PSET 5

Kevin Lin

5/9/2025

1

A. Near the surface at the center of the tropical cyclone, the winds spiral inward, thus the divergence of the horizontal wind field is negative as the winds are converging.

B. At the center and top of the tropical cyclone, the winds spiral outward, thus the divergence of the horizontal wind field is positive as the winds are diverging.

C. From the surface center of the cyclone, we know the winds are converging, and thus is drawing additional air mass into the cyclone. This air mass is then transported upward by the vertically convecting eye wall, and consequently must "leave" the cyclone and diverges at the top. Thus, the sign of divergence at the surface must be opposite to the sign at the top of the cyclone.

D.

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$\vec{\nabla} \cdot \vec{v} \sim \frac{m/s}{m} + \frac{m/s}{m} = \frac{1}{s}$$

E. Let the radial speed wind in the eyewall be $V_r = -1\ \text{ms}^{-1}$ (negative because converging). Because the diameter of the cyclone is 50 km, the radius of the cyclone is 25 km = 25000 m. The radial speed wind can be expressed in Cartesian vector form as:

$$\vec{V_r} = V_r \cos(\theta)\hat{i} + V_r \sin(\theta)\hat{j}$$

From polar coordinates, we know that:

$$\cos(\theta) = \frac{x}{r},\ \sin(\theta) = \frac{y}{r}$$

where $r = \sqrt{x^2 + y^2}$. Thus, we can express the radial speed wind in Cartesian coordinates as:

$$u = V_r \cos(\theta) = V_r \frac{x}{r} = -\frac{x}{r}$$
$$v = V_r \sin(\theta) = V_r \frac{y}{r} = -\frac{y}{r}$$

Thus, the divergence is:

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial u}{\partial x}\left(-\frac{x}{r}\right) + \frac{\partial v}{\partial y}\left(-\frac{y}{r}\right)$$
$$\frac{\partial u}{\partial x}\left(-\frac{x}{r}\right) = -\frac{\partial u}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = -\left(\frac{r - x\frac{2x}{2r}}{r^2}\right) = -\left(\frac{1}{r} - \frac{x^2}{r^3}\right)$$
$$\frac{\partial v}{\partial y}\left(-\frac{y}{r}\right) = -\left(\frac{1}{r} - \frac{y^2}{r^3}\right)$$
$$\vec{\nabla} \cdot \vec{v} = -\left(\frac{1}{r} - \frac{x^2}{r^3}\right) - \left(\frac{1}{r} - \frac{y^2}{r^3}\right)$$
$$\vec{\nabla} \cdot \vec{v} = -\left(\frac{2}{r} - \frac{x^2+y^2}{r^3}\right)$$
$$\vec{\nabla} \cdot \vec{v} = -\left(\frac{2}{r} - \frac{r^2}{r^3}\right) = -\frac{1}{r} = -\frac{1\text{ ms}^{-1}}{25000\text{m}} = -4 \times 10^{-5}\text{ s}^{-1}$$

F. We know that northern hemisphere cyclones spin counterclockwise, thus the vorticity at the center of the cyclone is positive.

G.

$$\vec{\nabla} \times \vec{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$
$$\vec{\nabla} \times \vec{v} \sim \frac{m/s}{m} - \frac{m/s}{m} = \frac{1}{s}$$

H. The cyclonic wind speed is tangential to the rotating cyclone and thus perpendicular to the radial wind speed. Following our calculations from E, we can modify our Cartesian coordinates:

$$u = -V_\theta \sin(\theta) = -V_\theta \frac{y}{r}$$
$$v = V_\theta \cos(\theta) = V_\theta \frac{x}{r}$$

We make u negative in order to maintain the counterclockwise rotation of the cyclone. Additionally, tangential velocity scales with the radius, so adjusting V_θ accordingly, we get:

$$u = -\frac{V_\theta r}{R}\frac{y}{r} = -\frac{V_\theta}{R} y$$
$$v = \frac{V_\theta r}{R}\frac{x}{r} = \frac{V_\theta}{R} x$$

Thus, our vorticity is:

$$\vec{\nabla} \times \vec{v} = \frac{\partial v}{\partial x}\left(\frac{V_\theta}{R}x\right) - \frac{\partial u}{\partial y}\left(-\frac{V_\theta}{R}y\right)$$
$$\vec{\nabla} \times \vec{v} = \frac{V_\theta}{R} + \frac{V_\theta}{R}$$
$$\vec{\nabla} \times \vec{v} = \frac{2V_\theta}{R} = \frac{2 \times 60 \text{ ms}^{-1}}{25000 \text{ m}} = 4.8 \times 10^{-3} \text{ s}^{-1}$$

I. If the hurricane is at a latitude of 20° N, and $\Omega = 7.3 \times 10^{-5}$ s^{-1}, we can calculate the planetary vorticity:

$$f = 2 \times 7.3 \times 10^{-5} \text{ s}^{-1} \sin(20°)$$
$$f = 5.13 \times 10^{-5} \text{ s}^{-1}$$

This makes the vorticity of the cyclone about 93 times stronger than the vorticity of the planet.

J.

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y}\left(2u_0 r_0^{\frac{1}{2}}(x^2+y^2)^{\frac{1}{4}}\right)$$
$$u = -2u_0 r_0^{\frac{1}{2}}\left(\frac{1}{4}(x^2+y^2)^{-\frac{3}{4}}(2y)\right) = -\frac{u_0 r_0^{\frac{1}{2}} y}{(x^2+y^2)^{\frac{3}{4}}}$$
$$v = \frac{\partial \psi}{\partial x} = \frac{u_0 r_0^{\frac{1}{2}} x}{(x^2+y^2)^{\frac{3}{4}}}$$

K. Note that $x^2 + y^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$, so:

$$u = -\frac{u_0 r_0^{\frac{1}{2}} r\sin(\theta)}{(r^2)^{\frac{3}{4}}} = -\frac{u_0 r_0^{\frac{1}{2}} \sin(\theta)}{r^{\frac{1}{2}}}$$
$$v = \frac{u_0 r_0^{\frac{1}{2}} r\cos(\theta)}{(r^2)^{\frac{3}{4}}} = \frac{u_0 r_0^{\frac{1}{2}} \cos(\theta)}{r^{\frac{1}{2}}}$$

L. The velocities decay by a factor of $r^{-\frac{1}{2}}$ as the distance from the center of the cyclone increases.