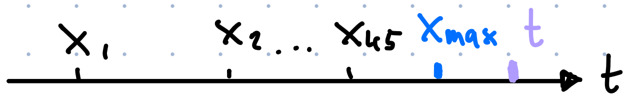
Распределение max и min

$X_1, \dots, X_n \sim iid\ F_X \qquad F_X(t) \quad f_X(t)$

$Y = X_{min} \sim ?$

$Z = X_{max} \sim ?$

$$F_{X_{max}}(t) = \mathbb{P}(X_{max} \leq t) = \mathbb{P}(X_1 \leq t \cap X_2 \leq t \cap \dots \cap X_n \leq t) =$$



$$= \underbrace{\mathbb{P}(X_1 \leq t)}_{F_X(t)} \cdot \underbrace{\mathbb{P}(X_2 \leq t)}_{F_X(t)} \cdot \ldots \cdot \underbrace{\mathbb{P}(X_n \leq t)}_{F_X(t)} = [F_X(t)]^n$$

$$f_{max}(t) = n \cdot (F_X(t))^{n-1} \cdot f_X(t)$$

Как сгенерировать:

$F_U(t) = t$

X_{max} из 100 равномерных сл. вел. $U[0;1]$?

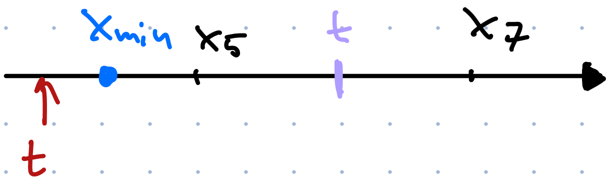
$F_{X_{max}}(X_{max}) \sim U[0;1]$ $\qquad F_{X_{max}}(t) = t^n$

$y_1, \dots, y_n \sim U[0;1]$ $\qquad F^{-1}(t) = \sqrt[n]{t}$

$z_i = F^{-1}_{X_{max}}(y_i) = \sqrt[100]{y_i}$

$z_1, \dots, z_n \sim F_{X_{max}}$

$$F_{X_{min}}(t) = \mathbb{P}(X_{min} \le t) = 1 - \mathbb{P}(X_{min} > t) =$$



$$= 1 - \mathbb{P}(X_1 > t) \cdot \ldots \cdot \mathbb{P}(X_n > t) =$$

$$= 1 - (1 - F_X(t)) \cdot (1 - F_X(t)) \cdot \ldots \cdot (1 - F_X(t)) =$$

$$= 1 - (1 - F_X(t))^n$$

Упражнение

$X, Y, Z \sim iid\ U[0;1]$

$$F_X(x) = \begin{cases} 1 & x \ge 1 \\ x, & x \in [0;1] \\ 0, & x < 0 \end{cases}$$

$(X \cdot Y)^Z \sim ?$

$W = Z(\ln X + \ln Y) \qquad e^W$

$V = \ln X \sim ?$

$$F_V(v) = \mathbb{P}(\ln X \le v) = \mathbb{P}(X \le e^v) = F_X(e^v) = e^v$$

$x \quad [0;1]$

$v \quad (-\infty\ 0]$

$$f_V(v) = \begin{cases} e^v, & (-\infty;0] \\ 0, & \text{иначе} \end{cases}$$

$$\int_{-\infty}^{0} e^v dv = e^0 - e^{-\infty} = 1$$

$$R = \ln X + \ln Y = V + S \sim ?$$

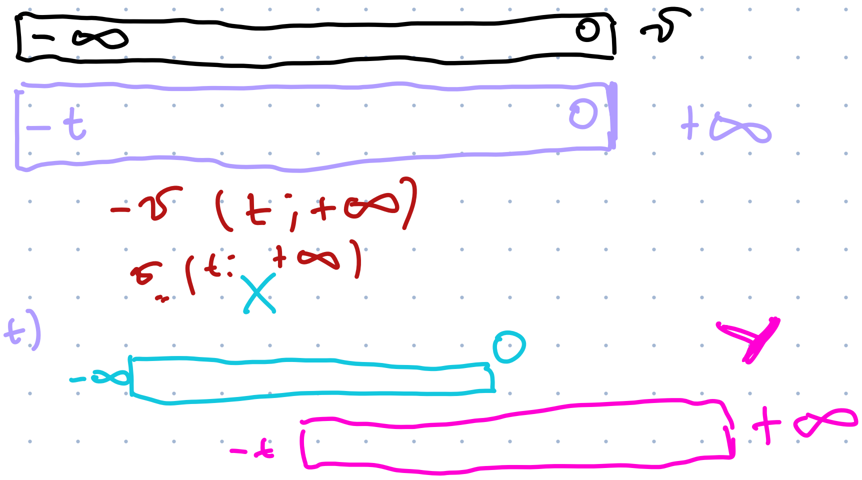
$(-\infty;0] \qquad (-\infty;0]\ (-\infty;0]$

X и Y нез.

$Z = X + Y$

$$f_Z(t) = \int_{-\infty}^{+\infty} f_X(x) \cdot f_Y(t-x)\, dx$$

$$f_R(r) = \int_{-\infty}^{+\infty} e^{\vartheta} \cdot e^{t-\vartheta}\, d\vartheta$$



$t-\vartheta \in (-\infty;0]$

$+\infty \qquad t-(-t)$

$$= \int_{-t}^{0} e^{\vartheta} \cdot e^{t-\vartheta}\, d\vartheta$$

$$= \int_{-t}^{0} e^{t}\, d\vartheta = e^{t} \cdot \vartheta \Big|_{-t}^{0} = e^{t}\big(0-(-t)\big) = t e^{t}$$

$$f_R(r) = \begin{cases} t e^{t}, & t \in (-\infty;0] \\ 0, & \text{иначе} \end{cases}$$

Надо вывести аналог формулы свёртки для произведения

$$M = Z \cdot R \sim \ ?$$

$(-\infty;0] \quad [0;1] \quad (-\infty;0]$

$$F_M(m) = \mathbb{P}(Z \cdot R \le m) = (*)$$

$m \geq z \cdot r$

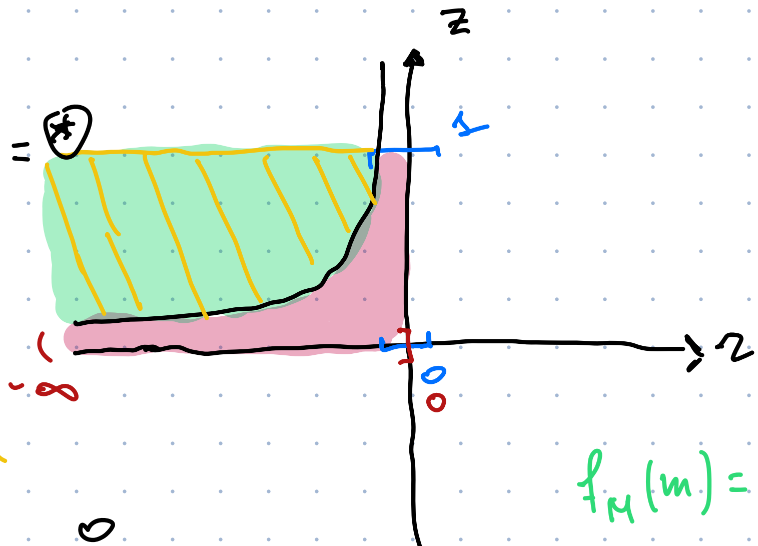
$\frac{m}{z} \geq r$

$z \geq \frac{r}{m}$

$m \geq z \cdot r$

$z \leq \frac{m}{r}$

ОШИБКА



$$(*) = \int_{-\infty}^{0} \int_{r/m}^{1} r e^{r} \cdot 1 \, dz\, dr = \int_{-\infty}^{0} r e^{r} \cdot z \Big|_{r/m}^{1} dr =$$

$f_M(m) = -\frac{2}{m^2} \cdot (-1)$

$= 2/m^2$

$-\infty;0$

$$= \int_{-\infty}^{0} r e^{r}\left(1-\frac{r}{m}\right) dr = \underbrace{\int_{-\infty}^{0} r e^{r}\, dr}_{①} - \underbrace{\int_{-\infty}^{0} r^2 e^{r}\cdot \frac{1}{m}\, dr}_{②} = 1 - \frac{2}{m}$$

① $$\int_{-\infty}^{0} r e^{r} dr = \left\| \begin{matrix} t=-r \\ a=-(-\infty)=+\infty \\ b=-0=0 \end{matrix} \right\| = -\int_{0}^{+\infty} -t e^{-t} d(-t) = \int_{0}^{+\infty} t e^{-t} dt =$$

интегр. по частям

мат. ож. Exp(1)

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$= -\int_0^{+\infty} t\, de^{-t} = -te^{-t}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-t} dt = -e^{-t}\Big|_0^{+\infty} = 0 - (-e^0) = 1$$

$$-\lim_{t\to\infty}\frac{t}{e^t} = 0$$

$$e^t >> t$$

$u'(t)dt \quad u(t) \quad v(t)$

$\quad '' \qquad\quad u' = u'(t)$

$du(t)$

$$(uv)'_t = u'\cdot v + u\cdot v'$$

$$\int_a^b (u(t)\cdot v(t))'_t\, dt = \int_a^b u'(t)\, v(t)\, dt + \int_a^b u(t)\cdot v'(t)\, dt$$

$$u(t)\cdot v(t)\Big|_a^b = \int_a^b v(t)\, du(t) + \int_a^b u(t)\, dv(t)$$

$$uv\Big|_a^b = \int_a^b v\,du + \int_a^b u\,dv$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

② $\int_{-\infty}^{0} r^2 e^r\,dr = \int_{-\infty}^{0} r^2\,de^r = \cancel{r^2 e^r \Big|_{-\infty}^{0}} \to 0 - \int_{-\infty}^{0} e^r\,dr^2 =$

$= -\int_{-\infty}^{0} e^r \cdot 2r\,dr = -2\int_{-\infty}^{0} r\,de^r = \cancel{-2\cdot r e^r \Big|_{-\infty}^{0}} \to 0 + 2\int_{-\infty}^{0} e^r\,dr =$

$= 2\cdot e^r \Big|_{-\infty}^{0} = 2\cdot(1 - e^{-\infty}) = 2$

есть ошибка

$$\boxed{\begin{array}{l} U = -Z\cdot R \qquad Z \sim U[0;1] \qquad R \sim \begin{cases} te^{-t}, & (0;+\infty) \\ 0, & \text{иначе} \end{cases} \\ \int_0^1 \int_0^{1/2} z e^{-z}\cdot 1\,dz\,dr = 1 - e^{-r} \end{array}}$$

$\int_{-\infty}^{0} \frac{2}{m^2}\,dm = 2\cdot\frac{m^{-2+1}}{-2+1}\Big|_{-\infty}^{0} = 2\cdot(-1)\cdot\frac{1}{m}\Big|_{-\infty}^{0} = \infty$ расх.

$e^U \qquad (XY)^Z$

$F(e^{-U} \le u) = \mathbb{P}(-U \le \ln u) = 1 - \mathbb{P}(U \le -\ln u) =$

$= 1 - (1 - e^{-\ln u}) = u$

$\Rightarrow (XY)^Z \sim U[0;1]$

Упражнение

$N \sim Poiss(\lambda)$ $\qquad \mathbb{P}(N=k) = \dfrac{e^{-\lambda} \cdot \lambda^k}{k!}$ $\qquad \mathbb{E}(N) = \lambda$, $\quad Var(N) = \lambda$

X – число посет. купили помить $\qquad p \qquad X|N \sim Bin(p, N)$

$$\mathbb{P}(X=k|N) = C_n^k p^k (1-p)^{N-k}$$

Y – не стали

$N = X + Y$

$$\mathbb{E}(X|N) = p \cdot N$$

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|N)) = \mathbb{E}(p \cdot N) = p \cdot \lambda$$

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} \mathbb{E}(X|N=n) \cdot \mathbb{P}(N=n) = \sum_{n=0}^{\infty} n \cdot p \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n!} = p \cdot \mathbb{E}(N) = p \cdot \lambda$$

$X \sim ?$

$$f_X(t) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_{-\infty}^{+\infty} f(x|y) \cdot f(y)\,dy$$

$$\mathbb{P}(X=k) = \sum_{n=0}^{\infty} \overbrace{\mathbb{P}(X=k \mid N=n) \cdot \mathbb{P}(N=n)}^{\mathbb{P}(X=k \,\cap\, N=n)} =$$

$$= \sum_{n=k}^{\infty} \underbrace{C_n^k}_{= \frac{\cancel{n!}}{k!(n-k)!}} p^k (1-p)^{n-k} \cdot \frac{e^{-\lambda} \cdot \lambda^n}{\cancel{n!}} = \frac{e^{-\lambda} \cdot (\lambda p)^k}{k!} \sum_{n=k}^{\infty} \frac{[\lambda(1-p)]^{n-k}}{(n-k)!} \underset{\substack{t = n-k \\ x = \lambda(1-p)}}{=}$$

$$\sum_{t=0}^{\infty} \frac{x^t}{t!} = e^x$$

$$= \frac{e^{-\lambda} (\lambda p)^k}{k!} \cdot e^{\lambda(1-p)} = \frac{(\lambda p)^k}{k!} e^{-\lambda p}$$

$X \sim Poiss(\lambda p)$

$Y \sim Poiss(\lambda(1-p))$

$X + Y \sim \text{Poiss}(\lambda)$

$X \sim \text{Poiss}(\lambda_1)$

$Y \sim \text{Poiss}(\lambda_2)$

$Z = X + Y \sim \text{Poiss}(\lambda_1 + \lambda_2)$