① $\begin{cases} f(x,y,z) = x+2y+3z \\ \ln x + \ln y + \ln z = 0 \end{cases}$

$L = x+2y+3z+\lambda(\ln x + \ln y + \ln z)$

$\begin{cases} \frac{\partial L}{\partial x} = 1 + \frac{\lambda}{x} = 0 \\ \frac{\partial L}{\partial y} = 2 + \frac{\lambda}{y} = 0 \\ \frac{\partial L}{\partial z} = 3 + \frac{\lambda}{z} = 0 \\ \ln x + \ln y + \ln z = 0 \end{cases}$ $\quad$ $\frac{\lambda}{x} = -1 \Rightarrow \lambda = -x$; $\frac{\lambda}{y} = -2 \Rightarrow \lambda = -2y$; $\frac{\lambda}{z} = -3 \Rightarrow \lambda = -3z$

$\begin{cases} x = 2y \\ x = 3z \\ \ln x + \ln y + \ln z = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{x}{2} \\ z = \frac{x}{3} \\ \ln x + \ln\frac{x}{2} + \ln\frac{x}{3} = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{x}{2} \\ z = \frac{x}{3} \\ 3\ln x = \ln 2 + \ln 3 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} y = \frac{x}{2} \\ z = \frac{x}{3} \\ \ln x = \frac{\ln 6}{3} \end{cases} \Rightarrow \begin{cases} y = \frac{\sqrt[3]{6}}{2} \\ z = \frac{\sqrt[3]{6}}{3} \\ x = \sqrt[3]{6} \end{cases} \Rightarrow \lambda = -x = -\sqrt[3]{6}$

$\frac{\partial^2 L}{\partial x^2} = -\frac{\lambda}{x^2}$; $\frac{\partial^2 L}{\partial y^2} = -\frac{\lambda}{y^2}$; $\frac{\partial^2 L}{\partial z^2} = -\frac{\lambda}{z^2}$; $\frac{\partial L^2}{\partial x \partial y} = \frac{\partial L^2}{\partial x \partial z} = 0$;

$\frac{\partial^2 L}{\partial y \partial x} = \frac{\partial^2 L}{\partial y \partial z} = 0$; $\frac{\partial^2 L}{\partial z \partial x} = \frac{\partial^2 L}{\partial z \partial y} = 0 \Rightarrow$

$$H = \begin{pmatrix} -\frac{\lambda}{x^2} & 0 & 0 \\ 0 & -\frac{\lambda}{y^2} & 0 \\ 0 & 0 & -\frac{\lambda}{z^2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt[3]{6}}{(\sqrt[3]{6})^2} & 0 & 0 \\ 0 & \frac{\sqrt[3]{6}\cdot 4}{(\sqrt[3]{6})^2} & 0 \\ 0 & 0 & \frac{\sqrt[3]{6}\cdot 9}{(\sqrt[3]{6})^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt[3]{6}} & 0 & 0 \\ 0 & \frac{4}{\sqrt[3]{6}} & 0 \\ 0 & 0 & \frac{9}{\sqrt[3]{6}} \end{pmatrix}$$

$\Delta_1 = \frac{1}{\sqrt[3]{6}} > 0$; $\Delta_2 = \frac{4}{(\sqrt[3]{6})^2} > 0$, $\Delta_3 = \frac{36}{6} = 6 > 0 \Rightarrow$

$\Rightarrow$ т. $\left(\sqrt[3]{6}; \frac{\sqrt[3]{6}}{2}; \frac{\sqrt[3]{6}}{3}\right)$ является минимумом

② $\begin{cases} f(x,y,z) = x^2 + 2y^2 + 3z \\ x - y + z = -1 \\ -2x + 12y + 3z = +7 \end{cases}$

$$L = x^2+2y^2-3z+\lambda_1(x-y+z+1)+\lambda_2(-2x+12y+3z-7)$$

$$\begin{cases} \dfrac{\partial L}{\partial x} = 2x+\lambda_1-2\lambda_2=0 \\ \dfrac{\partial L}{\partial y} = 4y-\lambda_1+12\lambda_2=0 \\ \dfrac{\partial L}{\partial z} = -3+\lambda_1+3\lambda_2=0 \\ x-y+z=-1 \\ -2x+12y+3z=7 \end{cases} \Rightarrow \begin{cases} 2x+3-5\lambda_2=0 \\ 4y-3+15\lambda_2=0 \\ \lambda_1=3-3\lambda_2 \\ x-y+z=-1 \\ -2x+12y+3z=7 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \lambda_2 = \dfrac{2x+3}{5} \\ \lambda_2 = \dfrac{3-4y}{15} \\ \lambda_1 = 3-\dfrac{3-4y}{5} \\ x-y+z=-1 \\ -2x+12y+3z=7 \end{cases} \Rightarrow \begin{cases} 6x+9=3-4y \\ \lambda_1 = \dfrac{4y-3}{5}+3 \\ x-y+z=-1 \\ -2x+12y+3z=7 \end{cases} \Rightarrow \begin{cases} y=-\dfrac{6x+6}{4} \\ \lambda_1=\dfrac{4y-3}{5}+3 \\ 4x+6x+6+4z=-4 \\ -2x-18x-18+3z=7 \end{cases}$$

$$\Rightarrow \begin{cases} y=\dfrac{-3x-3}{2} \\ \lambda_1=\dfrac{4y-3}{5}+3 \\ -\dfrac{10x-10}{4}=z=\dfrac{-5x-5}{2} \\ 20x+25=-\dfrac{3}{2}(5x+5) \end{cases} \Rightarrow \begin{cases} y=\dfrac{-3x-3}{2} \\ \lambda_1=\dfrac{4y-3}{5}+3 \\ z=\dfrac{-5x-5}{2} \\ 40x+50=-3(5x+5) \end{cases} \Rightarrow \begin{cases} y=\dfrac{3}{11} \\ \lambda_1=\dfrac{144}{55} \\ z=\dfrac{5}{11} \\ x=-\dfrac{13}{11} \\ \lambda_2=\dfrac{7}{55} \end{cases}$$

$\dfrac{\partial^2 L}{\partial x^2}=2$, $\dfrac{\partial^2 L}{\partial y^2}=4$; $\dfrac{\partial^2 L}{\partial z^2}=0$ Все остальные вторые производные равны 0, т.е.

$$H=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \Delta_1=2>0 \\ \Delta_2=8>0 \\ \Delta_3=0 \Rightarrow \text{т.}\left(-\frac{13}{11};\frac{3}{11};\frac{5}{11}\right) - \end{array}$$

является седлом

(3) $\begin{cases} f(x,y,z)=x^2+3y^2+5z^2 \to \min \\ x+y+z \le -23 \end{cases}$

$$L = x^2+3y^2+5z^2+\mu(x+y+z+23)$$

$$\begin{cases} L'_x = 2x+\mu=0 \Rightarrow \mu=-2x \\ L'_y = 6y+\mu=0 \Rightarrow \mu=-6y \\ L'_z = 10z+\mu=0 \Rightarrow \mu=-10z \\ x+y+z+23\le 0 \\ \mu\ge 0 \end{cases}$$

1) $\begin{cases} x+y+z+23\le 0 \\ \mu=0 \Rightarrow x=y=z=0 \end{cases} \Rightarrow \begin{cases} 0+0+0+23\le 0 \text{ ← неверно} \\ \mu=x=y=z=0 \end{cases}$

2) $\begin{cases} x+y+z+23=0 \\ \mu\ge 0 \\ -2x=-6y \Rightarrow x=3y \\ -6y=-10z \Rightarrow z=\frac{3}{5}y \end{cases} \Rightarrow \begin{cases} 3y+y+\frac{3}{5}y+23=0 \Rightarrow y=-5 \\ \mu\ge 0 \Rightarrow \mu=-2\cdot(-15)=+30\ge 0 \\ x=-15 \\ z=-3 \end{cases}$

~~т. (-15; -5; -3) экстремум~~

$H=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 10 \end{pmatrix}$ $\quad \Delta_1=2\ge 0 \quad \Delta_2=12\ge 0 \quad \Delta_3=120\ge 0 \Rightarrow$ т. $(-15; -5; -3)$ минимум

$f(-15; -5; -3)=225+75+45=345$