HW 2

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Let X and Y be bivariate Gaussian with mean (μ_X, μ_Y), common variance σ^2, and correlation coefficient $\rho > 0$. The bivariate mean is random and determined by Z, where

$$(\mu_X, \mu_Y) = \begin{cases} (-\mu, \mu) & Z = -1, \\ (\mu, -\mu) & Z = 1 \end{cases}$$

with equal probability for $Z = -1$ and $Z = 1$.

1

(a) The conditional expectation $E[Y|X, Z]$ can be calculated as:

$$E[Y|X, Z] = \mu_Y + \frac{Cov(X, Y)}{Var(X)}(X - \mu_X)$$

Since $\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$, we have:

$$\rho = \frac{Cov(X, Y)}{\sqrt{\sigma^2 \dot{\sigma}^2}}$$
$$Cov(X, Y) = \rho\sigma^2$$

Substituting this back into our equation:

$$E[Y|X, Z] = \mu_Y + \frac{\rho\sigma^2}{\sigma^2}(X - \mu_X)$$
$$E[Y|X, Z] = \mu_Y + \rho(X - \mu_X)$$

$\therefore$

$$E[Y|X, Z] = \begin{cases} \mu - \rho(X + \mu) & Z = -1, \\ -\mu + \rho(X - \mu) & Z = 1 \end{cases}$$

The CEF consequently is linear in X for each fixed Z.
The conditional expectation $E[Y|Z]$ is:

$$E[Y|Z] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(E[X|Z] - \mu_X)$$
$$E[Y|Z] = \mu_Y + \rho(\mu_X - mu_X)$$
$$E[Y|Z] = \mu_Y$$
$$E[Y|Z] = \begin{cases} \mu & Z = -1, \\ -\mu & Z = 1 \end{cases}$$

The CEF consequently is constant for each fixed Z.
The conditional expectation $E[Y|X]$ is:

$$E[Y|X] = E[E[Y|X,Z]|X] = [-\mu + \rho(X - \mu)] \cdot P(Z = 1|X) + [\mu - \rho(X + \mu)] \cdot P(Z = -1|X)$$
$$P(Z = 1|X) = \frac{P(X|Z = 1)P(Z = 1)}{P(X|Z = 1)P(Z = 1) + P(X|Z = -1)P(Z = -1)}$$
$$P(Z = 1|X) = \frac{\exp\left(-\frac{(X-\mu)^2}{2\sigma^2}\right) \cdot \frac{1}{2}}{\exp\left(-\frac{(X-\mu)^2}{2\sigma^2}\right) \cdot \frac{1}{2} + \exp\left(-\frac{(X+\mu)^2}{2\sigma^2}\right) \cdot \frac{1}{2}}$$
$$P(Z = 1|X) = \frac{1}{1 + \exp\left(-\frac{2\mu}{\sigma^2}X\right)}$$
$$P(Z = -1|X) = 1 - P(Z = 1|X) \therefore$$
$$E[Y|X] = [-\mu + \rho(X - \mu)] \cdot \frac{1}{1 + \exp\left(-\frac{2\mu}{\sigma^2}X\right)} + [\mu - \rho(X + \mu)] \cdot \left(1 - \frac{1}{1 + \exp\left(-\frac{2\mu}{\sigma^2}X\right)}\right)$$
$$\text{Let } p = \frac{1}{1 + \exp\left(-\frac{2\mu}{\sigma^2}X\right)} \therefore$$
$$E[Y|X] = [-\mu + \rho(X - \mu)] \cdot p + [\mu - \rho(X + \mu)] \cdot (1 - p)$$
$$E[Y|x] = -\mu p + \rho X p - \rho\mu p + \mu - \mu p - \rho X + \rho X p - \rho\mu + \rho\mu p$$
$$E[Y|X] = \rho X + \mu(1 - \rho)(1 - 2p)$$
$$E[Y|X] = \rho X + \mu(1 - \rho)\left(1 - 2 \cdot \frac{1}{1 + \exp\left(-\frac{2\mu}{\sigma^2}X\right)}\right)$$

The CEF consequently is not linear in X.

(b) See attached Jupyter notebook.

2

(a) The best linear approximations for the CEFs $E[Y|X,Z], E[Y|Z]$ are their literal lines as they are already linear, and respectively thus are:

$$E^*[Y|X,Z] = E[Y|X,Z] = \begin{cases} \mu - \rho(X+\mu) = 1 - \frac{1}{2}(X+1) & Z = -1, \\ -\mu + \rho(X-\mu) = -1 + \frac{1}{2}(X-1) & Z = 1 \end{cases}$$

$$E^*[Y|Z] = E[Y|Z] = \begin{cases} \mu = 1 & Z = -1, \\ -\mu = -1 & Z = 1 \end{cases}$$

The best linear approximation for the CEF $E[Y|X]$ can be found as:

$$a + bX$$

$$b = \frac{Cov(X, E[Y|X])}{Var(X)} = \frac{E[Cov(X,Y|Z)] + Cov(E[X|Z], E[Y|Z])}{Var(X)}$$

$$= \frac{\rho\sigma^2 - \mu^2}{\sigma^2 + \mu^2}$$

$$a = E[E[Y|X]] - bE[X] = 0 - b \cdot 0 = 0$$

$$\therefore$$

$$E^*[Y|X] = \frac{\rho\sigma^2 - \mu^2}{\sigma^2 + \mu^2} X$$

(b) See attached Jupyter notebook.

3

(a) The conditional residual variance $Var[Y - E[Y|X,Z]|X,Z]$ can be calculated as:

$$\begin{aligned} Var[Y - E[Y|X,Z]|X,Z] &= Var[Y|X,Z] \\ &= Var[Y] - \frac{Cov(X,Y)^2}{Var(X)} \\ &= \sigma^2 - \frac{(\rho\sigma^2)^2}{\sigma^2} \\ &= \sigma^2(1-\rho^2) \end{aligned}$$

The conditional residual variance $Var[Y - E[Y|Z]|Z]$ can be calculated as:

$$\begin{aligned} Var[Y - E[Y|Z]|Z] &= Var[Y|Z] \\ &= Var[Y] = \sigma^2 \end{aligned}$$

The conditional residual variance $Var[Y - E[Y|X]|X]$ can be calculated as:

$$\begin{aligned}
Var[Y - E[Y|X]|X] &= E[Var[Y|X,Z]|X] + Var[E[Y|X,Z] - E[Y|X]|X] \\
&= E[Var[Y|X,Z]|X] + Var[E[Y|X,Z]|X] \\
&= \sigma^2(1-\rho^2) + p(1-p)(E[Y|X,Z=1] - E[Y|X,Z=-1])^2 \\
&= \sigma^2(1-\rho^2) + p(1-p)(2\mu(1+\rho))^2 \\
&\text{where } p \text{ is defined as in (1a)}
\end{aligned}$$

(b) See attached Jupyter notebook.

4

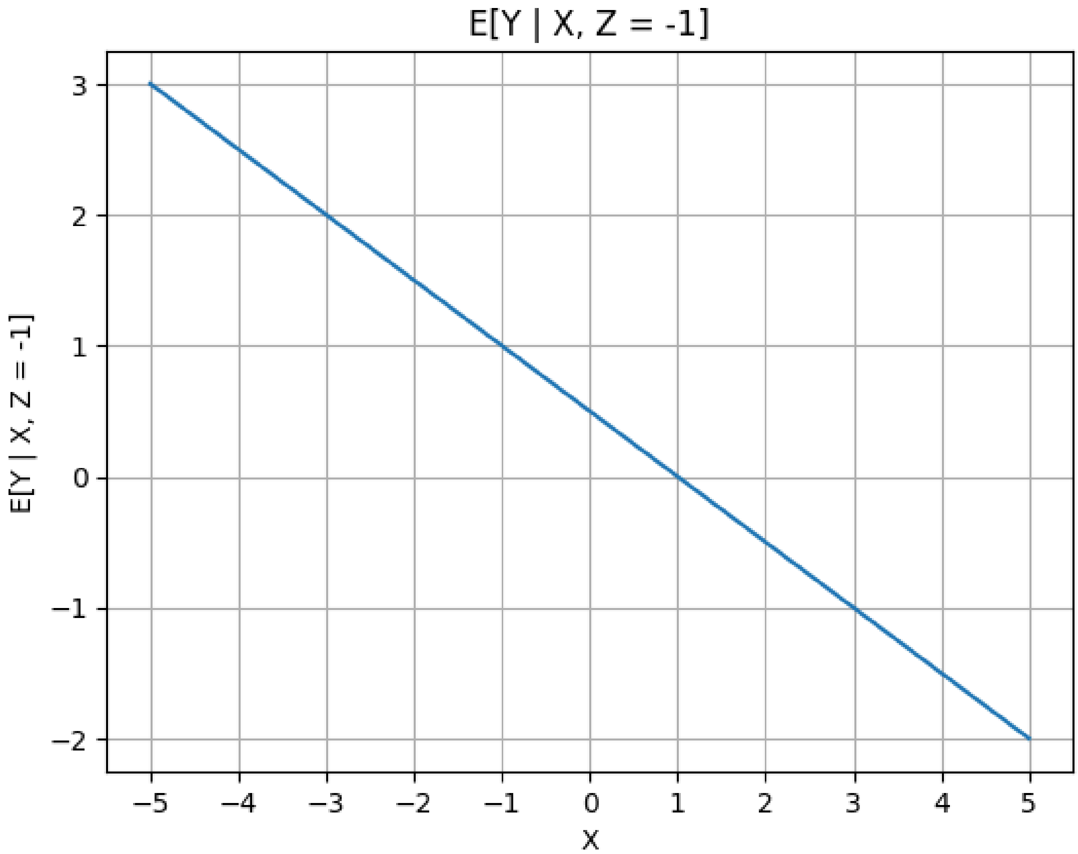
(a) When ignoring Z and looking at each subgroup, the best linear predictor coincides with each CEF, showing a seemingly perfect linear relationship between X and Y for each fixed Z. However, when you actually condition on Z, we realize that the opposite is true: there is no linear relationship between X and Y and we see a negative slope which extremely poorly fits the data. Z is the confounding variable that causes this paradoxical situation.

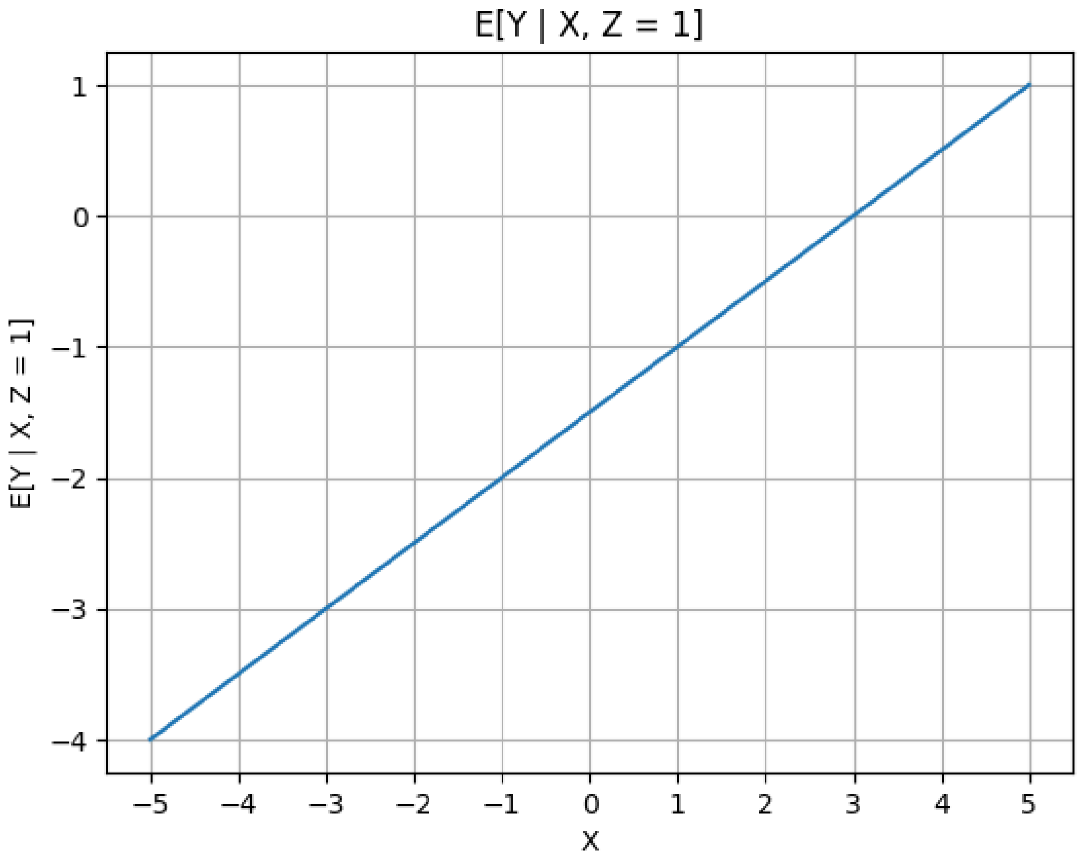
(b) As we can already see when calculating the best linear predictor for $E[Y|X]$, the slope becomes negative even though the data trend is positive. Thus we can conclude that Simpson's paradox arises when $\mu^2 > \rho\sigma^2$ and $\rho > 0$.

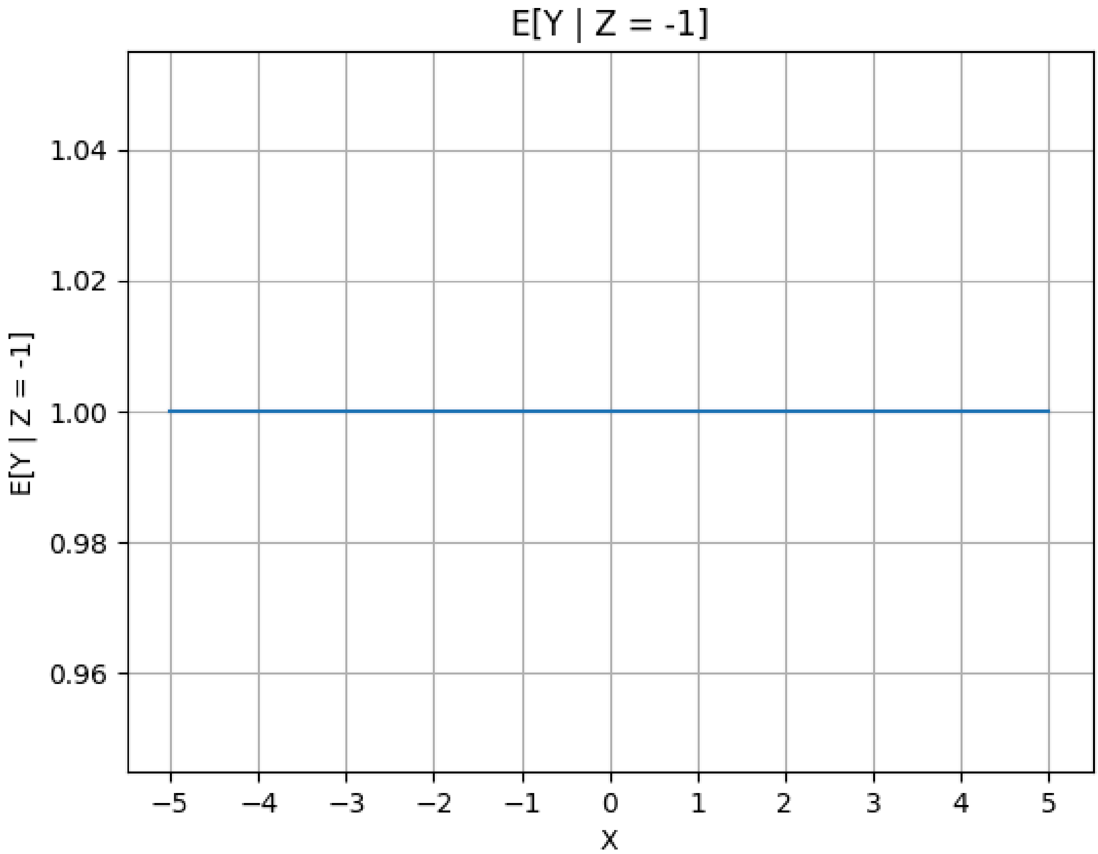
In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

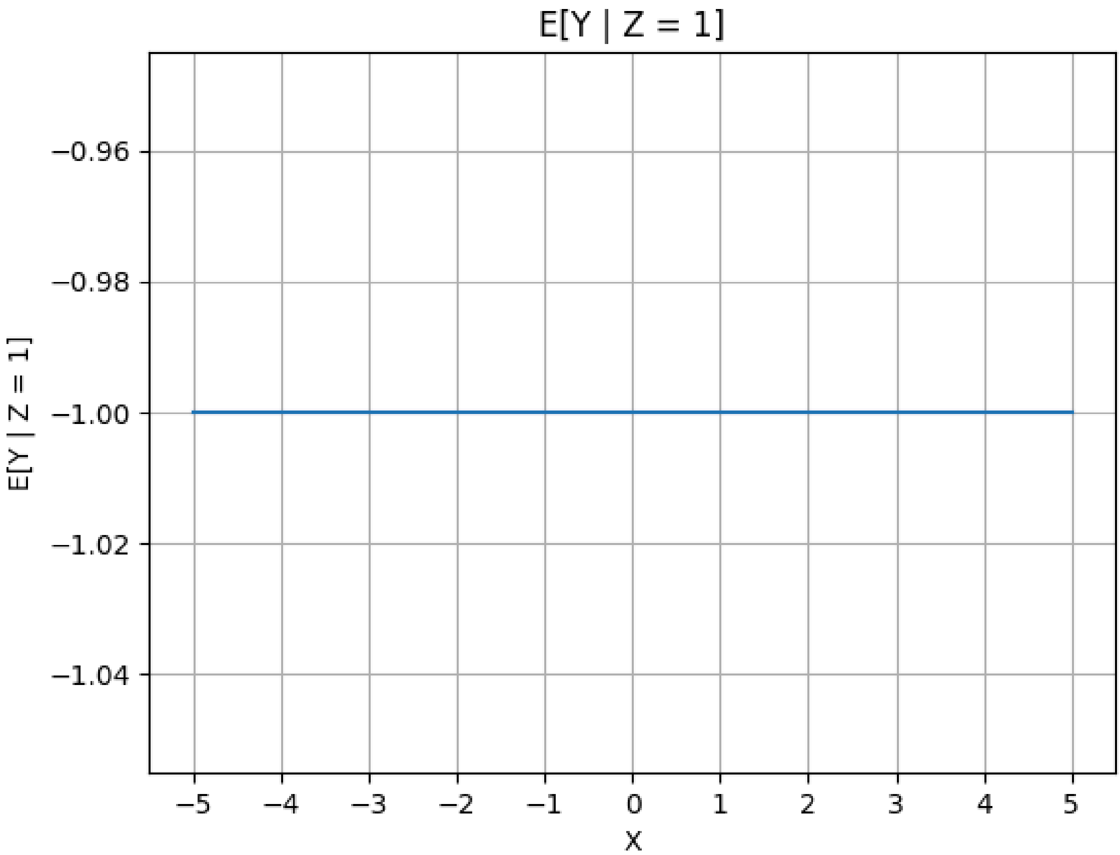
1b

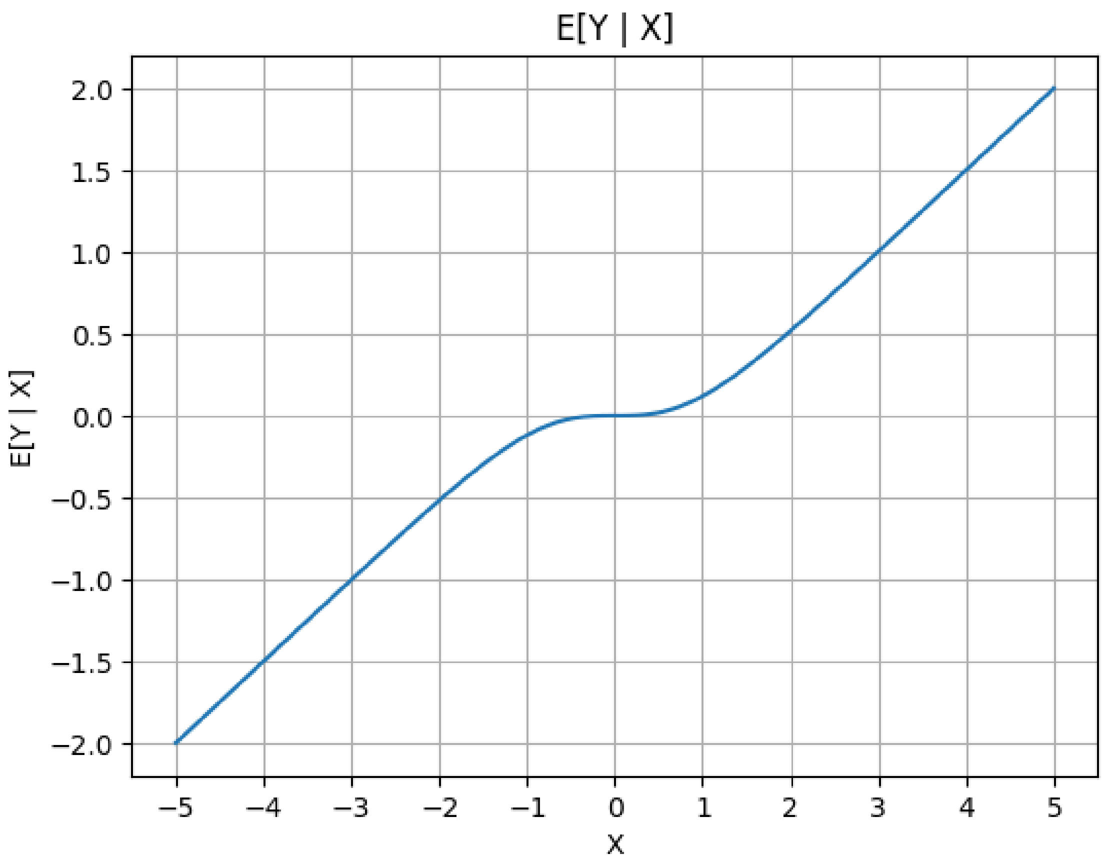
In [4]:
```
mu = 1
sigma_sq = 1
rho = 0.5

X = np.linspace(-5, 5, 100)
E_Y_given_X_Zneg1 = mu - rho * (X + mu)
E_Y_given_X_Z1 = -mu + rho * (X - mu)

E_Y_given_Zneg1 = mu
E_Y_given_Z = -mu

p = (1 / (1 + np.exp(-2 * X)))
E_Y_given_X = rho * X + mu * (1 - rho) * (1 - 2 * p)

def plot_cef(x, y, title):
    plt.plot(x, y)
    plt.title(title)
    plt.xlabel("X")
    plt.xticks(np.arange(min(x), max(x) + 1, 1))
    plt.ylabel(title)
    plt.grid(True)
    plt.show()

plot_cef(X, E_Y_given_X_Zneg1, "E[Y | X, Z = -1]")
plot_cef(X, E_Y_given_X_Z1, "E[Y | X, Z = 1]")
plot_cef(X, E_Y_given_Zneg1 * np.ones_like(X), "E[Y | Z = -1]")
plot_cef(X, E_Y_given_Z * np.ones_like(X), "E[Y | Z = 1]")
plot_cef(X, E_Y_given_X, "E[Y | X]")
```


E[Y | X, Z = -1]
E[Y | X, Z = -1]
X
3
2
1
0
−1
−2
−5
−4
−3
−2
−1
0
1
2
3
4
5


E[Y | X, Z = 1]
E[Y | X, Z = 1]
X
1
0
−1
−2
−3
−4
−5
−4
−3
−2
−1
0
1
2
3
4
5

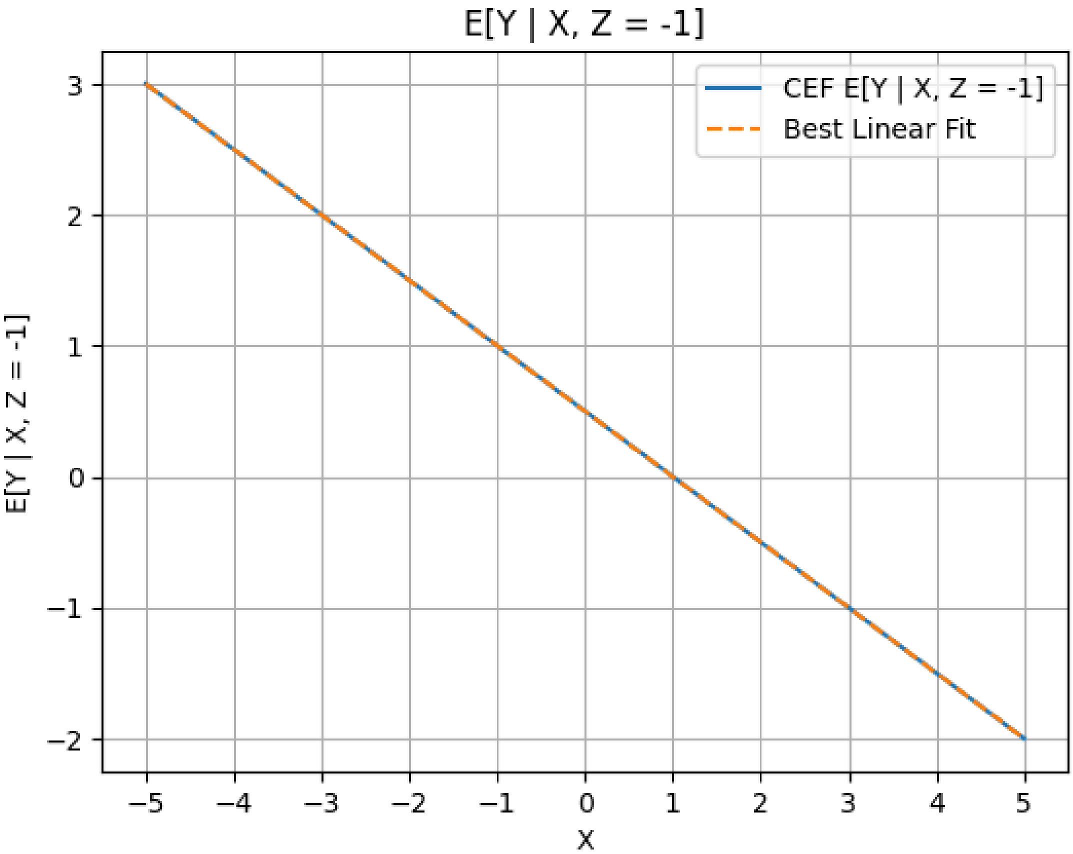

E[Y | Z = -1]
1.04
1.02
1.00
0.98
0.96
E[Y | Z = -1]
−5 −4 −3 −2 −1 0 1 2 3 4 5
X

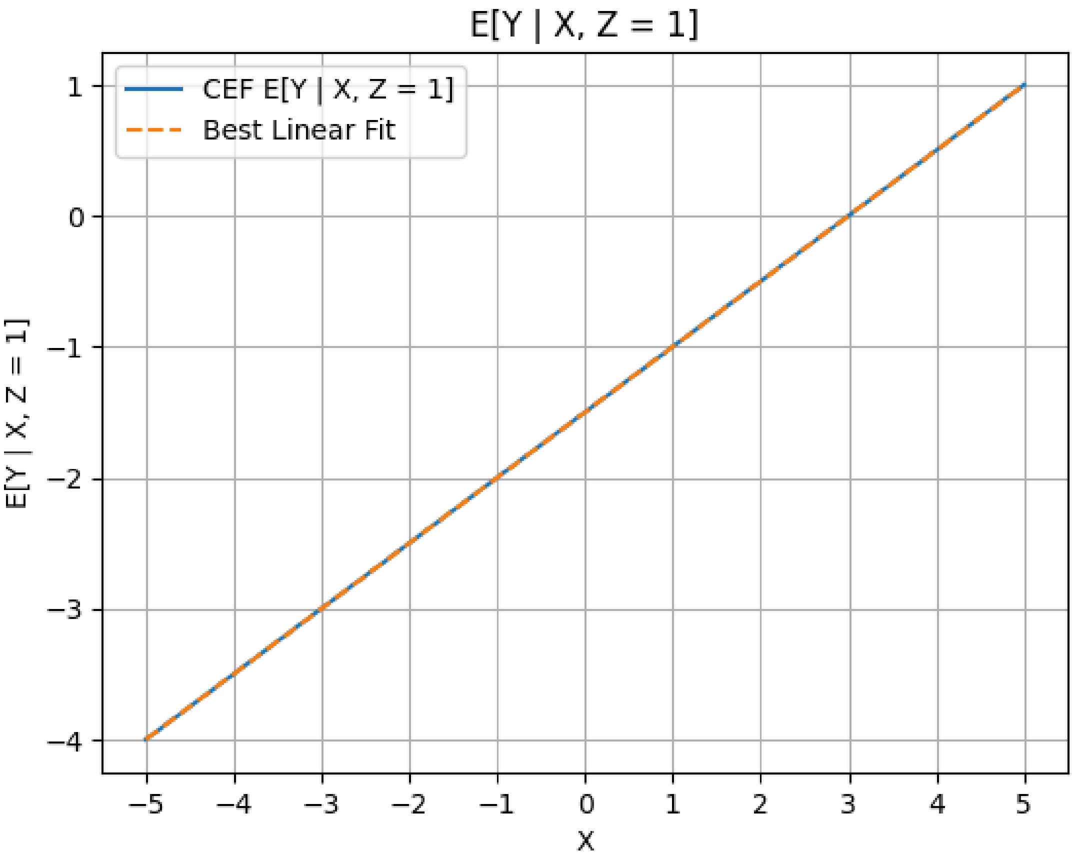

E[Y | Z = 1]
−0.96
−0.98
−1.00
−1.02
−1.04
E[Y | Z = 1]
−5 −4 −3 −2 −1 0 1 2 3 4 5
X

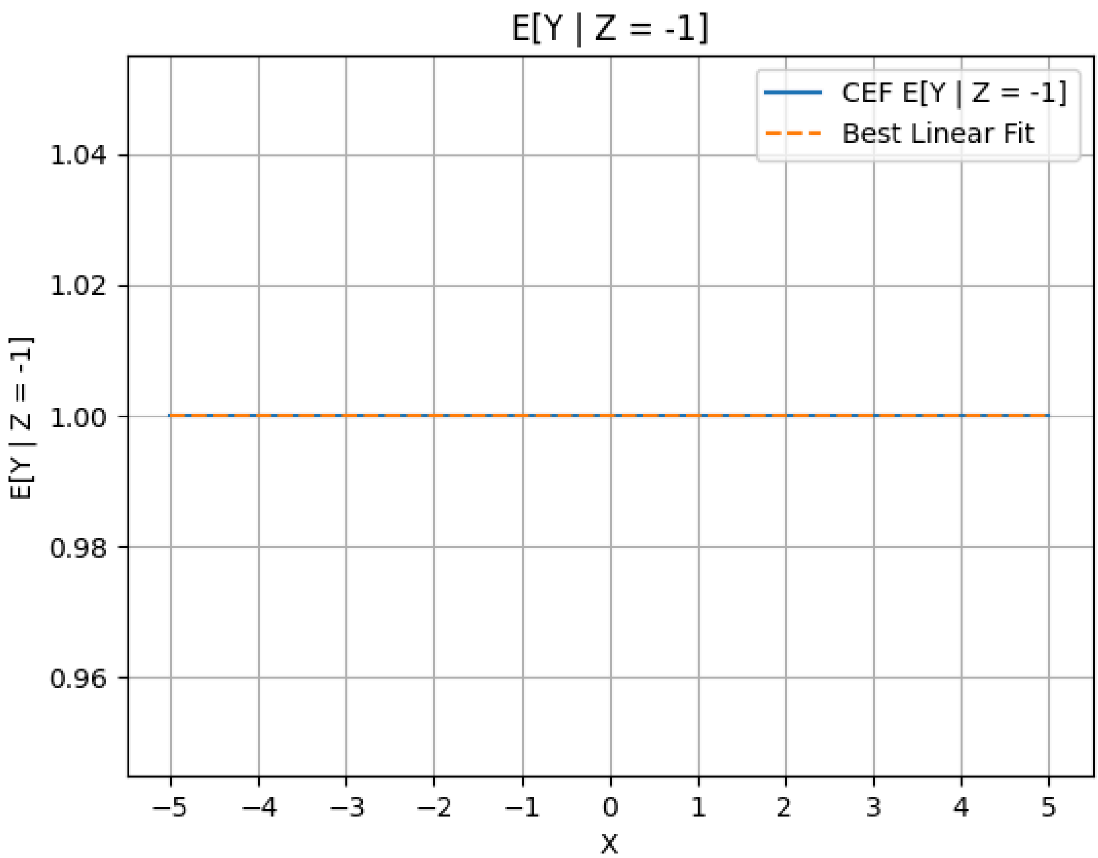


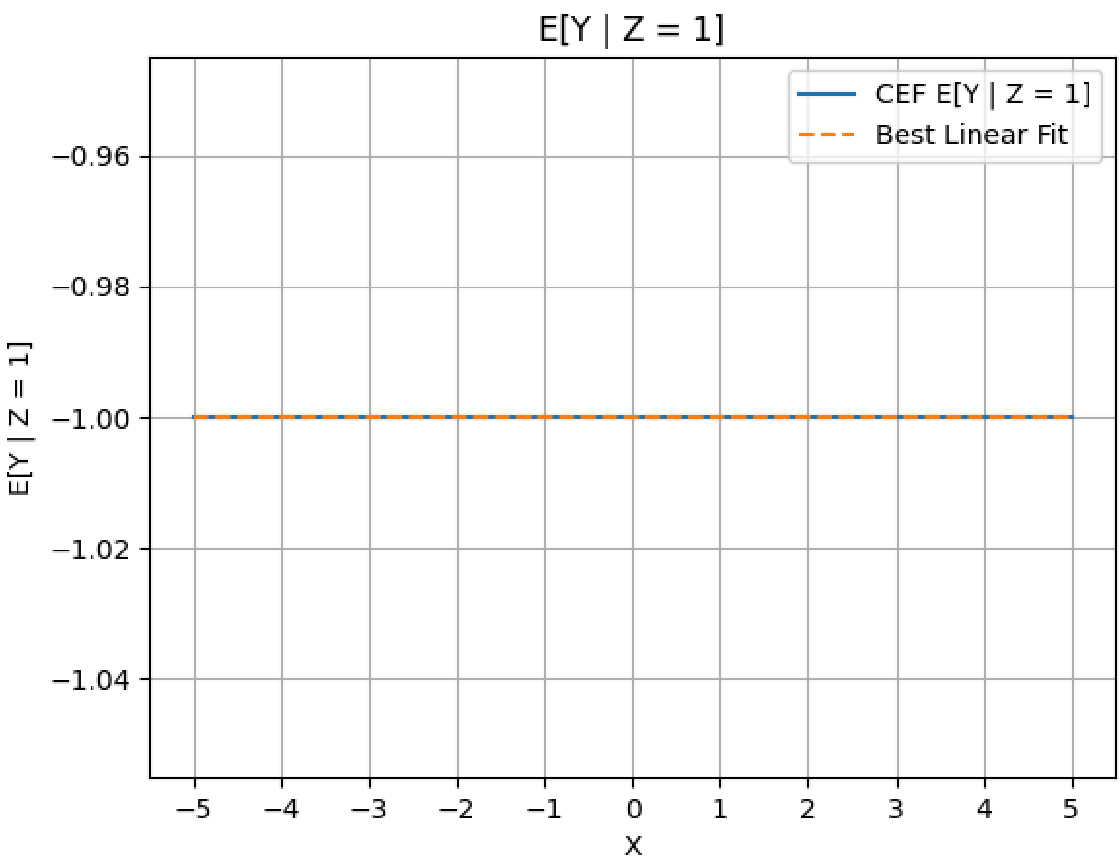
2b

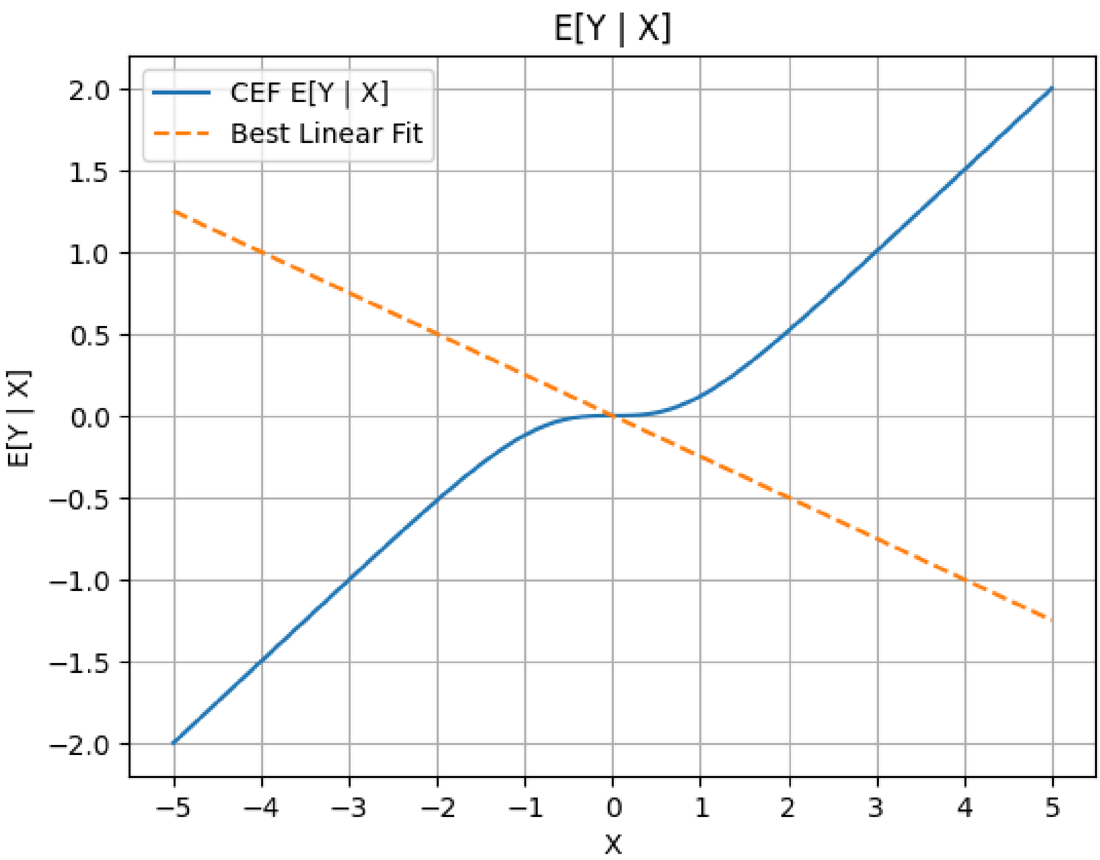
In [5]:
```
def plot_cef_bestfit(x, y, bestfit, title):
    plt.plot(x, y, label="CEF " + title)
    plt.plot(x, bestfit, linestyle='--', label="Best Linear Fit")
    plt.title(title)
    plt.xlabel("X")
    plt.xticks(np.arange(min(x), max(x) + 1, 1))
    plt.ylabel(title)
    plt.grid(True)
    plt.legend()
    plt.show()

plot_cef_bestfit(X, E_Y_given_X_Zneg1, 1 - 0.5 * (X + 1), "E[Y | X, Z = -1]")
plot_cef_bestfit(X, E_Y_given_X_Z1, -1 + 0.5 * (X - 1), "E[Y | X, Z = 1]")
plot_cef_bestfit(X, E_Y_given_Zneg1 * np.ones_like(X), np.ones_like(X), "E[Y | Z =
plot_cef_bestfit(X, E_Y_given_Z * np.ones_like(X), -np.ones_like(X), "E[Y | Z = 1]"
plot_cef_bestfit(X, E_Y_given_X, (rho * sigma_sq - mu**2) * X / (sigma_sq + mu**2),
```


E[Y | X, Z = -1]
CEF E[Y | X, Z = -1]
Best Linear Fit
E[Y | X, Z = -1]
X
3
2
1
0
−1
−2
−5
−4
−3
−2
−1
0
1
2
3
4
5


E[Y | X, Z = 1]
CEF E[Y | X, Z = 1]
Best Linear Fit
E[Y | X, Z = 1]
X
1
0
−1
−2
−3
−4
−5
−4
−3
−2
−1
0
1
2
3
4
5


E[Y | Z = -1]
CEF E[Y | Z = -1]
Best Linear Fit
1.04
1.02
1.00
0.98
0.96
E[Y | Z = -1]
−5
−4
−3
−2
−1
0
1
2
3
4
5
X

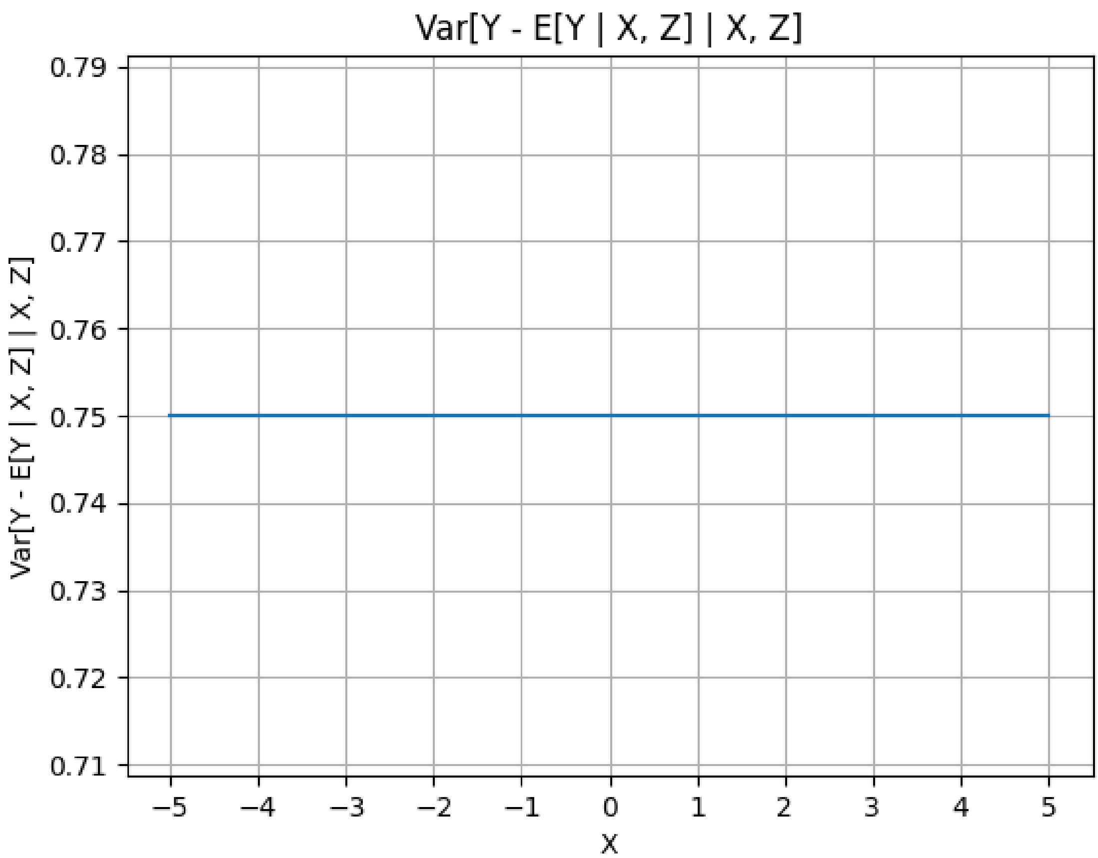

E[Y | Z = 1]
CEF E[Y | Z = 1]
Best Linear Fit
−0.96
−0.98
−1.00
−1.02
−1.04
E[Y | Z = 1]
−5
−4
−3
−2
−1
0
1
2
3
4
5
X

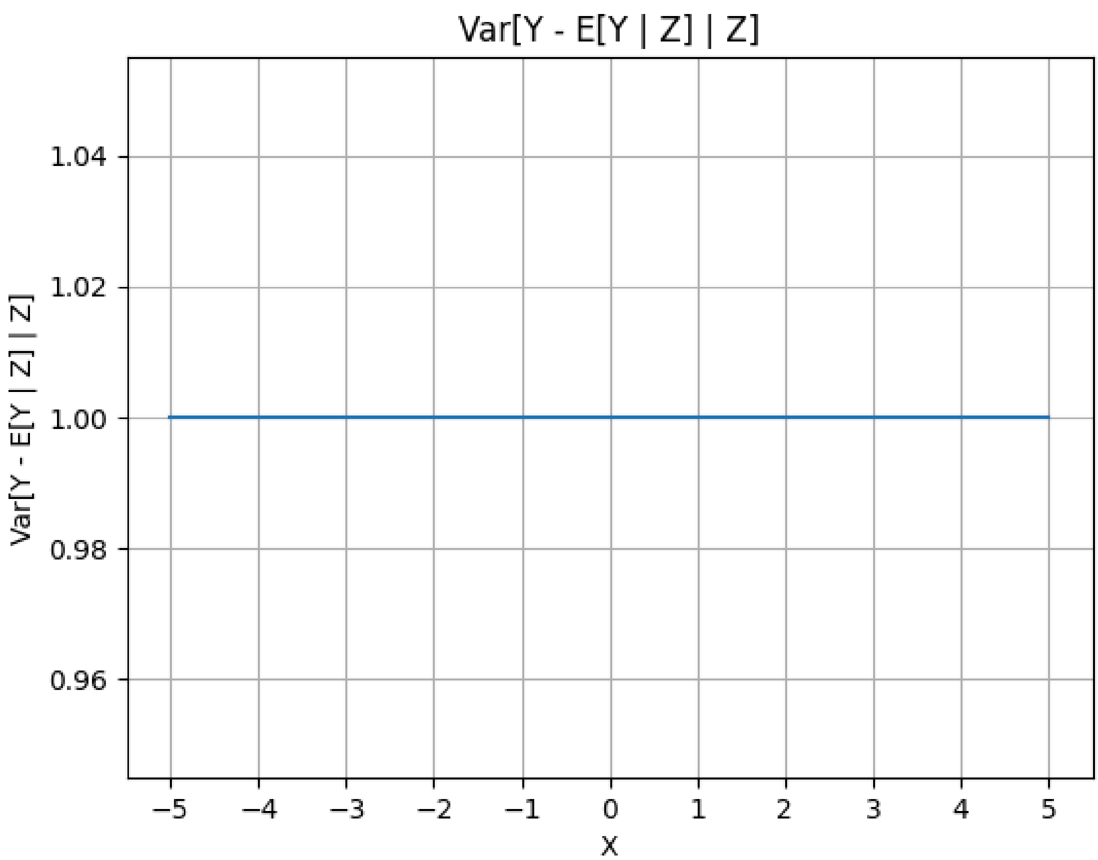


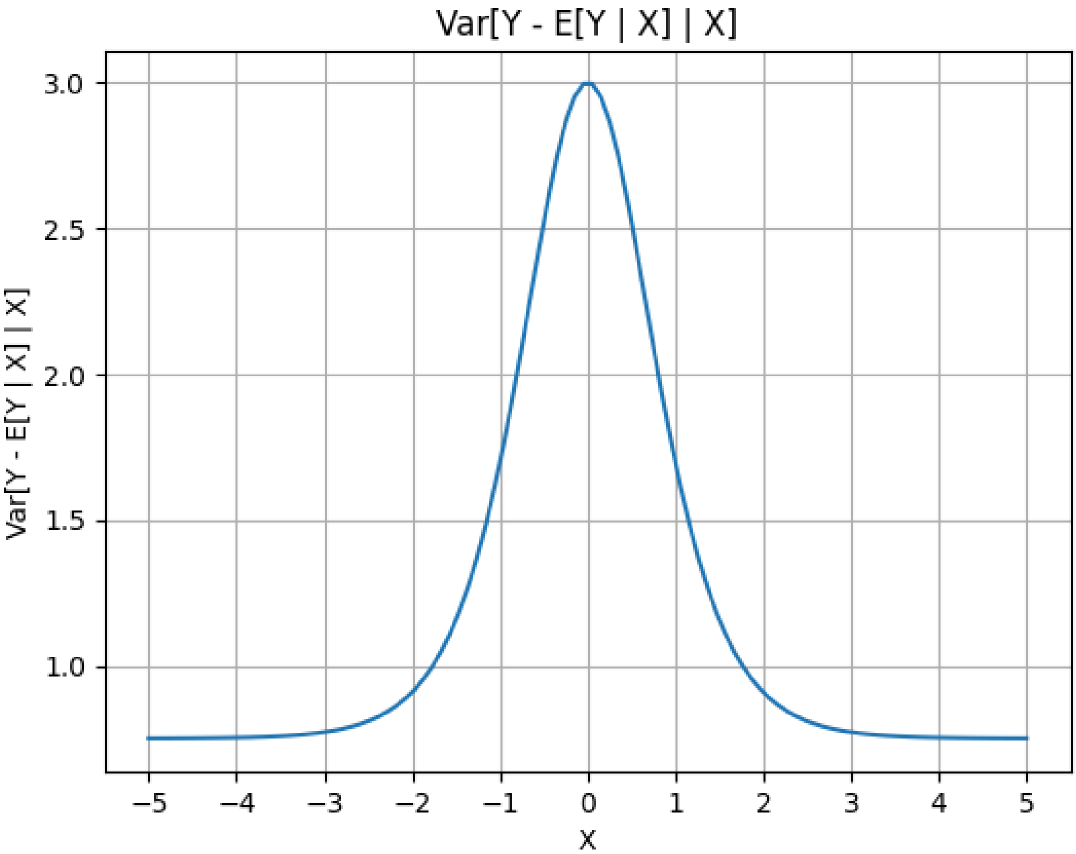
All of the fits make sense and are perfect except for $E[Y|X]$. This is indicative of Simpson's paradox.

3b

In [6]:
```
def plot_cef_var(x, var, title):
    plt.plot(x, var)
    plt.title(title)
    plt.xlabel("X")
    plt.xticks(np.arange(min(x), max(x) + 1, 1))
    plt.ylabel(title)
    plt.grid(True)
    plt.show()

plot_cef_var(X, sigma_sq * (1 - rho**2) * np.ones_like(X), "Var[Y - E[Y | X, Z] | X
plot_cef_var(X, sigma_sq * np.ones_like(X), "Var[Y - E[Y | Z] | Z]")
plot_cef_var(X, sigma_sq * (1 - rho**2) + p*(1 - p)*(2*mu*(1 + rho))**2, "Var[Y - E
```


Var[Y - E[Y | X, Z] | X, Z]
0.79
0.78
0.77
0.76
0.75
0.74
0.73
0.72
0.71
Var[Y - E[Y | X, Z] | X, Z]
−5
−4
−3
−2
−1
0
1
2
3
4
5
X


Var[Y - E[Y | Z] | Z]
1.04
1.02
1.00
0.98
0.96
Var[Y - E[Y | Z] | Z]
−5
−4
−3
−2
−1
0
1
2
3
4
5
X



Plots make sense as the variance shouldn't vary for linear CEF's, but non linear CEF should have variable variance around 0 which quickly reaches asymptotic nature as we go further out with X as seen in the last plot for 1a.

LLM Usage: All work was done in VSCode with GitHub Copilot integration. The integration "provides code suggestions, explanations, and automated implementations based on natural language prompts and existing code context," and also offers autonomous coding and an in-IDE chat interface that is able to interact with the current codebase. Only the Copilot provided automatic inline suggestions for both LaTex and Python in `.tex` and `.ipynb` Jupyter notebook files respectively were taken into account / used.