2025 HSC Mathematics Extension 1 / Extension 2

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Chapter 1

Vectors

Vectors have both magnitude and direction. A vector from point A to point B is written $\vec{AB}$.

1.1 Magnitude

- $|x\underset{\sim}{i} + y\underset{\sim}{j}| = \sqrt{x^2 + y^2}$

1.2 Scalar Multiplication

- $x\underset{\sim}{i} + y\underset{\sim}{j} = \sqrt{x^2 + y^2}$

Chapter 2

Complex Numbers

2.1 The Imaginary Number

$i = \sqrt{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, perpendicular to the real number line $\lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\forall x \in \mathfrak{R} : \sqrt{-x} = i\sqrt{x}$. $\begin{bmatrix} x \\ y \end{bmatrix} = x + yi, z = \begin{bmatrix} \mathfrak{Re}(z) \\ \mathfrak{Im}(z) \end{bmatrix}$.

2.1.1 Examples

Express $z = \sqrt{-9}$ in terms of i.

1. $z = \sqrt{-9}$
2. $z = \sqrt{9}\sqrt{-1}$
3. $z = 3i$

Simplify $z = i^3$.

1. $z = i^3$
2. $z = i^2 i$
3. $z = -i$

Solve $x^2 + 1 = 0$ for x.

1. $x^2 + 1 = 0$
2. $x^2 = -1$
3. $x = \pm\sqrt{-1}$
4. $x = \pm i$

2.2 Complex Conjugate

$\overline{z} = \mathfrak{Re}(z) - \mathfrak{Im}(z)$. A vector and its conjugate form a conjugate pair. $z + \overline{z} \in \mathfrak{R}, z\overline{z} \in \mathfrak{R}$.

2.2.1 Examples

State the complex conjugate of $z = \frac{-1+i\sqrt{3}}{2}$

1. $z = \frac{-1+i\sqrt{3}}{2}$
2. $\overline{z} = \frac{-1-i\sqrt{3}}{2}$

State the complex conjugate of $z = \frac{2x-5i-ix+3y}{x^2+y^2}$

1. $z = \frac{2x-5i-ix+3y}{x^2+y^2}$
2. $z = \frac{2x-i(5+x)+3y}{x^2+y^2}$
3. $\overline{z} = \frac{2x+i(5+x)+3y}{x^2+y^2}$

Prove $z\overline{z} \in \Re \quad where \quad z = a+ib \wedge a,b \in \Re$

1. $z\overline{z} = (a+ib)(a-ib) \quad where \quad a,b \in \Re$
2. $z\overline{z} = a^2 + abi - abi - i^2b^2$
3. $z\overline{z} = a^2 + b^2$

2.3 Realising the denominator

To realise the denominator of a complex number $\frac{1}{z}$, multiply it by $\frac{\overline{z}}{\overline{z}}$.

2.3.1 Examples

Simplify $z = \frac{1}{1+i}$

1. $z = \frac{1}{1+i}$
2. $z = \frac{1}{1+i}\frac{1-i}{1-i}$
3. $z = \frac{1(1-i)}{(1+i)(1-i)}$
4. $z = \frac{1-i}{1+i-i-i^2}$
5. $z = \frac{1-i}{2}$
6. $z = \frac{1}{2} - i\frac{1}{2}$

Simplify $z = \frac{2+2i\sqrt{3}}{\sqrt{3}-i}$

1. $z = \frac{2+2i\sqrt{3}}{\sqrt{3}-i}$
2. $z = \frac{2+2i\sqrt{3}}{\sqrt{3}-i}\frac{\sqrt{3}+i}{\sqrt{3}+i}$
3. $z = \frac{(2+2i\sqrt{3})(\sqrt{3}+i)}{(\sqrt{3}-i)(\sqrt{3}+i)}$
4. $z = \frac{2\sqrt{3}+2i3+2i+2i^2\sqrt{3}}{4}$
5. $z = \frac{2\sqrt{3}+8i-2\sqrt{3}}{4}$
6. $z = \frac{8i}{4}$
7. $z = 2i$

Simplify $z = \frac{1}{\sqrt{2}+i\sqrt{2}} + \frac{1}{1-i}$

1. $z = \frac{1}{\sqrt{2}+i\sqrt{2}} + \frac{1}{1-i}$
2. $z = \frac{1}{\sqrt{2}+i\sqrt{2}}\frac{\sqrt{2}-i\sqrt{2}}{\sqrt{2}-i\sqrt{2}} + \frac{1}{1-i}\frac{1+i}{1+i}$
3. $z = \frac{\sqrt{2}-i\sqrt{2}}{4} + \frac{1+i}{2}$
4. $z = \frac{\sqrt{2}-i\sqrt{2}+2+2i}{4}$
5. $z = \frac{\sqrt{2}+2}{4} + i\frac{2-\sqrt{2}}{4}$

2.4 Addition

- $p + q = \mathfrak{Re}(p) + \mathfrak{Re}(q) + i(\mathfrak{Im}(p) + \mathfrak{Im}(q))$

2.4.1 Examples

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2.5 Multiplication

- $pq = \mathfrak{Re}(p)\mathfrak{Re}(q) + i(\mathfrak{Re}(p)\mathfrak{Im}(q) + \mathfrak{Re}(q)\mathfrak{Re}(p)) - \mathfrak{Im}(p)\mathfrak{Im}(q)$
- $pq = p\mathfrak{Re}(q) + ip\mathfrak{Im}(q) \quad where \quad p \in \mathfrak{R}$
- $pq = |p||q|(cos(p_\theta + q_\theta) + isin(p_\theta + q_\theta))$

2.5.1 Examples

Simplify $z = (3 - 4i)(7 + 3i)$

1. $z = (3 - 4i)(7 + 3i)$
2. $z = 3 \times 7 - 4i7 + 3i3 - 4i3i$
3. $z = 21 + 12 - 28i + 9i$
4. $z = 33 - 19i$

Simplify $z = 18(4 - 5i)$

1. $z = 18(4 - 5i)$
2. $z = 72 - 90i$

Simplify $z = 2(cos(\frac{\pi}{5}) + isin(\frac{\pi}{5}))5(cos(\frac{\pi}{7}) + isin(\frac{\pi}{7}))$

1. $z = 2(cos(\frac{\pi}{5}) + isin(\frac{\pi}{5}))5(cos(\frac{\pi}{7}) + isin(\frac{\pi}{7}))$
2. $z = 10(cos(\frac{\pi}{5} + \frac{\pi}{7}) + isin(\frac{\pi}{5} + \frac{\pi}{7}))$
3. $z = 10(cos(\frac{7\pi+5\pi}{35}) + isin(\frac{7\pi+5\pi}{35}))$
4. $z = 10(cos(\frac{12\pi}{35}) + isin(\frac{12\pi}{35}))$

2.6 Modulus

- $|z| = \sqrt{\mathfrak{Re}(z)^2 + \mathfrak{Im}(z)^2}$

2.6.1 Examples

Simplify $z = |2 - i\sqrt{3}|$

1. $z = |2 - i\sqrt{3}|$
2. $z = \sqrt{(2)^2 + (-\sqrt{3})^2}$
3. $z = \sqrt{4 + \sqrt{9}}$
4. $z = \sqrt{7}$
5. $z = 7$

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2.7 Argument

- $arg(z) = f(x) = tan^{-1}(\frac{\mathfrak{Im}(z)}{\mathfrak{Re}(z)})$

2.7.1 Examples

Simplify $z = arg(1 - i\sqrt{3})$

1. $z = arg(1 - i\sqrt{3})$
2. $z = f(x) = tan^{-1}(\frac{-\sqrt{3}}{1})$
3. $z = f(x) = tan^{-1}(-\sqrt{3})$
4. $z = -\frac{\pi}{3}$

2.8 Square Root

- $\sqrt{z} = \pm(\sqrt{\frac{|z| + \mathfrak{Re}(z)}{2}} + isgn(\mathfrak{Im}(z))\sqrt{\frac{|z| - \mathfrak{Re}(z)}{2}}) = \frac{b}{2d} \pm i\sqrt{\frac{b^2}{4d^2}}$

2.8.1 Examples

Simplify $\sqrt{5 + 12i} = a + ib$

1. $\sqrt{5 + 12i} = a + ib$
2. $5 + 12i = (a + ib)^2$
3. $5 + 12i = a^2 - b^2 + i2ab$
4. $5 = a^2 - b^2$
5. $12 = 2ab$
6. $ab = 6$
7. $(a = 2, b = 3) \vee (a = -2, b = -3)$ // From inspection.

Simplify $\sqrt{4i-3} = a + ib \quad where \quad a, b \in \Re$

1. $\sqrt{4i-3} = a + ib$
2. $4i - 3 = (a + ib)^2$
3. $4i - 3 = a^2 - b^2 + i2ab$
4. $a^2 - b^2 = -3$
5. $2ab = 4$
6. $ab = 2$
7. $a = \frac{2}{b}$
8. $\frac{4}{b^2} - b^2 = -3$
9. $-b^2 + 3 + 4b^{-2} = 0$
10. $b^4 - 3b^2 - 4 = 0$
11. $(b^2 - 4)(b^2 + 1) = 0$
12. $b^2 = 4, -1$
13. $b = \pm 2$
14. $a = \frac{2}{\pm 2}$
15. $a = \pm 1$
16. $\sqrt{4i-3} = 1 + 2i$

Solve $iz^2 - z + 2i = 0$ **for** z

1. $iz^2 - z + 2i = 0$

2.9 Conversion to Vector or Cartesian form

- $z = \begin{bmatrix} \mathfrak{Re}(z) \\ \mathfrak{Im}(z) \end{bmatrix}$

2.9.1 Examples

Express $z = a + ib$ **in vector form.**

1. $z = a + ib = \begin{bmatrix} a \\ b \end{bmatrix}$

2.10 Conversion to Modulus-Argument Form

- $z = |z|, arg(z)$

2.11 Conversion to Polar Form

- $r = |z|$
- $\theta = f(x) = tan^{-1}(\frac{\mathfrak{Im}(z)}{\mathfrak{Re}(z)})$
- $z = r(cos(\theta) + isin(\theta))$

2.11.1 Examples

Express $z = -\sqrt{2} + i\sqrt{2}$ in polar form.

1. $z = -\sqrt{2} + i\sqrt{2}$
2. $|z| = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2}$
3. $|z| = \sqrt{2+2}$
4. $|z| = 2$
5. $arg(z) = f(x) = tan^{-1}()$

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2.12 Polar Quotient

- $\frac{p}{q} = \frac{|p|}{|q|}(cos(p_\theta - q_\theta) + isin(p_\theta - q_\theta))$

2.12.1 Examples

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2.13 Polar Exponentiation

- $z^n = r^n(cos(n\theta) + isin(n\theta))$

2.13.1 Examples

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2.14 Polar Reciprocal

- $z^{-1} = \frac{cos(\theta) - isin(\theta)}{r}$

2.14.1 Examples

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2.15 Polar Unit Vector Reciprocal

- $|z| = 1 \implies z^{-1} = \overline{z}$

2.15.1 Examples

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2.16 Product of Conjugate pairs

- $z\overline{z} = |z|^2$

2.16.1 Examples

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2.17 Sum of Conjugate pairs

- $z + \overline{z} = 2\mathfrak{Re}(z)$

2.17.1 Examples

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2.18 Difference of Conjugate pairs

- $z - \overline{z} = 2i\mathfrak{Im}(z)$

2.18.1 Examples

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2.19 Conjugate Argument

- $\overline{z} = arg(\overline{z}) = -arg(z)$

2.19.1 Examples

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2.20 Associativity of Conjugate Pairs

- $\overline{p} + \overline{q} = \overline{p + q}$
- $\overline{pq} = \overline{pq}$

2.20.1 Examples

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2.21 Triangle Inequality

- $|p + q| \leq |p| + |q|$

2.21.1 Examples

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2.22 Euler's Formula

- $e^{i\theta} = cos(\theta) + isin(\theta)$
- $z = re^{i\theta}$

2.22.1 Examples

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2.23 Practice Set

2.23.1 Express each complex number in terms of i.

$\sqrt{-25}$

1. $\sqrt{25}\sqrt{-1}$
2. $5i$

$\sqrt{-18}$

1. $\sqrt{18}\sqrt{-1}$
2. $3\sqrt{2}i$

$\sqrt{-\frac{8}{9}}$

1. $\sqrt{\frac{8}{9}}\sqrt{-1}$
2. $\frac{2}{3}\sqrt{2}i$

2.23.2 Simplify each expression.

2.23.3 Solve each equation.

2.23.4 Solve each equation using the quadratic formula.

2.23.5 Solve each equation by completing the square.

2.23.6 Factorise each expression in the equation as a difference of 2 squares, then solve the equation.

2.23.7 State $\mathfrak{Re}(z)$ and $\mathfrak{Im}(z)$ for each complex number.

2.23.8 State the complex conjugate of each complex number.

2.23.9 If $z = p - 3iq$ where $p, q \in \Re$, prove that:

2.23.10 If x and y are real, solve each equation for x and y.

2.23.11 If $V = \frac{2x+2yi-5+3ix-2y+7i}{x^2+y^2}$ is always real, where x and

Chapter 3

Proof

3.1 Symbols

$\implies$	Implication
$\iff$	Equivalence
$\neg$	Negation
$\wedge$	Conjunction
$\vee$	Inclusive Disjunction
$\veebar$	Exclusive Disjunction
$\top$	Truth
$\bot$	False
$\forall$	Universal Quantification
$\exists$	Existential Quantification
$\exists!$	Unique Quantification
$()$	Precedence

Chapter 4

3D Vectors

4.1 Points

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4.2 Addition

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4.3 Scalar Multiplication

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4.4

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4.5 Magnitude

- $|x\underset{\sim}{i} + y\underset{\sim}{j}z| = \sqrt{x^2 + y^2 + z^2}$

4.6 Unit Vector

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4.7 Angle Between Vectors

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4.8 Scalar / Dot Product

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4.9 Parallel Vectors

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4.10 Perpendicular Vectors

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4.11 Midpoint of Vectors

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4.12 Points

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4.13 Parametric Vector Equations of Curves

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4.14

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