Формула свёртки, функция случ. арг.

Упражнение

$$f(x,y) = \begin{cases} \overset{=4}{c} \cdot x \cdot y, & x \in [0;1] \quad y \in [0;1] \\ 0, & \text{иначе} \end{cases}$$

а) c – ?

б) X, Y – незав. ?

в) $\mathbb{E}(X \cdot Y)$

г) $\mathbb{P}\begin{pmatrix} 0 \le X \le 0.5 \\ 0 \le Y \le 0{,}5 \end{pmatrix}$

д) $F(x, y)$

е) $f_X(x)$, $f_Y(y)$

ж) $\mathrm{Cov}(X, Y)$

з) $\mathrm{Corr}(X, Y)$

а) $$\iint\limits_{D} f(x,y)\,dx\,dy = \int_0^1\int_0^1 c \cdot x \cdot y\,dx\,dy = c \cdot \int_0^1 x\,dx \cdot \int_0^1 y\,dy =$$

$$= c \cdot \frac{x^2}{2}\Big|_0^1 \cdot \frac{y^2}{2}\Big|_0^1 = c \cdot \frac{1}{4} = 1 \Rightarrow c = 4$$

е) $$f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 f(x|y) \cdot f(y)\,dy = \mathbb{E}_Y\big(f(x|y)\big)$$

$$f_X(x) = \int_0^1 4xy\,dy = 4x \cdot \frac{y^2}{2}\Big|_0^1 = 2x; \quad x \in [0;1]$$

$$f_Y(y) = \int_0^1 4xy\,dx = 2y, \quad y \in [0;1]$$

д) $f(x,y) = f_X(x) \cdot f_Y(y)$

$4xy = 2x \cdot 2y \qquad x \in [0;1] \quad y \in [0;1]$

$0 = 0 \cdot 0 \qquad$ иначе

$\Rightarrow$ Да, X и Y нез.-ые

е) $\mathbb{E}(X \cdot Y) = \iint\limits_{D} x \cdot y \cdot f(x,y)\, dx\, dy$

$\mathbb{E}(X \cdot Y) = \int_0^1 \int_0^1 x \cdot y \cdot 4xy \, dx\, dy = 4 \cdot \frac{x^3}{3}\Big|_0^1 \cdot \frac{y^3}{3}\Big|_0^1 = \frac{4}{9}$

ж) $\mathbb{E}(X) = \int_0^1 x \cdot 2x \, dx = 2 \cdot \frac{x^3}{3}\Big|_0^1 = 2/3$

$\mathbb{E}(Y) = 2/3 \qquad \mathbb{E}(x) = \int_{-\infty}^{+\infty} t \cdot f(t)\, dt$

$\operatorname{Cov}(X,Y) = \mathbb{E}(X \cdot Y) - \mathbb{E}(X) \cdot \mathbb{E}(Y) = 0$

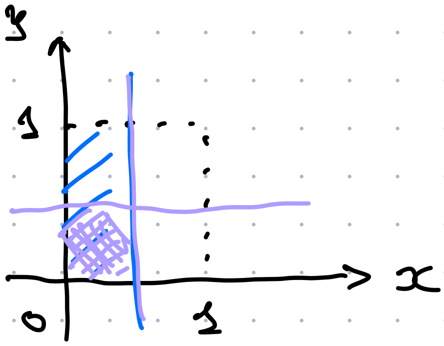
з) $\operatorname{Corr}(X,Y) = 0$

$\mathbb{E}(X^2) = \int_0^1 x^2 \cdot 2x\, dx = 2 \cdot \frac{x^4}{4}\Big|_0^1 = \frac{1}{2}$

$\operatorname{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$

г) $\mathbb{P}(\underline{0 \le X \le 0.5}) = \int_0^{1/2} \underbrace{f(x)}_{2x} dx = 2 \cdot \frac{x^2}{2}\Big|_0^{1/2} = \frac{1}{4}$

$\int_0^{1/2} \underbrace{\int_0^1 f(x,y)\, dy}_{f_X(x)} \, dx$

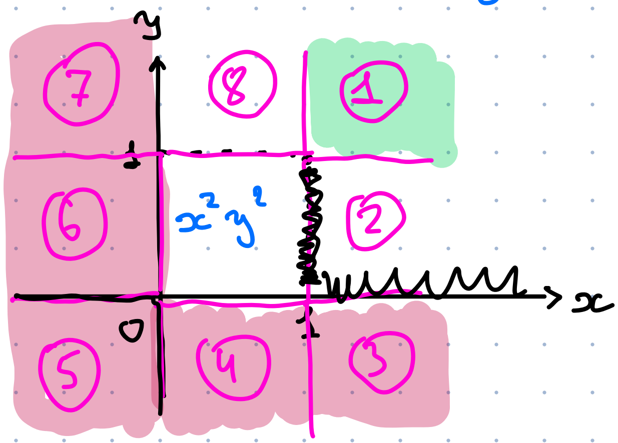


$$\mathbb{P}\left(X\in[0;\tfrac12]\cap Y\in[0;\tfrac12]\right)=\int_0^{1/2}\int_0^{1/2}4xy\,dx\,dy=$$

$$=4\cdot\frac{x^2}{2}\Big|_0^{1/2}\cdot\frac{y^2}{2}\Big|_0^{1/2}=\frac14\cdot\frac14=1/16$$

g) $F(x,y)=\int_0^x\int_0^y 4xy\,dx\,dy=4\cdot\frac{x^2}{2}\Big|_0^x\cdot\frac{y^2}{2}\Big|_0^y=x^2y^2$

это правда внутри квадрата $[0;1]^2$



⑤ $\mathbb{P}(X\le x\cap Y\le y)=F(-\infty,-\infty)=0$

7, 6, 4, 3 $\quad F(x,-\infty)=0$

$F(-\infty,y)=0$

① $F(+\infty,+\infty)=1$

② $\mathbb{P}(X\le+\infty,\ Y\le y)=F(y)=\int_0^1 dx\int_0^y 4xy\,dx\,dy=\frac{y^2}{2}$

8 $F(x)$

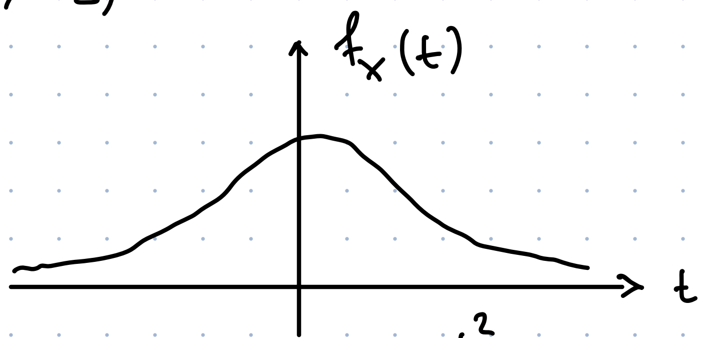
$$F(x,y) = \begin{cases} x^2y^2, & \square \\ 0, & 3\ 4\ 5\ 6\ 7 \\ 1, & 1 \\ x^2/2, & 2 \\ y^2/2, & 8 \end{cases}$$

Функция случайного аргумента

$f_X(x) \quad X \quad F_X(x)$

$Y = \varphi(X) \quad F_Y(y) - ?$

$f_Y(y) - ?$



Упражнение

$X \sim \mathcal{N}(0, 1)$

$\mathbb{E}(X) \quad Var(X)$

$$f_X(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

$Y = X^2 \sim \chi_1^2$

$f_Y(y) - ?$

$f_Y(y) = F_Y'(y)$

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) =$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = \left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right)'_y \qquad [g(f(t))]' = g'(f(t)) \cdot f'(t)$$

$$(F_X(\sqrt{y}))'_y = F_X'(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{2\pi y}} \cdot e^{-\frac{y}{2}}$$

$$(F_X(-\sqrt{y}))'_y = f_X(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}} = \frac{1}{2\sqrt{2\pi y}} \cdot e^{-\frac{y}{2}} \cdot (-1)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}$$

$$x \in (-\infty; +\infty)$$

$$y = x^2 = [0; +\infty)$$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}, & y \geq 0 \\ 0, & \text{иначе} \end{cases}$$

Формула из учебников:

$$Y = \varphi(X) \qquad \varphi(t) \text{ монотонна}$$

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\varphi(X) \leq y) = \mathbb{P}(X \leq \varphi^{-1}(y))$$

$$f_Y(y) = F_X'(\varphi^{-1}(y)) = f_X(\varphi^{-1}(y)) \cdot (\varphi^{-1}(y))'_y = f_X(\varphi^{-1}(y)) \cdot \frac{1}{|\varphi'(y)|}$$

Упражнение

$$X \sim U[0;1] \qquad f_X(t) = \begin{cases} 1, & t \in [0;1] \\ 0, & \text{иначе} \end{cases}$$

$$Y = -\frac{1}{\lambda} \ln(1-X) \qquad F_X(t) = \begin{cases} t, & t \in [0;1) \\ 0, & t < 0 \\ 1, & t \geq 1 \end{cases} = \mathbb{P}(X \leq t)$$

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}\left(-\frac{1}{\alpha}\ln(1-X) \le y\right) =$$

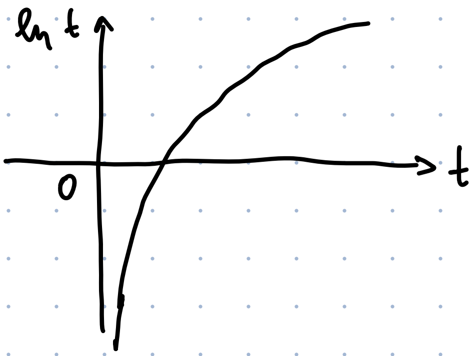
$$= \mathbb{P}(\ln(1-X) \ge -\alpha y) = \mathbb{P}(1-X \ge e^{-\alpha y}) =$$

$$= \mathbb{P}(X \le 1-e^{-\alpha y}) = F_X(1-e^{-\alpha y}) = 1-e^{-\alpha y}$$

$x \in [0; 1]$

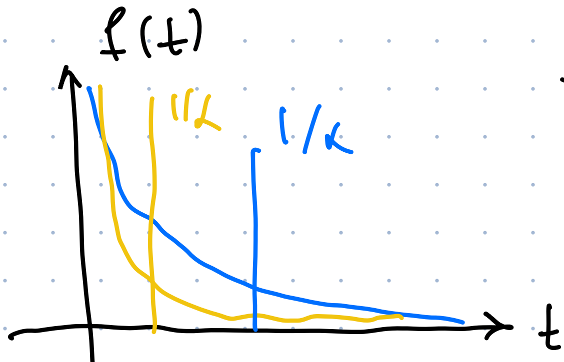
$$-\frac{1}{\alpha}\ln(1-0) = \ln 1 = 0$$

$$\lim_{t\to 1}\left(-\frac{1}{\alpha}\ln(1-t)\right) = -\frac{1}{\alpha}\lim_{t\to 0}\ln 0 = -1\cdot(-\infty) = +\infty$$



$y \in [0; +\infty)$

$$f_Y(y) = \begin{cases} \alpha e^{-\alpha y}, & y \ge 0 \\ 0, & \text{иначе} \end{cases}$$



$\frac{1}{\alpha} < \frac{1}{\alpha}$

$\alpha > \alpha$

$Y \sim Exp(\alpha)$

$\mathbb{E}(Y) = \frac{1}{\alpha} \qquad Var(Y) = \frac{1}{\alpha^2}$

Экспоненциальное р.-е.

III. (Квантильное преобразование)

$X \sim F_X(t)$ $\qquad$ $Y = F_X(X)$

$Y \sim ?$ $\qquad$ $Y \sim U[0;1]$

$$F_Y(t) = \mathbb{P}(Y \leq t) = \mathbb{P}(F_X(X) \leq t) =$$
$$= \mathbb{P}(X \leq F_X^{-1}(t)) = F_X(F_X^{-1}(t)) = t$$

$x \in (-\infty; +\infty)$

$y \in [0; 1]$

① $X \sim U[0;1]$

② $F_Y^{-1}(x)$

$x_1, \ldots, x_n \sim \text{iid } U[0;1]$

$y_i = -\frac{1}{\lambda} \ln(1 - x_i)$

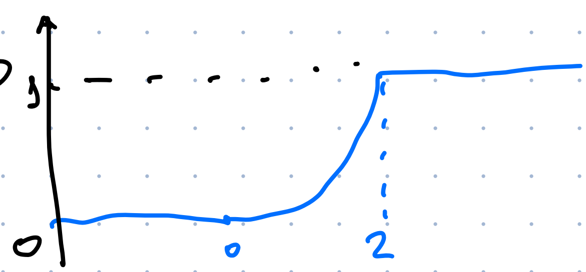
$y_1, \ldots, y_n \sim \text{iid } Exp(\lambda)$

$$F_Y(t) = 1 - e^{-\lambda t}$$

$$F_Y^{-1}(t) = -\frac{1}{\lambda} \ln(1 - t)$$

Упражнение

$$X \qquad F_X(t) = \begin{cases} t^2/4 & ; t \in [0;2] \\ 1 & ; t \geq 2 \\ 0 & ; t < 0, \end{cases}$$



Как сгенерить на компе?

$w = \frac{t^2}{4} \qquad 4w = t^2 \qquad \sqrt{4w} = t$

$F_X^{-1}(w) = \sqrt{4w} = 2\sqrt{w} \qquad F_X(t) = \frac{t^2}{4}$

$w \in [0; 1] \qquad t \in [0; 2]$

$w_1, \ldots, w_n \sim iid\ U[0; 1]$

$x_i = 2\sqrt{w_i}$

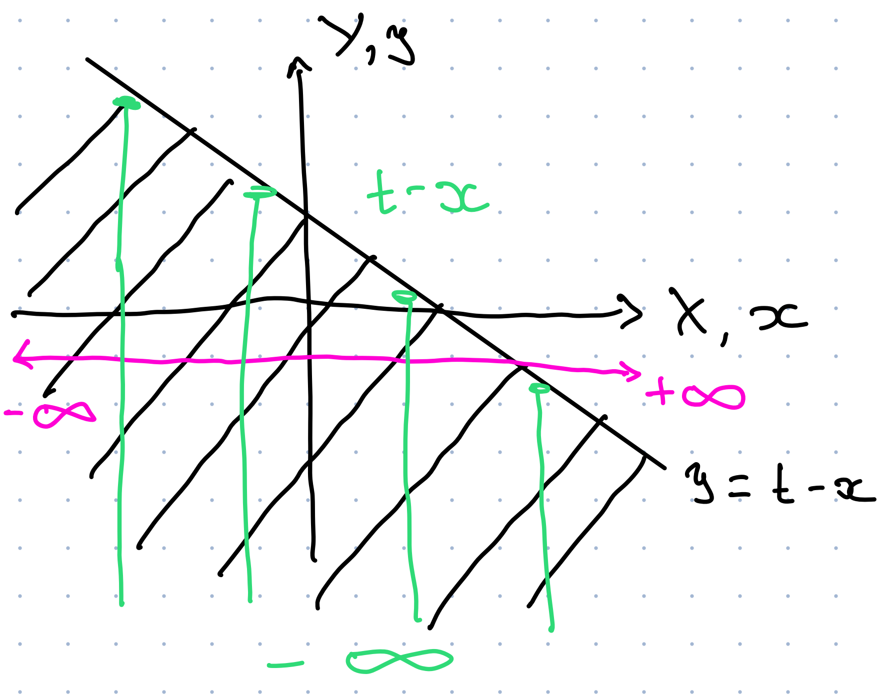
$x_1, \ldots, x_n \sim iid\ F_X(t)$

Формула свёртки

Y, X - независимые $\quad f_X(x) \quad f_Y(y)$

$Z = Y + X \qquad f_Z(t) - ?$

$F_Z(t) = \mathbb{P}(Z \leq t) = \mathbb{P}(X + Y \leq t) =$



$Y \leq t - X$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{t-x} f(x, y)\, dx\, dy$$

$$\parallel$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{t-x} f_X(x) \cdot f_Y(y)\, dx\, dy = (*)$$

$$(*) = \int_{-\infty}^{+\infty} f_X(x) \cdot \int_{-\infty}^{t-x} f_Y(x)\, dx\, dy = F_Z(t) \qquad f_Z(t)$$

$$f_Z(t) = F_Z'(t) = \int_{-\infty}^{+\infty} f_X(x) \left(\int_{-\infty}^{t-x} f_Y(x)\, dy \right)'_t \cdot dx =$$

$$\frac{d}{dt} \int_{\alpha(t)}^{\beta(t)} f(x,t)\, dx = \int_{\alpha(t)}^{\beta(t)} \underbrace{\frac{df(x,t)}{dt}}_{=0} dx + f(\beta(t),t) \cdot \frac{d\beta(t)}{dt} - \underbrace{f(\alpha(t),t) \cdot \frac{d\alpha(t)}{dt}}_{=0}$$

$$= \int_{-\infty}^{+\infty} f_X(x) \cdot f_Y(t-x) \cdot 1\, dx$$

$$F'(x) = \left[\int_0^x f(x)\, dx \right]'_x = f(x)$$

Итог:

X и Y нез. сл. вел. $\qquad Z = X + Y$

$$f_Z(t) = \int_{-\infty}^{+\infty} f_X(x)\, f_Y(t-x)\, dx$$

$$f_Z(t) = \int_{-\infty}^{+\infty} f_X(t-y) \cdot f_Y(y)\, dy$$