Daffodil International University

NUMERICAL METHODS WITH ALGORITHM

A few words.

Allah is almighty, by His grace, I am able to write few words and make lecture sheet for my beloved students as they will get benefits from it in their respective line. I am also thankful to my colleague **Mr Mohiuddin** to help me in designing the chapter interpolation & RK Method.It is only designed for the students in the Daffodil International University at computer science & engineering department according to their syllabus.

All valuable suggestion for the improvment of this lecture sheet will be highly appreciated and gratefully received.

Mohammad Abdul Halim

Senior Lecturer in Mathematics

Department of General Educational Development (GED)

Faculty of Science and Information Technology (FSIT)

Daffodil International University (DIU), Dhaka, Bangladesh

Contents of Numerical Methods

ERROR ANALYSIS CH - 00

Numerical methods are mathematical techniques used for solving mathematical problems that cannot be solved or are difficult to solve. Such as finding the roots of the equation $x^2 - 4\sin x = 0$ in normal calculation in so difficult but easy to find in iterative technique as Bisection method, Newton Raphson method etc. The numerical solution is an approximate numerical value that means value with certain level of accuracy for the solution. Although numerical solutions are an approximation, they can be very accurate.

After reading this chapter, you should be able to:

- find the true and relative true error,
- find the approximate and relative approximate error,
- relate the absolute relative approximate error to the number of significant digits at least correct in your answers
- know the concept of significant digits.

Error is the deviation of a quantity from its true value and it arises in numerical computation due to Round off and truncation. Truncation is due to the fact that in scientific computing we are often making a discrete approximation to something continuous. Numerical round-off error arises because computers cannot represent some numbers exactly.

There are two types of numbers such as:

- Exact Number: A number with true value is called exact number. For example: The number π is an exact number.
- Approximate Number: An approximate number is a number having slightly error. For example: The number $\pi = 3.1416$ is an approximate number.

Significant digit of a number: A digit in a number is called significant digit if it has an importance in it. For example: In the number 3.023 ,the digit 0 have an importance because in absence of it the number converted into a different number 3.23.

Error: The error (true error) of a quantity is the difference between its true value and approximate value. It is denoted by E. If the true value is X and approximate value is x then the error of the quantity is given by,

$$E = X - x$$

There are many types of errors in computing but we have discussed here the following errors only:

- Absolute Error
- Relative Error
- Percentage Error

Absolute Error: The absolute error of a quantity is the absolute value of the difference between the true value X and the approximate value x. It is denoted by E_A.

$$i.e,\ E_A = |X - x|$$

Relative Error: The relative error of a quantity is the ratio of it's absolute error to it's true value. It is denoted by E_R.

$$i.e,\ E_R = \frac{E_A}{X}$$

Percentage Error: The percentage error of a quantity is 100 times of its relative error. It is denoted by E_p.

$$i.e,\ E_p = 100E_R$$

Note: If the number X is rounded to N decimal places, then $E_A = \frac{1}{2}\left(10^{-N}\right)$

Worked-Out problems

Problem-01: An approximate value of π is 3.1428571 and true value is 3.1415926. Find the absolute, relative and percentage errors.

Solution: We have, true value $X = 3.1415926$ and approximate value $x = 3.1428571$

The absolute error is,

$$E_A = |X - x| = |3.1415926 - 3.1428571|$$
$$= |-0.0012645| = 0.0012645$$

The relative error is,

$$E_R = \frac{E_A}{X} = \frac{0.0012645}{3.1415926} = 0.000402$$

The percentage error is,

$$E_p = 100E_R = 100 \times 0.000402 = 0.0402$$

Problem-02: Find the absolute, relative and percentage errors of the number 8.6 if both of its digits are correct.

Solution: The given number is $X = 8.6$

Since both digits are correct so $N = 1$

The absolute error is,

$$E_A = \frac{1}{2}\left(10^{-1}\right) = 0.05$$

The relative error is,

$$E_R = \frac{E_A}{X} = \frac{0.05}{8.6} = 0.0058$$

The percentage error is,

$$E_p = 100E_R = 100 \times 0.0058 = 0.58$$

Problem-03: Evaluate the sum $S = \sqrt{2} + \sqrt{3} + \sqrt{5}$ to 4 significant digits and find its absolute, relative and percentage errors.

Solution: we have, $\sqrt{2} = 1.414, \quad \sqrt{3} = 1.732, \quad \sqrt{5} = 2.236$

$$\therefore \; S = 1.414 + 1.732 + 2.236 = 5.382$$

Since the values of $\sqrt{2}, \sqrt{3}$ *and* $\sqrt{5}$ are rounded of three decimal places, so $N = 3$ .e

The absolute error is,

$$E_A = \frac{1}{2}\left(10^{-3}\right) + \frac{1}{2}\left(10^{-3}\right) + \frac{1}{2}\left(10^{-3}\right) = 0.0005 + 0.0005 + 0.0005 = 0.0015$$

The absolute error shows that the sum is correct to 3 significant digits only.

Hence, we take $S = 5.38$

The relative error is,

$$E_R = \frac{E_A}{X} = \frac{0.0015}{5.38} = 0.00028$$

The percentage error is,

$$E_p = 100E_R = 100 \times 0.00028 = 0.028$$

Problem 04:

The derivative of a function $f(x)$ at a particular value of x can be approximately calculated by $f'(x) \approx \dfrac{f(x+h)-f(x)}{h}$ where $f(x)=7e^{0.5x}$ and $h=0.3$,then find the followings:

(a) The approximate value of $f'(2)$

(b) The true value of $f'(2)$

(c) The true error for part (a)

Solution: We have, $f'(x) \approx \dfrac{f(x+h)-f(x)}{h}$ where $f(x)=7e^{0.5x}$ and $h=0.3$

Now, $f'(x) \approx \dfrac{f(x+h)-f(x)}{h}$

Putting $x=2$ and $h=0.3$, we get

$$f'(2) \approx \frac{f(2+0.3)-f(2)}{0.3} = \frac{f(2.3)-f(2)}{0.3} = \frac{7e^{0.5\times 2.3}-7e^{0.5\times 2}}{0.3} = \frac{22.3107-19.028}{0.3} = 10.265$$

Now differentiating $f(x)$ w.r.to x we get,

$$f'(x) = 7\times 0.5\ e^{0.5x}$$

After putting $x=2$, we get the true value $f'(2)=7\times 0.5\ e^{0.5\times 2}=3.5\times e^{1.0}=9.5140$.

Finally, the true error, $E_t = X - x = 9.5140 - 10.265 = -0.75061$ (Ans)

Questions

1. What is error?
2. Write down the names of the three errors.
3. Define the terms absolute error, relative error & Percentage error.
4. Explain the errors with examples.
5. What is significant digit?
6. Three approximations of the number $\frac{1}{3}$ are given as 0.32, 0.33 and 0.34. Find the best approximation among the three.
7. Find the absolute, relative and percentage errors of the number 7.8 if both of its digits are correct.
8. Evaluate the sum $S=\sqrt{2}+\sqrt{5}+\sqrt{7}$ to 4 significant digits and find its E_A , E_R & E_P .
9. Estimate the absolute and relative errors for the function $f(x)=\sqrt{x}+x$ for $x_a = 4.000$.
10. The derivative of a function $f(x)$ at a particular value of x can be approximately calculated by $f'(x) \approx \dfrac{f(x+h)-f(x)}{h}$ where $f(x)=7e^{0.5x}$,then find the relative error in calculating $f'(2)$ using values from $h=0.3$ and $h=0.15$.

THE SOLUTION OF ALGEBRAIC & TRANSCENDENTAL EQUATIONS

CH 01

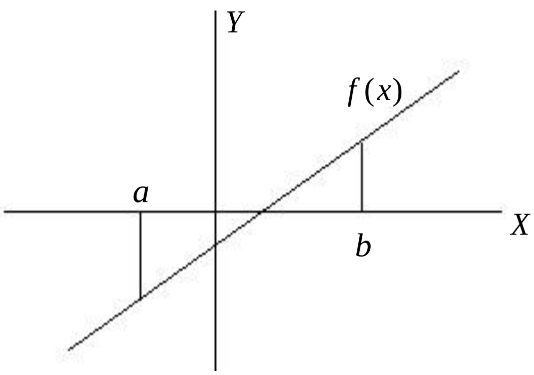
In this Lesson, we have discussed about the solution of equations $f(x)=0$ where $f(x)$ linear, non-linear, algebraic or transcendental function. We get the solution of the equation $f(x)=0$ by using Bisection method, Newton- Raphson method and method of false position. Those methods are established based on **Intermediate Value Theorem.**

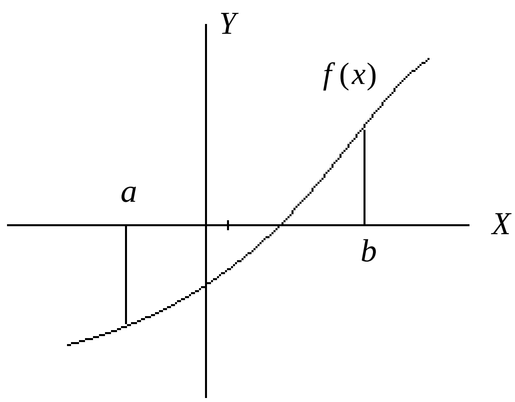
After reading this chapter, you should be able to:

1. derive the Bisection, Newton-Rapshon & Iteration Method,
2. develop the algorithm of the Bisection, Newton-Rapshon & Iteration Method,
3. use the the Bisection, Newton-Rapshon & Iteration Method to solve a nonlinear equation, and
4. discuss the drawbacks and Advantages of the the Bisection, Newton-Rapshon & Iteration Method.

Statement of Intermediate Value Theorem:

If f(x) is continuous in the interval (a, b) and if f (a) and f(b) are of opposite signs, then the equation f(x) = 0 will have at least one real root between a and b.





Algebraic equation:

An **algebraic equation** is an equation that includes one or more variables such as $x^2+xy-z=0$.

Transcendental equation:

An equation together with algebraic, trigonometrical, exponential or logarithmic function etc. is called transcendental equation such as $e^x+2\sin x-5x=0$.

Solution/root:

A solution/root of an equation is the value of the variable or variables that satisfies the equation.

Iteration:

Iteration is the repeated process of calculation until the desired result or approximate numerical value has come. Each repetition of the process is also called iteration and the result of one iteration is used as the starting point for the next iteration.

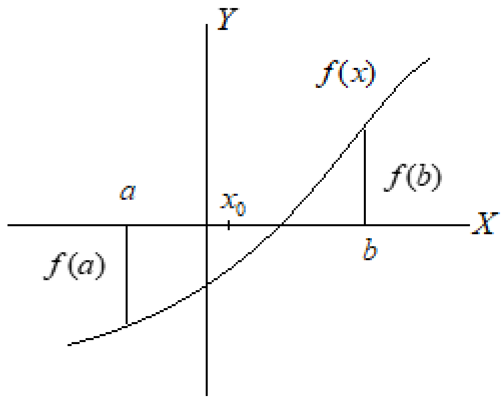
We are capable to find the root of algebraic or transcendental function by using following methods:

1. Bisection method
2. Newton Rapshon method (Newton's Iteration method)
3. Iteration method (Method of successive approximation/Fixed-point Iteration Method)

4. Regular-Falsi method (The method of False position)
5. The secant method
6. Muller's method
7. Ramanujan's method
8. Horner's method

Bisection Method:

Let us suppose we have an equation of the form $f(x)=0$ in which solution lies between in the range (a,b) where $a<b$. Also $f(x)$ is continuous and it can be algebraic or transcendental. If $f(a)$ and $f(b)$ are opposite signs, then there exist at least one real root between a and b. Let $f(a)$ be positive and $f(b)$ negative. Which implies at least one root exits between a and b.We assume that root to be $x_o=\frac{a+b}{2}$. Check the sign of $f(x_o)$. If $f(x_o)$ is negative, the root lies between a and x_0. If $f(x_o)$ is positive, the root lies between x_0 and b.



Subsequently any one of this case occur. $x_1=\frac{a+x_0}{2}$ Or $x_1=\frac{b+x_0}{2}$. When $f(x_1)$ is negative, the root lies between x_0 and x_1 and let the root be $x_2=\frac{x_1+x_0}{2}$. Again $f(x_2)$ negative then the root lies between x_0 and x_2, let $x_3=\frac{x_0+x_2}{2}$ and so on. Repeat the process $x_0, x_1, x_2, \ldots, x_{k-1}, x_k$ whose limit of convergence is the exact root. We have to stop the iteration when the value of two successive iterations are approximately equal. That is $x_{k-1}\approx x_k$ or $|x_k-x_{k-1}|\approx 0$.

Advantages of the bisection method:

1. It is always convergent.
2. The error bound decreases by half with each iteration i.e., error can be controlled.
3. It is well suited to electronic Computers.
4. It is very simple method.

Disadvantages/draw-back of the bisection method:

1. The bisection method converges very slowly
2. It requires large number of iterations
3. The bisection method cannot detect multiple roots
4. Choosing a guess close to the root may result in needing many iterations to converge.
5. Cannot find roots of some equations such as $y=x^2=0$ because upper guess and lower guess always produce positive value.
6. May seek a singularity point as a root as the equation like $y=\frac{1}{x}=0$

Algorithm for Bisection method:

Steps	Task
01	Define $f(x)$
02	Read a 'The lower bound of the desired roots'
03	Read b 'The upper bound of the desired roots'
04	Set $k=1$
05	Calculate $x_k=\dfrac{a+b}{2}$
06	Calculate $f_k=f\left(x_k\right)$
07	Print k, x_k, f_k

08	If $\left\|x_k-x_{k-1}\right\| \approx 0.0001$ then
	GOTO Step 11
	elseif
	$f\left(a\right).f_k<0$ then $b=x_k$.
	Else
	$f\left(b\right).f_k<0$ then $a=x_k$.
	Endif

09	Set $k=k+1$
10	GOTO Step 05
11	Print 'Required root, x_k'
12	STOP

Problem 01:

Find a root of the equation $x^2-4x-10=0$ using Bisection method.

Solution:

Let $f(x)=x^2-4x-10$

Here,

$f(-2)=4+8-10=2>0$ and $f(-1)=1+4-10=-5<0$.

Since $f(-2)$ is positive and $f(-1)$ is negative so at least one real root lies between -2 and -1.

$$\therefore\ x=\frac{-2-1}{2}=\frac{-3}{2}=-1.5$$

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.

Iteration	Value of a (+)	Value of b (-)	$x=\dfrac{a+b}{2}$	Sign of $f(x)=x^2-4x-10$
1	-2	-1	-1.5	-1.75 < 0
2	-2	-1.5	-1.75	0.0625 > 0
3	-1.75	-1.5	-1.625	-0.859 < 0
4	-1.75	-1.625	-1.6875	-0.40 < 0

5	-1.75	-1.6875	-1.7188	-0.1705 < 0
6	-1.75	-1.7188	-1.7344	-0.054 < 0
7	-1.75	-1.7344	-1.7422	0.004 > 0
8	-1.7422	-1.7344	-1.7383	-0.025 < 0
9	-1.7422	-1.7383	-1.7402	-0.0109 < 0
10	-1.7422	-1.7402	-1.7412	-0.003 < 0

The approximate root of the given equation is $= -1.7412$ because $f(-1.7412) = -0.003 \approx 0$.

Problem 02:

Find the root of the equation $x^3 - x - 1 = 0$ by using Bisection method correct up to two decimal places.

Solution:Let $f(x) = x^3 - x - 1$

Here, $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 8 - 2 - 1 = 5 > 0$

Since $f(1)$ and $f(2)$ are of opposite sign so at least one real root lies between 1 and 2.

$$\therefore\ x = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.

Iteration	Value of a (+)	Value of b (-)	$x = \frac{a+b}{2}$	Sign of $f(x) = x^3 - x - 1$
1.	2	1	1.5	0.875 > 0
2.	1.5	1	1.25	-0.297 < 0
3.	1.5	1.25	1.375	0.2246 > 0
4.	1.375	1.25	1.3125	-0.0515 < 0
5.	1.375	1.3125	1.34375	0.08626 > 0
6.	1.34375	1.3125	1.3281	0.018447 > 0
7.	1.3281	1.3125	1.3203	-0.019 < 0
8.	1.3281	1.3203	1.3242	-0.002 < 0
9.	1.3281	1.3242	1.3261	0.005970 > 0
10.	1.3261	1.3242	1.3251	0.00162 < 0

It is evident that from the above table, the difference between two successive iterative values of x is $|1.3261 - 1.3251| = 0.001$ which the accuracy condition for the solution exact. So, the required root of the given equation up to the two decimal places is 1.32.

Problem 03:

Find the root of the equation $xe^x = 1$ by using Bisection method correct up to three decimal places on the interval (0, 1).

Solution:

Let $f(x) = xe^x - 1$

Here, $f(0) = 0.e^0 - 1 = -1 < 0$ and $f(1) = 1.e^1 - 1 = 1.7182 > 0$

Since $f(0)$ and $f(1)$ are of opposite sign so at least one real root lies between 0 and 1.

$$\therefore\ x = \frac{0+1}{2} = \frac{1}{2} = 0.5$$

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.

Iteration	Value of a (+)	Value of b (-)	$x = \frac{a+b}{2}$	Sign of $f(x) = xe^x - 1$
1.	1	0	0.5	-0.1756 < 0
2.	1	0.5	0.75	0.5877 > 0
3.	0.75	0.5	0.625	0.1676 > 0
4.	0.625	0.5	0.5625	-0.0127 < 0
5.	0.625	0.5625	0.59375	0.0751 > 0
6.	0.59375	0.5625	0.578125	0.0306 > 0
7.	0.578125	0.5625	0.5703125	0.00877 > 0
8.	0.5703125	0.5625	0.56640625	-0.0023 < 0
9.	0.5703125	0.56640625	0.5683594	0.00336 > 0
10.	0.5683594	0.56640625	0.5673828	0.000662 > 0

It is evident that from the above table, the difference between two successive iterative values of x is $|0.5683594 - 0.5673828| \approx 0.001$ which the accuracy condition for the solution exact. So, the required root of the given equation up to the three decimal places is ≈ 0.567.

Problem 04:

Find the root of the equation $4\sin x - e^x = 0$ by using Bisection method correct up to four decimal places.

Solution:

Consider that, $f(x) = 4\sin x - e^x$

Here,

$f(0) = 4\sin(0) - e^0 = -1 < 0$ and $f(1) = 4\sin(1) - e^1 = 0.64 > 0$ **[Change calculator in radian Mode]**

Since $f(0)$ and $f(1)$ are of opposite sign so at least one real root lies between 0 and 1.

$$\therefore\ x = \frac{0+1}{2} = \frac{1}{2} = 0.5$$

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.

Iteration	Value of a (+)	Value of b (-)	$x = \frac{a+b}{2}$	Sign of *f(x)*
1.	1	0	0.5	0.268 > 0
2.	0.5	0	0.25	-0.294 < 0
3.	0.5	0.25	0.375	0.0101 > 0
4.	0.375	0.25	0.3125	-0.1371 < 0
5.	0.375	0.3125	0.34375	-0.0621 < 0
6.	0.375	0.34375	0.359375	-0.0256 < 0
7.	0.375	0.359375	0.3671875	-0.0077 < 0
8.	0.375	0.3671875	0.3710937	-0.00122 < 0
9.	0.375	0.3710937	0.373046	0.00566 > 0
10.	0.373046	0.3710937	0.372070	-0.00344 < 0
11.	0.373046	0.372070	0.372558	0.00455 > 0
12.	0.372558	0.372070	0.372279	0.0039 > 0
13.	0.372279	0.372070	0.372174	0.0036 > 0
14	0.372174	0.372070	0.372122	0.0036 > 0

It is evident that from the above table, the difference between two successive iterative values of x is $|0.372174 - 0.372122| \approx 0.0001$ which the accuracy condition for the solution exact. So, the required root of the given equation up to the three decimal places is ≈ 0.3721.

Note: To determine the value of the trigonometrical function f(x), we have to change our calculator in radian mode.

Try yourself:

TYPE01:

To find the root of the following equations using Bisection method by your own choosing interval

1. $2^x - 5x + 2 = 0$
2. $e^{2x} - e^x - 2 = 0$
3. $x^3 + x^2 - 1 = 0$
4. $2x + \cos x - 3 = 0$
5. $\cos x - \ln x = 0$
6. $x^2 - 4x - 10 = 0$
7. $2x = 1 + \sin x$
8. $x^3 - 2x^2 - 4 = 0$
9. $\sin^2 x = x^2 - 1$
10. $x \sin x = 1$
11. $\cos x - xe^x = 0$
12. $4x = \tan x$
13. $e^x \tan x = 1$

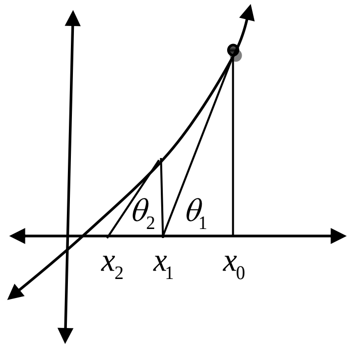
TYPE02:

1. Apply bisection method to find real root of $x^3 - 3x + 1 = 0$ that lies in (0, 1).
2. Find the real root of the equation $x + \ln x - 2 = 0$ belonging to the interval (1, 2) using Bisection Method.
3. Find the real root of the equation $x + \log_{10} x - 1.2 = 0$ belonging to the interval (2, 3) using Bisection Method.
4. Find the real root of the equation $e^{-x^2} - \cos x = 0$ belonging to the interval (1, 3) using Bisection Method.
5. Find the real root of the equation $e^{-x^2} - 4\sin x = 0$ belonging to the interval (1, 2) using Bisection Method.
6. Find the real root of $e^x + 4x^2 = 0$ that lies in (0, 1).
7. Find the real root of the equation $x^3 - 3x - 5 = 0$ belonging to the interval (2, 3) using Bisection Method.
8. Find the real root of the equation $x^3 - x - 1 = 0$ belonging to the interval (2, 3) using Bisection Method.
9. Find the positive real root of the equation $x^3 - 3x + 1.06 = 0$ by Bisection Method correct to four decimal places.
10. Find the positive real root of the equation $x^4 + x^2 - 80 = 0$ by Bisection Method correct to three decimal places.
11. Use bisection method to find the real root of the equation $xe^x = 1$ between 0 and 1 to four significant figures.
12. Compute a root of the equation $e^x = x^2$ to an accuracy of 10^{-5} using bisection method.
13. Compute one root of $e^x - 3x = 0$ correct to two decimal places.
14. Solve the equation $x - \exp\left(\frac{1}{x}\right) = 0$ by bisection method.

TYPE03:

1. Discuss the Bisection Method to find a real root of the equation $f(x) = 0$ in the interval [a,b].
2. Write down an Algorithm for Bisection Method.
3. Mention the Merits and demerits of Bisection Method.
4. What is the draw-backs of the Bisection Method?

Newton Raphson Method:



Suppose we want to find a real root of the given equation $f(x)=0$ that lies in (a,b).Consider $x_0 \in (a,b)$ be an arbitrary point which is very close to the desired root of the given equation $f(x)=0$.Draw a tangent to the curve $f(x)=0$ at $x=x_0$.Suppose this tangent makes an angle θ_1 with x-axis at the point $(x_1,0)$ where x_1 is the first approximation of the desired root.

$$\tan\theta_1 = \frac{f(x_0)}{x_0 - x_1} \cdots\cdots\cdots\cdots\cdots (i)$$

On the other hand, the slope of the curve $f(x)=0$ at $x=x_0$ is $f'(x_0)$.

$$\tan\theta_1 = f'(x_0) \cdots\cdots\cdots\cdots\cdots (ii)$$

Therefore, from equation (i) and (ii) we have

$$\frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If $f(x_1)=0$ we say that x_1 is the desired root of the given equation $f(x)=0$.

Suppose that $f(x_1) \neq 0$ .Now a draw a tangent to the curve $f(x)=0$ at $x=x_1$ which makes an angle θ_2 with x-axis at the point $(x_2,0)$ where x_2 is the second approximation of the desired root .Consequently we have

$$\tan\theta_2 = f'(x_1) = \frac{f(x_1)}{x_1 - x_2}$$

On Simplification we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general the k-th approximation x_k can be computed by using the following iterative

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}, k = 3,4,5,......etc.$$

We shall continue this iterative process until the value of two successive approximation are approximately equal.i.e $x_k \approx x_{k-1}$ or $f\left(x_k\right) \approx 0$.

Newton Rapson formula from Taylor series:

Let x_0 be an approximate value of the desired root of the equation $f(x)=0$ and if x_1 is the exact root of $f(x)=0$ then $f(x_1)=0$ .Let the small quantity h be the correction of the approximation x_0 so that $x_1 = x_0 + h \quad \cdots(*)$ .For this reason the given equation $f(x_1)=0$ reduces to the form as ,

$$f(x_1) = f(x_0+h) = 0 \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ (1)$$

From Taylor's theorem, we have

$$f(x_0+h) = f(x_0) + \frac{h}{1!} f^{'}(x_0) + \frac{h^2}{2!} f^{''}(x_0) + \frac{h^3}{3!} f^{'''}(x_0) + \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ (2)$$

Comparing equation (1) and (2) we find,

$$f(x_0) + \frac{h}{1!} f^{'}(x_0) + \frac{h^2}{2!} f^{''}(x_0) + \frac{h^3}{3!} f^{'''}(x_0) + \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots = 0 \ \cdots\cdots\cdots \ (3)$$

In the above equation the quantity h is relatively so small, so we may neglect the higher power of h more than one and we get from the equation (3)

$$f(x_0) + \frac{h}{1!} f^{'}(x_0) = 0$$

$$f(x_0) + \frac{h}{1} f^{'}(x_0) = 0$$

$$f(x_0) + h \ f^{'}(x_0) = 0$$

$$h = -\frac{f(x_0)}{f^{'}(x_0)}$$

Now from (*) we get the improved value of the root is of the following

$$x_1 = x_0 - \frac{f(x_0)}{f^{'}(x_0)}$$

If we use x_1 is the approximate value, then the next approximation to the root is as follows,

$$x_2 = x_1 - \frac{f(x_1)}{f^{'}(x_1)}$$

Proceeding in this way we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)} \quad \text{where } n = 0,1,2,3,\ldots\ldots\ etc.$$

Therefore, the formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f^{'}(x_n)}$ is known as the Newton Rapson Formula or Tangential Method.

Advantage of Newton Rapshon method:

1. Converge fast if it converge to the root compare to another method.
2. Requires only one guess.
3. Convergence to the root quadratically.
4. Easy to convert to multiple dimentions.
5. Can be to polish a root found by another methods.

Dis-advantage / drawback of Newton Rapshon method:

1. Must find the derivative.
2. Poor global convergence properties.
3. It takes more computing time
4. It should never be used when the graph of $f(x)=0$ is nearly horizontal where it crosses the x-axis.
5. Dependent on initial guess

- May be too far from local root
- May encounter a zero derivative
- May loop indefinitely

Algorithm for Newton Rapson method:

Steps	Task
01	Define $f(x)$
02	Define $f'(x)$
03	Read x_0
04	Set $k=0$
05	$k=k+1$
06	Calculate $x_k = x_{k-1} - \dfrac{f(x_{k-1})}{f'(x_{k-1})}$
07	If $\lvert x_k - x_{k-1}\rvert \approx 0.0001$ then GOTO Step 8 elseif GOTO Step 5
08	Print x_k ,the desired root
09	STOP

Problem 01:

Find the root of the equation $x^3-3x-5=0$ by Newton-Rapshon Method correct to four decimal places.

Solution:

Let $f(x)=x^3-3x-5$ then $f^{'}(x)=3x^2-3$.

Here $f(2)=8-6-5=-3<0$ and $f(3)=27-9-5=13>0$

Since $f(2)$ and $f(3)$ are of opposite sign so at least one real root lies between 2 and 3.

we know that from Newton-Rapshon method ,

$$x_{n+1}=x_n-\frac{f(x_n)}{f^{'}(x_n)}$$

$$x_{n+1}=x_n-\frac{x_n^3-3x_n-5}{3x_n^2-3} \qquad [\text{ putting values}]$$

$$x_{n+1}=\frac{3x_n^3-3x_n-x_n^3+3x_n+5}{3x_n^2-3}$$

$$x_{n+1}=\frac{2x_n^3+5}{3x_n^2-3} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

Choosing an initial guess $x_0=2$ and putting $n=0$ and $x_0=2$ in above mentioned equation (1), we are capable to find the sccessive improved approrimations are as follows:

$$x_1=\frac{2x_0^3+5}{3x_0^2-3}=\frac{2\times 2^3+5}{3\times 2^2-3}=2.333$$

$$x_2 = \frac{2x_1^3+5}{3x_1^2-3} = \frac{2\times(2.333)^3+5}{3\times(2.333)^2-3} = 2.2806$$

$$x_3 = \frac{2x_2^3+5}{3x_2^2-3} = \frac{2\times(2.2806)^3+5}{3\times(2.2806)^2-3} = 2.2790$$

$$x_4 = \frac{2x_3^3+5}{3x_3^2-3} = \frac{2\times(2.2790)^3+5}{3\times(2.2790)^2-3} = 2.2790$$

Since $x_4 = x_3$ so the Newton Rapshon method gives no new values of x and the approximate root is correct to four decimal places.Hence the require root is 2.2790 .

Problem 02:

Using Newton-Rapshon method ,find the root of the equation $x^4 - x - 10 = 0$ which is nearer to $x = 2$,correct to three decimal places.

Solution:

Let $f(x) = x^4 - x - 10$ then $f^{'}(x) = 4x^3 - 1$.

Here $f(1) = 1-1-10 = -10 < 0$ and $f(2) = 16-2-10 = 4 > 0$

Since $f(1)$ and $f(2)$ are of opposite sign so at least one real root lies between 1 and 2.

we know that from Newton-Rapshon method ,

$$x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^4 - x_n - 10}{4x_n^3 - 1}$$

$$x_{n+1} = \frac{4x_n^4 - x_n + x_n^4 + x_n + 10}{4x_n^3 - 1}$$

$$x_{n+1} = \frac{3x_n^4 + 10}{4x_n^3 - 1} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

Choosing an initial guess $x_0 = 1.9$ and putting $n = 0$ and $x_0 = 1.9$ in above mentioned equation (1), we are capable to find the sccessive improved appromations are as follows:

$$x_1 = \frac{3x_0^4+10}{4x_0^3-1} = \frac{3\times(1.9)^4+10}{4\times(1.9)^3-1} = 1.8$$

$$x_2 = \frac{3x_1^4+10}{4x_1^3-1} = \frac{3\times(1.8)^4+10}{4\times(1.8)^3-1} = 1.85556$$

$$x_3 = \frac{3x_2^4+10}{4x_2^3-1} = \frac{3\times(1.85556)^4+10}{4\times(1.85556)^3-1} = 1.85556$$

Since $x_2 = x_3$ so the Newton Rapshon method gives no new values of x and the approximate root is correct to five decimal places.Hence the require root is 1.85556.

Problem 03:

Find the real root of the equation $x^2 - 4\sin x = 0$ correct to four decimal places using Newton-Rapshon method.

Solution:

Let $f(x) = x^2 + 4\sin x$ then $f^{'}(x) = 2x + 4\cos x$.

Here $f(-1)=(-1)^2+4\sin(-1)=-2.36<0$ and $f(-2)=(-2)^2+4\sin(-2)=0.36<0$.

Hints: Calculator must be in radian Mode.

Since $f(-1)$ and $f(-2)$ are of opposite sign so at least one real root lies between -2 and -1.

we know that from Newton-Rapshon method ,

$$x_{n+1}=x_n-\frac{f(x_n)}{f^{'}(x_n)}$$

$$x_{n+1}=x_n-\frac{x_n^2+4\sin x_n}{2x_n+4\cos x_n}$$

$$x_{n+1}=\frac{2x_n^2+4x_n\cos x_n-x_n^2-4\sin x_n}{2x_n+4\cos x_n}$$

$$x_{n+1}=\frac{x_n^2+4x_n\cos x_n-4\sin x_n}{2x_n+4\cos x_n} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

Choosing an initial guess $x_0=-1.9$ and putting $n=0$ and $x_0=-1.9$ in above mentioned equation (1), we are capable to find the sccessive improved approrimations are as follows:

$$x_1=\frac{x_0^2+4x_0\cos x_0-4\sin x_0}{2x_0+4\cos x_0}$$

$$=\frac{(-1.9)^2+4\times(-1.9)\cos(-1.9)-4\sin(-1.9)}{2\times(-1.9)+4\cos(-1.9)}=-1.93$$

$$x_2=\frac{x_1^2+4x_1\cos x_1-4\sin x_1}{2x_1+4\cos x_1}$$

$$=\frac{(-1.93)^2+4\times(-1.93)\cos(-1.93)-4\sin(-1.93)}{2\times(-1.93)+4\cos(-1.93)}=-1.9338$$

Since $x_1\approx x_2$ so the Newton Rapshon method gives no new values of x and the approximate root is correct to two decimal places.Hence the require root is -1.93.

Problem 04:

Find the root of the equation $x\sin x+\cos x=0$,using Newton-Rapshom method.

Solution:

Let $f(x)=x\sin x+\cos x$ then $f^{'}(x)=1.\sin x+x\cos x+(-\sin x)=x\cos x$.

Here $f(2)=2\sin 2+\cos 2=1.40>0$ and $f(3)=3\sin 3+\cos 3=-0.56>0$.

Hints: Calculator must be in radian Mode.

Since $f(2)$ and $f(3)$ are of opposite sign so at least one real root lies between 2 and 3.

we know that from Newton-Rapshon method ,

$$x_{n+1}=x_n-\frac{f(x_n)}{f^{'}(x_n)}$$

$$x_{n+1}=x_n-\frac{x_n\sin x_n+\cos x_n}{x_n\cos x_n}$$

$$x_{n+1}=\frac{x_n^2\cos x_n-x_n\sin x_n-\cos x_n}{x_n\cos x_n}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Choosing an initial guess $x_0=2.79$ and putting $n=0$ and $x_0=2.79$ in above mentioned equation (1), we are capable to find the sccessive improved approrimations are as follows:

$$x_1=\frac{x_0^2\cos x_0-x_0\sin x_0-\cos x_0}{x_0\cos x_0}=\frac{(2.79)^2\cos(2.79)-2.79\sin(2.79)-\cos(2.79)}{2.79\cos(2.79)}=2.7984$$

$$x_2=\frac{x_1^2\cos x_1-x_1\sin x_1-\cos x_1}{x_1\cos x_1}=\frac{(2.7984)^2\cos(2.7984)-2.7984\sin(2.7984)-\cos(2.7984)}{2.7984\cos(2.7984)}$$

$$=2.79834$$

Since $x_1\approx x_2$ so the Newton Rapshon method gives no new values of x and the approximate root is correct to three decimal places.Hence the require root is 2.7984.

Try yourself:

TYPE01:

To find the root of the following equations using Newton-Rapshon method by taking your own guess:

1. $x+\log x=2$	**2.** $2x=\log_{10}x+7$	**3.** $e^x=4x$	**4.** $x^3+x^2-1=0$
5. $3x-\cos x-1=0$	**6.** $\sin x=1-x^2$	**7.** $\sin x=1-x^2$	**8.** $\sin^2 x=x^2-1$
9. $3x+\sin x=e^x$	**10.** $e^x\tan x=1$	**11.** $\cos x-xe^x=0$	**12.** $x\sin x=1$

TYPE02:

1. Find the real root of $2x-\log_{10}x-7=0$ using Newton Raphson in (3, 4).
2. Using Newton Raphson Method find root of the equation $e^x-4x^2=0$ that lies in (4, 5).
3. Using Newton Raphson Method find root of the equation $e^{x^2}-4\sin x=0$ in the interval (0, 1).
4. Find a real root of the equation $x^3+x^2-1=0$ by using Newton Raphson Method correct up to four decimal places.
5. Use Newton Raphson's Method to find the root of $x^3-x-2=0$ with $x_0=3$.
6. Find a real root of the equation $x^4+x^2-80=0$ by using Newton Raphson Method correct up to three decimal places.
7. Find a real root of the equation $x^2+\ln x-2=0$ in [1,2] by using Newton Raphson Method correct up to five decimal places.
8. By using Newton Method find a real root of the equation $x^4-x-10=0$ which is near to x=2 correct up to three decimal places.
9. Find a real root of the equation $x^3+x^2+3x+4=0$ by using Newton Raphson Method correct up to four decimal places.
10. Find by using Newton Method the real root of the equation $e^x=4x$ which is approximately 2 correct to three places of decimals.

TYPE03:

1. Using Newton Raphson Method establish the formula $x_{n+1}=\frac{1}{2}\left(x_n+\frac{N}{x_n}\right)$ to calculate the square root of N. Hence find the square root of 5 correct to four places of decimals.
2. Show that the iterative formula for finding the reciprocal of N is $x_{n+1}=x_n\left(2-Nx_n\right)$ and hence find the value of $\frac{1}{31}$.
3. Derive the iterative formula for Newton Raphson method to solve the equation $f(x)=0$.
4. Write down the merits and demerits of Newton Raphson Method.
5. Write down the algorithm for Newton Raphson Method.
6. When does Newton Raphson Method Fails.

Fixed Point Iteration Method:

Let us consider an equation $f(x)=0$ whose roots are to be determined in the interval (a,b). The equation $f(x)=0$ can be expressed as

$$x=\varphi(x) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

We assume that $\varphi(x)$ must be such that $|\varphi'(x)|<1, \forall x \in (a,b)$.

Let x_0 is an initial solution or approximation for the equation $f(x)=0$ .we substitute the value of x_0 in the right-hand side of the equation (1) and obtain a better approximation x_1 given by the equation $x_1=\varphi(x_0)$.

Again, substituting $x=x_1$ in the equation (1), we get next approximation as $x_2=\varphi(x_1)$.

Preceding in this way we can find the following successive approximations,

$$x_3=\varphi(x_2)$$
$$x_4=\varphi(x_3)$$
$$\vdots$$
$$x_n=\varphi(x_{n-1})$$

Therefore, the iterative formula for successive approximation method is,

$$x_n=\varphi(x_{n-1}), n=1,2,3,4,...etc.$$

Here x_n is the n-th approximation of the desired root of $f(x)=0$.

We shall continue this iterative cycle until the values of two successive approximations are almost equal. This above-mentioned method is known as **Iteration method. Or Method of successive approximation or Fixed-point Iteration.**

Algorithm for Iteration method:

Steps	Task
01	Define $\varphi(x)$
02	Read x_0
03	Set $k=1$
04	$x_n=\varphi(x_{n-1})$

05	If $\lvert x_n-x_{n-1}\rvert \approx 0.0001$ then
	GOTO Step 6
	else
	$n=n+1$
	GOTO Step 04

06	Print x_n ,the desired root
07	STOP

Problem 01:

Find the real root of the equation $x^3+x^2-1=0$ on the interval [0, 1] with an accuracy of 10^{-4} .

Solution:

Let $f(x)=x^3+x^2-1$

$$\therefore f(0)=0^3+0^2-1<-1 \; and f(1)=1^3+1^2-1=1>1$$

Since $f(0)$ and $f(1)$ are of opposite sign so at least one real root lies between 0 and 1.

The given equation can be expressed as $x^3+x^2-1=0$

$$x^2(x+1)=1$$

$$x^2 = \frac{1}{(x+1)}$$

$$x = \frac{1}{\sqrt{(x+1)}} = \varphi(x)\ [say]$$

$$\therefore \varphi(x) = \frac{1}{\sqrt{(x+1)}} \Rightarrow \varphi'(x) = -\frac{1}{2}\frac{1}{(x+1)^{\frac{3}{2}}}$$

Also, $|\varphi'(x)| = \left|-\frac{1}{2}\frac{1}{(x+1)^{\frac{3}{2}}}\right| = \frac{1}{2}\left|\frac{1}{(x+1)^{\frac{3}{2}}}\right| < 1$ for $x \in (0,1)$.

Therefore, the iteration method is applicable for the given function.

Assume $x_0 = 0.7$ is an initial solution or approximation for the equation $f(x) = 0$.

So successive approximations are,

$$x_1 = \varphi(x_0) = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{0.7+1}} = 0.76697$$

$$x_2 = \varphi(x_1) = \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{0.76697+1}} = 0.75229$$

$$x_3 = \varphi(x_2) = \frac{1}{\sqrt{x_2+1}} = \frac{1}{\sqrt{0.75229+1}} = 0.75543$$

$$x_4 = \varphi(x_3) = \frac{1}{\sqrt{x_3+1}} = \frac{1}{\sqrt{0.75543+1}} = 0.75476$$

$$x_5 = \varphi(x_4) = \frac{1}{\sqrt{x_4+1}} = \frac{1}{\sqrt{0..75543+1}} = 0.75490$$

$$x_6 = \varphi(x_5) = \frac{1}{\sqrt{x_5+1}} = \frac{1}{\sqrt{0.75543+1}} = 0.75487$$

$$x_7 = \varphi(x_6) = \frac{1}{\sqrt{x_6+1}} = \frac{1}{\sqrt{0.75487+1}} = 0.75488$$

$$x_8 = \varphi(x_7) = \frac{1}{\sqrt{x_7+1}} = \frac{1}{\sqrt{0.75488+1}} = 0.75488$$

Since $x_7 \approx x_8$ so the Iteration method gives no new values of x and the approximate root is correct to four decimal places.Hence the require root is 0.7548.

Problem 02:

Find the real root of the equation $x - \ln x - 2 = 0$ that lies on [3, 4] using fixed point iteration method.

Solution:

Let $f(x) = x - \ln x - 2 = 0$

Now $f(3) = 3 - \ln 3 - 2 = -0.0986$

$f(4) = 4 - \ln 4 - 2 = 0.6137$

$f(3.5) = 3.5 - \ln 3.5 - 2 = 0.2472$

Hence there exist a root in (3, 3.5).

Now we rewrite the given equation $f(x) = 0$ in the following form:

$$x = \ln x - 2 = \varphi(x)\ [say]$$

$$\varphi'(x) = \frac{1}{x}$$

Now Max $(|\varphi'(3)|, |\varphi'(3.5)|) = (0.333, 0.2857) < 1$

Therefore $\varphi'(x) < 1$ in (3, 3.5).

Then the iterative technique for fixed point iteration method is

$x_n = \varphi(x_{n-1})$, where $n = 1, 2, 3, ..., etc.$

Now let us start with the initial guess $x_0 = 3$ then successive approximation using fixed point iteration method are tabulated below.

Values of n	Values of x_{n-1}	$x_n = \ln(x_{k-1}) + 2$
1.	3	3.098612289
2.	3.098612	3.130954362
3.	3.130954	3.141337866
4.	3.141338	3.144648781
5.	3.144649	3.145702209
6.	3.145702	3.146037143
7.	3.146037	3.146143611
8.	3.146144	3.146177452

Since $|x_8 - x_7| \approx 0.000 = 0$.

Hence the root of the given equation $x - \ln x - 2 = 0$ is equal to 3.1461.

Problem 03:

Find the real root of the equation $x + \ln x - 2 = 0$ that lies on [1, 2] using fixed point iteration method.

Solution:

Let $f(x) = x + \ln x - 2 = 0$

Now $f(1) = 1 + \ln 1 - 2 = -1$

$f(2) = 2 + \ln 2 - 2 = 0.6931$

$f(1.5) = 1.5 + \ln 1.5 - 2 = -0.09453$

$f(1.7) = 1.7 + \ln 1.7 - 2 = 0.230628$

Hence there exist a root in (1.5, 1.7).

Now we rewrite the given equation $f(x) = 0$ in the following form:

$$x = 2 - \ln x = \varphi(x)\ [say]$$

$$\varphi'(x) = -\frac{1}{x}$$

Now $|\varphi'(x)| = \left|-\frac{1}{x}\right| = \left|\frac{1}{x}\right| < 1$

Therefore $\varphi'(x) < 1$ in (1.5, 1.7).).

Then the iterative technique for fixed point iteration method is

$x_n = \varphi(x_{n-1})$, where $n = 1, 2, 3, ..., etc.$

Now let us start with the initial guess $x_0 = 1.5$ then successive approximation using fixed point iteration method are tabulated below.

Values of n	Values of x_{n-1}	$x_n = 2 - \ln\left(x_{k-1}\right)$
1.	1.5	1.594534892
2.	1.594535	1.53341791
3.	1.533418	1.572500828
4.	1.572501	1.547332764
5.	1.547333	1.563467349
6.	1.563467	1.553093986
7.	1.553094	1.559750939
8.	1.559751	1.555473846
9.	1.555474	1.558219777
10.	1.55822	1.556455999
11.	1.556456	1.557588559
12.	1.557589	1.556861171
13.	1.556861	1.557328276
14.	1.557328	1.557028291

Since $\left|x_{14} - x_{13}\right| \approx 0.0001 = 0$.

Hence the root of the given equation $x - \ln x - 2 = 0$ is equal to 1.557328.

Problem 04:

Find the real root of the equation $\sin x - 5x + 2 = 0$ that lies on [0, 1] using fixed point iteration method.

Solution:

Let $f(x) = \sin x - 5x + 2 = 0$

Now $f(0) = 2$

$$f(0.5) = \sin\left(0.5\right) - 5 \times 0.5 + 2 = -0.20$$

$$f(1) = \sin\left(1\right) - 5.1 + 2 = -2.98$$

Note: Change the calculator in radian mode before calculation

Hence there exist a root in (0, 0.5).

Now we rewrite the given equation $f(x) = 0$ in the following form:

$$x = \frac{\sin x + 2}{5} = \varphi(x) \ [say]$$

$$\varphi'(x) = \frac{\cos x}{5}$$

Now $|\varphi'(x)| = \left|\frac{\cos x}{5}\right| < 1$

Therefore $\varphi'(x) < 1$ in (0, 0.5).

Then the iterative technique for fixed point iteration method is

$x_n = \varphi(x_{n-1})$, where $n = 1, 2, 3, \ldots.., etc.$

Now let us start with the initial guess $x_0 = 0.5$ then successive approximation using fixed point iteration method are tabulated below.

Values of n	Values of x_{n-1}	$x_n = \frac{\sin x_{n-1} + 2}{5}$
1.	0.5	0.327026799
2.	0.327026799	0.377578359
3.	0.377578359	0.341867524
4.	0.341867524	0.338731938
5.	0.338731938	0.338983858
6.	0.338983858	0.338223295

Since $|x_6 - x_5| \approx 0.0002 = 0$.

Hence the root of the given equation $\sin x - 5x + 2 = 0$ is equal to 0.338983858.

Try yourself:

TYPE01:

To find the root of the following equations using Iteration method by taking your own guess:

1. $x^3 - 5x + 1 = 0$	**2.** $x^3 - 9x + 1 = 0$	**3.** $2x - \log x = 7$	**4.** $3x - \log_{10} x - 16 = 0$
5. $e^x = 4x$	**6.** $3x = 1 + \cos x$	**7.** $e^x \tan x = 1$	**8.** $x + \ln x - 2 = 0$
9. $x^3 - 2x^2 - 4 = 0$	**10.** $4\sin x + x^2 = 0$	**11.** $\sin x = 1 - x^2$	**12.** $3x + \sin x = e^x$

TYPE02:

1. Find the root of $x^2 + x - 1 = 0$ by iteration method given that root lies near 1.
2. Find a real root $\cos x = 3x - 1$ correct to three decimal places.
3. Find by iteration method the root near 3.8 of equation $2x - \log_{10} x = 7$ correct to four decimal places.
4. Solve the equation $x^3 - 2x^2 - 5 = 0$ by iteration method.
5. Find the real root the equation $x^3 + x^2 - 100 = 0$ by the method of successive approximations.
6. Find the root of $x^2 = \sin x$ which lies between 0.5 and 1 correct to four decimal places.
7. Show that the equation $\log_e x = x^2 - 1$ has exactly two real roots $\alpha_1 = 0.45$ and $\alpha_2 = 1$.
8. Find the positive real root of $x^3 + 3x^2 - 12x - 11 = 0$ correct to three decimal places.
9. Solve the equation $x \sin x = 1$ for the positive root by iteration method.
10. Determine the real root of the equation $\tan x = x$ by iteration method.
11. Find the root of $x^2 + \ln x - 2 = 0$ in [1,2] by fixed point iteration method taking $x_0 = 1$ correct to five decimal places.
12. Use the method of iteration to find a positive root between 0 and 1 of the equation $xe^x = 1$.
13. Compute a root of the equation $e^x = x^2$ to an accuracy of 10^{-5}, using the iteration method.

TYPE03:

1. Derive the fixed-point iteration method to solve the equation $f(x)=0$.
2. Write down the algorithm for fixed point iteration Method.
3. When does fixed point iteration Method Fails.

SOLUTION TO SYSTEM OF LINEAR EQUATIONS

CH 02

In this chapter, we have a concentration to solve the system of linear equations in various methods. Methods are as follows:

- Gauss Elimination Method
- LU Decomposition Method
- Gauss Jacobi & Gauss Seidel Method

System of linear Equations:

A system of linear equations is a collection of equations involving the same set of variables. That are the equations of the form

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots\cdots + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots\cdots + a_{2n}x_n = b_2\\
&a_{31}x_1 + a_{32}x_2 + \cdots\cdots\cdots\cdots + a_{3n}x_n = b_3\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots\cdots\cdots + a_{mn}x_n = b_m
\end{aligned}$$

We can write this system of equations as matrix form as follows

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & \ldots & \ldots & a_{1n} \\ a_{21} & a_{23} & a_{23} & \ldots & \ldots & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & \ldots & \ldots & a_{mn} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

We simply write AX=B

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & \ldots & \ldots & a_{1n} \\ a_{21} & a_{23} & a_{23} & \ldots & \ldots & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & \ldots & \ldots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$

Here A is coefficient matrix while

$$(A|B) = \left[\begin{array}{ccccccc|c} a_{11} & a_{12} & a_{13} & \cdots & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & a_{23} & a_{23} & \cdots & \cdots & \cdots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & \ldots & \ldots & a_{mn} & b_n \end{array}\right]$$ is called an augmented matrix.

Solution of linear systems:
A solution of a system of linear equation s is an n-tuple that satisfies all equations in the system. Here the word system indicates that the equations are to be considered collectively. In other way we define it as a solution to a linear system is an assignment of numbers to the variables such that all the equations are simultaneously satisfied. The solution of linear system of equations can be accomplished by a numerical method which falls in one of two categories:

Direct method:
These methods require no knowledge of an initial approximation and are used for solving polynomial equations. Direct methods involve a certain amount of fixed computation and they are exact solutions. In practice while using a computer direct methods lead to very poor and useless results. This is because of the various types of errors involved in numerical approximations.

Name of Some direct methods are,

1. **Gauss elimination Method**
2. **Gauss Jordan Method**
3. **LU Decomposition Method (Method of Factorization/Triangularisation Method)**
4. **Choleski's Decomposition Method**
5. **Crout's Method**
6. **Cramer's Rule / Determinant Method**
7. **Matrix Inverse Method**

Iterative/Indirect method:
Iterative methods are those in which the solution is got by successive approximations. But the method of iteration is not applicable to all system of equations. Some iterative methods may actually diverge and some others may converge so slowly that they are computationally useless. The iterative methods are suited for use in computers because of simplicity and uniformity of the operations to be performed.

Name of Some Indirect methods are,

1. **Gauss Jacobi's Method**
2. **Gauss Seidel Method**
3. **Relaxaton Method**

There are many such methods such as Newton-Raphson method, Iterative method, Bisection method etc.
Generally, solutions are two types according to solving system:

1. **Analytical solution:**

A solution which is obtained by using direct method is called analytical solution. Analytical solutions are calculated using techniques that provide exact solutions. It may not always be possible to calculate the solution using analytical techniques. This is when a solution can be approximated using numerical techniques.

2. **Numerical solution:**

A solution which is obtained by using approximation or iterative method is called numerical solution. In this case repeat the process till you get as close as you want to the required solution. Numerical methods sometimes are the only way to solve problems such as: sin(x) + x - 0.5 = 0

Direct Methods
In this Chapter we describe four direct methods namely Gauss elimination Method, Gauss Jordan Method, LU Decomposition Method, Choleski's Decomposition Method and also two iterative methods namely Gauss Jacobi's Method, Gauss Seidel Method.

Definition of strictly diagonally dominant matrix:
A matrix $A = (a_{ij})_{n\times n}$ is called strictly diagonally dominant if $\sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}| < |a_{ii}|, i = 1,2,\ldots\ldots,n.$

Problem:
Check the system whether it is strictly diagonally dominant.

$$20x_1 + 2x_2 + 6x_3 = 28$$
$$x_1 + 20x_2 + 9x_3 = -23$$
$$2x_1 - 7x_2 - 20x_3 = -57$$

Mohammad Abdul Halim, Senior Lecturer in Mathematics, Department of General Educational Development (GED)
Faculty of Science and Information Technology (FSIT), Daffodil International University (DIU), Dhaka, Bangladesh

Solution:
Here $|a_{11}| = 20, |a_{22}| = 20, |a_{33}| = 20.$

We have

$$|a_{12}|+|a_{13}| = 2+6 = 8 < 20 = |a_{11}|$$
$$|a_{21}|+|a_{23}| = 1+9 = 10 < 20 = |a_{22}|$$
$$|a_{31}|+|a_{32}| = 2+7 = 9 < 20 = |a_{33}|$$

So $\qquad \sum_{\substack{j\neq i\\ j=1}}^{3} |a_{ij}| < |a_{ii}|, i = 1,2,3.$

Hence the system is strictly diagonally dominant.

Gauss elimination method:

The objective of gauss elimination method is to transform (reduce) the given system in to an equivalent upper triangular system with unit diagonal elements by forward elimination from which the solution can easily be obtained by back substitution.

Consider the system of n linear equations in n unknowns as

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots\cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots\cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots\cdots\cdots\cdots + a_{3n}x_n = b_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots\cdots\cdots + a_{mn}x_n = b_m \end{array}\right\} \cdots\cdots\cdots\cdots\cdots\cdots (i)$$

First stage:

We transform the given system into an upper-triangular form by forward elimination. This is done in the following steps.

Step-1: suppose that $a_{11} \neq 0$. Now eliminate x_1 from all but the first equation.

This is done as follows:

Divide the first equation by a_{11} and then multiply this first equation successively by $-a_{21}, -a_{31}, \cdots\cdots, -a_{n1}$ and add respectively with second, third, … , n^{th} equations of the system(i).Then the new system would be of the following form:

$$\begin{array}{l} x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \cdots\cdots + a_{1n}^{(1)}x_n = b_1^1 \\ \qquad a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \cdots\cdots + a_{2n}^{(1)}x_n = b_2^1 \\ \qquad a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 + \cdots\cdots + a_{3n}^{(1)}x_n = b_3^1 \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad a_{n2}^{(1)}x_2 + a_{n3}^{(1)}x_3 + \ldots\ldots\ldots + a_{nn}^{(1)}x_n = b_n^1 \end{array} \qquad \cdots\cdots\cdots\cdots\cdots\cdots (\text{ii})$$

Where the new coefficients and constants in $k^{th}, k \geq 2$ equation are given by

$$\left.\begin{array}{l} a_{kj}^{(1)} = a_{kj} + a_{1j}^{(1)} \times (-a_{k1}),\ where\ a_{1j}^{(1)} = \dfrac{a_{1j}}{a_{11}} \\ b_k^{(1)} = b_k + b_1^{(1)} \times (-a_{k1}),\ where\, b_1^{(1)} = \dfrac{b_1}{a_{11}} \end{array}\right\} \ k \geq 2,\ j = 1,2,\ldots,n$$

Step-2: Let us assume that $a_{22}^{(1)} \neq 0$.Eliminate x_2 from the third equation through to the last equation in the new system (ii).

This is done as follows:

Normalize the 2nd equation by dividing it by $a_{22}^{(1)}$ and then multiply this normalized equation successively by $-a_{32}^{(1)}, -a_{42}^{(1)}, \ldots\ldots\ldots, -a_{n2}^{(1)}$ and add respectively with third, fourth, ………., n^{th} equations of the system (ii). Then the new system would be of the following form:

$$\left.\begin{aligned}
x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \cdots\cdots\cdots + a_{1n}^{(1)}x_n &= b_1^{(1)} \\
x_2 + a_{23}^{(2)}x_3 + \cdots\cdots\cdots + a_{2n}^{(2)}x_n &= b_2^{(2)} \\
a_{33}^{(2)}x_3 + \cdots\cdots\cdots + a_{3n}^{(2)}x_n &= b_3^{(2)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{n3}^{(2)}x_3 + \cdots\cdots\cdots + a_{nn}^{(2)}x_n &= b_n^{(2)}
\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(iii)$$

Where the new coefficients and constants in $k^{th}(k \geq 3)$ equation given by

$$\left.\begin{aligned}
a_{kj}^{(2)} &= a_{kj}^{(1)} + a_{2j}^{(2)} \times \left(-a_{k2}^{(1)}\right), \text{where } a_{2j}^{(2)} = \frac{a_{2j}^{(1)}}{a_{22}^{(1)}} \\
b_k^{(2)} &= b_k^{(1)} + b_2^{(2)} \times \left(-a_{k2}^{(1)}\right), where\ b_2^{(2)} = \frac{b_2^{(1)}}{a_{22}^{(1)}}
\end{aligned}\right\} k \geq 3 and\ j = 2, 3, 4, \ldots\ldots\ldots, n.$$

Step-3: Assume that $a_{33}^{(2)} \neq 0$. now repeating the same process, eliminate x_3 from 4th equation through to the n^{th} equation. Then we can find another new system of equations.

Last step of elimination:

This elimination process is repeated until we get a triangular system of equations or the n^{th} equation containing only one unknown, namely x_n. Then the triangular form of equations will be as follows:

$$\left.\begin{aligned}
x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \cdots\cdots + a_{1n}^{(1)}x_n &= b_1^{(1)} \\
x_2 + a_{23}^{(2)}x_3 + \cdots\cdots + a_{2n}^{(2)}x_n &= b_2^{(2)} \\
x_3 + \cdots\cdots + a_{3n}^{(3)}x_n &= b_3^{(3)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
x_n &= b_n^{(n)}
\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(iv)$$

Second stage (*Backward substitution*):

We obtain a unique solution for $x_n, x_{n-1}, \ldots\ldots, x_3, x_2, x_1$ from the above triangular system by back substitution. The solution is as follows:

$x_k = b_k^{(k)}$, for k=n and $x_k = \left[b_k^{(k)} - \sum_{j=k+1}^{n} a_{kj}^{(k)} x_j\right]$, for k=n-1, n-2… 3,2,1.

Restriction:

We now come to the important case to ensure that small values (especially zeros) do not appear on the diagonal and, if they do, to remove them by rearranging the matrix and vectors. If the pivot is zero, the entire process fails and if it is close to zero, round off errors may occur. These problems can be avoided by adopting a procedure called pivoting. If a_{11} is either zero or very small compared to the other coefficients of the equation, then we find the largest available coefficient in the columns below the pivot equation and then interchange the two rows. In this way, we obtain a new pivot equation with a nonzero pivot. Such a process is called partial pivoting. If we search both columns and rows for the largest element, the procedure is called complete pivoting. It is obvious that complete pivoting involves more complexity in computations since interchange of columns means change of order of unknowns which invariably requires more programming effort. In comparison, partial pivoting, i.e. row interchanges, is easily adopted in programming. Due to this reason, complete pivoting is rarely used.

Advantage:
We can use this process for many variables compare to the above Method. When we perform the elimination there is a lot of writing. It is especially inconvenient to carry on the symbols of variables.

Problem01:
Solve the following system of linear equations with the help of Gaussian elimination method.

$$3x+y+2z=13$$
$$2x+3y+4z=19$$
$$x+4y+3z=15$$

Solution:
Let us consider the following system

$$3x+y+2z=13$$
$$2x+3y+4z=19$$
$$x+4y+3z=15$$

We see that in the last equation the x coefficient is 1. It is very convenient because it would be an equation to use to eliminate x from the other equations. Let us move it to the front by changing the order of equations.

$$x+4y+3z=15$$
$$3x+y+2z=13$$
$$2x+3y+4z=19$$

Now the elimination starts. We add the first equation multiplied by (−3) to the second equation.

$$x+4y+3z=15$$
$$-11y-7z=-32$$
$$2x+3y+4z=19$$

We add the first equation multiplied by (−2) to the third equation.

$$x+4y+3z=15$$
$$-11y-7z=-32$$
$$-5y-2z=-11$$

The system still does not fit to back-substitution. We need to eliminate the variable y from the last equation. In order to do it we need first to multiply the third equation by (−11).

$$x+4y+3z=15$$
$$-11y-7z=-32$$
$$55y+22z=121$$

Then we add the second equation multiplied by 5 to third equation.

$$x+4y+3z=15$$
$$-11y-7z=-32$$
$$-22z=121$$

Now we can use the back-substitution Method. It gives us the solution (2,1 ,3).

Problem02:

Solve the following system of linear equations

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

Solution: The given system of linear equations is,

$$\left.\begin{aligned} 2x - y + 3z = 8 \\ -x + 2y + z = 4 \\ 3x + y - 4z = 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots (i)$$

The system (i) can be written as following Matrices form

$$\begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix}$$, where $A = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix}$.

The augmented matrix is,

$$[A, B] = \begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ -1 & 2 & 1 & \vdots & 4 \\ 3 & 1 & -4 & \vdots & 0 \end{pmatrix}$$

$$\approx \begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 5 & -17 & \vdots & -24 \end{pmatrix} \begin{matrix} R_2' \to 2R_2 + R_1 \\ R_3' \to 2R_3 - 3R_1 \end{matrix}$$

$$\approx \begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 0 & -76 & \vdots & -152 \end{pmatrix} R_3' \to 3R_3 - 5R_2$$

Above Augmented matrix is in row echelon form.

Therefore, the reduced system is,

$$\left.\begin{aligned} 2x - y + 3z &= 8 \\ 3y + 5z &= 16 \\ -76z &= -152 \end{aligned}\right\}$$

By back substitution we get, $z = 2$, $y = 2$, $x = 2$.

Hence the given system is consistent and the solution is,

$$z = 2,\ y = 2,\ x = 2.$$

Problem03:

Solve by the Gauss Elimination Method the equations

$$2x_1 + x_2 + 4x_3 = 12$$
$$8x_1 - 3x_2 + 4x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$

Solution: The given system of linear equations is,

$$2x_1 + x_2 + 4x_3 = 12$$
$$8x_1 - 3x_2 + 4x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$

Let us rearrange the given system to make it diagonally dominant,

$$8x_1 - 3x_2 + 4x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$
$$2x_1 + x_2 + 4x_3 = 12$$

Converting the given system of linear equations into upper triangular form by using successive operations we get,

$$\approx \left.\begin{aligned} 8x_1 - 3x_2 + 4x_3 &= 20 \\ 0 \quad + 25x_2 - 6x_3 &= 46 \\ 0 \quad + 7x_2 + 12x_3 &= 28 \end{aligned}\right\} \qquad \begin{aligned} L_2{}' &= 2L_2 - L_1 \\ L_3{}' &= 4L_3 - L_1 \end{aligned}$$

$$\approx \left.\begin{aligned} 8x_1 - 3x_2 + 4x_3 &= 20 \\ 0 \quad + 25x_2 - 6x_3 &= 46 \\ 0 \quad + \ 0 \quad + 342x_3 &= 378 \end{aligned}\right\} \qquad L_3{}' = 25L_3 - 7L_2$$

Finally using backward substitution, we get

$$x_3 = \frac{378}{342} = \frac{19}{21}, \quad x_2 = \frac{40}{19} \text{ and } x_1 = \frac{52}{19}$$

(As desired)

Try Yourself

Mathematical Problems

1. Use Gaussian elimination Method to Solve the following system

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

2. Solve the system

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

by the Gauss elimination method.

3. Solve the following system of linear equations

$$3x + 5y - 7z = 13$$
$$4x + y - 12z = 6$$
$$2x + 9y - 3z = 20$$

4.Solve the following system of linear equations

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

5.Solve the following system of linear equations

$$2x-3y+4z=1$$
$$3x+4y-5z=1$$
$$5x-7y+2z=3$$

6.Using the Gauss Elimination Method solve the following system of equations

$$5x_1-2x_2+x_3=4$$
$$7x_1+x_2-5x_3=8$$
$$3x_1+7x_2+4x_3=10$$

Retain the results in the form of $\frac{p}{q}$ if necessary.

7. Using the Gauss Elimination Method solve the following system of equations

$$3x_1+2x_2+4x_3=7$$
$$2x_1+x_2+x_3=7$$
$$x_1+3x_2+5x_3=2$$

8.Solve the following system of equations by Gauss Elimination Method

$$x_1+3x_2+2x_3=5$$
$$2x_1-x_2+x_3=-1$$
$$x_1+2x_2+3x_3=2$$

9.Solve by Gauss Elimination Method

$$2x_1+2x_2+4x_3=18$$
$$x_1+3x_2+2x_3=13$$
$$3x_1+x_2+3x_3=14$$

10. Using the Gauss Elimination Method solve

$$x+\frac{y}{2}+\frac{z}{3}=1$$
$$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=0$$
$$\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=0$$

11.Explain/Discuss the Gauss Elimination Method to solve a set of n linear algebraic equations with n unknowns considering the case of without zero division.

12.When does a system said to be Strictly diagonally dominant?

13.Write an algorithm for the method you discussed in above.

Definition of LU Decomposition:

The nonsingular matrix **A** has an LU-factorization if it can be expressed as the product of a lower-triangular matrix **L** and an upper triangular matrix **U: A=LU**

When this is possible we say that **A** has an **LU**-decomposition. It turns out that this factorization (when it exists) is not unique. If **L** has 1's on its diagonal, then it is called a Doolittle factorization. If **U** has 1's on its diagonal, then it is called a Crout factorization. When $U=L^T$ (or $L=U^T$), it is called a Cholesky decomposition.

Theorem (A = LU; Factorization with NO Pivoting):
Assume that A has a Doolittle, Crout or Cholesky factorization. The solution X to the linear system AX=B is found in three steps:

1. Construct the matrices L and U, if possible.
2. Solve LY=B for Y using forward substitution.
3. Solve UX=Y for X using back substitution.

LU Decomposition Method (Factorization Method), Doolittle Factorization Method or Method of Triangularisation:
In this method we use the fact that a square matrix A can be factorized into the form A=LU where L is unit lower triangular matrix and U is upper triangular matrix, if all the principal minors of A are non-singular, i.e., if $a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, etc.$

Let us consider a system of linear equations:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \qquad \cdots(1)$$

This can be put in the form AX=B ... (2)
Let A=LU

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ $and\ U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

From (2) we have LUX=B ...(3)
Setting UX=Y, then equation (3) becomes
LY=B. ...(4)
The equation (4) is equivalent to the system

$$\left.\begin{aligned} y_1 &= b_1 \\ l_{21}y_1 + y_2 &= b_2 \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_3 \end{aligned}\right\} \qquad ....(5)$$

By forward substitution we get the values y_1, y_2, y_3.
When we know Y, the system UX=Y gives:

$$\left.\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1 \\ u_{22}x_2 + u_{23}x_3 &= y_2 \\ u_{33}x_3 &= y_3 \end{aligned}\right\} \qquad ....(5)$$

By the backward substitution we get the values of $x_1, x_2 and x_3$.
Now we shall discuss the procedure of computing the matrices L and U . From the relation A= LU, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$\Rightarrow \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Equating the corresponding components, we get

$$u_{11}=a_{11} \qquad u_{12}=a_{12} \qquad u_{13}=a_{13} \qquad ....(1)$$

$$l_{21}u_{11}=a_{21} \qquad l_{21}u_{12}+u_{22}=a_{22} \qquad l_{21}u_{13}+u_{23}=a_{23} \qquad ....(2)$$

$$l_{31}u_{11}=a_{31} \qquad l_{31}u_{12}+l_{32}u_{22}=a_{32} \qquad l_{31}u_{13}+l_{32}u_{23}+u_{33}=a_{33} \qquad ....(3)$$

From (1), we get

$$u_{11}=a_{11} \qquad u_{12}=a_{12} \qquad u_{13}=a_{13}$$

From (2), we get

$$l_{21}=\frac{a_{21}}{u_{11}}=\frac{a_{21}}{a_{11}} \qquad l_{21}u_{12}+u_{22}=a_{22}\Rightarrow u_{22}=a_{22}-l_{21}u_{12}=a_{22}-\frac{a_{21}}{a_{11}}.a_{12}$$

$$and \quad l_{21}u_{13}+u_{23}=a_{23}\Rightarrow u_{23}=a_{23}-l_{21}u_{13}=a_{23}-\frac{a_{21}}{a_{11}}.a_{13}$$

From (3), we get

$$l_{31}=\frac{a_{31}}{u_{11}}=\frac{a_{31}}{a_{11}} \qquad l_{31}u_{12}+l_{32}u_{22}=a_{32}\Rightarrow l_{32}=\frac{(a_{32}-l_{31}u_{12})}{u_{22}}=\frac{(a_{32}-\frac{a_{31}}{a_{11}}.a_{12})}{a_{22}-\frac{a_{21}}{a_{11}}.a_{12}}=\frac{a_{32}.a_{11}-a_{31}.a_{12}}{a_{22}.a_{11}-a_{21}.a_{12}}$$

and

$$l_{31}u_{13}+l_{32}u_{23}+u_{33}=a_{33}\Rightarrow u_{33}=a_{33}-(l_{31}u_{13}+l_{32}u_{23})=a_{33}-\{\frac{a_{31}}{a_{11}}.a_{13}+\frac{a_{32}.a_{11}-a_{31}.a_{12}}{a_{22}.a_{11}-a_{21}.a_{12}}.(a_{23}-\frac{a_{21}}{a_{11}}.a_{13})\}$$

Hence in a systematic way the elements of L and U can be evaluated.

Problem:

Solve the equations

$$2x+3y+z=9$$
$$x+2y+3z=6$$
$$3x+y+2z=8$$

by the Factorization Method.

Solution: Given System

$$\mathbf{2x+3y+z=9}$$
$$\mathbf{x+2y+3z=6}$$
$$\mathbf{3x+y+2z=8}$$

Here $A=\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}$. and

Let A=LU where $L=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix} and U=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$

Then form A=LU we have

$$\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}=\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Now equating the like component from both-sides of the matrices, we get

$$u_{11}=2 \quad , \quad u_{12}=3 \quad , \quad u_{13}=1$$

$$l_{21}u_{11}=1 \Rightarrow l_{21}=\frac{1}{u_{11}}=\frac{1}{2} \quad , \quad l_{21}u_{12}+u_{22}=2 \Rightarrow 0.5\times 3+u_{22}=2 \Rightarrow u_{22}=2-1.5=0.5$$

$$l_{21}u_{13}+u_{23}=3 \Rightarrow 0.5\times 1+u_{23}=3 \Rightarrow u_{23}=3-0.5=2.5 \quad , \quad l_{31}u_{11}=3 \Rightarrow l_{31}=\frac{3}{u_{11}}=\frac{3}{2}=1.5$$

$$l_{31}u_{12}+l_{32}u_{22}=1 \Rightarrow 1.5\times 3+l_{32}\times 0.5=1 \Rightarrow l_{32}\times 0.5=1-4.5=-3.5 \Rightarrow l_{32}=-\frac{3.5}{0.5}=-7$$

$$l_{31}u_{13}+l_{32}u_{23}+u_{33}=2 \Rightarrow 1.5\times 1+(-7)\times 2.5+u_{33}=2 \Rightarrow u_{33}=2-1.5+17.5=18$$

Therefore

$$L=\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \; and \; U=\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

Now calculation for $LY=B$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}.$$

$$\begin{bmatrix} y_1 \\ \frac{1}{2}y_1+y_2 \\ \frac{3}{2}y_1-7y_2+y_3 \end{bmatrix}=\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

This system is equivalent to

$$y_1=9$$

$$\tfrac{1}{2}y_1+y_2=6 \Rightarrow y_2=\tfrac{3}{2}$$

$$\tfrac{3}{2}y_1-7y_2+y_3=8 \Rightarrow y_3=5.$$

Now calculation for $UX=Y$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3y+z \\ 0+\dfrac{1}{2}y+\dfrac{5}{2}z \\ 0+0+18z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3y+z \\ \dfrac{1}{2}y+\dfrac{5}{2}z \\ 18z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

This system is equivalent to

$$\left.\begin{aligned} 2x+3y+z &= 9 \\ \frac{1}{2}y+\frac{5}{2}z &= \frac{3}{2} \\ 18z &= 5 \end{aligned}\right\}$$

Solving this system by back substitution, we get $x = 35/18,\ y = 29/18,\ z = 5/18$.

Try Yourself
Mathematical Problems

1.Use LU Method to Solve the following system

$$\begin{aligned} 2x+y+z &= 10 \\ 3x+2y+3z &= 18 \\ x+4y+9z &= 16 \end{aligned}$$

2.Solve the system

$$\begin{aligned} x+5y+\ \ z &= 21 \\ 2x+y+3z &= 20 \\ 3x+y+4z &= 26 \end{aligned}$$ by LU method.

3. Solve the following system of linear equations

$$\begin{aligned} 3x+5y-7z &= 13 \\ 4x+y-12z &= 6 \\ 2x+9y-3z &= 20 \end{aligned}$$

4.Solve the following system of linear equations

$$\begin{aligned} 2x+2y+4z &= 14 \\ 3x-\ \ y+2z &= 2 \\ 5x+2y-2z &= 2 \end{aligned}$$

5. Apply Triangularization(Factorization) Method to solve the equations

$$\begin{aligned} 2x-3y-5z &= 11 \\ 5x+2y-7z &= -12 \\ -4x+3y+z &= 5 \end{aligned}$$

6.Explain the technique of LU factorization Method to solve a system of linear equations $AX = B$.

7.What are the decomposition methods for solving simultaneous algebraic equations? Illustrate any one of these with the help of suitable example.

8.Decompose the matrix $\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into the form LU and hence solve the system $AX = B$ where $B = \begin{bmatrix} 4 & 8 & 10 \end{bmatrix}^T$ .Determine also L^{-1} and U^{-1} and hence find A^{-1}.

9.Discuss the method to solve the tridiagonal system.

Iterative Method

Definition of iteration:

A computational procedure in which a cycle of operations is repeated, often to approximate the desired result more closely.

Gauss-Jacobi iteration method:

We consider the system Ax=b where A=(a_{ij}) is non-singular and x=$(x_i)_{n\times 1}$; $b = (b_i)_{n\times 1}$

Now we shall write the system in detail:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots\dots\dots + a_{3n}x_n &= b_3 \\
\dots \quad \dots \quad \dots \quad \dots \quad \dots & \quad \dots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots\dots + a_{nn}x_n &= b_n
\end{aligned} \qquad \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

If necessary, we re-arrange the given system (1) making strictly diagonally dominant, such that

$$|a_{ii}| > \sum_{\substack{j=1 \\ j=i}}^{n} |a_{ij}|, (i = 1,2,3,\dots\dots\dots\dots, n).$$

Suppose that the system (1) is (strictly) diagonally dominant. Now we re-write the system as follows:

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2 + a_{13}x_3 + \ \dots\dots\dots + a_{1n}x_n\right)\right] \\
x_2 &= \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1 + a_{23}x_3 + \dots\dots\dots + a_{2n}x_n\right)\right] \\
x_3 &= \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1 + a_{32}x_2 + \dots\dots\dots + a_{3n}x_n\right)\right] \\
\dots \quad & \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
x_n &= \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{n(n-1)}x_{n-1}\right)\right]
\end{aligned} \qquad \dots\dots\dots\dots\dots\dots (2)$$

The set of equations in (2) can be written as :

$$x_i = \frac{1}{a_{ij}}\left[b_i - \left(\sum_{\substack{j=1 \\ j=i}}^{n} a_{ij}x_j\right)\right]. \ (i = 1,2,3,\dots\dots\dots, n).$$

Then the solution of system (1) by Gauss-Jacobi iteration method is given by following iterative formulae:

$$x_1^{(k)} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(k-1)} + a_{13}x_3^{(k-1)} + \dots\dots\dots + a_{1n}x_n^{(k-1)}\right)\right]$$

$$x_2^{(k)} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(k-1)} + a_{23}x_3^{(k-1)} + \dots\dots\dots + a_{2n}x_n^{(k-1)}\right)\right]$$

$$x_3^{(k)} = \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1^{(k-1)} + a_{32}x_2^{(k-1)} + \dots\dots\dots + a_{3n}x_n^{(k-1)}\right)\right]$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$x_n^{(k)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(k-1)} + a_{n2}x_2^{(k-1)} + \dots\dots\dots + a_{n(n-1)}x_{n-1}^{(k-1)}\right)\right] \qquad \dots\dots\dots(3)$$

Where $x_i^{(k)}(i = 1,2,3,\dots\dots\dots,n)$denote the values of$x_i at k^{th}$ iteration, and $x_i^{(0)}(i = 1,2,3,\dots\dots\dots,n)$are the initial guesses being taken arbitrarily.

We shall continue the until the values of x_i at two successive iteration are approximately equal that is until $x_i^{(k)} \cong x_i^{(k-1)}$ for any values of k.

The equations in system (3) can be written as follows:

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j\neq 1\\ j=1}} a_{ij}x_j^{(k-1)}\right] i = 1,2,3,\dots\dots\dots,n.$$

Remarks:

The sufficient condition for the convergence for Gauss-Jacobi method is that the system of equations must be strictly diagonally dominant that is the coefficient matrix $A = \left(a_{kj}\right)_{n-1}$ be such that

$$\left|a_{kk}\right| > \sum_{\substack{j=1\\ j\neq i}}^{n}\left|a_{ij}\right|, \left(i = 1,2,3,\dots\dots\dots,\ n\right).$$

Gauss-Seidel iteration method:

We consider the system Ax=b where A=$\left(a_{ij}\right)$ is non-singular and x=$(x_i)_{n\times 1}$; $b = (b_i)_{n\times 1}$

Now we shall write the system in detail:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots\dots\dots + a_{3n}x_n = b_3$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots\dots + a_{nn}x_n = b_n \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

If necessary, we re-arrange the given system (1) making strictly diagonally dominant, such that

$$\left|a_{ii}\right| > \sum_{\substack{j=1\\ j=i}}^{n}\left|a_{ij}\right|, \left(i = 1,2,3,\dots\dots\dots\dots,n\right).$$

Suppose that the system (1) is (strictly) diagonally dominant. Now we re-write the system as follows:

$$x_1 = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2 + a_{13}x_3 + \;.........+ a_{1n}x_n\right)\right]$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1 + a_{23}x_3 + ...+ a_{2n}x_n\right)\right]$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1 + a_{32}x_2 + ...+ a_{3n}x_n\right)\right]$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$x_n = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1 + a_{n2}x_2 + ...+ a_{n(n-1)}x_{n-1}\right)\right] \qquad .... (2)$$

The set of equations in (2) can be written as :

$$x_i = \frac{1}{a_{ij}}\left[b_i - \left(\sum_{\substack{j=1 \\ j=i}}^{n} a_{ij}x_j\right)\right]. \left(i = 1,2,3,.........., n\right).$$

Then the solution of system (1) by Gauss-Seidel iteration method is given by following iterative formulae:

$$x_1^{(k)} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(k-1)} + a_{13}x_3^{(k-1)} + ...+ a_{1n}x_n^{(k-1)}\right)\right]$$

$$x_2^{(k)} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(k)} + a_{23}x_3^{(k-1)} + ...+ a_{2n}x_n^{(k-1)}\right)\right]$$

$$x_3^{(k)} = \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1^{(k)} + a_{32}x_2^{(k)} + ...+ a_{3n}x_n^{(k-1)}\right)\right]$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$x_n^{(k)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + ...+ a_{n(n-1)}x_{n-1}^{(k)}\right)\right] \qquad .... (3)$$

Where $x_i^{(k)}(i = 1,2,3, \ldots\ldots, n)$ denote the values of x_i at k^{th} iteration and $x_i^{(0)}$ $(i = 1,2,3, \ldots\ldots, n)$ are the initial guesses being taken arbitrarily.

We shall continue the iteration until the values of x_i at two successive iteration are approximately equal that is until $x_i^{(k)} \cong x_i^{(k-1)}$ for any *i*.

The equations in system (3) can be written as follows:

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \left(\sum_{\substack{i>j \\ i}}^{n} a_{ij}x_j^{(k)} + \sum_{i<j} a_{ij}x_j^{(k-1)}\right)\right] i,\; j = 1,2,3,........., n.$$

Remarks:

The sufficient condition for the convergence for Gauss-Seidel method is that the system of equations must be strictly diagonally dominant that is the coefficient matrix $A = \left(a_{kj}\right)_{n-1}$ be such that

$$\left|a_{kk}\right| > \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right|, \left(i = 1,2,3,........,\; n\right).$$

Convergence of Gauss-Jacobi and Gauss-Seidel:

The Gauss-Jacobi and Gauss-Seidel methods converge for any choice of the initial guess $x_i^{(0)}(i = 1,2,3,\ldots\ldots\ldots,n)$if every equation of the system (2) satisfies the condition that the some of the absolute values of the co-efficients a_{ij}/a_{ii} is almost equal to or in at least one equation less than unity that is , provided that

$$\sum_{\substack{j=1\\ j\neq i}}^{n}\left|\frac{a_{ij}}{a_{ii}}\right| \leq 1, i = 1,2,\ldots\ldots.., n$$

Where the` <' sign should be valid in the case of at least one equation.

Problem:

Solve the following system of equations by Gauss-Jacobi and Gauss-Seidel method.

$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

Solution: The given systems of equations are

$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

We observe that the coefficient matrix of (1) is not diagonally dominant because $|3| > |1| + |10|$

$$A = \begin{pmatrix} 28 & 4 & -1 \\ 1 & 3 & 10 \\ 2 & 17 & 4 \end{pmatrix} R_2 \leftrightarrow R_3 \begin{pmatrix} 28 & 4 & -1 \\ 2 & 17 & 4 \\ 1 & 3 & 10 \end{pmatrix}$$

Now

This is diagonally dominant.

So Gauss-Jacobi and Gauss-Seidel iteration method is applicable.

$\therefore$ The given system (1) becomes

$$28x + 4y - z = 32\ldots\ldots\ldots\ldots\ldots(1)$$
$$2x + 17y + 4z = 35\ldots\ldots\ldots\ldots\ldots(2)$$
$$x + 3y + 10z = 24\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\therefore (1) \Rightarrow x = \frac{1}{28}(-4y + z + 32)\ldots\ldots\ldots..(14)$$

$$(2) \Rightarrow y = \frac{1}{17}(-2x - 4z + 36)\ldots\ldots\ldots\ldots..(5)$$

$$(3) \Rightarrow z = \frac{1}{10}(-x - 3y + 24)\ldots\ldots\ldots\ldots\ldots..(6)$$

Gauss-Jacobi method:

First iteration: Let $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$ be the initial values **of** x,y,z respectively. Using these values in (4) (5) and (6), we get

$$x^{(1)} = \frac{1}{28}(0+0+32) = 1.1429$$

$$y^{(1)} = \frac{1}{27}(00+35) = 2.0588$$

$$z^{(1)} = \frac{1}{10}(-0-9+24) = 2.4$$

Second iteration:

$$x^{(2)} = \frac{1}{28}(-4\times 2.0588+2.4+32) = 0.9345$$

$$y^{(2)} = \frac{1}{17}(-2\times 1.1429-4\times 2.4+35) = 1.3597$$

$$z^{(2)} = \frac{1}{10}(-1.1429-3\times 2.0588+24) = 1.6681$$

Third iteration:

$$x^{(3)} = \frac{1}{28}(-4\times 1.3597+1.6681+32) = 1.0082$$

$$y^{(3)} = \frac{1}{17}(-2\times 0.9345-4\times 1.6681+35) = 1.5564$$

$$z^{(3)} = \frac{1}{10}(-0.9345-2\times 1.3597+24) = 1.8986$$

Fourth iteration:

$$x^{(4)} = \frac{1}{28}(-4\times 1.5564+1.8986+32) = 0.9883$$

$$y^{(4)} = \frac{1}{17}(-2\times 1.0082-4\times 1.8986+35) = 1.4935$$

$$z^{(4)} = \frac{1}{10}(-1.0082-3\times 1.5564+24) = 1.8323$$

Fifth iteration:

$$x^{(5)} = \frac{1}{28}(-4\times 1.4935+1.8323+32) = 0.9949$$

$$y^{(5)} = \frac{1}{17}(-2\times 0.9883-3\times 1.8323+35) = 1.5114$$

$$z^{(5)} = \frac{1}{10}(-0.9883-3\times 1.4935+24) = 1.8531$$

Sixth iteration:

$$x^{(6)} = \frac{1}{28}\left(-4\times 1.5114 + 1.8531 + 32\right) = 0.9931$$

$$y^{(6)} = \frac{1}{17}\left(-2\times 0.9949 - 4\times 1.8531 + 35\right) = 1.5058$$

$$z^{(6)} = \frac{1}{10}\left(-0.9949 - 3\times 1.5114 + 24\right) = 1.8471$$

Seventh iteration:

$$x^{(7)} = \frac{1}{28}\left(-4\times 1.5058 + 1.8471 + 32\right) = 0.9937$$

$$y^{(7)} = \frac{1}{17}\left(-2\times 0.9931 - 4\times 1.8471 + 35\right) = 1.5074$$

$$z^{(7)} = \frac{1}{10}\left(-0.9931 - 3\times 1.5058 + 24\right) = 1.8490$$

Eighth iteration:

$$x^{(8)} = \frac{1}{28}\left(-4\times 1.5074 + 1.8490 + 32\right) = 0.9936$$

$$y^{(8)} = \frac{1}{17}\left(-2\times 0.9937 - 4\times 1.8490 + 35\right) = 1.5069$$

$$z^{(8)} = \frac{1}{10}\left(-0.9937 - 3\times 1.5074 + 24\right) = 1.8484$$

Ninth iteration:

$$x^{(9)} = \frac{1}{28}\left(-4\times 1.5069 + 1.8484 + 32\right) = 0.9936$$

$$y^{(9)} = \frac{1}{17}\left(-2\times 0.9936 - 4\times 1.8484 + 35\right) = 1.5070$$

$$z^{(9)} = \frac{1}{10}\left(-0.9936 - 3\times 1.5069 + 24\right) = 1.8486$$

After ninth iteration the difference between 8^{th} and 9^{th} iteration is very negligible. Hence the required solution is x=0.9936, y=1.5070 , z=1.8486.

Gauss-Seidel Method:

Let $y^{(0)} = 0 \text{ and } z^{(0)} = 0$ be the initial values of y and z respectively.

First iteration:

$$x^{(1)} = \frac{1}{28}\left(-4y^{(0)} + z^{(0)} + 32\right) = \frac{1}{28}\left(-4.0 + 0 + 32\right) = 1.1429$$

$$y^{(1)} = \frac{1}{27}\left(-2x^{(1)} - 4z^{(0)} + 35\right) = \frac{1}{27}\left(-2\times 1.1429 - 4.0 + 35\right) = 1.2116$$

$$z^{(1)} = \frac{1}{10}\left(-x^{(1)} - 3y^{(1)} + 24\right) = \frac{1}{10}\left(-1.1429 - 3\times 1.2116 + 24\right) = 1.9322$$

Second iteration:

$$x^{(2)} = \frac{1}{28}(-4\times1.2116+1.9322+32)=1.0388$$

$$y^{(2)} = \frac{1}{17}(-2\times1.0388-4\times1.9322+35)=1.4820$$

$$z^{(2)} = \frac{1}{10}(-1.0388-3\times1.4820+24)=1.8515$$

Third iteration:

$$x^{(3)} = \frac{1}{28}(-4\times1.4820+1.8515+32)=0.9973$$

$$y^{(3)} = \frac{1}{17}(-2\times0.9973-4\times1.8515+35)=1.5059$$

$$z^{(3)} = \frac{1}{10}(-0.9973-2\times1.5059+24)=1.8485$$

Fourth iteration:

$$x^{(4)} = \frac{1}{28}(-4\times1.5059+1.8485+32)=0.9938$$

$$y^{(4)} = \frac{1}{17}(-2\times0.8838-4\times4\times1.8485+35)=1.5070$$

$$z^{(4)} = \frac{1}{10}(-0.9938-3\times1.5070+24)=1.8485$$

Fifth iteration:

$$x^{(5)} = \frac{1}{28}(-4\times1.5070+1.8485+32)=0.9936$$

$$y^{(5)} = \frac{1}{17}(-2\times0.9936-4\times1.8485+35)=1.5070$$

$$z^{(5)} = \frac{1}{10}(-0.9936-3\times1.5070+24)=1.8485$$

After five iterations the difference between 4th and 5th iteration is very negligible. Hence the solution of the given system of equations by Gauss-Seidel method is x=0.994, y=1.507, z=1.849 correct up to three decimal places.

Try Yourself

Mathematical Problems

1.Find the solution to three decimal places of the system

$$\begin{aligned}83x+11y-4z&=95\\7x+52y+13z&=104\\3x+8y+29z&=71\end{aligned}$$

using Jacobi and Gauss Seidel methods.

2. Solve by Gauss Jacobi's Method

$$\begin{aligned}x+5y+\ \ z&=21\\2x+y+3z&=20\\3x+y+4z&=26\end{aligned}$$

3. Solve by Gauss Jacobi's Method

$$5x+2y+z=12$$
$$x+4y+2z=15$$
$$x+2y+5z=20$$

4.Solve by Gauss-Seidel Method of Iteration the equations

$$10x+y+z=12$$
$$2x+10y+z=13$$
$$2x+2y+10z=14$$

5. Solve by Gauss-Seidel Method of Iteration the equations

$$x+10y+z=6$$
$$10x+y+z=6$$
$$x+y+10z=6$$

6.Derive the technique of Gauss Seidel Method for Numerical solution of the system of linear equations $AX=B$.

7. Derive the technique of Gauss Jacobi Method for Numerical solution of the system of linear equations $AX=B$.

8.Discuss the advantage and disadvantage of the Method over Jacobi's iterative Method.

9.Make a comparative study of Gauss Jacobi and Gauss Seidel Methods and both of them which method is more accurate and why?

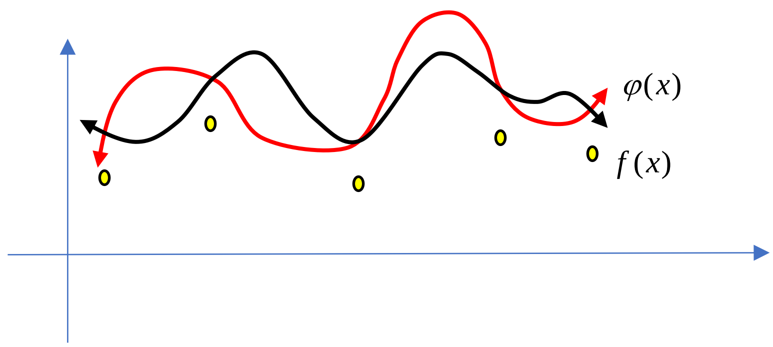
CH 03

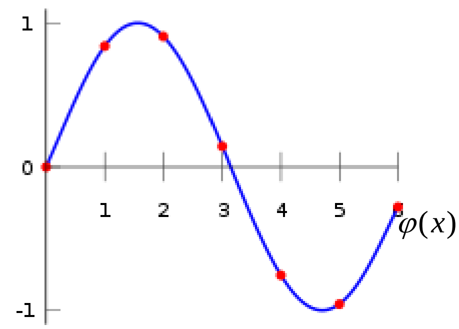
INTERPOLATION

Interpolation is the process of computing the values of a function $y = f(x)$ for any values of the independent variable x within an interval for which some values are given. And also, the process of finding the values of a function $y = f(x)$ for any values of the independent variable x outside the interval for which some values are given is called extrapolation and the curve obtained is called as Interpolating curve.

Interpolating function:
Let $y = f(x)$ is satisfied by the tabular points (nodes, pivotal points) $(x_i, y_i), i = 1,2,3, \dots \dots \dots \dots, n$ where the explicit nature of the given function $f(x)$ is not known. If the form of $f(x)$ is known, we can easily calculate the value of y corresponding to any value of x. But most of the practical cases the exact form of $f(x)$ is not known. In such case, it is required to find a simpler function say $\varphi(x)$ such that $f(x)$ and $\varphi(x)$ gives the same values for the set of tabulated points. Such a process is called interpolation. The function $\varphi(x)$ is called an interpolating function or interpolating formula.

1. when $\varphi(x)$ is a linear function then the process of representing $f(x)$ by $\varphi(x)$ is called linear interpolation.
2. when $\varphi(x)$ is a polynomial function then the process of representing $f(x)$ by $\varphi(x)$ is called polynomial interpolation.
3. when $\varphi(x)$ is a finite trigonometric series then the process of representing $f(x)$ by $\varphi(x)$ is called trigonometric interpolation.





Finite Differences:

The Interpolation depends upon finite difference concept, so first we work with the concept of finite difference.In this section frirstly, we will discuss about several difference operators such as forward difference operator, backward difference operator, central difference operator, shifting operator etc.

Forward difference operator: The operator Δ is called forward difference operator and defined as,

$$\Delta y_{r-1} = y_r - y_{r-1} \text{ where } y_r\,;\ r = 0,1,2,\ \dots\ \dots\ \dots, n \text{ are values of } y\,.$$

Backward difference operator: The operator ∇ is called backward difference operator and defined as,

$$\nabla y_r = y_r - y_{r-1} \text{ where } y_r\,;\ r = 0,1,2,\ \dots\ \dots\ \dots, n \text{ are values of } y\,.$$

Central difference operator: The operator δ is called central difference operator and defined as,

$$\delta y_{(2r-1)/2} = y_r - y_{r-1} \text{ where } y_r\,;\ r = 0,1,2,\ \dots\ \dots\ \dots, n \text{ are values of } y\,.$$

Shifting operator: The operator E is called shifting operator and defined as,

$$E y_{r-1} = y_r \text{ where } y_r\,;\ r = 0,1,2,\ \dots\ \dots\ \dots, n \text{ are values of } y\,.$$

Which shows that the effect of E is to shift the functional value of y to it's next higher value.

Averaging operator: The operator μ is called averaging operator and defined as,

$$\mu y_r = \frac{1}{2}\left(y_{(2r+1)/2} - y_{(2r-1)/2} \right) \text{where } y_r\,;\; r = 0,1,2,\; \dots\; \dots\; \dots, n \text{ are values of } y\,.$$

Differential operator: The operator D is called differential operator and defined as,

$$Dy_r = \frac{d}{dx}\left(y_r \right) \text{ where } y_r\,;\; r = 0,1,2,\; \dots\; \dots\; \dots, n \text{ are values of } y\,.$$

Unit operator: The unit operator 1 is defined by,

$$1.y_r = y_r \text{ where } y_r\,;\; r = 0,1,2,\; \dots\; \dots\; \dots, n \text{ are values of } y\,.$$

Forward Differences: If $y_0\,, y_1\,, y_2\,, \cdots\cdots\cdots\,, y_n$ denote a set of values of y , then $y_1 - y_0\,, y_2 - y_1, \cdots\cdots\cdots\,, y_n - y_{n-1}$ are called the differences of y . If these differences are denoted as follows,

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\cdots\; \cdots\; \cdots\; \cdots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

where Δ is called the forward difference operator and $\Delta y_0,\, \Delta y_1,\, \cdots\cdots,\, \Delta y_{n-1}$ are called first forward differences.

The second forward differences are,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$
$$\cdots\; \cdots\; \cdots\; \cdots\; \cdots$$
$$\Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$$

Similarly, we can determine kth forward differences. $i.e,\; \Delta^k y_{n-1} = \Delta^{k-1} y_n - \Delta^{k-1} y_{n-1}$

Forward Difference Table:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

Backward Differences: If $y_0, y_1, y_2, \cdots\cdots\cdots, y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1,, \cdots\cdots\cdots, y_n - y_{n-1}$ are called the differences of y. If these differences are denoted as follows,

$$\nabla y_1 = y_1 - y_0$$
$$\nabla y_2 = y_2 - y_1$$
$$\cdots \; \cdots \; \cdots \; \cdots$$
$$\nabla y_n = y_n - y_{n-1}$$

where ∇ is called the backward difference operator and $\nabla y_1, \nabla y_2, \cdots\cdots, \nabla y_n$ are called first backward differences.

The second backward differences are,

$$\nabla^2 y_1 = \nabla y_1 - \nabla y_0$$
$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$
$$\cdots \; \cdots \; \cdots \; \cdots \; \cdots$$
$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

Similarly, we can determine kth backward differences. $i.e, \; \nabla^k y_n = \nabla^{k-1} y_n - \nabla^{k-1} y_{n-1}$

Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		∇y_1				
			$\nabla^2 y_2$			
x_1	y_1			$\nabla^3 y_3$		
		∇y_2			$\nabla^4 y_4$	
			$\nabla^2 y_3$			$\nabla^5 y_5$
x_2	y_2			$\nabla^3 y_4$		
		∇y_3			$\nabla^4 y_5$	
			$\nabla^2 y_4$			
x_3	y_3			$\nabla^3 y_5$		
		∇y_4				
			$\nabla^2 y_5$			
x_4	y_4					
		∇y_5				
x_5	y_5					

Central Differences: If $y_0, y_1, y_2, \cdots\cdots\cdots, y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1, \cdots\cdots\cdots, y_n - y_{n-1}$ are called the differences of y. If these differences are denoted as follows,

$$\delta y_{1/2} = y_1 - y_0$$
$$\delta y_{3/2} = y_2 - y_1$$
$$\cdots \; \cdots \; \cdots \; \cdots$$
$$\delta y_{(2n-1)/2} = y_n - y_{n-1}$$

Where δ is called the central difference operator and $\delta y_{1/2}, \delta y_{3/2}, \cdots\cdots, \delta y_{(2n-1)/2}$ are called first central differences.

The second central differences are,

$$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$$
$$\delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}$$
$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots$$
$$\delta^2 y_n = \delta y_{(2n+1)/2} - \delta y_{(2n-1)/2}$$

Similarly, we can determine kth central differences. $i.e,\ \delta^k y_n = \delta^{k-1} y_{(2n+1)/2} - \delta^{k-1} y_{(2n-1)/2}$

Central Difference Table

x	y	δ	δ^2	δ^3	δ^4	δ^5
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$				
x_2	y_2		$\delta^2 y_2$	$\delta^3 y_{3/2}$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	$\delta^4 y_2$	
x_3	y_3		$\delta^2 y_3$			
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$	$\delta^4 y_3$	$\delta^5 y_{5/2}$
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

Interpolation

Interpolation is a numerical technique which is used to estimate unknown values of a function by using known values. For example**,** If we are to find out the population of Bangladesh in 1978 when we know the population of Bangladesh in the year 1971, 1975, 1979, 1984, 1988, 1992 and so on, then the process of finding the population of 1978 is known as interpolation.

Mathematically, let $y = f(x)$ be a function which gives $y_0, y_1, y_2, \cdots\cdots, y_n$ for $x_0, x_1, x_2, \cdots\cdots, x_n$ respectively. The method of finding $f(x)$ for $x = \alpha$ where α lies in the given range is called an interpolation and if α lies outside the given range is called an extrapolation.

Assumption for interpolation: For the application of the methods of interpolation, the following fundamental assumptions are required.

a) In the interval under consideration, the values of the function cannot be jumped or fallen down suddenly.
b) In the absence of any evidence to the contrary, the rise and fall in the values of the function must be uniform.
c) The data can be expressed as a polynomial function so that the method of finite difference be applicable.

Methods of interpolation: The various methods of interpolation are as follows:

a) Method of graph
b) Method of curve fitting
c) Method for finite differences.

In this chapter we shall discuss only interpolation formulae for finite differences. These formulae can be separated as follows:

a) Interpolation formulae for equal intervals
b) Interpolation formulae for unequal intervals
c) Interpolation formulae for central difference.

Interpolation formulae for equal intervals: The interpolation formulae for equal intervals are given bellow:

I. Newton's forward interpolation formula: Suppose, $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots \cdots \cdots, (x_n, y_n)$ be a set of $(n+1)$ values of x and y. Let the values of x be equidistant, $i.e, \quad x_r = x_0 + rh \qquad ; r = 0,1,2,\cdots\cdots, n$ where h is difference between the points. Let $y_n(x)$ be a polynomial of nth degree such that y and $y_n(x)$ agree at the tabulated points, which is to be determined. It can be written as,

$$y_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots\cdots + a_n(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1}) \qquad \cdots(1)$$

where the constants $a_0, a_1, a_2, ..., a_n$ can be determine as follows:

Putting $x = x_0$ in Eq.(1) we have, $a_0 = y_0$

Putting $x = x_1$ in Eq.(1) we have, $y_1 = a_0 + a_1(x_1 - x_0)$

$$or,\ y_1 = y_0 + a_1 h \Rightarrow a_1 = \frac{y_1 - y_0}{h}$$

$$\therefore\ a_1 = \frac{\Delta y_0}{h}$$

Putting $x = x_2$ in Eq.(1) we have,

$$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$or,\ y_2 = y_0 + \frac{\Delta y_0}{h}(2h) + a_2(2h)(h)$$

$$or,\ y_2 = y_0 + 2(y_1 - y_0) + 2a_2 h^2$$

$$or,\ y_2 = 2y_1 - y_0 + 2a_2 h^2$$

$$or,\ a_2 = \frac{y_2 - 2y_1 + y_0}{2h^2}$$

$$or,\ a_2 = \frac{(y_2 - y_1) - (y_1 - y_0)}{2h^2}$$

$$or,\ a_2 = \frac{\Delta y_1 - \Delta y_0}{2h^2}$$

$$\therefore\ a_2 = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly,

$$a_3 = \frac{\Delta^3 y_0}{3!h^3}$$

$$a_4 = \frac{\Delta^4 y_0}{4!h^4}$$

$$\cdots\cdots\cdots$$

$$a_n = \frac{\Delta^n y_0}{n!h^n}$$

Using these values in Eq.(1) we have,

$$y_n(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{\Delta^2 y_0}{2!h^2}(x-x_0)(x-x_1) + \cdots\cdots + \frac{\Delta^n y_0}{n!h^n}(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1}) \qquad \cdots(2)$$

Setting $x = x_0 + ph$ we have, $x - x_0 = ph$

$$x - x_1 = x - x_0 - x_1 + x_0$$
$$= (x - x_0) - (x_1 - x_0)$$
$$= ph - h$$
$$= (p-1)h$$

Similarly, $x - x_2 = (p-2)h$

$$x - x_3 = (p-3)h$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$x - x_{n-1} = (p-n+1)h$$

Equation (2) becomes,

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \cdots\cdots + \frac{p(p-1)\cdots\cdots(p-n+1)}{n!}\Delta^n y_0 \qquad \cdots(3)$$

This is called Newton's forward interpolation formula.

Note: Newton's forward interpolation formula is used to interpolate the values of *y* near the beginning of a set of tabular values.

II. Newton's backward interpolation formula: Suppose, $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots\ \cdots\ \cdots, (x_n, y_n)$ be a set of $(n+1)$ values of x and y. Let the values of x be equidistant,

$$i.e, \quad x_r = x_0 + rh \qquad ; r = 0, 1, 2, \cdots\cdots, n$$

where h is difference between the points.

Let $y_n(x)$ be a polynomial of nth degree such that y and $y_n(x)$ agree at the tabulated points, which is to be determined. It can be written as,

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \cdots\cdots + a_n(x - x_n)(x - x_{n-1})\cdots\cdots(x - x_0) \qquad \cdots(1)$$

where the constants $a_0, a_1, a_2, ..., a_n$ can be determine as follows:

Putting $x = x_n$ in Eq.(1) we have, $a_0 = y_n$

Putting $x = x_{n-1}$ in Eq.(1) we have,

$$y_{n-1} = y_n + a_1(x_{n-1} - x_n)$$
$$or,\ y_{n-1} = y_n + a_1(-h)$$
$$or,\ a_1 = \frac{y_n - y_{n-1}}{h}$$
$$\therefore\ a_1 = \frac{\nabla y_n}{h}$$

Putting $x = x_{n-2}$ in Eq.(1) we have,

$$y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$
$$or,\ y_{n-2} = y_n + \frac{\nabla y_n}{h}(-2h) + a_2(-2h)(-h)$$
$$or,\ y_{n-2} = y_n - 2(y_n - y_{n-1}) + 2a_2h^2$$
$$or,\ y_{n-2} = -y_n + 2y_{n-1} + 2a_2h^2$$
$$or,\ a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2}$$

$$or,\ a_2 = \frac{(y_n - y_{n-1}) - (y_{n-1} - y_{n-2})}{2h^2}$$
$$or,\ a_2 = \frac{\nabla y_n - \nabla y_{n-1}}{2h^2}$$
$$\therefore\ a_2 = \frac{\nabla^2 y_n}{2!h^2}$$

Similarly,

$$a_3 = \frac{\nabla^3 y_n}{3!h^3}$$
$$a_4 = \frac{\nabla^4 y_n}{4!h^4}$$
$$\cdots\cdots\cdots$$
$$a_n = \frac{\nabla^n y_n}{n!h^n}$$

Using these values in Eq.(1) we have,

$$y_n(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!h^2}(x - x_n)(x - x_{n-1}) + \cdots\cdots + \frac{\nabla^n y_n}{n!h^n}(x - x_n)(x - x_{n-1})\cdots\cdots(x - x_1) \qquad \cdots(2)$$

Setting $x = x_n + ph$ we have, $x - x_n = ph$

$$\begin{aligned} x - x_{n-1} &= x - x_n + x_n - x_{n-1} \\ &= (x - x_n) + (x_n - x_{n-1}) \\ &= ph + h \\ &= (p+1)h \end{aligned}$$

Similarly, $x - x_{n-2} = (p+2)h$

$$x - x_{n-3} = (p+3)h$$
$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots$$
$$x - x_1 = (p+n-1)h$$

Equation (2) becomes,

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \cdots\cdots + \frac{p(p+1)\cdots\cdots(p+n-1)}{n!}\nabla^n y_n \qquad \cdots(3)$$

This is called Newton's backward interpolation formula.

Note: Newton's backward interpolation formula is used to interpolate the values of *y* near the end of a set of tabular values.

Problem-01: Construct a difference table to find the polynomial of the data $(1,1),(2,8),(3,27),(4,64),(5,125),(6,216),$ $(7,343),(8,512)$ considering appropriate method. Also find r, where $(9,r)$ is given.

Solution:

We may construct any one of forward, backward and central difference tables. Since we also have to find r for $x = 9$ which is nearer at the end of the set of given tabular values, so we will construct the backward difference table.The backward difference table of the given data is as follows:

x	y	∇	∇^2	∇^3	∇^4
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		**0**
		127		**6**	
7	343		**42**		
		169			
8	**512**				

This is the required difference table.

Here $x_n = 8,\ h = 1, y_n = 512,\ \nabla y_n = 169,\ \nabla^2 y_n = 42,\ \nabla^3 y_n = 6,\ \nabla^4 y_n = 0.$

$$\therefore\ p = \frac{x - x_n}{h} = \frac{x-8}{1} = (x-8)$$

By Newton's backward formula we get,

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \cdots\cdots + \frac{p(p+1)\ \cdots\cdots(p+n-1)}{n!}\nabla^n y_n$$

$$= 512 + 169(x-8) + \frac{(x-8)(x-8+1)}{2!} \times 42 + \frac{(x-8)(x-8+1)(x-8+2)}{3!} \times 6$$

$$= 512 + 169x - 1352 + 21x^2 - 315x + 1176 + x^3 - 21x^2 + 146x - 336$$

$$= x^3$$

This is the required polynomial.

2nd part: For $x = 9$ we get, $y(9) = 9^3\ \therefore\ \ r = 729$ **(Ans.)**

Problem-02: From the following table of yearly premiums for policies maturing at quinquennial ages, estimate the premiums foe policies maturing at the age of 46 years.

Age(x)	45	50	55	60	65
Premium(y)	2.871	2.404	2.083	1.862	1.712

Solution: Since $x = 46$ is nearer at the beginning of the set of given tabular values, so we have to construct the forward difference table. The forward difference table of the given data is as follows:

Age(x)	Premium(y)	Δ	Δ^2	Δ^3	Δ^4
45	**2.871**				
		-0.467			
50	2.404		**0.146**		
		-0.321		**-0.046**	
55	2.083		0.100		**0.017**
		-0.221		-0.029	
60	1.862		0.071		
		-0.150			
65	1.712				

Here $x = 46, h = 5,\ x_0 = 45,\ y_0 = 45,\ \Delta y_0 = -0.467,\ \Delta^2 y_0 = 0.146,\ \Delta^3 y_0 = -0.046,\ \Delta^4 y_0 = 0.017$

$$\therefore\ p = \frac{x - x_0}{h} = \frac{46-45}{5} = \frac{1}{5} = 0.2$$

By Newton's forward formula we get,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \cdots\cdots + \frac{p(p-1)\cdots\cdots(p-n+1)}{n!}\Delta^n y_0$$

$$\therefore\ y(46) = 2.871 + 0.2\times(-0.467) + \frac{0.2(0.2-1)}{2!}\times(0.146) + \frac{0.2(0.2-1)(0.2-2)}{3!}\times(-0.046) + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4!}\times(0.017)$$
$$= 2.8710 - 0.0934 - 0.01168 - 0.002208 - 0.0005712$$
$$= 2.763(approx.)$$

Problem-03: The values of $\sin x$ are given below for different values of x, find the value of $\sin 38^0$.

x	15	20	25	30	35	40
$y=\sin x$	0.2588190	0.3420201	0.4226183	0.5	0.5735764	0.6427876

Solution: Since $x = 38^0$ is nearer at the end of the set of given tabular values, so we have to construct the backward difference table.
The backward difference table of the given data is as follows:

x	$y=\sin x$	∇	∇^2	∇^3	∇^4	∇^5
15	0.2588190					
		0.0832011				
20	0.3420201		-0.0026029			
		0.0805982		-0.0006136		
25	0.4226183		-0.0032165		0.0000248	
		0.0773817		-0.0005888		**0.0000041**
30	0.5		-0.0038053		**0.0000289**	
		0.0735764		**-0.0005599**		
35	0.5735764		**-0.0043652**			
		0.0692112				
40	**0.6427875**					

Here $x = 38,\ x_n = 40,\ h = 5,\ y_n = 0.6427875,\ \nabla y_n = 0.0692112,\ \nabla^2 y_n = -0.0043652,\ \nabla^3 y_n = -0.0005599,$

$\nabla^4 y_n = 0.0000289, \nabla^5 y_n = 0.0000041$.

$$\therefore\ p = \frac{x - x_n}{h} = \frac{38-40}{5} = -\frac{2}{5} = -0.4$$

By Newton's backward formula we get,

$$y(38) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \cdots\cdots + \frac{p(p+1)\cdots\cdots(p+n-1)}{n!}\nabla^n y_n$$
$$= 0.6427876 + (-0.4)\times 0.0692112 + \frac{(-0.4)(-0.4+1)}{2!}\times(-0.0043652) + \frac{(-0.4)(-0.4+1)(-0.4+2)}{3!}\times(-0.0005599)$$
$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{4!}\times(0.0000289) + \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)(-0.4+4)}{5!}\times(0.0000041)$$
$$= 0.6427876 - 0.02768448 + 0.00052382 + 0.00003583 - 0.00000120$$
$$= 0.6156614(approx.)$$

Problem-04: In an examination the number of candidates who obtained marks between certain limits were as follows:

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

Find the number of candidates whose scores lie between 45 and 50.

Solution: First of all we construct a cumulative frequency table for the given data.

Upper limits of the class intervals	40	50	60	70	80
Cumulative frequency	31	73	124	159	190

Since $x = 45$ is nearer at the beginning of the set of values in cumulative frequency table, so we have to construct the forward difference table.

The forward difference table of the given data is as follows:

Marks(x)	Cumulative frequencies(y)	Δ	Δ^2	Δ^3	Δ^4
40	**31**				
		42			
50	73		**9**		
		51		**-25**	
60	124		-16		**37**
		35		12	
70	159		-4		
		31			
80	190				

Here $x = 45,\ x_0 = 40,\ h = 10,\ y_0 = 31,\ \Delta y_0 = 42,\ \Delta^2 y_0 = 9,\ \Delta^3 y_0 = -25,\ \Delta^4 y_0 = 37$.

$$\therefore\ p = \frac{x - x_0}{h} = \frac{45-40}{10} = \frac{5}{10} = 0.5$$

By Newton's forward formula we get,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \cdots\cdots + \frac{p(p-1)\cdots\cdots(p-n+1)}{n!}\Delta^n y_0$$

$$\therefore\ y(45) = 31 + 0.5\times 42 + \frac{0.5(0.5-1)}{2!}\times 9 + \frac{0.5(0.5-1)(0.5-2)}{3!}\times(-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}\times 37$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4452$$

$$= 47.8673$$

$$= 48\,(approx.)$$

Problem-05: The population of a town in the last six censuses was as given below. Estimate the population for the year 1946.

Year(x)	1911	1921	1931	1941	1951	1961
Population in thousands(y)	12	15	20	27	39	52

Solution: Since $x = 1946$ is nearer at the end of the set of given tabular values, so we have to construct the backward difference table

The backward difference table of the given data is as follows:

$Year(x)$	$Populations(y)$	∇	∇^2	∇^3	∇^4	∇^5
1911	12					
		3				
1921	15		2			
		5		0		
1931	20		2		3	
		7		3		**-10**
1941	27		5		**-7**	
		12		**-4**		
1951	39		**1**			
		13				
1961	**52**					

Here $x = 1946,\ x_n = 1961,\ h = 10,\ y_n = 52,\ \nabla y_n = 13,\ \nabla^2 y_n = 1,\ \nabla^3 y_n = -4, \nabla^4 y_n = -7, \nabla^5 y_n = -10$.

$$\therefore\ p = \frac{x - x_n}{h} = \frac{1946-1961}{10} = -\frac{15}{10} = -1.5$$

By Newton's backward formula we get,

$$y(x)=y_n+p\nabla y_n+\frac{p(p+1)}{2!}\nabla^2 y_n+\frac{p(p+1)(p+2)}{3!}\nabla^3 y_n\cdots\cdots+\frac{p(p+1)\cdots\cdots(p+n-1)}{n!}\nabla^n y_n$$

$$\therefore\ y(1946)=52+(-1.5)\times 13+\frac{(-1.5)(-1.5+1)}{2!}\times 1+\frac{(-1.5)(-1.5+1)(-1.5+2)}{3!}\times(-4)+\frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)}{4!}\times(-7)$$
$$+\frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)(-1.5+4)}{5!}\times(-10)$$

$$=32.3438\,(approx.)$$

Exercise:

Problem-01: The population of a village in the last six censuses was recorded as follows. Estimate the population for the year 1945.

Year(x)	1941	1951	1961	1971	1981	1991
Population(y)	2500	2800	3200	3700	4350	5225

Problem-02: In a company the number of persons whose daily wage are as follows:

Daily wage in Tk.	0-20	20-40	40-60	60-80	80-100
No. of persons	120	145	200	250	150

Find the number of persons whose daily wage is between TK. 40 and TK.50.

Problem-03: The population of a town in decennial census was recorded as follows. Estimate the population for the year 1985.

Year(x)	1951	1961	1971	1981	1991
Population in thousands(y)	98.752	132.285	168.076	195.690	246.05

Problem-04: The population of a town in decennial census was recorded as follows. Estimate the population for the year 1895.

Year(x)	1891	1901	1911	1921	1931
Population in thousands(y)	46	66	81	93	101

Problem-05: Estimate the production of cotton in the year 1935 from the data given below.

Year(x)	1931	1932	1934	1936	1938
Production in millions of bales(y)	17.1	13.0	14.0	9.6	12.4

Interpolation formulae for unequal intervals: The interpolation formulae for unequal intervals are given bellow:

1) Newton's Interpolation formula for unequal intervals.
2) Lagrange's Interpolation formula for unequal intervals.

Divided Differences: Let $y=f(x)$ be a polynomial which gives $f(x_0),\ f(x_1),\ f(x_2),\ldots,\ f(x_n)$ at the points $x_0,x_1,x_2,\ldots,x_n$ (which are not equally spaced) respectively. Then the first divided difference for the arguments x_0 and x_1 is denoted by $f(x_0,x_1)$ or $\Delta f(x)$ and defined as,

$$f[x_0,x_1]=\frac{f(x_0)-f(x_1)}{x_0-x_1}$$

The second divided difference for the arguments x_0,x_1 *and* x_2 is denoted by $f(x_0,x_1,x_2)$ or $\Delta^2 f(x)$ and defined as,

$$f(x_0,x_1,x_2)=\frac{f(x_0,x_1)-f(x_1,x_2)}{x_0-x_2}$$

The third divided difference for the arguments x_0,x_1,x_2 *and* x_3 is denoted by $f(x_0,x_1,x_2,x_3)$ or $\Delta^3 f(x)$ and defined as,

$$f(x_0,x_1,x_2,x_3)=\frac{f(x_0,x_1,x_2)-f(x_1,x_2,x_3)}{x_0-x_3}$$

Similarly, the nth divided difference for the arguments $x_0,x_1,x_2,\ldots,x_n$ is denoted by $f(x_0,x_1,x_2,\cdots\cdots,x_n)$ or $\Delta^n f(x)$ and defined as,

$$f(x_0,x_1,x_2,\cdots\cdots,x_n)=\frac{f(x_0,x_1,\cdots\cdots,x_{n-1})-f(x_1,x_2,\cdots\cdots,x_n)}{x_0-x_n}$$

Divided Difference Table

x	$f(x)$	Δ	Δ^2	Δ^3
x_0	$f(x_0)$			
		$f(x_0,x_1)$		
x_1	$f(x_1)$		$f(x_0,x_1,x_2)$	
		$f(x_1,x_2)$		$f(x_0,x_1,x_2,x_3)$
x_2	$f(x_2)$		$f(x_1,x_2,x_3)$	
		$f(x_2,x_3)$		
x_3	$f(x_3)$			

Properties of Divided Differences: The properties are given as follows:

1) The divided differences are symmetric. $i.e,\ f(x_0,x_1)=f(x_1,x_0)$.
2) The nth divided differences of a polynomial of the nth degree are constant.
3) The nth divided differences can be expressed as the quotient of two determinants each of order $(n+1)$.

I). **Newton's Interpolation formula for unequal intervals:** Let $y=f(x)$ be a polynomial which gives $f(x_0), f(x_1),..., f(x_n)$ at the points $x_0,x_1,x_2,...,x_n$ (which are not equally spaced) respectively. Then the first divided difference for the arguments x and x_0 is given by,

$$f(x,x_0)=\frac{f(x)-f(x_0)}{x-x_0}$$

$$or,\ \ f(x)=f(x_0)+(x-x_0)f(x,x_0)\ \cdots\ \cdots(1)$$

The second divided difference for the arguments $x, x_0\, and\ x_1$ is given by,

$$f(x,x_0,x_1)=\frac{f(x,x_0)-f(x_0,x_1)}{x-x_1}$$

$$or,\ \ f(x,x_0)=f(x_0,x_1)+(x-x_1)f(x,x_0,x_1)\ \cdots\ \cdots(2)$$

The third divided difference for the arguments $x, x_0, x_1\ and\ x_2$ is given by,

$$f(x,x_0,x_1,x_2)=\frac{f(x,x_0,x_1)-f(x_0,x_1,x_2)}{x-x_2}$$

$$or,\ \ f(x,x_0,x_1)=f(x_0,x_1,x_2)+(x-x_2)f(x,x_0,x_1,x_2)\ \cdots\ \cdots(3)$$

The nth divided difference for the arguments $x, x_0, x_1,..., x_n$, is given by,

$$f(x,x_0,x_1,\cdots\cdots,x_{n-1})=f(x_0,x_1,\cdots\cdots,x_n)+(x-x_n)f(x,x_0,x_1,\cdots\cdots,x_{n-1})\ \cdots\ \cdots(4)$$

Multiplying Eq.(2) by $(x-x_0)$, Eq.(3) by $(x-x_0)\ (x-x_1)$ and so on and finally the Eq. (4) by $(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})$ and adding with Eq.(1) we get,

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$
$$\cdots\ \cdots+(x-x_0)(x-x_1)(x-x_2)\cdots\cdots(x-x_{n-1})f(x_0,x_1,x_2,\cdots\cdots x_n)+R_n\cdots\cdots(5)$$

where $R_n=(x-x_0)(x-x_1)(x-x_2)\cdots\cdots(x-x_n)f(x,x_0,x_1,x_2,\cdots\cdots x_n)$

If $f(x)$ be a polynomial of degree n, then the $(n+1)$ th divided difference of $f(x)$ will be zero.

$$\therefore\ f(x,x_0,x_1,\cdots\cdots,\ x_n)=0$$

Then the Eq. (5) can be written as,

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$

$$\cdots\ \cdots + (x-x_0)(x-x_1)(x-x_2)\cdots\cdots(x-x_{n-1})\, f(x_0,x_1,x_2,\cdots\cdots x_n)\cdots\cdots(6)$$

This formula is called Newton's divided difference interpolation formula for unequal intervals.

II). **Lagrange's Interpolation formula for unequal intervals:**

Let $y=f(x)$ be a polynomial of degree n which gives $f(x_0), f(x_1), f(x_2),..., f(x_n)$ at the points $x_0,x_1,x_2,...,x_n$ (which are not equally spaced)respectively. This polynomial can be written as,

$$f(x)=a_0(x-x_1)(x-x_2)\cdots\cdots(x-x_n)+a_1(x-x_0)(x-x_2)\cdots\cdots(x-x_n)+a_2(x-x_0)(x-x_1)\cdots\cdots(x-x_n)$$
$$\cdots\cdots + a_n(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1}) \qquad \cdots(1)$$

putting $x=x_0$ in Eq.(1) we get,

$$f(x_0)=a_0(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)$$
$$or,\ a_0=\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)}$$

putting $x=x_1$ in Eq.(1) we get,

$$f(x_1)=a_1(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)$$
$$or,\ a_1=\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)}$$

Similarly putting $x=x_2,\ x=x_3\ ,\cdots\cdots x=x_n$ in Eq.(1) we get,

$$a_2=\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)\cdots\cdots(x_2-x_n)}$$
$$\cdots\ \cdots\ \cdots\ \ \cdots\cdots\cdots$$
$$a_n=\frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})}$$

Substituting the values of $a_0,a_1,a_2,\cdots\cdots,a_n$ in Eq.(1) we get,

$$f(x)=\frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)}f(x_1)+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots\cdots(x_2-x_n)}f(x_2)$$
$$\cdots\cdots+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})}\ f(x_n) \qquad \cdots(2)$$

This formula is called Lagrange's interpolation formula for unequal intervals.

NOTE: The calculation is more complicated in Lagrange's formula than Newton's formula. The application of the formula is not speedy and there is always a chance of committing some error due to the number of positive and negative signs in the numerator and denominator of each term.

Comparisons between Lagrange and Newton Interpolation:

1). The Lagrange and Newton interpolating formulas provide two different forms for an interpolating polynomial, even though the interpolating polynomial is unique.

2). Lagrange method is numerically unstable but Newton's method is usually numerically stable and computationally efficient.

3). Newton formula is much better for computation than the Lagrange formula.

4). Lagrange form is most often used for deriving formulas for approximating derivatives and integrals

5). Lagrange's form is more efficient then the Newton's formula when you have to interpolate several data sets on the same data points.

Problem-01: Using Newton's divided difference estimate $f(8)$ & $f(15)$ from the following table.

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution: The divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4}=52$			
5	100		$\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	
7	294		$\frac{202-97}{10-5}=21$		0
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	
10	900		$\frac{310-202}{11-7}=27$		0
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
11	1210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2028				

Here, $x_0=4,\ x_1=5,\ x_2=7,\ x_3=10,\ x_4=11,\ x_5=13$

$$f(x_0)=48,\ f(x_0,x_1)=52,\ f(x_0,x_1,x_2)=15,\ f(x_0,x_1,x_2,x_3)=1$$

By Newton's divided difference formula we get,

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$
$$\cdots\ \cdots+(x-x_0)(x-x_1)(x-x_2)\cdots\cdots(x-x_{n-1})f(x_0,x_1,x_2,\cdots\cdots x_n)$$
$$=48+(x-4)\times 52+(x-4)(x-5)\times 15+(x-4)(x-5)(x-7)\times 1$$
$$=48+52x-208+15x^2-135x+300+x^3-16x^2+83x-140$$
$$\therefore\ f(x)=x^3-x^2$$

Now $f(8)=8^3-8^2=512-64=448\ \ (Ans.)$

And $f(15)=(15)^3-(15)^2=3375-225=3150\ \ (Ans.)$

Problem-02: Using Newton's divided difference estimate $f(x)$ from the following table.

x	-1	0	2	3	4
$f(x)$	-16	-7	-1	8	29

Solution: The divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-1	-16				
		$\frac{-7+16}{0+1}=9$			
0	-7		$\frac{3-9}{2+1}=-2$		
		$\frac{-1+7}{2-0}=3$		$\frac{2+2}{3+1}=1$	
2	-1		$\frac{9-3}{3-0}=2$		0
		$\frac{8+1}{3-1}=9$		$\frac{6-2}{4-0}=1$	
3	8		$\frac{21-9}{4-2}=6$		
		$\frac{29-8}{4-3}=21$			
4	29				

Here, $x_0=-1,\ x_1=0,\ x_2=2,\ x_3=3,\ x_4=4$

$$f(x_0)=-16,\ f(x_0,x_1)=9,\ f(x_0,x_1,x_2)=-2,\ f(x_0,x_1,x_2,x_3)=1$$

By Newton's divided difference formula we get,

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$

$$\cdots\ \cdots+(x-x_0)(x-x_1)(x-x_2)\cdots\cdots(x-x_{n-1})f(x_0,x_1,x_2,\cdots\cdots x_n)$$

$$=-16+(x+1)\times 9+x(x+1)\times(-2)+x(x+1)(x-2)\times 1$$

$$=-16+9x+9-2x^2-2x+x^3-2x^2+x^2-2x$$

$$\therefore\ f(x)=x^3-3x^2+5x-7 \quad (Ans.)$$

Problem-03: Using Lagrange's formula estimate $f(x)$ from the following table.

x	0	1	2	5
$f(x)$	2	3	12	147

Solution: Here, $x_0=0,\ x_1=1,\ x_2=2,\ x_3=5$

$$f(x_0)=2,\ f(x_1)=3,\ f(x_2)=12,\ f(x_3)=147$$

By Lagrange's formula we get,

$$f(x)=\frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)}f(x_1)+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots\cdots(x_2-x_n)}f(x_2)$$

$$\cdots\cdots+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})}\ f(x_n)$$

$$=\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}\times 2+\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)\cdots\cdots(1-5)}\times 3+\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}\times 12+\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}\times 147$$

$$=\frac{x^3-8x^2+17x-10}{-10}\times 2+\frac{x^3-7x^2+10x}{4}\times 3+\frac{x^3-6x^2+5x}{-6}\times 12+\frac{x^3-3x^2+2x}{60}\times 147$$

$$=\frac{-x^3+8x^2-17x+10}{5}+\frac{3x^3-21x^2+30x}{4}-2x^3+12x^2-10x+\frac{49x^3-147x^2+98x}{20}$$

$$=\frac{1}{20}(-4x^3+32x^2-68x+40+15x^3-105x^2+150x-2x^3+12x^2-10x+49x^3-147x^2+98x)$$

$$=\frac{1}{20}(20x^3+20x^2-20x+40)$$

$$=x^3+x^2-x+2$$

$$\therefore\ f(x)=x^3+x^2-x+2 \quad (Ans.)$$

Problem-04: Using Lagrange's formula estimate $f(10)$ from the following table.

x	5	6	9	11
$f(x)$	12	13	14	16

Solution: Here, $x_0=5,\ x_1=6,\ x_2=9,\ x_3=11\ \&\ x=10$

$$f(x_0)=12,\ f(x_1)=13,\ f(x_2)=14,\ f(x_3)=16$$

By Lagrange's formula we get,

$$f(x)=\frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)}f(x_1)+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots\cdots(x_2-x_n)}f(x_2)$$

$$\cdots\cdots + \frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})}\ f(x_n)$$

$$=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}\times12+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}\times13+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(5-11)}\times14+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\times16$$

$$=\frac{-4}{-24}\times12+\frac{-5}{15}\times13+\frac{-20}{-24}\times14+\frac{20}{60}\times16$$

$$=2-4.333+11.667+5.333$$

$$=14.667\ \ (Ans.)$$

Problem-05: Using Lagrange's formula estimate $\sqrt[3]{55}$ from the following table.

x	50	52	54	56
$f(x)=\sqrt[3]{x}$	3.684	3.732	3.779	3.825

Solution: Here, $x_0=50,\ x_1=52,\ x_2=54,\ x_3=56\ \&\ x=55$

$$f(x_0)=3.684,\ f(x_1)=3.732,\ f(x_2)=3.779,\ f(x_3)=3.825$$

By Lagrange's formula we get,

$$f(x)=\frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)}\,f(x_0)+\frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)}\,f(x_1)+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots\cdots(x_2-x_n)}\,f(x_2)$$

$$\cdots\cdots + \frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})}\ f(x_n)$$

$$=\frac{(55-52)(55-54)(55-56)}{(50-52)(50-54)(50-56)}\times3.684+\frac{(55-50)(55-54)(55-56)}{(52-50)(52-54)(52-56)}\times3.732+\frac{(55-50)(55-52)(55-56)}{(54-50)(54-52)(54-56)}\times3.779+\frac{(55-50)(55-52)(55-54)}{(56-50)(56-52)(56-54)}\times3.825$$

$$=\frac{-3}{48}\times3.684+\frac{-5}{16}\times3.732+\frac{-15}{-16}\times3.779+\frac{15}{48}\times3.825$$

$$=-0.23025-1.16625+3.5428125+1.1953125$$

$$=3.341625\ (Ans.)$$

Exercise:

Problem-01: Using Lagrange's formula estimate $\sin 39^0$ from the following table.

x	0	10	20	30	40
$f(x)=\sin x$	0	1.1736	0.3420	0.5000	0.6428

Problem-02: Using Lagrange's formula estimate $\log 5.15$ from the following table.

x	5.1	5.2	5.3	5.4	5.5
$f(x)=\log x$	0.7076	0.7160	0.7243	0.7324	0.7404

Problem-03: The following table gives the sales of a concern for the five years. Using Lagrange's formula estimate the sales for the years $1986\ \&1992$.

Year	1985	1987	1989	1991	1993
Sales	40	43	48	52	57

Problem-04: Using Lagrange's formula estimate $\sqrt{151}$ from the following table.

x	150	152	154	156
$f(x)=\sqrt{x}$	12.247	12.329	12.410	12.490

Problem-05: Using Lagrange's formula estimate $\tan(0.15)$ from the following table.

x	0.10	0.13	0.20	0.22	0.30
$f(x)=\tan x$	0.1003	0.1307	0.2027	0.2236	0.3093

Problem-06: Using Newton's divided difference formula estimate $f(8)$ from the following table.

x	4	5	7	10	11
$f(x)$	48	100	294	900	1210

Problem-07: Using Newton's divided difference formula estimate $f(x)$ from the following table.

x	0	1	4	5
$f(x)$	8	11	68	123

Problem-08: Using Newton's divided difference formula estimate $f(x)$ in powers of $(x-5)$ from the following table.

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

Problem-09: Using Newton's divided difference formula estimate $f(6)$ from the following table.

x	5	7	11	13	21
$f(x)$	150	392	1452	2366	9702

Problem-010: Using Newton's divided difference formula estimate $\tan(0.12)$ from the following table.

x	0.10	0.13	0.20	0.22	0.30
$f(x)=\tan x$	0.1003	0.1307	0.2027	0.2236	0.3093

CURVE FITTING

CH 04

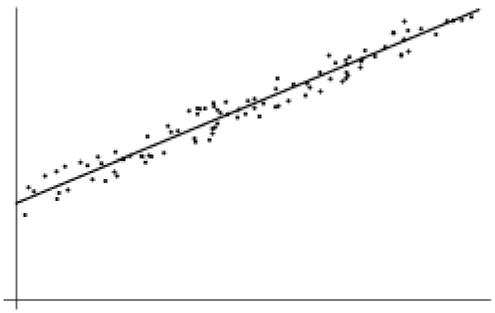
Curve fitting is the process of constructing a curve (named as an approximating curve) or mathematical function that has the best fit to a given set of data points, possibly subject to constraints. In this case the curve drawn is such that the discrepancy between the data points and the curve is least. The method of least squares is most commonly used in fitting curve.

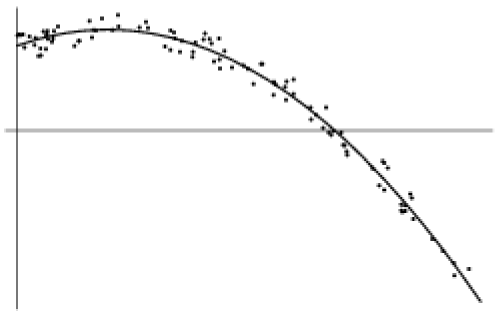
Or

Curve Fitting is most often used by scientists and engineers to visualize and plot the curve that best describes the shape & behavior of their data. Curve fitting is the procedure in finding a curve which matches a series of data points and possibly other constraints.

Or

A procedure in which the basic problem is to pass a curve through a set of points, representing experimental data, in such a way that the curve shows as well as possible the relationship between the two quantities plotted. It is always possible to pass some smooth curve through all the points plotted, but since there is assumed to be some experimental error present, such a procedure would ordinarily not be desirable.





The method curve fitting was suggested early in the 19th century by the French mathematician Adrien Legendre.

Least Squares Method:

The least squares method is the most systematic procedure to fit a unique curve through the given data points and its widely used universally in practical computations. The method of least squares assumes that the best-fit curve of a given type is the curve that has the minimal sum of the deviations squared (*least square error*) from a given set of data.

Suppose that the data points are $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots\dots\dots\dots\dots\dots, (x_n, y_n)$ where x is the independent variable and y is the dependent variable. The fitting curve $y = f(x)$ has the deviation (error/residual) d from each data point, i.e., $d_1 = y_1 - f(x_1), d_2 = y_2 - f(x_2), \dots\dots\dots\dots\dots\dots\dots, d_n = y_n - f(x_n)$. It is clear that some of the residuals will be positive and the remaining will be negative. Hence to give equal importance to positive and negative residuals we square each of them and form the sum of squares.

Now the sum of the squares of the errors or deviations is,

$$S = d_1^{\ 2} + d_2^{\ 2} + \cdots\cdots\cdots\cdots\cdots\cdots + d_n^{\ 2}$$

$$S = \{y_1 - f(x_1)\}^2 + \{y_2 - f(x_2)\}^2 + \cdots\cdots\cdots\cdots\cdots\cdots + \{y_n - f(x_n)\}^2$$

$$S = \sum_{i=1}^{n} \{y_i - f(x_i)\}^2$$

[Must be Minimum for Best Fitting]

The quantity S provides a measure of the goodness of fit of the curve to the given data if it is very minimum and if it is large then the curve fitting is bad. For $S = 0$each of the given points lies on $y = f(x)$ and it will decrease in value depending on the closeness of the points to the curves. Therefore, the best representative curve to the given data set of points is that for which the sum of squares of the errors S is minimum. This is known as the least Square method /Criterion or the principle of least squares.

Note:

Least squares curves fitting is of two types such as linear and nonlinear least squares fitting to given data $(x_i, y_i), i = 1,2, \dots\dots, n$ according to the choice of approximating curves $f(x)$ as linear or nonlinear. The constant occurring in the equation $y = f(x)$ of the approximating curve can be found by several methods mentioned in the followings:

1.Graphical Method
2.The method of group average
3.Method of least squares

Linear curve fitting

Fitting a straight line:

Fitting a straight line means finding the values of the parameters a and b of the straight line $y = ax + b$ as well as actually constructing the line itself. The graphical method and least square method are two useful methods for finding a straight line.

Let us consider n data points $(x_i, y_i), i = 1,2, \dots \dots, n$ and a linear function $y = ax + b$ in x and y that represents a straight line best fit to the given data. We have to find the constants a and b. For any x_i the expected value of y (Value calculated from the equations) is $a + bx_i$and observed value of y is y_i.

Therefore, the deviation/error/residual $d_i = y_i - (ax_i + b)$,by giving values $i = 1,2, \dots \dots, n$ we get the various residuals.

Now the sum of the squares of the errors or deviations is,

$$S = {d_1}^2 + {d_2}^2 + \cdots\cdots\cdots\cdots\cdots\cdots + {d_n}^2$$

$$S = \{y_1 - (ax_1 + b)\}^2 + \{y_2 - (ax_2 + b)\}^2 + \cdots\cdots\cdots\cdots\cdots\cdots + \{y_n - (ax_n + b)\}^2$$

$$S = \sum_{i=1}^{n} \{y_i - (ax_i + b)\}^2$$

[Must be Minimum for Best Fitting]

The quantity S provides a measure of the goodness of fit of the curve to the given data if it is very minimum.

For S to be minimum the conditions are $\frac{\partial S}{\partial a} = 0$ and $\frac{\partial S}{\partial b} = 0$.

Partially differentiating S with respect to a and b, we get

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} 2\{y_i - (ax_i + b)\}(-x_i)$$

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} (-2x_i)\{y_i - (ax_i + b)\}$$

And

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} 2\{y_i - (ax_i + b)\}(-1)$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} (-2)\{y_i - (ax_i + b)\}$$

For satisfying the conditions above equation equating with zero, we find

$$\sum_{i=1}^{n} (-2x_i)\{y_i - (ax_i + b)\} = 0$$

$$-2\sum_{i=1}^{n} x_i \{y_i - (ax_i + b)\} = 0$$

$$\sum_{i=1}^{n} x_i \{y_i - (ax_i + b)\} = 0$$

$$\sum_{i=1}^{n} \left\{x_i y_i - (a{x_i}^2 + bx_i)\right\} = 0$$

$$\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} (a{x_i}^2 + bx_i) = 0$$

$$a\sum_{i=1}^{n} {x_i}^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (i)$$

And

$$\sum_{i=1}^{n}(-2)\{y_i-(ax_i+b)\}=0$$

$$(-2)\sum_{i=1}^{n}\{y_i-(ax_i+b)\}=0$$

$$\sum_{i=1}^{n}\{y_i-(ax_i+b)\}=0$$

$$\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}(ax_i+b)=0$$

$$\sum_{i=1}^{n}(ax_i+b)=\sum_{i=1}^{n}y_i$$

$$\sum_{i=1}^{n}ax_i+\sum_{i=1}^{n}b=\sum_{i=1}^{n}y_i$$

$$a\sum_{i=1}^{n}x_i+bn=\sum_{i=1}^{n}y_i\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(ii)$$

We represent the equations (i) and (ii) in matrix form

$$\begin{bmatrix}\sum_{i=1}^{n}x_i^{\,2} & \sum_{i=1}^{n}x_i\\ \sum_{i=1}^{n}x_i & n\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{n}x_iy_i\\ \sum_{i=1}^{n}y_i\end{bmatrix}$$

$$\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{n}x_i^{\,2} & \sum_{i=1}^{n}x_i\\ \sum_{i=1}^{n}x_i & n\end{bmatrix}^{-1}\begin{bmatrix}\sum_{i=1}^{n}x_iy_i\\ \sum_{i=1}^{n}y_i\end{bmatrix}$$

$$\begin{bmatrix}a\\ b\end{bmatrix}=\frac{1}{n\sum_{i=1}^{n}x_i^{\,2}-\sum_{i=1}^{n}x_i\sum_{i=1}^{n}x_i}\begin{bmatrix}n & -\sum_{i=1}^{n}x_i\\ -\sum_{i=1}^{n}x_i & \sum_{i=1}^{n}x_i^{\,2}\end{bmatrix}\begin{bmatrix}\sum_{i=1}^{n}x_iy_i\\ \sum_{i=1}^{n}y_i\end{bmatrix}\qquad\because\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix},ad-bc\neq 0$$

$$\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}\dfrac{n\sum_{i=1}^{n}x_iy_i-\sum_{i=1}^{n}x_i\sum_{i=1}^{n}y_i}{n\sum_{i=1}^{n}x_i^{\,2}-\sum_{i=1}^{n}x_i\sum_{i=1}^{n}x_i}\\ \dfrac{\sum_{i=1}^{n}x_i^{\,2}\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}x_i\sum_{i=1}^{n}x_iy_i}{n\sum_{i=1}^{n}x_i^{\,2}-\sum_{i=1}^{n}x_i\sum_{i=1}^{n}x_i}\end{bmatrix}$$

Now equating the two equal matrices we get,

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i}$$

$$= \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$= \frac{n\sum_{i=1}^{n} x_i y_i - n\bar{x}\, n\bar{y}}{n\sum_{i=1}^{n} x_i^2 - (n\bar{x})^2} \qquad \left[\because \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}\right]$$

$$= \frac{n\sum_{i=1}^{n} x_i y_i - n^2\bar{x}\,\bar{y}}{n\sum_{i=1}^{n} x_i^2 - n^2\bar{x}^2} = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\,\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\,\bar{x}^2}$$

And

$$b = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$= \frac{n\bar{y}\sum_{i=1}^{n} x_i^2 - n\bar{x}\sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - (n\bar{x})^2}$$

$$= \frac{\bar{y}\sum_{i=1}^{n} x_i^2 - \bar{x}\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2 - n\,\bar{x}^2}$$

Putting this value of a and b in the equation $y = ax + b$we get the equation of the line best fitting the data as$y = ax + b$.

Note:

The equations $a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$ and $a\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i$ are called the normal equations.

Dropping of the suffices above equation can be written as $a\sum x^2 + b\sum x = \sum xy$ and $a\sum x + bn = \sum y$

Mathematical problem on Linear Curve Fitting

Problem 01: Use the method of least squares to fit a straight line to the following data:

x	*0*	*5*	*10*	*15*	*20*
y	*7*	*11*	*16*	*20*	*26*

Estimate the value of y when x=25.

Solution:
Assume that the least square straight line to be fitted to the given data be $y = ax + b$.
Then we have the normal equations are

$$a\sum x^2 + b\sum x = \sum xy \cdots\cdots\cdots\cdots\cdots\cdots(i)$$

and

$$a\sum x + bn = \sum y \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(ii)$$

Here the number of data points n= 8.
Calculation for finding the coefficients a and b of the least square line.

x	y	xy	x^2
0	7	0	0
5	11	55	25
10	16	160	100
15	20	300	225
20	26	520	400
$\sum x = 50$	$\sum y = 80$	$\sum xy = 1035$	$\sum x^2 = 750$

Now putting these values in the above equations (i) and (ii) we get

$750a + 50b = 1035$ and $50a + 5b = 80$

Solving above two equations by calculator, we get values of $a = 0.94$ and $b = 6.6$.
Putting these values in the equation $y = ax + b$ we get the required line as $y = 0.94x + 6.6$.
Expected value of $y = 0.94 \times 25 + 6.6 = 30.1$ as $x = 25$. **(As desired)**

Problem 02: Find the least square line $y = ax + b$ for the data points $(-1,10)$,$(0,9)$,$(1,7)$,$(2,5)$,$(3,4)$,$(4,3)$ $(5,0)$ and $(6,-1)$.

Solution:
Given that least square straight is $y = ax + b$ and number of data points n= 8.
Then we have the normal equations are

$$a\sum x^2 + b\sum x = \sum xy \cdots\cdots\cdots\cdots\cdots\cdots(i)$$

and

$$a\sum x + bn = \sum y \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(ii)$$

Calculation for finding the coefficients a and b of the least square line.

x	y	xy	x^2
-1	10	-10	1
0	9	0	0
1	7	7	1
2	5	10	4
3	4	12	9
4	3	12	16
5	0	0	25
6	-1	-6	36
$\sum x = 20$	$\sum y = 37$	$\sum xy = 25$	$\sum x^2 = 92$

Now putting these values in the above equations (i) and (ii) we get

$92a + 20b = 25$and $20a + 8b = 37$

Solving above two equations by calculator, we get values of $a = -1.60714$ and $b = 8.64286$.
Putting these values in the equation $y = ax + b$ we get the required line as $y = -1.60714x + 8.64286$.
(As desired)

Another Method:
Use the following steps to find the equation of line of best fit for a set of ordered pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.
Step 1: Calculate the mean of the x-values and the mean of the y-values.

$$\overline{X} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\overline{Y} = \frac{\sum_{i=1}^{n} y_i}{n}$$

Step2:The following formula gives the slope of the line of best fit:

$$m = \frac{\sum_{i=1}^{n}\left(x_i - \overline{X}\right)\left(y_i - \overline{Y}\right)}{\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2}$$

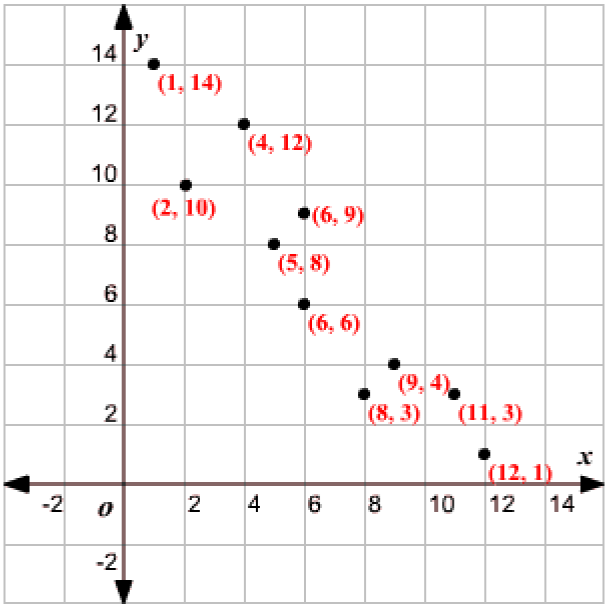
Step 3: Compute the y-intercept of the line by using the formula:

$$b = \overline{Y} - m\overline{X}$$

Step 4: Use the slope m and the y-intercept b to form the equation of the line.
Problem 03: Use the least square method to determine the equation of line of best fit for the data. Then plot the line.

x	8	2	11	6	5	4	12	9	6	1
y	3	10	3	6	8	12	1	4	9	14

Solution:
Plot the points on a coordinate plane.



Calculate the means of the *x*-values and the *y*-values.

$$\overline{X} = \frac{8+2+11+6+5+4+12+9+6+1}{10} = 6.4$$

$$\overline{Y} = \frac{3+10+3+6+8+12+1+4+9+14}{10} = 7$$

Now calculate $x_i - \overline{X}, y_i - \overline{Y}, (x_i - \overline{X})(y_i - \overline{Y})$ and $(x_i - \overline{X})^2$ for each *i*.

i	x_i	y_i	$x_i - \overline{X}$	$y_i - \overline{Y}$	$(x_i - \overline{X})(y_i - \overline{Y})$	$(x_i - \overline{X})^2$
1	8	3	1.6	−4	−6.4	2.56
2	2	10	−4.4	3	−13.2	19.36
3	11	3	4.6	−4	−18.4	21.16
4	6	6	−0.4	−1	0.4	0.16
5	5	8	−1.4	1	−1.4	1.96
6	4	12	−2.4	5	−12	5.76
7	12	1	5.6	−6	−33.6	31.36
8	9	4	2.6	−3	−7.8	6.76
9	6	9	−0.4	2	−0.8	0.16
10	1	14	−5.4	7	−37.8	29.16
					$\sum_{i=1}^{n}(x_i - \overline{X})(y_i - \overline{Y}) = -131$	$\sum_{i=1}^{n}(x_i - \overline{X})^2 = 118.4$

Calculate the slope.

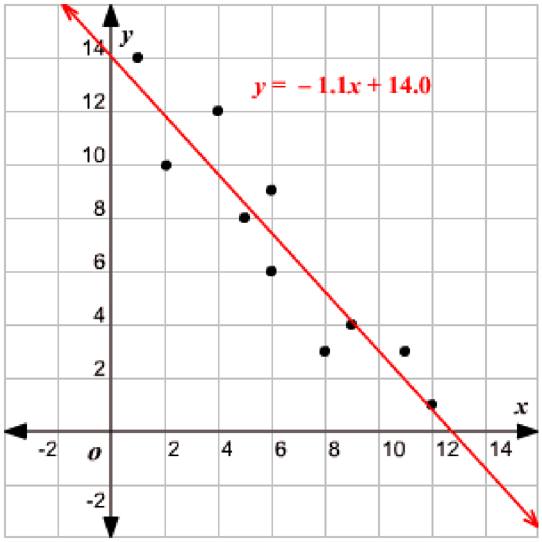
$$m = \frac{\sum_{i=1}^{n}(x_i - \overline{X})(y_i - \overline{Y})}{\sum_{i=1}^{n}(x_i - \overline{X})^2} = \frac{-131}{118.4} \approx -1.1$$

Calculate the *y*-intercept.
Use the formula to compute the *y*-intercept.

$$\begin{aligned} b &= \overline{Y} - m\overline{X} \\ &= 7 - (-1.1 \cdot 6.4) \\ &= 7 + 7.04 \\ &\approx 14.0 \end{aligned}$$

Use the slope and *y*-intercept to form the equation of the line of best fit.
The slope of the line is –1.1 and the *y* -intercept is 14.0.
Therefore, the equation is *y* = –1.1 *x* + 14.0.

Draw the line on the scatter plot.



Problems for practicing:

1.Find the least square line $y = ax + b$ for the data

x	-2	-1	0	1	2
y	1	2	3	3	4

2. Find the values of a_0 and a_1 so that $y = a_0 + a_1x$ fits the data given in the table:

x	0	1	2	3	4
y	1	2.9	4.8	6.7	8.6

3.Fit a straight line of the form $y = a_0 + a_1x$ to the data:

x	1	2	3	4	6	8
y	2.4	3.1	3.5	4.2	5	6

4.The table below gives the temperature T (in 0^0C) and length l (in mm) of a heated rod. If $l = a_0 + a_1T$ find the values of a_0 and a_1 using linear least squares

T	40	50	60	70	80
l	600.5	600.6	600.8	600.9	601

5.Find the least square line $y = ax + b$ for the data

x	-4	-2	0	2	4
y	1.2	2.8	6.2	7.8	13.2

6.Fit a straight line to the following data regarding x as the independent variable

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

7.Find the least square fit straight line of the form $y = ax + b$ for the data of fertilize application and yield of a plant

fertilizer	0	10	20	30	40	50
Yield (kg)	.8	.8	1.3	1.6	1.7	1.8

Non-linear Curve Fitting

Fitting a Parabola:

Fitting a Parabola means finding the values of the parameters a, b and c of the Parabola $y = ax^2 + bx + c$ as well as actually constructing the parabola itself. The graphical method and least square method are two useful methods for finding parabola.

Let us consider n data points $(x_i, y_i), i = 1,2, \dots \dots, n$ and a non-linear function $y = ax^2 + bx + c$ in x and y that represents a parabola best fit to the given data. We have to find the constants a, b and c. For any x_i the expected value of y (Value calculated from the equations) is $a + bx_i$and observed value of y is y_i.

Therefore, the deviation/error/residual $d_i = y_i - (ax_i^2 + bx_i + c)$,by giving values $i = 1,2, \dots \dots, n$ we get the various residuals.

Now the sum of the squares of the errors or deviations is,

$$S = d_1^{\,2} + d_2^{\,2} + \cdots\cdots\cdots\cdots\cdots + d_n^{\,2}$$

$$S = \left\{y_1 - (ax_1^{\,2} + bx_1 + c)\right\}^2 + \left\{y_2 - (ax_2^{\,2} + bx_2 + c)\right\}^2 + \cdots\cdots\cdots\cdots\cdots + \left\{y_n - (ax_n^{\,2} + bx_n + c)\right\}^2$$

$$S = \sum_{i=1}^{n} \left\{y_i - (ax_i^{\,2} + bx_i + c)\right\}^2$$

[Must be Minimum for Best Fitting]

The quantity S provides a measure of the goodness of fit of the curve to the given data if it is very minimum.

For S to be minimum the conditions are $\frac{\partial S}{\partial a} = 0$,$\frac{\partial S}{\partial b} = 0$ and $\frac{\partial S}{\partial c} = 0$.

Partially differentiating S with respect to a,b and c ,we get

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} 2\left\{y_i - (ax_i^{\,2} + bx_i + c)\right\}\left(-x_i^{\,2}\right)$$

$$\frac{\partial S}{\partial a} = -2\sum_{i=1}^{n} x_i^{\,2}\left\{y_i - (ax_i^{\,2} + bx_i + c)\right\}$$

And

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} 2\left\{y_i - (ax_i^{\,2} + bx_i + c)\right\}\left(-x_i\right)$$

$$\frac{\partial S}{\partial b} = -2\sum_{i=1}^{n} x_i\left\{y_i - (ax_i^{\,2} + bx_i + c)\right\}$$

And

$$\frac{\partial S}{\partial c} = \sum_{i=1}^{n} 2\left\{y_i - (ax_i^{\,2} + bx_i + c)\right\}(1)$$

$$\frac{\partial S}{\partial c} = 2\sum_{i=1}^{n} \left\{y_i - (ax_i^{\,2} + bx_i + c)\right\}$$

For satisfying the conditions above equation equating with zero, we find

$$\sum_{i=1}^{n} x_i^{\,2}\left\{y_i - (ax_i^{\,2} + bx_i + c)\right\} = 0$$

$$\sum_{i=1}^{n} \left\{x_i^{\,2} y_i - (ax_i^{\,4} + bx_i^{\,3} + cx_i^{\,2})\right\} = 0$$

$$\sum_{i=1}^{n} (ax_i^{\,4} + bx_i^{\,3} + cx_i^{\,2}) = \sum_{i=1}^{n} x_i^{\,2} y_i$$

$$a\sum_{i=1}^{n} x_i^{\ 4} + b\sum_{i=1}^{n} x_i^{\ 3} + c\sum_{i=1}^{n} x_i^{\ 2} = \sum_{i=1}^{n} x_i^{\ 2} y_i \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (i)$$

And

$$\sum_{i=1}^{n} x_i \left\{y_i - (ax_i^{\ 2} + bx_i + c)\right\} = 0$$

$$\sum_{i=1}^{n} \left\{x_i y_i - (ax_i^{\ 3} + bx_i^{\ 2} + cx_i)\right\} = 0$$

$$\sum_{i=1}^{n} (ax_i^{\ 3} + bx_i^{\ 2} + cx_i) = \sum_{i=1}^{n} x_i y_i$$

$$a\sum_{i=1}^{n} x_i^{\ 3} + b\sum_{i=1}^{n} x_i^{\ 2} + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (ii)$$

And

$$\sum_{i=1}^{n} \left\{y_i - (ax_i^{\ 2} + bx_i + c)\right\} = 0$$

$$\sum_{i=1}^{n} (ax_i^{\ 2} + bx_i + c) = \sum_{i=1}^{n} y_i$$

$$a\sum_{i=1}^{n} x_i^{\ 2} + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} 1 = \sum_{i=1}^{n} y_i$$

$$a\sum_{i=1}^{n} x_i^{\ 2} + b\sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (iii)$$

The above equations (i),(ii) and (iii) are called normal equations. Now dropping off the suffices normal equations becomes as

$$a\sum x_i^{\ 4} + b\sum x_i^{\ 3} + c\sum x_i^{\ 2} = \sum x_i^{\ 2} y_i$$

$$a\sum x_i^{\ 3} + b\sum x_i^{\ 2} + c\sum x_i = \sum x_i y_i$$

$$a\sum x_i^{\ 2} + b\sum x_i + cn = \sum y_i$$

By solving these above normal equations, we get the values of the parameters/constants a, b and c and hence get the equation to the best fitting parabola $y = ax^2 + bx + c$.

Polynomial of mth degree:

Polynomials are one of the most commonly used types of curves in regression. The applications of the method of least squares curve fitting using polynomials are briefly discussed as follows.

$$\begin{cases} \sum_{i=1}^{n} y_i = a_0\sum_{i=1}^{n} 1 + a_1\sum_{i=1}^{n} x_i + a_2\sum_{i=1}^{n} x_i^{\ 2} + \ldots + a_m\sum_{i=1}^{n} x_i^{\ m} \\ \sum_{i=1}^{n} x_i y_i = a_0\sum_{i=1}^{n} x_i + a_1\sum_{i=1}^{n} x_i^{\ 2} + a_2\sum_{i=1}^{n} x_i^{\ 3} + \ldots + a_m\sum_{i=1}^{n} x_i^{\ m+1} \\ \sum_{i=1}^{n} x_i^{\ 2} y_i = a_0\sum_{i=1}^{n} x_i^{\ 2} + a_1\sum_{i=1}^{n} x_i^{\ 3} + a_2\sum_{i=1}^{n} x_i^{\ 4} + \ldots + a_m\sum_{i=1}^{n} x_i^{\ m+2} \\ \vdots \\ \vdots \\ \sum_{i=1}^{n} x_i^{\ m} y_i = a_0\sum_{i=1}^{n} x_i^{\ m} + a_1\sum_{i=1}^{n} x_i^{\ m+1} + a_2\sum_{i=1}^{n} x_i^{\ m+2} + \ldots + a_m\sum_{i=1}^{n} x_i^{\ 2m} \end{cases}$$

Let us consider an m[th] degree polynomial

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$$

to approximate the given set of data points $(x_i, y_i), i = 1,2, \ldots \ldots, n$ where $n \geq m + 1$the best fitting curve $f(x)$ has the least square error, i.e.,

$$\Pi = \sum_{i=1}^{n} [y_i - f(x_i)]^2 = \sum_{i=1}^{n} \left[y_i - (a_0 + a_1 x_i + a_2 x_i^{\,2} + \ldots + a_m x_i^{\,m})\right]^2 = \min.$$

Mentionable that $a_0, a_1, a_2, \ldots \ldots \ldots, a_m$ are unknown coefficients while all $x_i$and $y_i$are given. To obtain the least square error, the unknown coefficients $a_0, a_1, a_2, \ldots \ldots \ldots, a_m$ must yield zero first partial derivatives.

$$\begin{cases} \dfrac{\partial \Pi}{\partial a_0} = 2\displaystyle\sum_{i=1}^{n} \left[y_i - (a_0 + a_1 x_i + a_2 x_i^{\,2} + \ldots + a_m x^m)\right] = 0 \\ \dfrac{\partial \Pi}{\partial a_1} = 2\displaystyle\sum_{i=1}^{n} x_i \left[y_i - (a_0 + a_1 x_i + a_2 x_i^{\,2} + \ldots + a_m x^m)\right] = 0 \\ \dfrac{\partial \Pi}{\partial a_2} = 2\displaystyle\sum_{i=1}^{n} x_i^{\,2} \left[y_i - (a_0 + a_1 x_i + a_2 x_i^{\,2} + \ldots + a_m x^m)\right] = 0 \\ \vdots \\ \vdots \\ \dfrac{\partial \Pi}{\partial a_m} = 2\displaystyle\sum_{i=1}^{n} x_i^{\,m} \left[y_i - (a_0 + a_1 x_i + a_2 x_i^{\,2} + \ldots + a_m x^m)\right] = 0 \end{cases}$$

Expanding the above equations, we get the normal equations as a system of linear equations as follows:

Solving this following system of linear equations, we get the values of the parameters or constants $a_0, a_1, a_2, \ldots \ldots \ldots, a_m$and putting these values we get the least square polynomial $y = a_0 + a_1 x + a_2 x^2 + \cdots \cdots \cdots \cdots \cdots + a_n x^m$.

Note that:

Dropping off the suffices we get the normal equations as follows:

$$\begin{aligned} \sum y &= m a_0 + a_1 \sum x + \cdots\cdots\cdots\cdots\cdots\cdots + a_m \sum x^m \\ \sum xy &= a_0 \sum x + a_1 \sum x^2 + \cdots\cdots\cdots\cdots\cdots\cdots + a_m \sum x^{m+1} \\ \sum x^2 y &= a_0 \sum x^2 + a_1 \sum x^3 + \cdots\cdots\cdots\cdots\cdots\cdots + a_m \sum x^{m+2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum x^m y &= a_0 \sum x^m + a_1 \sum x^{m+1} + \cdots\cdots\cdots\cdots\cdots\cdots + a_m \sum x^{2m} \end{aligned}$$

Particular Cases:

If n = 1 then the curve to be fitted is a straight line $y = ax + b$ and normal equations are

$$a\sum x^2 + b\sum x = \sum xy \quad \text{and} \quad a\sum x + bn = \sum y$$

If n = 2 then the curve to be fitted is a line $y = ax^2 + bx + c$ and normal equations are

$$\begin{aligned} a\sum x_i^{\,4} + b\sum x_i^{\,3} + c\sum x_i^{\,2} &= \sum x_i^{\,2} y_i \\ a\sum x_i^{\,3} + b\sum x_i^{\,2} + c\sum x_i &= \sum x_i y_i \\ a\sum x_i^{\,2} + b\sum x_i + cn &= \sum y_i \end{aligned}$$

Mathematical problem on No-Linear Curve Fitting

Problem 01:

Fit a second-degree parabola to the following data:

x	0	1	2	3	4
y	1	5	10	22	38

Solution:

Let the parabola to be fitted to the given data be $y = ax^2 + bx + c$ and number of data points n= 5.

Then the normal equations are as follows as a system of linear equations:

$$a\sum x_i^4 + b\sum x_i^3 + c\sum x_i^2 = \sum x_i^2 y_i$$
$$a\sum x_i^3 + b\sum x_i^2 + c\sum x_i = \sum x_i y_i$$
$$a\sum x_i^2 + b\sum x_i + cn = \sum y_i$$

Calculation for finding the coefficients a ,b and c.

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	5	1	1	1	5	5
2	10	4	8	16	20	40
3	22	9	27	81	66	198
4	38	16	64	256	152	608
$\sum x = 10$	$\sum y = 76$	$\sum x^2 = 30$	$\sum x^3 = 100$	$\sum x^4 = 354$	$\sum xy = 243$	$\sum x^2 y = 851$

Substituting the values in the above system of linear equations becomes

$$354a + 100b + 30c = 851$$
$$100a + 30b + 10c = 243$$
$$30a + 10b + 5c = 76$$

Solving the above system of linear equations by calculator we get

$$a = 2.21\ ,\ b = 0.24\ \ and\ \ c = 1.43$$

Hence the fitted required parabola is $y = 2.21x^2 + 0.24x + 1.43$ **(As desired)**

Problem 02:

Two quantities x and y are measured and corresponding values are given in the following table.

x	10	20	30	40	50	60
y	4.5	7.1	10.5	15.5	20.5	27.1

Fit a curve of the form $y = ax^2 + bx + c$.

Solution:

Let the parabola to be fitted to the given data be $y = ax^2 + bx + c$ and number of data points n= 6.

Then the normal equations are as follows as a system of linear equations:

$$a\sum x_i^4 + b\sum x_i^3 + c\sum x_i^2 = \sum x_i^2 y_i$$
$$a\sum x_i^3 + b\sum x_i^2 + c\sum x_i = \sum x_i y_i$$
$$a\sum x_i^2 + b\sum x_i + cn = \sum y_i$$

Calculation for finding the coefficients a ,b and c.

x	y	x^2	x^3	x^4	xy	x^2y
10	4.5	100	1000	10000	45	450
20	7.1	400	8000	160000	142	2840
30	10.5	900	27000	810000	315	9450
40	15.5	1600	64000	2560000	620	24800
50	20.5	2500	125000	6250000	1025	51250

60	27.1	3600	216000	12960000	1626	97560
$\sum x = 210$	$\sum y = 85.2$	$\sum x^2 = 9100$	$\sum x^3 = 441000$	$\sum x^4 = 22750000$	$\sum xy = 3773$	$\sum x^2 y = 186350$

Substituting the values in the above system of linear equations becomes

$$22750000a + 441000b + 9100c = 186350$$
$$441000a + 9100b + 210c = 3773$$
$$9100a + 210b + 6c = 85.2$$

Solving the above system of linear equations by calculator we get

$$a = 0.005 \text{ , } b = 0.122 \text{ and } c = 2.78$$

Hence the fitted required parabola is $y = 0.005x^2 + 0.122x + 2.78$ **(As desired)**

Problems for practicing:

1.Fit a second-degree parabola to the following:

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

2.Fit a parabolic curve to the following data:

x	1	2	3	4	5
y	1090	1220	1390	1625	1915

3.Fit a curve of the form $y = ax^2 + bx + c$ to the data:

x	87.5	84	77.8	63.7	46.7
y	292	283	270	235	197

4.The following table gives the levels of prices in certain quantities. Fit a second-degree parabola to the data:

Quantities	2	3	5	6	7
Prices	292	283	270	235	197

5.Fit a polynomial of the second degree to the data points given in the following table:

x	0	1	2
y	1	6	17

6.Find the values of a,b and c so that $y = ax^2 + bx + c$ is the best fit to the data :

x	0	1	2	3	4
y	1	0	3	10	21

NUMERICAL DIFFERENTIATION

CH 05

Numerical differentiation is the process of finding the numerical value of a derivative of a given function at a given point. First intention of this section is to derive the approximated polynomial to compute the derivatives. In deriving the polynomial for computing derivative, it is important to keep in mind the following information.
For equidistant tabular points, one uses either the Newton's Forward/ Backward Formula or Sterling's Formula; otherwise Lagrange's formula is used. Newton's Forward/ Backward formula is used depending upon the location of the point at which the derivative is to be computed. In case the given point is near the midpoint of the interval, Sterling's formula can be used. We illustrate the process by taking (i) Newton's Forward formula (ii) Sterling's formula.

Remember!! Newton's Interpolation formulae:

1. **Forward difference formulae**

$$y(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0\cdots\cdots+\frac{p(p-1)\cdots\cdots(p-n+1)}{n!}\Delta^n y_0$$

Where $x=x_0+ph$.

2. **Backward difference formulae**

$$y(x)=y_n+p\nabla y_n+\frac{p(p+1)}{2!}\nabla^2 y_n+\frac{p(p+1)(p+2)}{3!}\nabla^3 y_n\cdots\cdots+\frac{p(p+1)\cdots\cdots(p+n-1)}{n!}\nabla^n y_n$$

Where $x=x_n+ph$.

Calculation of Derivatives

Derivatives using Newton's forward difference formulae:

Let us consider the Newton's forward difference formulae for interpolation is

$$y(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0\cdots\cdots+\frac{p(p-1)\cdots\cdots(p-n+1)}{n!}\Delta^n y_0$$

Where $x=x_0+ph$.

By using chain rule, we have

$$\frac{dy}{dx}=\frac{dy}{dp}.\frac{dp}{dx}==\frac{1}{h}.\frac{dy}{dp}$$

$$=\frac{1}{h}.\frac{d}{dp}\left(y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\frac{p(p-1)(p-2)(\text{p}-3)}{4!}\Delta^4 y_0\cdots\cdots\right)$$

$$=\frac{1}{h}.\frac{d}{dp}\left(y_0+p\Delta y_0+\frac{p^2-p}{2}\Delta^2 y_0+\frac{p(p^2-3p+2)}{6}\Delta^3 y_0+\frac{(p^2-3p)(\text{p}^2-3p+2)}{4!}\Delta^4 y_0\cdots\cdots\right)$$

$$=\frac{1}{h}.\frac{d}{dp}\left(y_0+p\Delta y_0+\frac{p^2-p}{2}\Delta^2 y_0+\frac{(p^3-3p^2+2p)}{6}\Delta^3 y_0+\frac{(\text{p}^4-6p^3+11p^2+6\text{p})}{4!}\Delta^4 y_0\cdots\cdots\right)$$

$$=\frac{1}{h}.\frac{d}{dp}\left(y_0+p\Delta y_0+\frac{p^2-p}{2}\Delta^2 y_0+\frac{(p^3-3p^2+2p)}{6}\Delta^3 y_0+\frac{(\mathrm{p}^4-6p^3+11p^2+6\mathrm{p})}{24}\Delta^4 y_0\cdots\cdots\right)$$

$$=\frac{1}{h}.\left(0+\Delta y_0+\frac{2p-1}{2}\Delta^2 y_0+\frac{(3p^2-6p+2)}{6}\Delta^3 y_0+\frac{(4\mathrm{p}^3-18p^2+22p+6)}{24}\Delta^4 y_0\cdots\cdots\right)$$

$$=\frac{1}{h}.\left(\Delta y_0+\frac{2p-1}{2}\Delta^2 y_0+\frac{(3p^2-6p+2)}{6}\Delta^3 y_0+\frac{(4\mathrm{p}^3-18p^2+22p+6)}{24}\Delta^4 y_0\cdots\cdots\right)\quad\cdots\cdots\cdots(1)$$

[Formula for non-tabular values]

At $x=x_0$ we have $p=0$ then first derivative is,

$$\left.\frac{dy}{dx}\right|_{x=x_0}=\frac{1}{h}.\left(\Delta y_0+\frac{-1}{2}\Delta^2 y_0+\frac{2}{6}\Delta^3 y_0+\frac{6}{24}\Delta^4 y_0\cdots\cdots\right)$$

$$=\frac{1}{h}.\left(\Delta y_0-\frac{1}{2}\Delta^2 y_0+\frac{1}{3}\Delta^3 y_0-\frac{1}{4}\Delta^4 y_0\cdots\cdots+(-1)^{n+1}\frac{1}{n}\Delta^n y_0\right)$$

[Formula for tabular Values]

Again differentiating (1) with respect to "x" we get,

$$\frac{d^2y}{dx^2}=\frac{1}{h}.\frac{d}{dx}\left(\frac{dy}{dp}\right)=\frac{1}{h}.\frac{d}{dp}\left(\frac{dy}{dp}\right).\frac{dp}{dx}=\frac{1}{h^2}\frac{d}{dp}\left(\frac{dy}{dp}\right)$$

$$=\frac{1}{h^2}\frac{d}{dp}\left(\Delta y_0+\frac{2p-1}{2}\Delta^2 y_0+\frac{(3p^2-6p+2)}{6}\Delta^3 y_0+\frac{(4\mathrm{p}^3-18p^2+22p+6)}{24}\Delta^4 y_0\cdots\cdots\right)$$

$$=\frac{1}{h^2}\left(0+\frac{2-0}{2}\Delta^2 y_0+\frac{(6p-6+0)}{6}\Delta^3 y_0+\frac{(12\mathrm{p}^2-36p+22+0)}{24}\Delta^4 y_0+\cdots\cdots\right)$$

$$=\frac{1}{h^2}\left(\Delta^2 y_0+(p-1)\Delta^3 y_0+\frac{(6\mathrm{p}^2-18p+11)}{12}\Delta^4 y_0\cdots\cdots\right)$$

[Formula for non-tabular values]

At $x=x_0$ we have $p=0$ then first derivative is,

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0}=\frac{1}{h^2}\left(\Delta^2 y_0-\Delta^3 y_0+\frac{11}{12}\Delta^4 y_0+\cdots\cdots\right)$$

[Formula for non-tabular values]

Similarly, Newton's backward difference formulae for derivatives:

$$\left.\frac{dy}{dx}\right|_{x=x_0}=\frac{1}{h}.\left(\Delta y_0+\frac{-1}{2}\Delta^2 y_0+\frac{2}{6}\Delta^3 y_0+\frac{6}{24}\Delta^4 y_0\cdots\cdots\right)$$

$$= \frac{1}{h} . \left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n \cdots \cdots + \frac{1}{n}\Delta^n y_0 \right)$$

[Formula for tabular Values]

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left(\nabla^2 y_n + \Delta^3 y_n + \frac{11}{12}\Delta^4 y_n + \frac{5}{6}\Delta^5 y_n \cdots \cdots \right)$$

Worked-Out Problems

Problem 01: From the following table of values of x and y, Obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.2$ and $x = 2.0$.

Solution: The difference table is formed as follows:

x	y	△	△²	△³	△⁴	△⁵	△⁶
1.0	2.7183						
		0.6018					
1.2	**3.3201**		0.1333				
		0.7351		0.0294			
1.4	4.0552		**0.1627**		0.0067		
		0.8978		**0.0361**		**0.0013**	
1.6	4.9530		0.1988		**0.0080**		0.0001
		1.0966		**0.0441**		**0.0014**	
1.8	6.0496		**0.2429**		0.0094		
		1.3395		0.0535			
2.0	**7.3891**		0.2964				
		1.6359					
2.2	9.0250						

Here $x_0 = 1.2$, $y_0 = 3.3201$ and $h = 0.2$.

We have the numerical derivative formulae,

$$\left.\frac{dy}{dx}\right|_{x=1.2} = \frac{1}{0.2} . \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 \right)$$

$$= \frac{1}{0.2} . \left(\mathbf{0.7351} - \frac{1}{2}\times\mathbf{0.1627} + \frac{1}{3}\times\mathbf{0.0361} - \frac{1}{4}\times\mathbf{0.0080} + \frac{1}{5}\times\mathbf{0.0014} \right) = 3.3205$$

Again, $$\left.\frac{d^2y}{dx^2}\right|_{x=1.2} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 \right)$$

$$= \frac{1}{0.04}\left(\mathbf{0.1627} - \mathbf{0.0361} + \frac{11}{12}\times\mathbf{0.0080} - \frac{5}{6}\times\mathbf{0.0014} \right) = 3.318$$

(As desired)

Again, $$\left.\frac{dy}{dx}\right|_{x=2.0} = \frac{1}{0.2} . \left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n \right)$$

$$=\frac{1}{0.2}.\left(\mathbf{1.3395}+\frac{1}{2}\times\mathbf{0.2429}+\frac{1}{3}\times\mathbf{0.0441}+\frac{1}{4}\times\mathbf{0.0080}+\frac{1}{5}\times\mathbf{0.0013}\right)$$

$$=\frac{1}{0.2}.\left(1.3395+0.12145+0.0147+0.0002+0.00026\right)$$

$$=\frac{1}{0.2}.\left(1.3395+0.12145+0.0147+0.0002+0.00026\right)=7.38055$$

(As desired)

Problem 02: From the following data, find the maximum and minimum values of y.

x	0	2	4	6
f(x)	2	0	-50	-196

Solution: The difference table for the given data:

x	y=f(x)	△	△²	△³
0	2			
		-2		
2	0		-48	
		-50		-48
4	-50		-96	
		-146		
6	-196			

Here $x_0=0$, $y_0=2$ and $h=2$.

Now, $p=\dfrac{x-0}{2}=0.5x=p(x)$

We have Newton's forward difference formula as

$$y(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

Putting values from above mentioned table we get,

$$y(x)=2+p(-2)+\frac{p(p-1)}{2!}(-48)+\frac{p(p-1)(p-2)}{3!}(-48)$$
$$=2-2p-24(p^2-p)-8p(p-1)(p-2)$$
$$=2-2p-24(p^2-p)-8p(p^2-3p+2)$$
$$=2-2p-24p^2+24p-8p^3+24p^2-16p$$
$$=-8p^3+6p+2\cdots\cdots\cdots(1)$$

Now differentiating two times w.r.to p, we get

$$\frac{dy}{dp}=-24p^2+6\qquad\Rightarrow\frac{d^2y}{dp^2}=-48p$$

For maxima and minima, $\dfrac{dy}{dp}=0$

$$-24p^2+6=0 \quad \Rightarrow p^2=\frac{6}{24}=\frac{1}{4} \quad \Rightarrow p=0.5 \ \ or \ \ p=-0.5$$

When $p=0.5$ then $\frac{d^2y}{dp^2}=-48\times 0.5=-24<0$.

Therefore, y value is maximum at $p = 0.5$.

So, the maximum value, $y_{max} = -8\times\frac{1}{8}+6\times\frac{1}{2}+2=-1+3+2=4$.

Again,

When $p=-0.5$ then $\frac{d^2y}{dp^2}=-48\times -0.5=24>0$.

Therefore, y value is minimum at $p = -0.5$.

So, the maximum value, $y_{max} = -8\times\left(\frac{1}{-8}\right)+6\times\left(-\frac{1}{2}\right)+2=1-3+2=0$. (As desired)

Try yourself:

1. From the following table, find x for which y is maximum and also find maximum value of y.

x	3	4	5	6	7	8
$f(x)$	0.205	0.240	0.259	0.262	0.250	0.224

2. Find the maximum and the minimum values of the function $y = f(x)$ from the following data

x	0	1	2	3	4	5
$f(x)$	0	0.25	0	2.25	16	56.25

3. From the table below, for what value of x, y is minimum? Also find this value of y.

x	3	4	5	6	7	8
$f(x)$	0.205	0.240	0.259	0.262	0.250	0.224

Solution of differential Equations

Many analytical methods exist to solve first order and first-degree differential equations. But these methods can be applied to solve only a selected class of differential equations. Sometimes differential equations cannot be solved at all analytically at all or gives solutions which are so difficult to obtain. For solving such differential equations, numerical methods are used. In numerical methods we do not want to find a relation between x and y, but find the numerical values of the dependent variable for certain values of independent variable.

The aim of this chapter is to solve the differential equation $\frac{dy}{dx}=f(x,y)$, $y(x_0)=y_0$ numerically but not analytically. Various methods are known to all to solve the differential equation such as:

1. Euler Method
2. Euler modified Method
3. Runge -Kutta Method
4. Picard's Method
5. Taylor series Method

6. Predictor-Corrector Method
7. Milne's Method.
8. Adams-Moulton Method etc.

From above mentioned different methods we discuss in this chapter about Euler Method and Runge – Kutta Method of fourth order.

The numerical solution of first order differential equation will generally give result in one of the following two types:

i. A series of y in terms of x, from which the values of y can be obtained by direct substitution.
ii. A set of tabulated values of x and y.

The methods of Taylor and Picard belong to the first class (i) whereas the methods of Euler, Runge-Kutta etc. belong to class (ii).

Problem: Analytically solve the differential equation $\frac{dy}{dx}=10x$, $y(0)=1$ and find the value of $y(3.5)$.

Solution:

Given differential equation is $\frac{dy}{dx}=10x$, $y(0)=1$

Now rearranging the differential equation to variable separable form, we get

$$dy=10x\,dx$$

Integrating both sides we find the general solution of the given differential equation,

$$\int dy=\int 10x\,dx$$

$$y=10\times\frac{x^2}{2}+c$$

$$y=5x^2+c$$

Now applying the given condition $y(0)=1$ we find the value of " c " as

$$1=5.0^2+c$$

$$\therefore c=1$$

Now the particular solution of $\frac{dy}{dx}=10x$, $y(0)=1$ is $y=5x^2+1$.

Therefore, the value of $y(3.5)=5\times(3.5)^2+1=62.25$ (Ans)

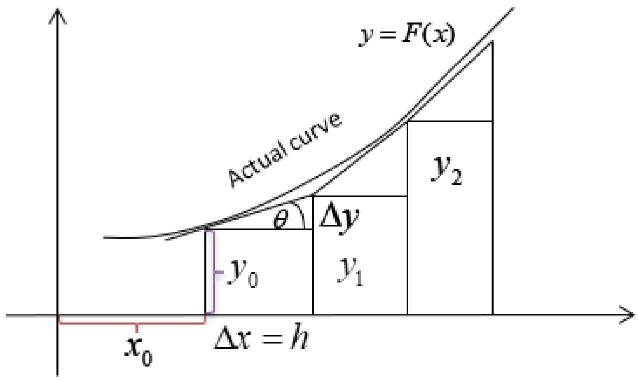
Euler's Method:

It is one of the oldest and simplest methods but also crudest and slowest. The use of Euler methods to solve the differential equation numerically is less efficient since it requires h to be very small for obtaining reasonable accuracy. If h is not small then the method is too inaccurate.

Derivation of Euler Method:

Let us consider a differential equation $\frac{dy}{dx}=f(x,y)$, $y(x_0)=y_0$ $\cdots\cdots\cdots\cdots(i)$.

Integrating (i) we get a relation between y and x which can be written in the form $y=F(x)$ .In the xy plane $y=F(x)$ represents a curve that shown in the following figure. Practically a smooth curve is straight for a short distance from any point.



Let $x_0, x_1, x_2, \ldots\ldots, x_n$ are equidistant value of x and the distance is h.

From the above figure we get,

$$\tan\theta \approx \frac{\Delta y}{\Delta x}$$

$$\therefore \Delta y \approx \Delta x \tan\theta$$

From the slope of a tangent to the curve $y = F(x)$ at (x_0, y_0) we have

$$\left(\frac{dy}{dx}\right)_{(x_0, y_0)} = f(x_0, y_0)$$

So from the above equation we find

$$\Delta y \approx \Delta x \left(\frac{dy}{dx}\right)_{(x_0, y_0)}$$

$$\Delta y \approx \Delta x \; f(x_0, y_0) = h\, f(x_0, y_0)$$

Now,

$$y_1 = y_0 + \Delta y$$

$$y_1 = y_0 + h\, f(x_0, y_0)$$

This y_1 is the approximate value of y for x_1. Simillarly we can write the followings approximations

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$y_2 = y_1 + h\, f(x_1, y_1)$$

....

....

....

In general, we obtain

$$y_{n+1} = y_n + h\, f(x_n, y_n) \quad where\ n = 0, 1, 2, \ldots\ldots, n$$

In this method actual solution curve is approximated by the sequence of short straight lines which some- times deviates from the solution curve significantly. All these considerations have led to a modification of Euler's method.

Problem: Solve the equation $\frac{dy}{dx} = 1 - y, y(0) = 0$ using Euler method and tabulate the solutions at $x = 0.1$,0.2.

Solution: Given IVP is $\frac{dy}{dx} = 1 - y, y(0) = 0$

Here, $f(x, y) = 1 - y,\ h = 0.1\ \&\ \ x_0 = 0, y_0 = 0$

Now, $x_1 = x_0 + h = 0 + 0.1 = 0.1 \quad \& \quad x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

Taking $n = 0$ in $y_{n+1} = y_n + h\ f\left(x_n, y_n\right)$ we get

$$y_1 = y_0 + h\ f\left(x_0, y_0\right) = 0 + h\ f\left(0, 0\right) = 0.1\left(1 - 0\right) = 0.1$$

Therefore, $y_1\left(0.1\right) = 0.1$.

Taking $n = 1$ in $y_{n+1} = y_n + h\ f\left(x_n, y_n\right)$ we get

$$y_2 = y_1 + h\ f\left(x_1, y_1\right) = 0.1 + 0.1\ f\left(0.1, 0.1\right) = 0.1 + 0.1\left(1 - 0.1\right) = 0.1 + 0.1 \times 0.9 = 0.1 + 0.11 = 0.19$$

Therefore, $y_2\left(0.2\right) = 0.19$ (As desired)

Try yourself:

1. Compute $y\left(0.2\right)$ by Euler's Method taking $h = 0.1$ in case of the equation $\frac{dy}{dx} = x^3 + y, y(0) = 1$.
2. Solve by Euler's Method the following differential equation $x = 0.1$ correct to four decimal places $\frac{dy}{dx} = \frac{y - x}{y + x}, y(0) = 1$.

Runge - Kutta method:

Consider an ordinary differential equation of first order first degree, $\frac{dy}{dx} = f(x, y)$ with the initial condition $y(x_0) = y_0$.

Let h be the wide length between equidistant values of x .

If $x_0,\ y_0$ denote the initial values then the first increment Δy in y is computed by,

$$y_1 = y_0 + \Delta y$$

$$or,\ y_1 = y_0 + \frac{1}{2}\left(k_1 + k_2\right) \quad \cdots \quad \cdots \quad \cdots (1)$$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf(x_0 + h, y_0 + k_1).$$

This is called second order Runge- Kutta method.

Again, if $x_0,\ y_0$ denote the initial values then the first increment Δy in y is computed by,

$$y_1 = y_0 + \Delta y$$

$$or,\ y_1 = y_0 + \frac{1}{6}\left(k_1 + 4k_2 + k_3\right) \quad \cdots \quad \cdots \quad \cdots (2)$$

Where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$

This is called third order Runge- Kutta method.

Again, if x_0, y_0 denote the initial values then the first increment Δy in y is computed by,

$$y_1 = y_0 + \Delta y$$

$$or,\ y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \cdots\ \cdots\ \cdots (3)$$

Where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

This is called fourth order Runge- Kutta method.

Problem-01: Find $y(0.1)$ & $y(0.2)$ by Runge-Kutta method of second, third & fourth order for the differential equation

$$\frac{dy}{dx} = -y\ ,\ y(0) = 1.$$

Solution: We have, $\frac{dy}{dx} = -y\ ,\ y(0) = 1.$

$$\therefore\ f(x, y) = -y,\ \ x_0 = 0\ ,\ y_0 = 1.$$

Let us take $h = 0.1$

(i). By second order Runge- Kutta method:

1st Step: For the 1st approximation, we have $x_0 = 0\ \&\ y_0 = 1$

$$\begin{aligned} k_1 &= hf(x_0\ ,\ y_0) \\ &= 0.1 \times (-1) \\ &= -0.1 \\ k_2 &= hf(x_0 + h\ ,\ y_0 + k_1) \\ &= 0.1 \times f(0 + 0.1\ ,\ 1 - 0.1) \\ &= 0.1 \times f(0.1\ ,\ 0.9) \\ &= 0.1 \times (-0.9) \\ &= -0.09 \end{aligned}$$

$$\begin{aligned} Now,\ y_1 &= y_0 + \frac{1}{2}(k_1 + k_2) \\ &= 1 + \frac{1}{2}(-0.1 - 0.09) \\ &= 1 - 0.095 \end{aligned}$$

$$\therefore\ y(0.1) = 0.905$$

(As desired)

2nd Step: For the 2nd approximation, we have $x_1 = 0.1\ \&\ y_1 = 0.905$

$$\begin{aligned} k_1 &= hf(x_1\ ,\ y_1) \\ &= 0.1 \times f(0.1\ ,\ 0.905) \end{aligned}$$

$$= 0.1 \times (-0.905)$$
$$= -0.0905$$
$$k_2 = hf(x_1 + h\ ,\ y_1 + k_1)$$
$$= 0.1 \times f(0.1 + 0.1\ ,\ 0.905 - 0.0905)$$
$$= 0.1 \times f(0.2\ ,\ 0.8145)$$
$$= 0.1 \times (-0.8145)$$
$$= -0.08145$$
$$Now,\ y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$
$$= 0.905 + \frac{1}{2}(-0.0905 - 0.08145)$$
$$= 0.905 - 0.08598$$
$$\therefore\ y(0.2) = 0.81902$$

(As desired)

(ii). By third order Runge- Kutta method:

1st Step: For the 1st approximation, we have $x_0 = 0$ & $y_0 = 1$

$$k_1 = hf(x_0\ ,\ y_0)$$
$$= 0.1 \times (-1)$$
$$= -0.1$$
$$k_2 = hf\left(x_0 + \frac{h}{2}\ ,\ y_0 + \frac{k_1}{2}\right)$$
$$= 0.1 \times f\left(0 + \frac{0.1}{2}\ , 1 - \frac{0.1}{2}\right)$$
$$= 0.1 \times f(0.05\ ,\ 0.95)$$
$$= 0.1 \times (-0.95)$$
$$= -0.095$$
$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$
$$= 0.1 \times f\left(0 + 0.1\ , 1 + 2(-0.095) - (-0.1)\right)$$
$$= 0.1 \times f(0.1\ ,\ 0.91)$$
$$= 0.1 \times (-0.91)$$
$$= -0.091$$
$$Now,\ y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$= 1 + \frac{1}{6}\left(-0.1 + 4(-0.095) + (-0.091)\right)$$
$$= 1 - 0.09517$$
$$\therefore\ y(0.1) = 0.90483$$

(As desired)

2nd Step: For the 2nd approximation, we have $x_1 = 0.1$ & $y_1 = 0.90483$

$$k_1 = hf(x_1\ ,\ y_1)$$

$$= 0.1\times f(0.1\ ,\ 0.90483)$$
$$= 0.1\times(-0.90483)$$
$$= -0.090483$$
$$k_2 = hf\left(x_1+\frac{h}{2}\ ,\ y_1+\frac{k_1}{2}\right)$$
$$= 0.1\times f\left(0.1+\frac{0.1}{2}\ ,\ 0.90483-\frac{0.090483}{2}\right)$$
$$= 0.1\times f(0.15\ ,\ 0.8596)$$
$$= 0.1\times(-0.8596)$$
$$= -0.08596$$
$$k_3 = hf(x_1+h, y_1+2k_2-k_1)$$
$$= 0.1\times f\left(0.1+0.1,\ 0.90483+2(-0.08596)-(-0.090483)\right)$$
$$= 0.1\times f(0.2\ ,\ 0.8234)$$
$$= 0.1\times(-0.8234)$$
$$= -0.08234$$
$$Now,\ y_2 = y_1+\frac{1}{6}\left(k_1+4k_2+k_3\right)$$
$$= 0.90483+\frac{1}{6}\left(-0.090483+4(-0.08596)+(-0.08234)\right)$$
$$= 0.90483-0.08611$$
$$= 0.81872$$
$$\therefore\ y(0.2) = 0.81872$$

(As desired)

(iii). By fourth order Runge- Kutta method:

1st Step: For the 1st approximation, we have $x_0 = 0\ \&\ y_0 = 1$

$$k_1 = hf(x_0\ ,\ y_0)$$
$$= 0.1\times(-1)$$
$$= -0.1$$
$$k_2 = hf\left(x_0+\frac{h}{2}\ ,\ y_0+\frac{k_1}{2}\right)$$
$$= 0.1\times f\left(0+\frac{0.1}{2}\ ,\ 1-\frac{0.1}{2}\right)$$
$$= 0.1\times f(0.05\ ,\ 0.95)$$
$$= 0.1\times(-0.95)$$
$$= -0.095$$
$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$
$$= 0.1\times f\left(0+\frac{0.1}{2}\ ,\ 1-\frac{0.095}{2}\right)$$

$$= 0.1\times f(0.05\ ,\ 0.9525)$$
$$= 0.1\times(-0.9525)$$
$$= -0.09525$$
$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$
$$= 0.1\times f\left(0+0.1\ ,\ 1-0.09525\right)$$
$$= 0.1\times f(0.1\ ,\ 0.9048)$$
$$= 0.1\times(-0.9048)$$
$$= -0.09048$$
$$Now,\ y_1 = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= 1 + \frac{1}{6}\left(-0.1 + 2(-0.095) + 2(-0.09525) - 0.09048\right)$$
$$= 1 - 0.09516$$
$$\therefore\ y(0.1) = 0.90484$$

(As desired)

2nd Step: For the 2nd approximation, we have $x_1 = 0.1\ \&\ y_1 = 0.90484$

$$k_1 = hf(x_1\ ,\ y_1)$$
$$= 0.1\times f(0.1\ ,\ 0.90484)$$
$$= 0.1\times(-0.90484)$$
$$= -0.090484$$
$$k_2 = hf\left(x_1 + \frac{h}{2}\ ,\ y_1 + \frac{k_1}{2}\right)$$
$$= 0.1\times f\left(0.1 + \frac{0.1}{2}\ ,\ 0.90484 - \frac{0.090484}{2}\right)$$
$$= 0.1\times f(0.15\ ,\ 0.8596)$$
$$= 0.1\times(-0.8596)$$
$$= -0.08596$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.1\times f\left(0.1 + \frac{0.1}{2}\ ,\ 0.90484 - \frac{0.08596}{2}\right)$$
$$= 0.1\times f(0.15\ ,\ 0.8619)$$
$$= 0.1\times(-0.8619)$$
$$= -0.08619$$
$$k_4 = hf\left(x_1 + h, y_1 + k_3\right)$$
$$= 0.1\times f\left(0.1+0.1\ ,\ 0.90484 - 0.08619\right)$$
$$= 0.1\times f(0.2\ ,\ 0.8187)$$
$$= 0.1\times(-0.8187)$$
$$= -0.08187$$

$$Now,\ y_2 = y_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= 0.90484 + \frac{1}{6}\left(-0.090484 + 2(-0.08596) + 2(-0.08619) - 0.08187\right)$$
$$= 0.90484 - 0.08611$$
$$= 0.81873$$
$$\therefore\ y(0.2) = 0.81873$$

(As desired)

Problem-02: Find $y(0.3)$ by Runge-Kutta method of fourth order for the differential equation $\frac{dy}{dx} + y + xy^2 = 0; y(0) = 1$.

Solution: We have, $\frac{dy}{dx} + y + xy^2 = 0; y(0) = 1$.

$$\therefore\ f(x, y) = -y - xy^2,\ x_0 = 0,\ y_0 = 1.$$

Let us take $h = 0.1$.

By fourth order Runge- Kutta method:

1st Step: For the 1st approximation, we have $x_0 = 0\ \&\ y_0 = 1$

$$k_1 = hf(x_0\ ,\ y_0)$$
$$= 0.1 \times (-1 - 0 \times 1^2)$$
$$= -0.1$$
$$k_2 = hf\left(x_0 + \frac{h}{2}\ ,\ y_0 + \frac{k_1}{2}\right)$$
$$= 0.1 \times f\left(0 + \frac{0.1}{2}\ , 1 - \frac{0.1}{2}\right)$$
$$= 0.1 \times f(0.05\ ,\ 0.95)$$
$$= 0.1 \times \left(-0.95 - 0.05 \times (0.95)^2\right)$$
$$= -0.09951$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1 \times f\left(0 + \frac{0.1}{2}\ , 1 - \frac{0.09951}{2}\right)$$
$$= 0.1 \times f(0.05\ ,\ 0.95025)$$
$$= 0.1 \times \left(-0.95025 - 0.05 \times (0.95025)^2\right)$$
$$= -0.09954$$
$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$
$$= 0.1 \times f\left(0 + 0.1\ , 1 - 0.09954\right)$$
$$= 0.1 \times f(0.1\ ,\ 0.90046)$$
$$= 0.1 \times \left(-0.90046 - 0.1 \times (0.90046)^2\right)$$
$$= -0.09815$$

$$Now,\ y_1 = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= 1 + \frac{1}{6}\left(-0.1 + 2(-0.09951) + 2(-0.09954) - 0.09815\right)$$

$$= 1 - 0.09938$$

$$\therefore\ y(0.1) = 0.90062$$

(As desired)

2nd Step: For the 2nd approximation, we have $x_1 = 0.1\ \&\ y_1 = 0.90062$

$$k_1 = hf(x_1\ ,\ y_1)$$

$$= 0.1 \times f(0.1\ ,\ 0.90062)$$

$$= 0.1 \times \left(-0.90062 - 0.1 \times (0.90062)^2\right)$$

$$= -0.0982$$

$$k_2 = hf\left(x_1 + \frac{h}{2}\ ,\ y_1 + \frac{k_1}{2}\right)$$

$$= 0.1 \times f\left(0.1 + \frac{0.1}{2}\ ,\ 0.90062 - \frac{0.0982}{2}\right)$$

$$= 0.1 \times f(0.15\ ,\ 0.8515)$$

$$= 0.1 \times \left(-0.8515 - 0.15 \times (0.8515)^2\right)$$

$$= -0.096$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1 \times f\left(0.1 + \frac{0.1}{2}\ ,\ 0.90062 - \frac{0.096}{2}\right)$$

$$= 0.1 \times f(0.15\ ,\ 0.8526)$$

$$= 0.1 \times \left(-0.8526 - 0.15 \times (0.8526)^2\right)$$

$$= -0.0962$$

$$k_4 = hf\left(x_1 + h, y_1 + k_3\right)$$

$$= 0.1 \times f\left(0.1 + 0.1\ ,\ 0.90062 - 0.0962\right)$$

$$= 0.1 \times f(0.2\ ,\ 0.8044)$$

$$= 0.1 \times \left(-0.8044 - 0.2 \times (0.8044)^2\right)$$

$$= -0.0934$$

$$Now,\ y_2 = y_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= 0.90062 + \frac{1}{6}\left(-0.0982 + 2(-0.096) + 2(-0.0962) - 0.0934\right)$$

$$= 0.90062 - 0.096$$

$$\therefore\ y(0.2) = 0.80462$$

(As desired)

3rd Step: For the 3rd approximation, we have $x_2 = 0.2 \;\&\; y_2 = 0.80462$

$$k_1 = hf(x_2\ ,\ y_2)$$
$$= 0.1\times f(0.2\ ,\ 0.80462)$$
$$= 0.1\times\left(-0.80462 - 0.2\times(0.80462)^2\right)$$
$$= -0.0934$$
$$k_2 = hf\left(x_2 + \frac{h}{2}\ ,\ y_2 + \frac{k_1}{2}\right)$$
$$= 0.1\times f\left(0.2 + \frac{0.1}{2}\ ,\ 0.80462 - \frac{0.0934}{2}\right)$$
$$= 0.1\times f(0.25\ ,\ 0.7579)$$
$$= 0.1\times\left(-0.7579 - 0.25\times(0.7579)^2\right)$$
$$= -0.0902$$
$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$
$$= 0.1\times f\left(0.2 + \frac{0.1}{2}\ ,\ 0.80462 - \frac{0.0902}{2}\right)$$
$$= 0.1\times f(0.25\ ,\ 0.7595)$$
$$= 0.1\times\left(-0.7595 - 0.25\times(0.7595)^2\right)$$
$$= -0.0904$$
$$k_4 = hf\left(x_2 + h, y_2 + k_3\right)$$
$$= 0.1\times f\left(0.2 + 0.1\ ,\ 0.80462 - 0.0904\right)$$
$$= 0.1\times f(0.3\ ,\ 0.7142)$$
$$= 0.1\times\left(-0.7142 - 0.3\times(0.7142)^2\right)$$
$$= -0.0867$$
$$Now,\ y_3 = y_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= 0.80462 + \frac{1}{6}\left(-0.0934 + 2(-0.0902) + 2(-0.0904) - 0.0867\right)$$
$$= 0.80462 - 0.09022$$
$$\therefore\ y(0.3) = 0.7144$$

(As desired)

Problem-03: Find $y(0.1)$ by Runge-Kutta method of fourth order for the differential equation $\frac{dy}{dx} = \frac{1}{x+y}; y(0) = 1$.

Solution: We have, $\frac{dy}{dx} = \frac{1}{x+y}; y(0) = 1$.

$$\therefore\ f(x,y) = \frac{1}{x+y},\ \ x_0 = 0\ ,\ y_0 = 1.$$

Let us take $h = 0.1$.

By fourth order Runge- Kutta method: we have $x_0 = 0$ & $y_0 = 1$

$$k_1 = hf(x_0\ ,\ y_0)$$
$$= 0.1 \times \left(\frac{1}{0+1}\right)$$
$$= 0.1$$
$$k_2 = hf\left(x_0 + \frac{h}{2}\ ,\ y_0 + \frac{k_1}{2}\right)$$
$$= 0.1 \times f\left(0 + \frac{0.1}{2}\ ,\ 1 + \frac{0.1}{2}\right)$$
$$= 0.1 \times f(0.05\ ,\ 1.05)$$
$$= 0.1 \times \left(\frac{1}{0.05 + 1.05}\right)$$
$$= 0.09091$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1 \times f\left(0 + \frac{0.1}{2}\ ,\ 1 + \frac{0.09091}{2}\right)$$
$$= 0.1 \times f(0.05\ ,\ 1.0455)$$
$$= 0.1 \times \left(\frac{1}{0.05 + 1.04545}\right)$$
$$= 0.09129$$
$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$
$$= 0.1 \times f\left(0 + 0.1\ ,\ 1 + 0.09129\right)$$
$$= 0.1 \times f(0.1\ ,\ 1.09129)$$
$$= 0.1 \times \left(\frac{1}{0.1 + 1.09129}\right)$$
$$= 0.08394$$
$$Now,\ y_1 = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= 1 + \frac{1}{6}\left(0.1 + 2 \times 0.09091 + 2 \times 0.09129 + 0.08394\right)$$
$$= 1 + 0.09139$$
$$\therefore\ y(0.1) = 1.09139$$

(As desired)

Exercise:

Problem-01: Find $y(0.2)$ by Runge-Kutta method of fourth order for the differential equation $\frac{dy}{dx}+\frac{y}{x}-\frac{1}{x^2}=0; y(1)=1$.

Problem-02: Find $y(1)$ & $y(2)$ by Runge-Kutta method of second order for the differential equation $\frac{dy}{dx}=\frac{y-x}{y+x}; y(0)=1$.

.

Problem-03: If $\frac{dy}{dx}=y^2+1$with $y(0)=0$, then find $y(0.2), y(0.4)$ & $y(0.6)$ by using Runge-Kutta fourth order method.

Problem-04: If $\frac{dy}{dx}=\frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$, then find $y(0.4)$ by using Runge-Kutta fourth order method.

Problem-05: If $\frac{dy}{dx}=x+y$ with $y(1)=1$, then find $y(1.5)$ by using Runge-Kutta third order method.

Problem-06: If $\frac{dy}{dx}=xy+y^2$ with $y(0)=1$, then find $y(0.2)$ & $y(0.3)$ by using Runge-Kutta fourth order method.

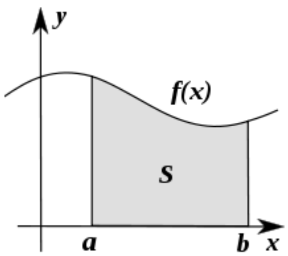
NUMERICAL INTEGRATION

CH 06

There are two main reasons for you to need to do numerical integration: analytical integration may be impossible or infeasible, or you may wish to integrate tabulated data rather than known functions. In this section, we outline the main approaches to numerical integration.
Numerical integration is the approximate computation of integral using numerical techniques. The numerical computation of an integral is sometimes called quadrature. Therefore, the basic problem in numerical integration is to compute an approximate value to a definite integral

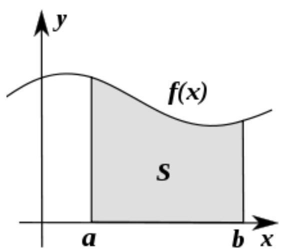
$$\int_a^b f(x)\,dx$$

to a given degree of accuracy.



Numerical Integration consists of finding numerical approximations for the value S.

Meaning of $\int_a^b f(x)dx$:



The definite integral $\int_a^b f(x)dx = \int_a^b y\,dx$ represents the area between the curve $y = f(x)$,the x-axis and the lines $x = a$and $x = b$.

Numerical Integration:

Numerical integration is the process by which we can find the value of definite integral $\int_a^b f(x)dx$ numerically by using some well-established formulae or rules. The exact value of a definite integral $\int_a^b f(x)dx$ can be computed only when the function $f(x)$ is integrable in finite terms ,whenever the function $y = f(x)$ cannot be exactly integrated in finite terms or the evaluation of its integral is too cumbersome, integration can be more conveniently performed by numerical method. Various methods have been derived to find the above area approximately, in this case when $f(x)$ is not easily integrable. Hence these methods of approximating an area are essential methods for approximating a definite integral. The developed approximating methods are as follows:

- **i) Trapezoidal Rule**
- **ii) Simpson's $\frac{1}{3}$ Rule**
- **iii) Simpson's $\frac{3}{8}$ Rule**
- **iv) Boole's Rule**
- **v) Weddle's Rule**
- **vi) Romberg's Integration Rule etc.**

General Formula for Numerical Integration:

Let us consider an integral $\int_a^b f(x)dx$ where $f(x)$ is continuous on [a,b] and be given for certain equidistant values of x. Our intention here is to find the approximate value of the definite integral $\int_a^b f(x)dx$.

Assume the partition of [a,b] with equal distance h is

Partition, $P: x_0 = a, x_1 = a+h, x_2 = a+2h, \ldots\ldots\ldots\ldots\ldots, x_n = a+nh = b$

That means $y_0, y_1, y_2, \ldots\ldots\ldots\ldots\ldots, y_n$ be a set of (n+1) values of the function $y = f(x)$ corresponding to the equidistant values $x_0, x_1, x_2, \ldots\ldots\ldots\ldots\ldots, x_n$ of the independent variable x.

Here $x_n = x_0 + nh \Rightarrow h = \dfrac{x_n - x_0}{n} = \dfrac{b-a}{n}$, where a is a lower bound of the interval [a,b] and where b is the upper bound of the interval [a,b] and n is the number of intervals.

Now,

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (i)$$

From Newton's Forward Interpolation formula, we have,

$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ where

$$u = \frac{x - x_0}{h} \Rightarrow x = x_0 + uh \qquad \therefore dx = h\,du$$

Limit Change:

When $x = x_0$ then $u = 0$

When $x = x_n$ then $u = n$

Therefore, above equation (i) takes the form,

$$\int_a^b f(x)\,dx = \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + upto(n+1)terms \right] h\,du$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + upto(n+1)terms \right] du$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{(u^2-u)}{2!}\Delta^2 y_0 + \frac{(u^2-u)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + upto(n+1)terms \right] du$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{(u^2-u)}{2!}\Delta^2 y_0 + \frac{(u^3-3u^2+2u)}{3!}\Delta^3 y_0 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + upto(n+1)terms \right] du$$

$$= h\left[y_0 u + \frac{u^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{u^4}{4} - u^3 + u^2\right)\Delta^3 y_0 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + upto(n+1)terms \right]_0^n$$

$$= h\left(ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + upto(n+1)terms \right)$$

This Formula is known as general quadrature formula or General formula for numerical integration and also known as General Gauss -Legendre integration formula for equidistant ordinates.

Note:

1. This formula is used to compute $\int_a^b f(x)dx$
2. Putting $n = 1$ in above equation we obtain Trapezoidal rule
3. Putting $n = 2$ in above equation we obtain Simpson's $\dfrac{1}{3}$ Rule
4. Putting $n = 3$ in above equation we obtain Simpson's $\dfrac{3}{8}$ Rule

5. Putting $n = 4$ in above equation we obtain Boole's Rule
6. Putting $n = 6$ in above equation we obtain Weddle's Rule

Trapezoidal Rule:

The general integration formula is

$$\int_{x_0}^{x_n} f(x)\,dx = h\left(ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\cdots + upto\,(n+1)terms\right)$$

Setting $n = 1$ in above equation we have the interval $[x_0, x_1]$ and neglecting the higher order differences more than one, we get

$$\int_{x_0}^{x_1} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right)$$

$$\int_{x_0}^{x_1} f(x)\,dx = h\left(y_0 + \frac{1}{2}(y_1 - y_0)\right)$$

$$\int_{x_0}^{x_1} f(x)\,dx = h\left(y_0 + \frac{1}{2}y_1 - \frac{1}{2}y_0\right)$$

$$\int_{x_0}^{x_1} f(x)\,dx = h\left(\frac{1}{2}y_0 + \frac{1}{2}y_1\right)$$

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(y_0 + y_1)$$

Similarly, we can get,

$$\int_{x_1}^{x_2} f(x)\,dx = \frac{h}{2}(y_1 + y_2)$$

$$\int_{x_2}^{x_3} f(x)\,dx = \frac{h}{2}(y_2 + y_3)$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\int_{x_{n-1}}^{x_n} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

By the rule of definite integral, we can write

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \cdots\cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \cdots\cdots + \frac{h}{2}(y_{n-1} + y_n)$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \cdots\cdots + y_{n-1} + y_n)$$

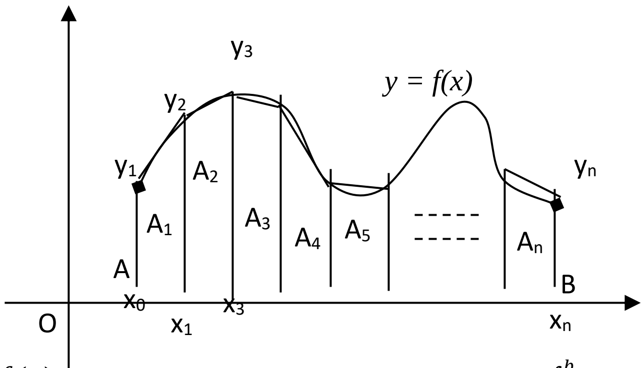
$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots\cdots + y_{n-1})]$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{k=1}^{n-1} y_k\right]$$

$$\int_{a}^{b} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{k=1}^{n-1} y_k\right]$$

The above formula is known as the trapezoidal rule for numerical integration.

Geometrical Proof:



Let $y = f(x)$ is a continuous curve on [a,b] and we have to find $\int_a^b f(x)dx$.Here $OA = a$ and $OB = b$.

Therefore $AB = OB - OA = b - a$ .Now divide the line segment AB into n equal parts with distance h so that $h = \frac{b-a}{n}$ $[say]$.

Then the area,

$$A_1 = \frac{1}{2}(y_0 + y_1)\times h = \frac{h}{2}(y_0 + y_1)$$

Similarly, we find,

$$A_2 = \frac{1}{2}(y_1 + y_2)\times h = \frac{h}{2}(y_1 + y_2)$$

$$A_3 = \frac{1}{2}(y_2 + y_3)\times h = \frac{h}{2}(y_2 + y_3)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_n = \frac{1}{2}(y_{n-1} + y_n)\times h = \frac{h}{2}(y_{n-1} + y_n)$$

Now,

$$\int_a^b f(x)\,dx = A_1 + A_2 + A_3 + \cdots\cdots\cdots\cdots\cdots + A_n$$

$$\int_a^b f(x)\,dx = \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \cdots\cdots\cdots\cdots\cdots + \frac{h}{2}(y_{n-1} + y_n)$$

$$\int_a^b f(x)\,dx = \frac{h}{2}(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \cdots\cdots\cdots\cdots + y_{n-1} + y_n)$$

$$\int_a^b f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots\cdots\cdots + y_{n-1})]$$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{k=1}^{n-1} y_k\right]$$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{k=1}^{n-1} y_k\right]$$

This is the well-known trapezoidal rule, so called because it approximates the integral by the sum of n trapezoidals. If the number of points of the line segments AB be increased a better approximation to the area well be obtained.

Simpson's $\frac{1}{3}$ Rule:

The general integration formula is

$$\int_{x_0}^{x_n} f(x)\,dx = h\left(ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\cdots\cdots + upto\,(n+1)terms\right)$$

Setting $n = 2$ in above equation we have the interval $[x_0, x_2]$ and neglecting the higher order differences more than two, we get

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + \frac{2^2}{2}\Delta y_0 + \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\frac{\Delta^2 y_0}{2!}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2\Delta y_0 + \left(\frac{8}{3} - 2\right)\frac{\Delta^2 y_0}{2}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2\Delta y_0 + \left(\frac{8-6}{3}\right)\frac{\Delta^2 y_0}{2}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2\Delta y_0 + \left(\frac{2}{3}\right)\frac{\Delta^2 y_0}{2}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2\Delta y_0 + \frac{\Delta^2 y_0}{3}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2(y_1 - y_0) + \frac{1}{3}\Delta(\Delta y_0)\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2(y_1 - y_0) + \frac{1}{3}\Delta(y_1 - y_0)\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2(y_1 - y_0) + \frac{1}{3}(\Delta y_1 - \Delta y_0)\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2(y_1 - y_0) + \frac{1}{3}\{(y_2 - y_1) - (y_1 - y_0)\}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2y_1 - 2y_0 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left(6y_0 + 6y_1 - 6y_0 + (y_2 - 2y_1 + y_0)\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Similarly, we can write,

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$$

$$\int_{x_4}^{x_6} f(x)\,dx = \frac{h}{3}\left(y_4 + 4y_5 + y_6\right)$$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

By the rule of definite integral, we can write

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \int_{x_4}^{x_6} f(x)\,dx + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) + \frac{h}{3}\left(y_2 + 4y_3 + y_4\right) + \frac{h}{3}\left(y_4 + 4y_5 + y_6\right) + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + y_{n-2} + 4y_{n-1} + y_n\right]$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + y_{n-2})\right]$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4\sum_{k=1,3,5,\ldots\ldots}^{n-1} y_k + 2\sum_{k=2,4,6,\ldots\ldots}^{n-2} y_k\right]$$

$$\int_{a}^{b} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4\sum_{k=1,3,5,\ldots\ldots}^{n-1} y_k + 2\sum_{k=2,4,6,\ldots\ldots}^{n-2} y_k\right]$$

This is the Simpson's one third rule for numerical integration.

Note:

This formula is used only when the number of partitions of the interval of integration is even.

Simpson's $\frac{3}{8}$ Rule:

The general integration formula is

$$\int_{x_0}^{x_n} f(x)\,dx = h\left(ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\cdots + upto\,(n+1)terms \right)$$

Setting $n=3$ in above equation we have the interval $[x_0, x_3]$ and neglecting the higher order differences more than three, we get

$$\int_{x_0}^{x_3} f(x)\,dx = h\left(3y_0 + \frac{3^2}{2}\Delta y_0 + \left(\frac{3^3}{3} - \frac{3^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{3^4}{4} - 3^3 + 3^2\right)\frac{\Delta^3 y_0}{3!} \right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = h\left(3y_0 + \frac{9}{2}\Delta y_0 + \left(9 - \frac{9}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{6} \right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = h\left(3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \left(\frac{81}{4} - 18\right)\frac{\Delta^3 y_0}{6} \right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = h\left(3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \left(\frac{81}{4} - 18\right)\frac{\Delta^3 y_0}{6} \right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = h\left(3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \frac{3}{8}\Delta^3 y_0 \right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12\Delta y_0 + 6\Delta^2 y_0 + \Delta^3 y_0\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12\Delta y_0 + 6\Delta^2 y_0 + \Delta^3 y_0\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6\Delta(y_1 - y_0) + \Delta^2(y_1 - y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6(\Delta y_1 - \Delta y_0) + \Delta(\Delta y_1 - \Delta y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6(y_2 - y_1 - y_1 + y_0) + \Delta(y_2 - y_1 - y_1 + y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + \Delta(y_2 - 2y_1 + y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (\Delta y_2 - 2\Delta y_1 + \Delta y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - y_2 - 2(y_2 - y_1) + y_1 - y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12y_1 - 12y_0 + 6y_2 - 12y_1 + 6y_0 + \left(y_3 - y_2 - 2y_2 + 2y_1 + y_1 - y_0\right)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12y_1 - 12y_0 + 6y_2 - 12y_1 + 6y_0 + \left(y_3 - 3y_2 + 3y_1 - y_0\right)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12y_1 - 12y_0 + 6y_2 - 12y_1 + 6y_0 + y_3 - 3y_2 + 3y_1 - y_0\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right)$$

Similarly, we can write,

$$\int_{x_3}^{x_6} f(x)\,dx = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

$$\int_{x_6}^{x_9} f(x)\,dx = \frac{3h}{8}\left(y_6 + 3y_7 + 3y_8 + y_9\right)$$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$$\int_{x_{n-3}}^{x_n} f(x)\,dx = \frac{3h}{8}\left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)$$

By the rule of definite integral, we can write

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \int_{x_6}^{x_9} f(x)\,dx + \cdots\cdots\cdots\cdots\cdots\cdots + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right) + \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right) + \frac{3h}{8}\left(y_6 + 3y_7 + 3y_8 + y_9\right) + \cdots\cdots\cdots\cdots\cdots\cdots + \frac{3h}{8}\left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left\{\left(y_0 + 3y_1 + 3y_2 + y_3\right) + \left(y_3 + 3y_4 + 3y_5 + y_6\right) + \left(y_6 + 3y_7 + 3y_8 + y_9\right) + \cdots\cdots\cdots\cdots\cdots\cdots + \left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)\right\}$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left\{y_0 + 3y_1 + 3y_2 + y_3 + y_3 + 3y_4 + 3y_5 + y_6 + y_6 + 3y_7 + 3y_8 + y_9 + \cdots\cdots\cdots\cdots\cdots\cdots + y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right\}$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left\{\left(y_0 + y_n\right) + 3\left(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 + \cdots\cdots\cdots + y_{n-2} + y_{n-1}\right) + 2\left(y_3 + y_6 + y_9 + \cdots\cdots\cdots\cdots + y_{n-3}\right)\right\}$$

$$\int_{a}^{b} f(x)\,dx = \frac{3h}{8}\left\{\left(y_0 + y_n\right) + 3\sum_{\substack{k \neq 3,6,9,\ldots \\ k=1}}^{n-1} y_k + 2\sum_{k=3,6,9,\ldots\ldots}^{n-3} y_k\right\}$$

This is the Simpson's three-eight rule for integration.

Note: This formula is used only when the number of partitions of the interval of integration is a multiple of the number 3.

Similarly, we can derive Boole's rule and Weddle's rule for numerical Integration for $n=4$ and $n=6$ respectively as follows:

Weddle's Rule:

$$\int_a^b f(x)\,dx = \frac{3h}{10}\left\{ \sum_{k=0,2,4,6,\ldots\ldots}^{n} y_k + 5\sum_{k=1,3,5,\ldots}^{n-1} y_k + \sum_{k=3,6,9\ldots}^{n-3} y_k \right\}$$

Note:

1.This formula requires at least seven consecutive values of the function.

2. This formula is used only when the number of partitions of the interval of integration is a multiple of the number 6.

Mathematical Problems

1.Compute the definite integral $\int_{0.2}^{1.4} \left(\sin x - \ln x + e^x\right)\,dx$ by using various rules using 6 equidistant subintervals correct up to three decimal places.

Solution:

Here upper limit is $b = 1.4$, lower limit is $a = 0.2$ and No. of subintervals $n = 6$ and also $y = f(x) = \sin x - \ln x + e^x$.

Now,

$$h = \frac{1.4 - 0.2}{6} = 0.2$$

The values of the function y at each subinterval are given in the tabular form:

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y	3.0295	2.7975	2.8975	3.1660	3.5597	4.0698	4.7041

- **Trapezoidal Rule:**

  We know that

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\sum_{k=1}^{n-1} y_k\right]$$

Now for $n = 6$ the above formula reduces to the following form,

$$\int_a^b f(x)\,dx = \frac{0.2}{2}\left[\left(y_0 + y_6\right) + 2\sum_{k=1}^{5} y_k\right]$$

$$\int_a^b f(x)\,dx = \frac{0.2}{2}\left[\left(y_0 + y_6\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5\right)\right]$$

$$\int_a^b f(x)\,dx = \frac{0.2}{2}\left[\left(3.0295 + 4.7041\right) + 2\left(2.7975 + 2.8975 + 3.1660 + 3.5597 + 4.0698\right)\right]$$

$$\int_a^b f(x)dx = \frac{0.2}{2}\left[7.7326 + 32.981\right]$$

$$\int_a^b f(x)dx = \frac{0.2}{2} \times 40.7136$$

$$\int_{0.2}^{1.4}\left(\sin x+\ln x-e^{x}\right)dx=4.07136$$

(As desired)

□ **Simpson's $\frac{1}{3}$ Rule:**

We know that

$$\int_{a}^{b} f(x)\,dx=\frac{h}{3}\left[\left(y_0+y_n\right)+4\sum_{k=1,3,5,\ldots\ldots}^{n-1} y_k+2\sum_{k=2,4,6,\ldots\ldots}^{n-2} y_k\right]$$

Now for $n=6$ the above formula reduces to the following form,

$$\int_{a}^{b} f(x)\,dx=\frac{0.2}{3}\left[\left(y_0+y_6\right)+4\sum_{k=1,3,5,\ldots\ldots}^{5} y_k+2\sum_{k=2,4,6,\ldots\ldots}^{4} y_k\right]$$

$$\int_{a}^{b} f(x)dx=\frac{0.2}{3}\left[\left(y_0+y_6\right)+4\left(y_1+y_3+y_5\right)+2\left(y_2+y_4\right)\right]$$

$$\int_{a}^{b} f(x)dx=\frac{0.2}{3}\left[\left(3.0295+4.7041\right)+4\left(2.7975+3.1660+4.0698\right)+2\left(2.8975+3.5597\right)\right]$$

$$\int_{0.2}^{1.4}\left(\sin x+\ln x-e^{x}\right)dx=\frac{0.2}{3}\left[7.7336+40.1332+12.9144\right]=4.05208$$

(As desired)

□ **Simpson's $\frac{3}{8}$ Rule:**

We know that

$$\int_{a}^{b} f(x)\,dx=\frac{3h}{8}\left\{\left(y_0+y_n\right)+3\sum_{\substack{k\neq 3,6,9,\ldots\\ k=1}}^{n-1} y_k+2\sum_{k=3,6,9,\ldots\ldots}^{n-3} y_k\right\}$$

Now for $n=6$ the above formula reduces to the following form,

$$\int_{a}^{b} f(x)\,dx=\frac{3\times 0.2}{8}\left\{\left(y_0+y_6\right)+3\sum_{\substack{k\neq 3,6,9,\ldots\\ k=1}}^{5} y_k+2\sum_{k=3,6,9,\ldots\ldots}^{3} y_k\right\}$$

$$\int_{a}^{b} f(x)\,dx=\frac{3\times 0.2}{8}\left\{\left(y_0+y_6\right)+3\left(y_1+y_2+y_4+y_5\right)+2y_3\right\}$$

$$\int_{a}^{b} f(x)\,dx=\frac{3\times 0.2}{8}\left\{\left(3.0295+4.7041\right)+3\left(2.7975+2.8975+3.5597+4.0698\right)+2\times 3.1660\right\}$$

$$\int_{a}^{b} f(x)\,dx=\frac{3\times 0.2}{8}\left\{7.7336+3\times 13.3245+2\times 3.1660\right\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{8}\times 54.0391$$

$$\int_{0.2}^{1.4}\left(\sin x+\ln x-e^x\right)dx = \frac{3\times 0.2}{8}\times 54.0391 = 4.0529$$

(As desired)

□ **Weddle's Rule:**

We know that

$$\int_a^b f(x)\,dx = \frac{3h}{10}\left\{\sum_{k=0,2,4,6,......}^{n} y_k + 5\sum_{k=1,3,5,...}^{n-1} y_k + \sum_{k=3,6,9...}^{n-3} y_k\right\}$$

Now for $n=6$ the above formula reduces to the following form,

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{10}\left\{\sum_{k=0,2,4,6,......}^{6} y_k + 5\sum_{k=1,3,5,...}^{5} y_k + \sum_{k=3,6,9...}^{3} y_k\right\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{10}\left\{y_0+y_2+y_4+y_6+5(y_1+y_3+y_5)+y_3\right\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{10}\left\{3.0295+2.8975+3.5597+4.7041+5(2.7975+3.1660+4.0698)+3.1660\right\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{10}\left\{14.1908+5\times 10.0333+3.1660\right\}$$

$$\int_{0.2}^{1.4}\left(\sin x+\ln x-e^x\right)dx = \frac{3\times 0.2}{10}\times 67.5233 = 4.051398$$

(As desired)

2.Evaluate $\int_0^6 f(x)\,dx$ by using trapezoidal rule where the values of $f(x)$ are given by the following table:

x	0	1	2	3	4	5	6
Y=f(x)	0.146	0.161	0.176	0.190	0.204	0.217	0.230

Solution:

Here upper limit is $b=6$, lower limit is $a=0$ and No. of subintervals $n=6$.

Now,

$$h=\frac{6-0}{6}=1$$

The values of the function y at each subinterval are given in the tabular form:

x	0	1	2	3	4	5	6
Y=f(x)	0.146	0.161	0.176	0.190	0.204	0.217	0.230

From trapezoidal rule we have

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{k=1}^{n-1} y_k\right]$$

Now for $n=6$ the above formula reduces to the following form,

$$\int_a^b f(x)\,dx = \frac{1}{2}\left[(y_0 + y_6) + 2\sum_{k=1}^{5} y_k\right]$$

$$\int_a^b f(x)\,dx = \frac{1}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$\int_a^b f(x)\,dx = \frac{1}{2}\left[(0.146 + 0.230) + 2(0.161 + 0.176 + 0.190 + 0.204 + 0.217)\right]$$

$$\int_0^6 f(x)\,dx = 1.136$$

(As desired)

3. Compute $\int_1^2 x^2 dx$ by Simpson's one third rule and compare with exact value.

Solution:

Here upper limit is $b=2$, lower limit is $a=1$ and No. of subintervals $n=4$ and also $f(x)=x^2$.

Now,

$$h = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

The values of the function y at each subinterval are given in the tabular form:

x	1	1.25	1.50	1.75	2
Y=f(x)	1	1.5625	2.25	3.0625	4

From Simpson's $\frac{1}{3}$ Rule we have,

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4\sum_{k=1,3,5,\ldots\ldots}^{n-1} y_k + 2\sum_{k=2,4,6,\ldots\ldots}^{n-2} y_k\right]$$

Now for $n=4$ the above formula reduces to the following form,

$$\int_a^b f(x)\,dx = \frac{0.25}{3}\left[(y_0 + y_n) + 4\sum_{k=1,3,5,\ldots\ldots}^{3} y_k + 2\sum_{k=2,4,6,\ldots\ldots}^{2} y_k\right]$$

$$\int_a^b f(x)\,dx = \frac{0.25}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right]$$

$$\int_a^b f(x)\,dx = \frac{0.25}{3}\left[(1+4) + 4(1.5625 + 3.0625) + 2\times 2.25\right]$$

$$\int_1^2 x^2\,dx = \frac{7}{3}$$

Now exact value is $\int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{1}{3}(2^3 - 1^3) = \frac{7}{3}$

It is shown that exact result and Simpson's $\frac{1}{3}$ Rule's result are exactly same so there is no error between two results.

(As desired)

4. Determine $\int_4^{5.2} ln\, x\, dx$ by Simpson's 3/8 rule and Weddle's rule considering the number of intervals six. Find true value and then compare and comment on it.

Solution:

Here upper limit is $b = 5.2$, lower limit is $a = 4$ and No. of subintervals $n = 6$ and also $f(x) = \ln x$.

Now,

$$h = \frac{5.2-4}{6} = \frac{1.2}{6} = 0.2$$

The values of the function y at each subinterval are given in the tabular form:

x	4	4.2	4.4	4.6	4.8	5.0	5.2
Y=f(x)	1.3862	1.4350	1.4816	1.5260	1.5686	1.6094	1.6486

Simpson's 3/8 rule:

We know that

$$\int_a^b f(x)dx = \frac{3h}{8}\left\{(y_0 + y_n) + 3\sum_{\substack{k \neq 3,6,9,\ldots \\ k=1}}^{n-1} y_k + 2\sum_{k=3,6,9,\ldots\ldots}^{n-3} y_k\right\}$$

Now for $n = 6$ the above formula reduces to the following form,

$$\int_a^b f(x)dx = \frac{3\times 0.2}{8}\left\{(y_0 + y_6) + 3\sum_{\substack{k \neq 3,6,9,\ldots \\ k=1}}^{5} y_k + 2\sum_{k=3,6,9,\ldots\ldots}^{3} y_k\right\}$$

$$\int_a^b f(x)dx = \frac{3\times 0.2}{8}\{(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\}$$

$$\int_a^b f(x)dx = \frac{3\times 0.2}{8}\{(1.3862 + 1.6486) + 3(1.4350 + 1.4816 + 1.5686 + 1.6094) + 2\times 1.5260\}$$

$$\int_4^{5.2} \ln x\ dx = \frac{3\times 0.2}{8}\{3.0348 + 3\times 6.0946 + 2\times 1.5260\} = 1.827795$$

Weddle's Rule:

We know that

$$\int_a^b f(x)dx = \frac{3h}{10}\left\{\sum_{k=0,2,4,6,\ldots\ldots}^{n} y_k + 5\sum_{k=1,3,5,\ldots}^{n-1} y_k + \sum_{k=3,6,9\ldots}^{n-3} y_k\right\}$$

Now for $n = 6$ the above formula reduces to the following form,

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{10}\left\{\sum_{k=0,2,4,6,......}^{6} y_k + 5\sum_{k=1,3,5,...}^{5} y_k + \sum_{k=3,6,9...}^{3} y_k\right\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{10}\{y_0 + y_2 + y_4 + y_6 + 5(y_1 + y_3 + y_5) + y_3\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{10}\{1.3862 + 1.4816 + 1.5686 + 1.6486 + 5(1.4350 + 1.5260 + 1.6094) + 1.5260\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{10}\{6.085 + 5\times 4.5704 + 1.5260\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{10}\{6.085 + 5\times 4.5704 + 1.5260\}$$

$$\int_4^{5.2} \ln x\,dx = 1.82778$$

True value is $\int_4^{5.2} ln\,x\,dx = [x\,ln\,x]_4^{5.2} - \int_4^{5.2}\left[\frac{d}{dx}(ln\,x)\int dx\right]dx$

$$= [x\,ln\,x]_4^{5.2} - \int_4^{5.2}\left[\frac{1}{x}.x\right]dx$$
$$= [x\,ln\,x]_4^{5.2} - \int_4^{5.2} dx$$
$$= [x\,ln\,x]_4^{5.2} - [x]_4^{5.2}$$
$$= (5.2\,ln\,5.2 - 4\,ln\,4) - (5.2 - 4)$$
$$= 1.827847409$$

Result on Simpson's 3/8 rule and Weddle rule are closer to one another and also to the true value. That means both methods work well.
(As desired)

Try Yourself

1.Derive newton's general quadrature formula for numerical integration.
2.Obtain the formula for Simpson's one-third rule from general quadrature formula.
3. Obtain the formula for Simpson's rule and Weddle's rule from general quadrature formula to find $\int_a^b f(x)dx$.

4.Using Simpson's 3/8 th rule find the value of $\int_0^3 e^{-2x}\,sin\,4\,xdx$ taking six sub-intervals.
5.Calculate the value of $\int_{1.2}^{1.8}\left(x+\frac{1}{x}\right)dx$,correct up 5D taking six sub-intervals by Simpson's 3/8 th rule. Also, find the percentage errors and compare to the exact solution.
6. Using Simpson's 3/8 th rule find the value of $\int_0^1 \frac{ln(1+x^2)}{1+x^2}dx$
7.Evaluate $\int_0^2 e^{2x}\,sin\,3\,xdx$,using Simpson's rule and Weddle's rule.
8.Using Simpson's three-eight rule evaluate the integrals $\int_{0.2}^{1.4}(sin\,x + e^{2x})dx$ and hence find the errors.
9.Discuss the necessity of numerical techniques of integration.
10.Calculate the value of the integral $I = \int_0^1 \frac{x\,dx}{1+x^2}$ by taking seven equidistant ordinates, using the Simpson's 1/3 rule and trapezoidal rule.
Find the exact value of I and then compare and comment on it.
11.Find $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's 1/3 and 3/8 rules. Hence obtain the approximate value of π in each case.

Semester: Fall-2018

Course Code: CSE 234 Credit Hours: 3.0

Course Title: Numerical Methods Marks: 100

Course Intended Learning Outcomes: Student will be able to...

- solve algebraic or transcendental equations using appropriate numerical methods
- approximate functions using appropriate numerical methods
- solve differential equations using appropriate numerical methods
- evaluate derivatives at a point using appropriate numerical methods
- solve system of linear equations using appropriate numerical methods
- perform error analysis for a given numerical method
- prove results for numerical root finding methods
- model engineering systems using first and second order differential equations, and solve the equations both analytically and numerically
- calculate definite integrals using appropriate numerical methods
- code numerical methods in a modern computer language

Theory Session Plan:

Week No	Topics	Expected Learning Outcome	Assessments(ASSN/CT/Mid/Final)
WK1	➢ Introduction and error analysis ➢ Bisection method to solve algebraic and transcendental equations with algorithm	◆ Appreciate the needs of numerical Analysis ◆ Visualize the applications ◆ Performs an error analysis for a given numerical method	
WK2	➢ Newton Raphson method to solve algebraic and transcendental equations with algorithm ➢ Iteration method to solve algebraic and transcendental equations with algorithm	◆ Prove results for various numerical root finding methods ◆ Perform an error analysis for a given numerical method ◆ Code a numerical method in a modern computer language	2/3 problems related to discussion in the class
WK3	➢ LU decomposition ➢ Iteration method to solve SLE	◆ Able to find solution of linear system ◆ Find the dominant Eigen-values	**CLASS TEST1** (Up-to last class of the week)
WK4	➢ Curve fitting: Least square method for linear and non-linear case	◆ Construct a curve or mathematical function that has the best fit to a series of data points.	2/3 problems related to discussion in the class
WK5	➢ Interpolation: Newton's Forward difference Method. ➢ Lagrange Interpolation Formula	◆ approximate a function using an appropriate numerical method ◆ able to use in cryptography	2/3 problems related to discussion in the class
WK6	➢ Interpolation: Newton's Backward difference Method	◆ approximate a function using an appropriate numerical method ◆ code a numerical method in a modern computer language	
WK7	----- Midterm Week ------	----- Midterm Week ------	**MIDTERMEXAM**
WK8	➢ Lagrange Interpolation Formula ➢ Numerical Differentiation	◆ approximate a function using an appropriate numerical method ◆ able to forecast missing data	
WK9	➢ Maximum and minimum value of a tabulated functions ➢ Review discussion	◆ able to find maximum and minimum value of a tabulated functions.	**CLASS TEST2**
WK10	➢ Bezier curves ➢ B- spline curves ➢ Applications of the methods	◆ Construct the curve or mathematical function that has the best fit to a series of data points.	**PRESENTATION**
WK11	➢ Derivation of General Formula Numerical Integration for Trapezoidal rule,Simpson's 1/3 rule, Simpson's 3/8 rule and	◆ calculate a definite integral using an appropriate numerical method	**CLASS TEST3**

	Weddle's rule		
WK12	➢ Numerical solution of ordinary differential equations: Runge-kutta method of 2nd, 4^{th} order	◆ solve a differential equation using an appropriate numerical method	
WK13	➢ Euler Method to solve ODE	◆ Solve ODE by appropriate numerical method	2/3 problems related to discussion in the class
WK14	----- final exam week------	----- final exam week------	**FINALEXAM**

Text Book(s):

(1) Numerical Analysis by Burden & Faires , 5th edition
(2) Introductory Methods of Numerical Analysis, S.S Sastry, 5th edition
(3) Numerical Methods in Engineering, J. Kiusalaas

Marks Distribution:

Attendance	07 %
Class Test	15 %
Assignment	05 %
Presentation	08%
Mid Term Exam	25 %
Final Exam	40 %
Total	100%

Grading Policy:

Marks out of 100	Letter	Grade Point
80 - 100	A+	4.00
75 - 79	A	3.75
70 - 74	A-	3.50
65 - 69	B+	3.25
60 - 64	B	3.00
55 - 59	B-	2.75
50 - 54	C+	2.50
45 - 49	C	2.25
40 – 44	D	2.00
00 - 39	F	0.00

Course Instructor's Details:

Course Tutor:	Office:
Mohammad Abdul Halim Senior Lecturer (Mathematics) Department of GED DIU, Dhanmondi, Dhaka.	Teachers Room, 1st Floor, Exam Building, DIU, Shukrabad. Cell: 01770425705 Mail: halim.ged@diu.edu.bd

Daffodil International University

Department of CSE

Faculty of Science & Information Technology

Mid-term Examination, Semester: Summer 2016

*[Answer any **four** of the following questions **Question 1** (Q1) is compulsory]*

Time: 1.30 hour **Full Marks: 25**

1. Illustrate Bisection method and define percentage error. Obtain a root, correct to four decimal places for the equation $2^x - 5x + 2 = 0$ using this method. 10

2. **a)** Given the table of values 5

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$y = \sin x$	0	0.70711	1

Find $y = \sin \frac{\pi}{6}$

3. Discuss the various finite differences with their corresponding difference tables. 5

4. Find the annual premium by Newton's backward interpolation formula at the age of 30 from the following table. 5

Age : 21 25 29 33

Premium : 14.27 15.81 17.72 19.96

5. Evaluate the first and second derivative of $\sqrt{x}$ at x=15 from the following table 5

x	15	17	19	21	23
y	3.873	4.123	4.354	4.583	4.796

Daffodil International University

Department of Computer Science and Engineering

Faculty of Science and Information Technology

Final Examination, Spring 16

Course Code: CSE 234 **Course Title:** Numerical Methods

Sec: All **Course Teachers**: All

Time: 2.0hours **Total marks: 40**

Answer any five from the following seven questions:

01. i) Derive the general formula for numerical integration.

ii) Determine the quadratic splines satisfying the data points (1, -8), (2,-1) and (3,18) satisfying the function y = f(x), find the linear splines satisfying the given data. Determine the approximate values of y(2.5)and $\frac{dy}{dx}$ at x=2

02. Calculate the value of the integral $I = \int_0^1 \frac{x}{1+x^2} dx$ by taking seven equidistant ordinates, using the

(i)Simpson's $1/3$ rule (ii)Trapezoidal rule . Find exact value of I and decide which method gives the best accuracy.

03. Determine $\int_4^{5.2} lnx dx$ by Simpson's 3/8 rule and Weddle's rule considering the number of intervals, $n = 6$. Find true value of I and then compare and comments on it.

04. Write algorithm of Runge-Kutta second order method to solve first order initial value problem.

Given that $\frac{dy}{dx} = y^2 + 1$ with $y(0) = 0$, find $y(0.2)$ by using Runge-Kutta fourth order method.

05. Given that $\frac{dy}{dx} = \frac{x^2}{y^2+1}$ with $y(0) = 0$, use Picard's method to obtain y for $x = 0.1, 0.2, 0.3, 0.8, 0.9$ and 1.0 correct to three decimal places.

06. Solve the following system of equations by factorization method.

$$x + 5y + z = 21$$
$$2x + y + 3z = 20$$
$$3x + y + 4z = 26$$

Daffodil International University

Department of CSE

Faculty of Science & Information Technology

Mid-term Improvement Examination, Semester: Spring 2015

*[Answer any **three** of the following questions **Question 1** (Q1) is compulsory]*

Time: 1.30 hour **Full Marks: 25**

1. Illustrate Bisection method. Obtain a root, correct to three decimal places for the equation $3\mathbf{x}^3 + \mathbf{2x}^2 - \mathbf{7} = \mathbf{0}$ using this method. 9

2. Illustrate Newton-Raphson method. Obtain a root correct to two decimal places for the equation $\mathbf{x}^4 + \mathbf{x}^2 - \mathbf{80} = \mathbf{0}$ using this method. 8

3. What is interpolation? Discuss the various finite differences with their corresponding difference tables. 8

4. Illustrate iteration method. Find a real root with an accuracy of 10^{-4} for the equation $5\mathbf{x}^3 = \mathbf{20x} - \mathbf{3}$ using iterative method. 8

Daffodil International University

Department of CSE

Faculty of Science & Information Technology

Final Examination, Semester: Spring 2015

*[Answer any **four** of the following questions. Figures in the right-hand margin indicate full marks.]*

Time: 2 hours **Full Marks: 40**

1. **a)** Discuss the method of false position. Obtain a root, correct to two decimal places for the equation $x^3+3x^2-3=0$ using false position method. 5+5

b) Solve the initial value problem $\frac{dy}{dx}=x+y,\ y(0)=1$, Find y (0.2) by Runge-Kutta method of order 4.

2. **a)** Illustrate the Iteration method. Find a real root; with accuracy of 10^{-4} of the equation $x^3 + x - 5 = 0$, using this method. 6+4

b) What is interpolation? Discuss the pros and cons of different interpolation methods.

3. 6+4

a) Evaluate the first and second derivative of $\sqrt{x}$ at x=15 from the following table

x	15	17	19	21	23
y	3.873	4.123	4.354	4.583	4.796

b) Illustrate Newton's forward difference formula for interpolation.

4. **b)** Given the table of values 5+5

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$y=\sin x$	0	0.70711	1

Find $y=\sin\frac{\pi}{6}$

b) Evaluate $\int_0^1 \frac{1}{1+x\hat{}2}$ by Trapezodal rule and Simpson 3/8 rule of Integration

5. **a)** Find the values of a_0 and a_1 so that $Y= a_0+a_1x$ fits the data given in the table. 4+6

x	0	2	5	7
y	-1	5	12	20
w	1	1	10	1

b) Solve $\frac{dy}{dx}=\frac{y-x}{y+x},\ y(0)=1$ Find $y(0.1)$ by Euler Method up to 5 iterations.

6. a) State the general equation of Simpson's 1/3 rule for numerical integration. 5+5

b) What is Trapezoidal Rule? Derive its general equation for numerical integration.

Daffodil International University

Department of Computer Science and Engineering
Faculty of Science and Information Technology
Final Examination, Fall 2016

Course Code: CSE 234 **Course Title:** Numerical Methods
Sec: All **Course Teachers**: All

Time: 2.0hours **Total marks: 40**

Answer any five from the following six questions:

1. Suppose a polynomial $p(x)$ is agreed with the data points $(3, 25.14), (3.5, 22.07), (4, 21), (4.5, 18.647), (5, 17.262), (5.5, 16.089)$.

a) Construct a related difference table to find $p(\text{x})$. 3

b) Find interpolating polynomial $p(x)$ to calculate $p(5)$. 3

c) Find extrapolated value $p(8)$. 2

2. Evaluate the following integral both numerically by Simpson's 3/8 rule, Weddle's rule and analytically using 6 equidistant sub-intervals and find percentage error. 5 2 1

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$$

3. a) What is numerical integration? Derive the general formula for numerical integration. 4

b) Find an approximate value of $\log_e 5$ by calculating $\int_0^5 \frac{dx}{4x+5}$ by Simpson's 1/3 rule of integration where the no. of ordinates is 11. 4

4. a) Compare Lagrange's and Newton's Interpolation Formula. Applying appropriate method find $\sqrt{155}$ 2 3

x:	150	152	154	157
$f(x) = \sqrt{x}$	12.247	12.329	12.410	12.529

b) Give all possible $x=\phi(x)$ connected with the equation $x^3+4x^2-10=0$ for fixed point iteration method and solve the equation by this method correct up to five decimal places. 3

5. a) Given IVP $\frac{dy}{dx}=1+y^2$, $y(0)=0$. Find the value of *y* approximately for $x=0.9$ by Runge-Kutta method of four order in three steps. Compare the result with the exact value. 3 2

b) The population of a country in the decennial census was as under. Estimate the population for the year 1995. 3

Year	X	1991	1996	2001	2006	2011
Population (in Crore)	Y	48	69	81	93	101

6. a) What is system of linear equations? Write down the condition when SLEs converge to its solutions. 2

b) Decompose the matrix A =LU where L and U has as usual meaning and solve the system AX=B where $A=\begin{bmatrix}5 & -2 & 1\\ 7 & 1 & -5\\ 3 & 7 & 4\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}4\\ 8\\ 10\end{bmatrix}$ 6

END