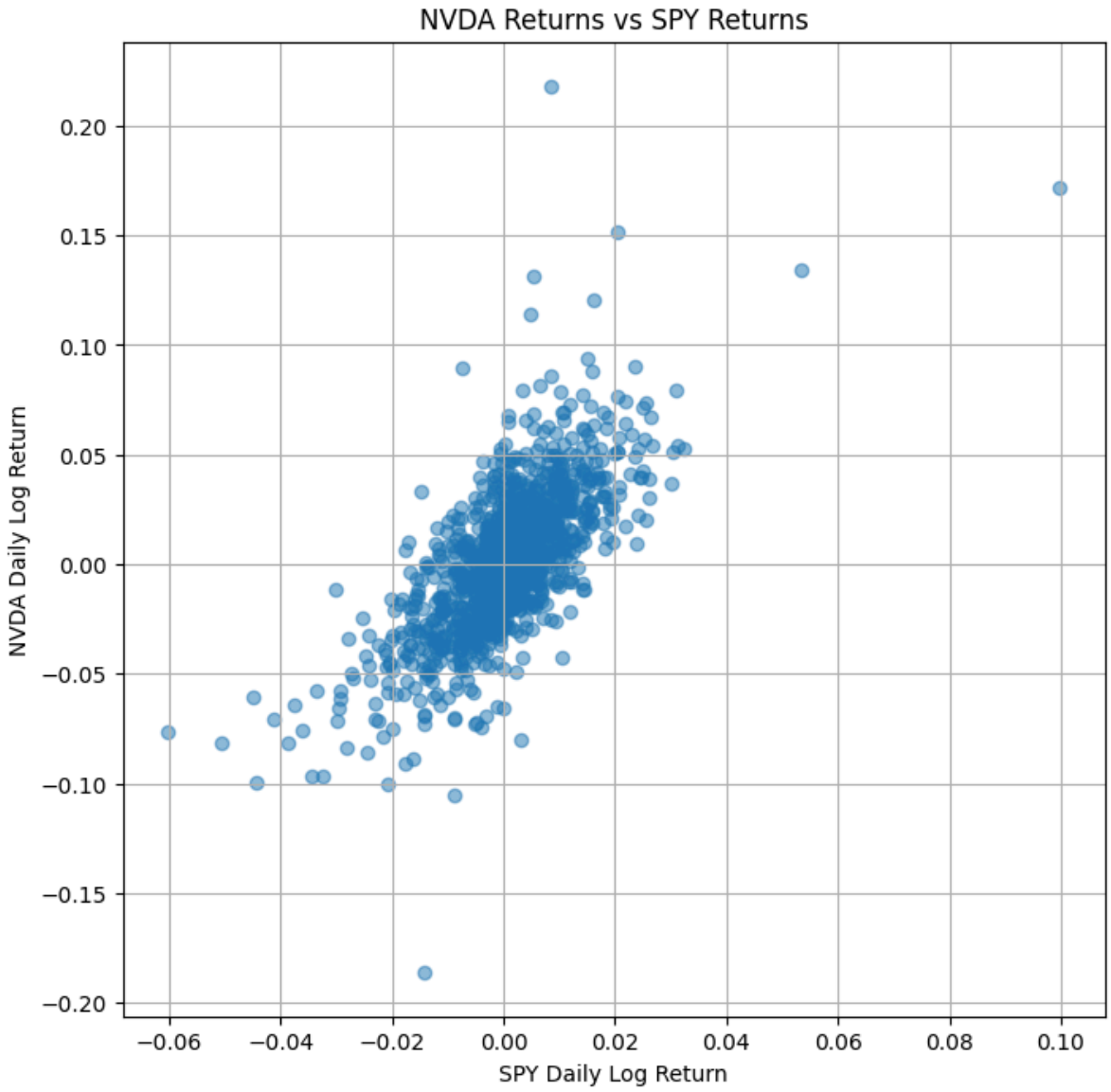
ii

In [6]:
```
# daily log returns of NVDA and SPY
NVDA_returns = np.log(NVDA / NVDA.shift(1))
SPY_returns = np.log(SPY / SPY.shift(1))

# plot daily log returns for both NVDA and SPY
plt.figure(figsize=(12, 6))
plt.plot(NVDA_returns.index, NVDA_returns.values, label='NVDA Returns', color='blue
plt.plot(SPY_returns.index, SPY_returns.values, label='SPY Returns', color='orange'
plt.title('Daily Log Returns of NVDA and SPY Over Last 5 Years')
plt.xlabel('Date')
plt.ylabel('Daily Log Return')
plt.legend()
plt.grid()
plt.show()

# scatter plot of NVDA returns versus SPY returns on the same trading day
plt.figure(figsize=(8, 8))
plt.scatter(SPY_returns.values, NVDA_returns.values, alpha=0.5)
plt.title('NVDA Returns vs SPY Returns')
plt.xlabel('SPY Daily Log Return')
plt.ylabel('NVDA Daily Log Return')
plt.grid()
plt.show()
```





In [7]:
```
# fit CAPM regression r_theta_NVDA = alpha + beta * r_theta_SPY + epsilon_t
X_returns = add_constant(SPY_returns.values[1:])  # exclude NaN from shift
model_returns = OLS(NVDA_returns.values[1:], X_returns).fit()
print(model_returns.summary())
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.495
Model:                            OLS   Adj. R-squared:                  0.495
Method:                 Least Squares   F-statistic:                     1228.
Date:                Tue, 03 Feb 2026   Prob (F-statistic):          5.00e-188
Time:                        17:10:53   Log-Likelihood:                 2944.5
No. Observations:                1254   AIC:                            -5885.
Df Residuals:                    1252   BIC:                            -5875.
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0010      0.001      1.457      0.145      -0.000       0.002
x1             2.1393      0.061     35.040      0.000       2.020       2.259
==============================================================================
Omnibus:                      257.416   Durbin-Watson:                   2.115
Prob(Omnibus):                  0.000   Jarque-Bera (JB):             2862.311
Skew:                           0.609   Prob(JB):                         0.00
Kurtosis:                      10.301   Cond. No.                         93.4
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly spe
cified.
```

Intercept: 0.0010, SE 0.001, p 0.142
This is the predicted value of daily return after accounting for market exposure. This is expected as if there is no market activity, there shouldn't be any returns. The p value also indicates it is statistically insignificant, which is also expected.

Coefficient: 2.1386, SE 0.061, p < 0.0001
For every 1% daily return in SPY, NVDA is projected to return 2.14%. With a low SE and low p-value, the coefficient is well estimated and statistically significant.

R-Squared & Adjusted: 0.495 About half of NVDA's daily returns variation is explained by SPY's returns.

There is no direct evidence of nonlinearity, even given a low R^2. The Durbin-Watson statistic is also satisfactory, indicating no autocorrelation. However, the omnibus and JB statistic still point to non-Gaussian residuals, but this doesn't directly point to nonlinearity issues.

iii

In [12]:
```
influence = model_returns.get_influence()
cooks_d = influence.cooks_distance[0]
influential_points = np.argsort(cooks_d)[-3:]
influential_dates = NVDA_returns.index[influential_points + 1]
print("Most influential observations (dates):")
```

```
for date in influential_dates:
    print(date.date())
```

```
Most influential observations (dates):
2025-01-27
2025-04-04
2025-04-09
```

2025-01-27: DeepSeek-R1 was released a week before, gaining global traction. Consequently, NVDA stocks dropped as investors were worried at how the low cost model would impact leading AI companies such as Nvidia. Reuters

2025-04-04: Trump announces tariffs, leading to massive drops in the American market. This obviously would affect SPY (S&P 500) companies disproportionately. NVDA would also feel pressure due to its global supply chain and from international revenue. Reuters

2025-04-09: The Trump administration announces the 90 day pause on a majority of the previously announced tariffs, leading to some market recovery as investors take this as a sign of deescalation / market relief. NPR

iv

From (ii): **Intercept**: 0.0010, SE 0.001, p 0.142
This is the predicted value of daily return after accounting for market exposure. This is expected as if there is no market activity, there shouldn't be any returns. The p value also indicates it is statistically insignificant, which is also expected.

Coefficient: 2.1386, SE 0.061, p < 0.0001
For every 1% daily return in SPY, NVDA is projected to return 2.14%. With a low SE and low p-value, the coefficient is well estimated and statistically significant.

We are given the CAPM model:

$$\log\left(\frac{P_t^{\text{NVDA}}}{P_{t-1}^{\text{NVDA}}}\right) = \hat{\alpha} + \hat{\beta}\log\left(\frac{P_t^{\text{SPY}}}{P_{t-1}^{\text{SPY}}}\right) + \epsilon_t$$

Exponentiating both sides gives:

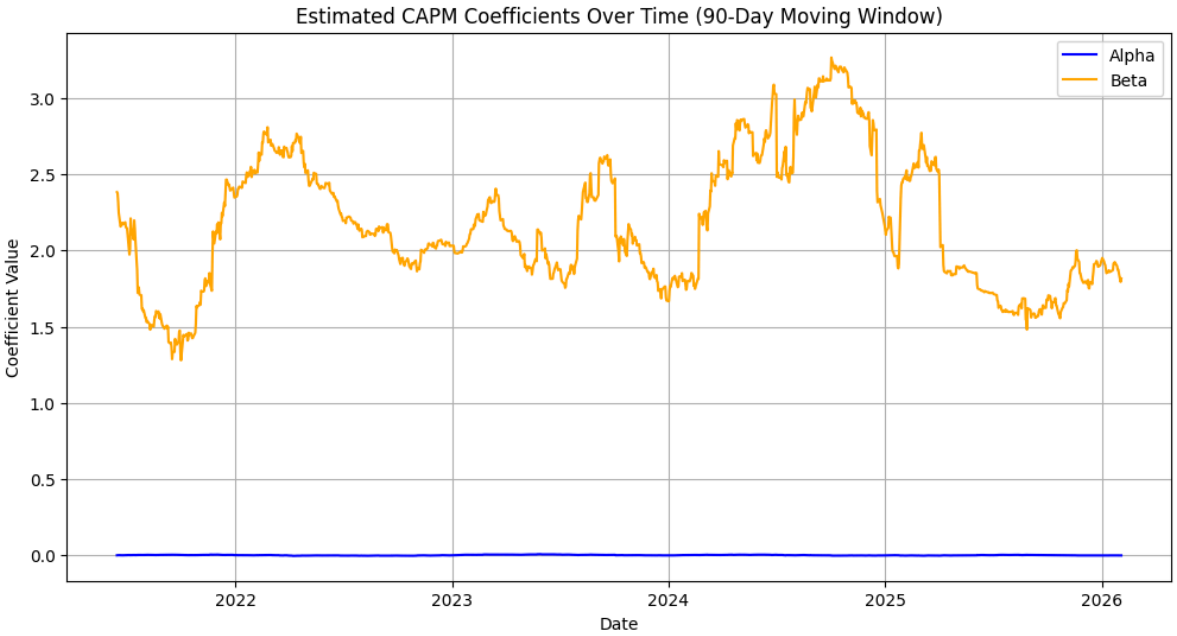
$$\frac{P_t^{\text{NVDA}}}{P_{t-1}^{\text{NVDA}}} = \exp(\hat{\alpha})\left(\frac{P_t^{\text{SPY}}}{P_{t-1}^{\text{SPY}}}\right)^{\hat{\beta}}\exp(\epsilon_t)$$

Because $\hat{\alpha} \approx 0$, $\exp(\hat{\alpha}) \approx 1$. Then on a price scale, $\frac{\partial \log P_t^{\text{NVDA}}}{\partial \log P_t^{\text{SPY}}} = \hat{\beta} \approx 2.14$, thus we can see where a 1% increase in SPY's daily log returns is associated with a 2.14% increase in NVDA's daily log returns. $\hat{\beta}$ becomes an elasticity factor comparing SPY and NVDA returns.

v

In [ ]:
```
window_size = 90
alphas = []
betas = []
dates = NVDA_returns.index[window_size:]
for i in range(len(NVDA_returns) - window_size):
    X_window = add_constant(SPY_returns.values[i+1:i+window_size+1])  # exclude NaN
    y_window = NVDA_returns.values[i+1:i+window_size+1]
    model_window = OLS(y_window, X_window).fit()
    alphas.append(model_window.params[0])
    betas.append(model_window.params[1])

plt.figure(figsize=(12, 6))
plt.plot(dates, alphas, label='Alpha', color='blue')
plt.plot(dates, betas, label='Beta', color='orange')
plt.title('Estimated CAPM Coefficients Over Time (90-Day Moving Window)')
plt.xlabel('Date')
plt.ylabel('Coefficient Value')
plt.legend()
plt.grid()
plt.show()
```



We see that the beta value fluctuates over time, varying around 2, rising during technology booms around AI, and declining during market stress (initial COVID, slow recovery, Trump takes office, market fluctuations as found in part (iii)). The alpha (intercept) remains around zero, consistent with transitory firm-specific effects rather than sustained abnormal returns. These results indicate that NVDA's relationship to the market is regime-dependent, violating the constant-beta assumption of the static CAPM and highlighting the importance of time-varying risk exposure.

LLM Usage: All work was done by myself in VSCode with GitHub Copilot integration. The integration "provides code suggestions, explanations, and automated implementations based on natural language prompts and existing code context," and also offers autonomous coding and an in-IDE chat interface that is able to interact with the current codebase. Only the Copilot provided automatic inline suggestions for both LaTex and Python in `.tex` and `.ipynb` Jupyter notebook files respectively were taken into account / used.