Ex: Determine $P(E|F)$ when a coin is tossed three times, where

i). E: head on third toss; F: heads on first two tosses.

ii). E: at least two heads; F: at most two heads

iii). E: at most two tails; F: at least one tail.

Sol: i) E: head on third toss

$\therefore$ Sample Space $= \{HHH, HTH, THH, TTH\}$

F: heads on first two tosses

$\therefore S = \{HHH, HHT\}$

Total No. of Cases $= 2^3 = 8$

$\therefore P(F) = \frac{2}{8} = \frac{1}{4}$

$P(E\cap F) = \frac{1}{8}$

$\therefore P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{\frac{1}{\cancel{8}\,2}}{\frac{1}{\cancel{4}}} = \frac{1}{2}$

ii). E: at least two heads

$\therefore S = \{HHT, HTH, THH, HHH\}$

F: at most two heads

$\therefore S = \{TTT, HTT, THT, TTH, HHT, HTH, THH\}$

$\therefore P(F) = 7/8$

$P(E\cap F) = 3/8$

$\therefore P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{3/8}{7/8} = 3/7.$

iii). E: at most two tails

$S = \{HHH, THH, HTH, HHT, HTT, THT, TTH\}$

F: at least one tail

$S = \{THH, HTH, HHT, HTT, THT, TTH, TTT\}$

$P(F) = 7/8$, $P(E\cap F) = 6/8$

$P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{6/8}{7/8} = 6/7$

Properties of conditional Probability:

Property 1: Let A and B be events of a sample space S of an experiment, then we have

$P(S|B) = P(B|B) = 1$

we know that $P(S|B) = \frac{P(S\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

Also $P(B|B) = \frac{P(B\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

Thus $P(S|B) = P(B|B) = 1$.

Property 2: If A and B are any two events of a sample space S and F is any event of S such that $P(B) \neq 0$, then

$P((A\cup B)|F) = P(A|F) + P(B|F) - P((A\cap B)|F)$

In particular, if A and B are disjoint events, then

$P((A\cup B)|F) = \frac{P((A\cup B)\cap F)}{P(F)}$

$= \frac{P[(A\cap F)\cup(B\cap F)]}{P(F)}$ By distributive law of union over intersection

$= \frac{P(A\cap F) + P(B\cap F) - P\{(A\cap B)\cap F\}}{P(F)}$

$= P(A|F) + P(B|F) - P((A\cap B)|F)$

when A and B are disjoint events, then

$P((A\cap B)|F) = 0$

$\therefore P((A\cup B)|F) = P(A|F) + P(B|F)$

Property 3 :- $P(E'|F) = 1 - P(E|F)$

Ex:- A pair of dice is rolled, find $P(A|B)$ if
A : 2 appears on atleast one dice
B : Sum of numbers appearing on dice is 6.

Sol:
$$A = \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(1,2),(3,2),(4,2),(5,2),(6,2)\}$$
$$B = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$
$$A \cap B = \{(2,4),(4,2)\}$$
$$P(A \cap B) = \frac{2}{36}$$
$$P(B) = 5/36$$
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/\cancel{36}}{5/\cancel{36}} = \frac{2}{5}.$$

Ex: A card is drawn from a well shuffled deck of 52 cards and then second card is drawn. Find the probability that the first card is spade and then second card is club if the first card is not replaced.

Sol: $P(\text{First card spade}) = P(S) = \frac{13}{52} = \frac{1}{4}$
After the event of drawing a spade the deck has 51 cards, 13 of which are club (c)
$$\therefore P(C|S) = \frac{13}{51}$$
Hence $P(S \text{ and } C) = P(S) \cdot P(C|S)$
$$= \frac{1}{4} \cdot \frac{13}{51} = \frac{13}{204}.$$

Ex: A couple has two children. Find the probability that both children are boys, if it is known that at least one of the children is a boy.

Sol: Let B_i, G_i stands for i^{th} child be a boy and girl respectively. Then the sample space is
$$S = \{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$$
Total No. of Cases = 4
Consider the following events
A = Both the children are boys
B = atleast one of children is a boy.
∴ sample space of $A = \{B_1B_2\}$
Total no. of cases in $A = 1$
Sample space of $B = \{B_1G_2, G_1B_2, B_1B_2\}$
Total no. of cases in $B = 3$
$$\therefore A \cap B = \{B_1B_2\}$$
$$P(B|A) = \frac{P(A \cap B)}{P(B)} = \frac{1/\cancel{4}}{3/\cancel{4}}$$
$$= \frac{1}{3}.$$

Ex:- The probability that a student selected at random from a class will pass in Mathematics is $\frac{4}{5}$ and the probability that he/she passes in Mathematics and Computer Science is $1/2$. What is the probability that he/she will pass in computer science, if it is known that he has passed in Mathematics?

Sol. Probability (Pass in Maths)

$$= \frac{4}{5} = P(M)$$

Probability (Pass in Maths and Computer Science)

$$= P(M \cap C) = \frac{1}{2}$$

$$P(C) = ?$$

$$P(C|M) = \frac{P(M \cap C)}{P(M)} = \frac{\frac{1}{2}}{\frac{4}{5}}$$

$$= \frac{1}{2} \times \frac{5}{4} = \frac{5}{8}$$