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Parameter Estimations Assignment

Q1. Let $(X_1, X_2 \ldots)$ be a random sample of size n taken from a normal population with parameters: mean $= \theta_1$ and variance $= \theta_2$. Find the maximum likelihood ~~parameters of~~ estimates of these two parameters.

Sol. For normal distribution,

$$PMF\, f(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

here $\mu = \theta_1$, $\sigma^2 = \theta_2$

$$\therefore f(x_i | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$\Rightarrow f(x_i | \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

Now likelihood function

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i | \theta_1, \theta_2)$$

$$\Rightarrow L(\theta_1,\theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1,\theta_2) = \prod_{i=1}^{n} (\theta_2)^{-1/2} \prod_{i=1}^{n} (2\pi)^{-1/2} \prod_{i=1}^{n} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1,\theta_2) = (\theta_2)^{-\frac{n}{2}} (2\pi)^{-\frac{n}{2}} e^{-\left(\frac{\sum_{i=1}^{n}(x_i-\theta_1)^2}{2\theta_2}\right)}$$

$$L(\theta_1,\theta_2) = (\theta_2)^{-\frac{n}{2}} (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}$$

taking log both sides

$$\ln L(\theta_1,\theta_2) = \ln\left[(\theta_2)^{-\frac{n}{2}} (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}\right]$$

$$= \ln(\theta_2)^{-\frac{n}{2}} + \ln(2\pi)^{-\frac{n}{2}} + \ln\left(e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}\right)$$

$$\ln L(\theta_1,\theta_2) = -\frac{n}{2}\ln\theta_2 - \frac{n}{2}\ln 2\pi - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2 \quad \text{—} (1)$$

differentiating both sides w.r.t θ_1

$$\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_1} = \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)$$

Now $\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_1} = 0$

$$\therefore \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1) = 0$$

$$\Rightarrow \frac{1}{\theta_2}\left(\sum_{i=1}^{n} x_i - n\theta_1\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - n\theta_1 = 0$$

$$\Rightarrow \theta_1 = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Rightarrow \theta_1 = \bar{x}_n$$

$$\therefore \boxed{\hat{\theta}_{1_{MLE}} = \bar{x}_n} \quad -(2)$$

differentiating eq (1) wrt θ_2

$$\frac{\partial l(\theta_1, \theta_2)}{\partial \theta_2} = \frac{-n}{2\theta_2} - \frac{1}{2(\theta_2)^2} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

now $\frac{\partial l(\theta_1, \theta_2)}{\partial \theta_2} = 0$

$$\therefore \frac{-n}{2\theta_2} - \frac{1}{2\theta_2^2} \sum_{i=1}^{n} (x_i - \theta_1)^2 = 0$$

$$\Rightarrow \frac{1}{2(\theta_2)^2} \sum_{i=1}^{n} (x_i - \theta_1)^2 = \frac{-n}{2\theta_2}$$

$$\Rightarrow \theta_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

from (2)

$$\boxed{\hat{\theta}_{2_{MLE}} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2}$$

Q2. Let $X_1, X_2, \ldots X_n$ be a random sample from $B(m, \theta)$ distribution where $\theta \in (0,1)$ is unknown and m is a known positive integer. compute value of θ using the M.L.E.

Sol. For binomial distribution

$$PMF(x_i) = {}^{m}C_{x_i}\, p^{x_i} (1-p)^{m-x_i}$$

Here $n = m$, $p = \theta$

$$\therefore f(x_i | m, \theta) = {}^{m}C_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i}$$

Likelihood function $\Rightarrow$

$$L(P) = \prod_{i=1}^{n} \left(P(x_i | m, \theta) \right)$$

$$= \prod_{i=1}^{n} \left({}^{m}C_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i} \right)$$

$$= \prod_{i=1}^{n} {}^{m}C_{x_i} \prod_{i=1}^{n} \theta^{x_i} \prod_{i=1}^{n} (1-\theta)^{m-x_i}$$

$$= \prod_{i=1}^{n} {}^{m}C_{x_i}\, \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{\sum_{i=1}^{n}(m-x_i)}$$

$$= \prod_{i=1}^{n} {}^{m}C_{x_i}\, \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{nm - \sum_{i=1}^{n} x_i}$$

taking ln both sides

$$\ln L(P) = \ln \left(\prod_{i=1}^{n} {}^{m}C_{x_i} \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{nm - \sum_{i=1}^{n} x_i} \right.$$

$$= \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\right) + \ln\left(\theta^{\sum_{i=1}^{n} x_i}\right) + \ln\left((1-\theta)^{nm-\sum_{i=1}^{n} x_i}\right)$$

$$= \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\right) + \ln(\theta)\cdot\sum_{i=1}^{n} x_i + \ln(1-\theta)\cdot\left(nm-\sum_{i=1}^{n} x_i\right)$$

differentiating wrt θ

$$\frac{\partial L(P)}{\partial \theta} = \frac{1}{\theta}\sum_{i=1}^{n} x_i + \left(\frac{-1}{1-\theta}\right) nm - \sum_{i=1}^{n} x_i$$

$$= \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\left(nm - \sum_{i=1}^{n} x_i\right)$$

now $\frac{\partial L(P)}{\partial \theta} = 0$

$$\therefore \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\left(nm - \sum_{i=1}^{n} x_i\right) = 0$$

$$\Rightarrow \frac{1}{\theta}\sum_{i=1}^{n} x_i = \frac{1}{1-\theta}\left(nm - \sum_{i=1}^{n} x_i\right)^*$$

$$\Rightarrow \frac{1-\theta}{\theta} = \frac{nm - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i}$$

$$\Rightarrow \frac{nm}{\sum_{i=1}^{n} x_i} - 1 = \frac{1}{\theta} - 1$$

$$\Rightarrow \theta = \frac{\sum_{i=1}^{n} x_i}{nm}$$

$$\Rightarrow \theta = \frac{\bar{x}_n}{m}$$

$$\therefore \boxed{\theta_{MLE} \in (0,1) = \frac{\bar{x}_n}{m}}$$