$$f(x)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f'(x)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}}\cdot\left(-\frac{2(x-\mu)}{2\sigma^2}\cdot\cancel{(-1)}\right)=$$

$$=\frac{\mu-x}{\sqrt{2\pi}\,\sigma^3}e^{\frac{-(x-\mu)^2}{2\sigma^2}}=0$$

$$\mu-x=0 \Rightarrow x=\mu$$

$$f''(x)=\frac{-1}{\sqrt{2\pi}\,\sigma^3}\cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}+\frac{\mu-x}{\sqrt{2\pi}\,\sigma^3}\cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}\cdot$$

$$\cdot\left(\frac{\mu-x}{\sigma^2}\right)=\frac{-1}{\sqrt{2\pi}\,\sigma^3}e^{\frac{-(x-\mu)^2}{2\sigma^2}}\left(1-\frac{(\mu-x)^2}{\sigma^2}\right)=0$$

$$1-\left(\frac{\mu-x}{\sigma}\right)^2=0$$

$\frac{\mu-x}{\sigma}=\pm1 \Rightarrow x=\mu\pm\sigma$ - точки перегиба

$$\cancel{f''(x)}\quad f''(\mu)=-\frac{1}{\sqrt{2\pi}\,\sigma^3}\cdot1\cdot(1-0)=-\frac{1}{\sqrt{2\pi}\,\sigma^3}<0$$

$\Rightarrow x=\mu$ - точка максимума

2) $\ln y\sim N(\mu,\sigma^2)$

$x=\ln y \Rightarrow y=e^x$, тогда

$$f(y)=f(\ln y)\cdot\frac{1}{y}=\frac{1}{\sqrt{2\pi}\,\sigma y}e^{\frac{-(\ln y-\mu)^2}{2\sigma^2}}$$

1) $$E(y)=\int_{-\infty}^{+\infty}\cancel{y}\frac{1}{\sqrt{2\pi}\,\sigma\cancel{y}}e^{\frac{-(\ln y-\mu)^2}{2\sigma^2}}dy=\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}dy=$$

$$=\left|\begin{array}{l}x=\ln y \Rightarrow dy=y\,dx=e^x dx\\ dx=\frac{dy}{y}\end{array}\right|=\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty}e^{\frac{-(x-\mu)^2}{2\sigma^2}}e^x dx=$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(x - \frac{(x-\mu)^2}{2\sigma^2}\right) dx \ominus$$

Вычисления $\left[x - \frac{(x-\mu)^2}{2\sigma^2} = \mu + x - \mu - \frac{(x-\mu)^2}{2\sigma^2} = \right.$

$$= \mu + \cancel{(x-\mu)\left(1 - \frac{x-\mu}{2\sigma^2}\right)} \frac{-1}{2\sigma^2}\left((x-\mu)^2 - 2\sigma^2(x-\mu)\right) =$$

$$= \mu - \frac{1}{2\sigma^2}\left((x-\mu)^2 - 2\sigma^2(x-\mu) + \sigma^4 - \sigma^4\right) =$$

$$\left. = \mu - \frac{1}{2\sigma^2}\left((x-\mu-\sigma^2)^2 - \sigma^4\right) = \mu + \frac{\sigma^2}{2} - \frac{(x-\mu-\sigma^2)^2}{2\sigma^2}\right]$$

$$\ominus e^{\mu + \frac{\sigma^2}{2}} \cdot \underbrace{\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-\mu-\sigma^2)^2}{2\sigma^2}\right) dx}_{=1} \Rightarrow$$

$$\Rightarrow E(y) = e^{\mu + \frac{\sigma^2}{2}}$$

2) $Var(y) = E(y^2) - (E(y))^2$

$$E(y^2) = \int_{-\infty}^{+\infty} \frac{y}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln y - \mu)^2}{2\sigma^2}\right) dy =$$

$$= \left[\begin{array}{l} x = \ln y \Rightarrow y = e^x \\ dx = \frac{dy}{y} \Rightarrow dy = e^x dx \end{array}\right] = \int_{-\infty}^{+\infty} \frac{e^{2x}}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \ominus$$

Вычисления $\left[2x - \frac{(x-\mu)^2}{2\sigma^2} = 2\mu + 2x - 2\mu - \frac{(x-\mu)^2}{2\sigma^2} = \right.$

$$= 2\mu - \frac{1}{2\sigma^2}\left((x-\mu)^2 - 4\sigma^2(x-\mu)\right) = 2\mu - \frac{1}{2\sigma^2}\left((x-\mu)^2 - \right.$$

$$\left. - 4\sigma^2(x-\mu) + 4\sigma^4 - 4\sigma^4\right) = 2\mu - \frac{1}{2\sigma^2}\left((x-\mu-2\sigma^2)^2 - 4\sigma^4\right.$$

$= 2\mu + 2\sigma^2 - \frac{(x-\mu-2\sigma^2)^2}{2\sigma^2} \Big] \Rightarrow E(y^2) = e^{2\mu+2\sigma^2}$

$Var(y) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$

3) $Med(y): \ P(y \le Med(y)) = 0{,}5$

$P(\ln y \le \ln Med\, y)) = P(X \le \ln Men(y))$

$P(X \le \mu) = 0{,}5 \Rightarrow \ln Med(y) = \mu \Rightarrow Med(y) = e^{\mu}$

4) $Mode(y)$:

$f(y) = \frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}$

$\ln f(y) = -\frac{(\ln y-\mu)^2}{2\sigma^2} - \ln y - \ln\sqrt{2\pi}\sigma$

$\frac{d}{dy}\ln f(y) = -\frac{(\ln y-\mu)}{\sigma^2 y} - \frac{\sigma^2}{y\sigma^2} \Rightarrow$

$\Rightarrow -(\ln y-\mu) - \sigma^2 = 0$

$\ln y = \mu - \sigma^2$

$y = e^{\mu-\sigma^2} \Rightarrow Mode(y) = e^{\mu-\sigma^2}$

3 1) $\frac{X_1^2 + X_2^2 + X_3^2 + X_4^2}{2}$, т.к. $X_1, X_2, X_3, X_4 \sim N(0,4)$,

то $X_i \sim N(0,\sigma^2) \Rightarrow \sigma = 2$

$X_i = 0 + 2N(0,1) \Rightarrow$

$\Rightarrow X_1^2 + X_2^2 + X_3^2 + X_4^2 = 4\underbrace{(Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2)}_{\sim \chi^2_4}$, где

$Z \sim N(0,1)$

6)

$$\Rightarrow \frac{X_1^2+X_2^2+X_3^2+X_4^2}{2} = \frac{4}{2}\chi_4^2 = 2\chi^2_4$$

2) $X_1 = 2Z_1$

$X_2^2 + X_3^2 = 4(Z_2^2+Z_3^2) \approx 4\chi^2_2$

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$$\Rightarrow \frac{X_1}{\sqrt{X_2^2+X_3^2}} = \frac{2Z_1}{2\sqrt{\chi^2_2}} = \frac{Z_1}{\sqrt{\chi^2_2/2}} \sim t(2)$$

3) $\dfrac{X_1^2+X_2^2}{X_3^2+X_4^2} = \dfrac{\chi^2_2/2}{\chi^2_2/2}$ – распределение Фишера

$= F(2,2)$

$X_1^2+X_2^2 \approx 4\chi^2_2$

$X_3^2+X_4^2 \approx 4\chi^2_2$

4) $$\frac{\sqrt{2}\,X_1}{\sqrt{X_2^2+X_3^2}} = \frac{2\sqrt{2}Z_1}{2\sqrt{\chi^2_2}} = \frac{\sqrt{2}Z_1}{2\sqrt{\chi^2_2/2}} \approx \sqrt{2}\,t(2)$$

4. $$\begin{pmatrix} x \\ y \end{pmatrix} \sim N\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}; \begin{pmatrix} 9 & -1 \\ -1 & 4 \end{pmatrix}\right]$$

1) $E(x) = 1$, $E(y) = -2$; 2) $Var(x) = 9$; $Var(y) = 4$;

$Cov(x,y) = -1$

3) $E(X+3Y-7) = E(X) + 3E(Y) - 7 = 1 - 6 - 7 = -12$

4) $Var(X+3Y-7) = Var(X) + 9Var(Y) + 2\cdot 3\, cov(X,Y) =$

$= 9 + 36 - 6 = 39$

5) $Cov(X-Y, 2X+3Y) = 2Var(X) - 3Var(Y) + 1Cov(X,Y)$

$= 18 - 12 - 1 = 5$

1. б) $\mathrm{Corr}(X-9,\ X+3Y)$

$$\mathrm{Corr}(X-9,\ X+3Y) = \frac{\mathrm{Cov}(X-9,\ X+3Y)}{\underbrace{\sqrt{\mathrm{Var}(X-9)}}_{=9}\cdot\underbrace{\sqrt{\mathrm{Var}(X+3Y)}}_{=39}} \circleddash$$

$$\mathrm{Cov}(\overset{=X}{X-9},\ X+3Y) = \mathrm{Var}(X) + 3\,\mathrm{Cov}(X,Y) = 9-3=6$$

$$\circleddash\ \frac{6}{3\sqrt{39}} = \frac{2}{\sqrt{39}}$$

2.1) $\mathbb{P}(X>5)\qquad X\sim N(1,9)\sim N(1,3^2)$

$$\mathbb{P}(X>5)=\mathbb{P}\left(\frac{X-1}{3}>\frac{5-1}{3}\right)=P\left(Z>\frac{4}{3}\right)\approx 0{,}0918$$

2.2) $\mathbb{P}(X+Y>5)\sim N(-1;(\sqrt{11})^2)$

$$E(X+Y)=1+(-2)=-1$$

$$\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)+2\,\mathrm{Cov}(X,Y)=9+4-2=11$$

$$\mathbb{P}(X+Y>5)=\mathbb{P}\left(\frac{X+Y+1}{\sqrt{11}}>\frac{5+1}{\sqrt{11}}\right)=\mathbb{P}\left(Z>\frac{6}{\sqrt{11}}\right)\approx 0{,}0351$$