Applied Multidimensional Girsanov Theorem

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July 14, 2012

Abstract

The present article is meant as a bridge between theory and practice concerning Girsanov theorem. In the first part we give theoretical results leading to a straightforward three step process allowing to express an asset's dynamics in a new probability measure. In the following sections we apply this three step process. The first application consists in expressing a foreign asset's dynamics in domestic currency which leads to the well known quanto adjustment formula. The second application is the expression of the forwards' dynamics under Libor Market Model using a unique probability measure. The final application consists in expressing foreign short rate Hull & White dynamics under domestic measure.

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Notations

The letters d and f, when used as subscripts or superscripts, will mean respectively "domestic" and "foreign".

Financial Notations

- r, r_d, r_f : short rates
- B, B_d, B_f : bank account numeraires
- $P(t,T), P_d(t,T), P_f(t,T)$: value at time t of zero coupons maturing at time T
- $X_{fd}(t)$: value at time t of foreign exchange rate from foreign currency f to domestic currency d. At time t, a unit of f is worth $X_{fd}(t)$ units of d
- $X_{fd}(t,T)$: value at time t of foreign exchange rate from f to d maturing at time T
- $L(T_1,T_2)$: IBOR rate fixing at time T_1 and whose tenor is $T_2 - T_1$
- $F(t,T_1,T_2)$: value at time t of the forward rate associated to IBOR rate $L(T_1,T_2)$
- $L_f(T_1,T_2)$: IBOR rate denominated in foreign currency f, fixing at time T_1 and whose tenor is $T_2 - T_1$
- $F_f(t,T_1,T_2)$: value at time t of the foreign forward rate associated to IBOR rate $L_f(T_1,T_2)$
- S_f : asset denominated in foreign currency f

Mathematical Notations

Mathematical notations are summerized in table hereafter. For the sake of simplicity, we used same notations for brownian motions and filtrations under risk neutral and forward measures.

	GENERAL	DOMESTIC	FOREIGN
risk neutral measure	$\mathbb{Q}$	$\mathbb{Q}^d$	$\mathbb{Q}^f$
numeraires associated	B	B_d	B_f
brownian motions	W	W^d	W^f
filtrations associated	$\mathcal{F}$	$\mathcal{F}^d$	$\mathcal{F}^f$
expectations	$\mathbb{E}^{\mathbb{Q}}$	$\mathbb{E}^{\mathbb{Q}^d}$	$\mathbb{E}^{\mathbb{Q}^f}$
T-forward measure	$\mathbb{Q}_T$	$\mathbb{Q}_T^d$	$\mathbb{Q}_T^f$
numeraires associated	$P(\cdot,T)$	$P_d(\cdot,T)$	$P_f(\cdot,T)$
brownian motions	W	W^d	W^f
filtrations associated	$\mathcal{F}$	$\mathcal{F}^d$	$\mathcal{F}^f$
expectations	$\mathbb{E}^{\mathbb{Q}_T}$	$\mathbb{E}^{\mathbb{Q}_T^d}$	$\mathbb{E}^{\mathbb{Q}_T^f}$

- Brownian motions W, W^d, W^f may be mutlidimensional, depending on the context. If it is the case, it will be of course explicitly mentioned.
- We use subscript t for conditional expectations. For instance $\mathbb{E}_t^{\mathbb{Q}}$ denotes expectation conditional on $\mathcal{F}(t)$.
- To distinguish brownian motions under a same measure we use the subscript linked to the asset. For instance W_S^f will denote asset's S brownian motion.

1 Mutlidimensional Girsanov Theorem

Let us start by reminding the multidimensional form of Girsanov theorem. Note that for simplicity, we do not bother with the detailed mathematical framework under which Girsanov theorem can be applied, nor with its proof. The interested reader may refer to [KS1991] section 3.5.

Let Z denote a one dimensional exponential martingale under probability measure $\mathbb{Q}$. Let us write Z dynamics under $\mathbb{Q}$ in the following form:

$$\frac{dZ(t)}{Z(t)} = \Theta(t)^\mathsf{T}\, dW(t)$$

where W is a n-dimensional standard brownian motion under $\mathbb{Q}$ and Θ a n-dimensional measurable process adapted to W filtration.[1]

Theorem 1.1. (Girsanov Theorem) *A new probability measure $\widetilde{\mathbb{Q}}$ is defined using Z as Radon-Nikodym derivative*[2]*:*

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\right] = Z(t)$$

Under the new measure $\widetilde{\mathbb{Q}}$, the process defined by

$$\widetilde{W}(t) = W(t) - \int_0^t \Theta(s)\, ds$$

is a n-dimensional standard brownian motion.

2 Girsanov Theorem in Financial Context

In this section we demonstrate how Girsanov theorem allows to express a process dynamics under a new probability measure. We define hereafter a simple three step process to apply whenever a change of measure is needed.

2.1 Notations and Framework

- $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ are probability measures linked through Radon-Nikodym derivative $\mathbb{E}_t^{\mathbb{Q}}\left[\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\right] = Z(t)$
- W denotes a n-dimensional standard brownian motion under $\mathbb{Q}$
- X denotes a general Itô process whose dynamics under $\mathbb{Q}$ are:

$$dX(t) = \mu_X(t, X(t))\, dt + \Sigma_X(t, X(t))^\mathsf{T}\, dW(t)$$

- S denotes an asset whose dynamics under $\mathbb{Q}$ have a more specific form than X, obtained with $\mu_X(t, X(t)) = S(t)\,\mu_S(t)$ and $\Sigma_X(t, X(t))^\mathsf{T} = S(t)\,\Sigma_S(t)^\mathsf{T}$:

$$\frac{dS(t)}{S(t)} = \mu_S(t)\, dt + \Sigma_S(t)^\mathsf{T}\, dW(t)$$

  which imply non null and non negative values for S. These dynamics include the specific case of log-normal dynamics.

- $\widetilde{\mu}_X$, respectively $\widetilde{\mu}_S$, denotes X, respectively S, drift under $\widetilde{\mathbb{Q}}$

[1] Note that vectors are represented as column vectors. Therefore $\Theta(t)$ is a $n \times 1$ column vector and $\Theta(t)^\mathsf{T}$ is a $1 \times n$ row vector.

[2] Recall that $\mathbb{E}_t^{\mathbb{Q}}\left[\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\right] = Z(t)$ means that for any $\mathcal{F}_t$ measurable set A: $\widetilde{\mathbb{Q}}(A) = \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_A Z(t)]$

2.2 Linking Drifts for General X Dynamics

Applying Girsanov theorem 1.1 allows to write X dynamics under $\widetilde{\mathbb{Q}}$:

$$\begin{aligned} dX(t) &= \mu_X(t, X(t))\, dt + \Sigma_X(t, X(t))^\mathsf{T} \left[d\widetilde{W}(t) + \Theta(t)\, dt \right] \\ &= \left[\mu_X(t, X(t)) + \Sigma_X(t, X(t))^\mathsf{T}\, \Theta(t) \right] dt + \Sigma_X(t, X(t))^\mathsf{T}\, d\widetilde{W}(t) \end{aligned}$$

where $\widetilde{W}$ is a brownian motion under $\widetilde{\mathbb{Q}}$.

The following proposition links X drifts under measures $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$:

Proposition 2.1. (Linking Drifts Under Different Measures) *S drifts μ_X and $\widetilde{\mu}_X$ are linked through formula:*

$$\widetilde{\mu}_X(t, X(t)) = \mu_X(t, X(t)) + \Sigma_X(t, X(t))^\mathsf{T}\, \Theta(t) \tag{1}$$

The previous “matricial” expression can be reformulated in a “scalar” way. The “scalar” relation expressed in corollary below will be used in practice to derive drifts under new measure.

Corollary 2.2. (Linking Drifts Under Different Measures Reformulation) *S drifts μ_X and $\widetilde{\mu}_X$ are linked through formula:*

$$\widetilde{\mu}_X(t, X(t)) = \mu_X(t, X(t)) + \frac{d\left\langle X,\, \ln(Z)\right\rangle(t)}{dt} \tag{2}$$

Proof. We want to proove the relation:

$$\Sigma_X(t, X(t))^\mathsf{T}\, \Theta(t) = \frac{d\left\langle X,\, \ln(Z)\right\rangle(t)}{dt}$$

To do so, let us first calculate $d\big\langle X,\, \ln(Z)\big\rangle(t)$.

$$\begin{aligned} d\big\langle X,\, \ln(Z)\big\rangle(t) &= \big\langle dX(t),\, d\ln(Z(t))\big\rangle \\ &= \left\langle dX(t),\, \frac{dZ(t)}{Z(t)} \right\rangle \\ &= \big\langle \Sigma_X(t, X(t))^\mathsf{T}\, dW(t),\, \Theta(t)^\mathsf{T}\, dW(t) \big\rangle \end{aligned}$$

Obviously, if the general equality $\big\langle A^\mathsf{T}\, dW(t),\, B^\mathsf{T}\, dW(t)\big\rangle = A^\mathsf{T}\, B\, dt$ holds, then the result is obtained just by replacing A^T with $\Sigma_X(t, X(t))^\mathsf{T}$ and B with $\Theta(t)$. Let us then proove this equality. We introduce the notations:

$$\begin{aligned} A^\mathsf{T} &= \left[a_1,\, \dots,\, a_n \right] \\ B^\mathsf{T} &= \left[b_1,\, \dots, b_n \right] \\ dW(t)^\mathsf{T} &= \left[dW_1(t), \dots, dW_n(t) \right] \end{aligned}$$

Using the above notations, we have

$$\begin{aligned} \big\langle A^\mathsf{T}\, dW(t),\, B^\mathsf{T}\, dW(t)\big\rangle &= \left\langle \sum_{i=1}^n a_i\, dW_i(t),\, \sum_{i=1}^n b_i\, dW_i(t) \right\rangle \\ &= \sum_{1 \le i,j \le n} a_i\, b_j\, \langle dW_i(t),\, dW_j(t)\rangle \\ &= \sum_{i=1}^n a_i\, b_i\, \langle dW_i(t),\, dW_i(t)\rangle \\ &= \sum_{i=1}^n a_i\, b_i\, dt \\ &= A^\mathsf{T}\, B\, dt \end{aligned}$$

As mentioned earlier, replacing A^T with $\Sigma_X(t, X(t))^\mathsf{T}$ and B with $\Theta(t)$ leads to the final formula.

□

2.3 Linking Drifts for S Dynamics

As metioned in subsection 2.1 using the specific form of drift and volatility $\mu_X(t, X(t)) = S(t)\,\mu_S(t)$ and $\Sigma_X(t, X(t))^\intercal = S(t)\,\Sigma_S(t)^\intercal$ leads to S dynamics. This allows to obtain the following results.

Proposition 2.3. (Linking Drifts Under Different Measures) *S drifts μ_S and $\widetilde{\mu}_S$ are linked through formula:*

$$\widetilde{\mu}_S(t) = \mu_S(t) + \Sigma_S(t)^\intercal\,\Theta(t) \tag{3}$$

Proof. Using $\mu_X(t, X(t)) = S(t)\,\mu_S(t)$ and $\Sigma_X(t, X(t))^\intercal = S(t)\,\Sigma_S(t)^\intercal$ into proposition 2.1 we obtain:

$$S(t)\,\widetilde{\mu}_S(t) = S(t)\,\mu_S(t) + S(t)\,\Sigma_S(t)^\intercal\,\Theta(t)$$

which simplifies into:

$$\widetilde{\mu}_S(t) = \mu_S(t) + \Sigma_S(t)^\intercal\,\Theta(t)$$

□

The previous "matricial" expression can be reformulated in a "scalar" way. The "scalar" relation expressed in corollary below will be used in practice to derive drifts under new measure.

Corollary 2.4. (Linking Drifts Under Different Measures Reformulation) *S drifts μ_S and $\widetilde{\mu}_S$ are linked through formula:*

$$\widetilde{\mu}_S(t) = \mu_S(t) + \frac{d\big\langle\, \ln(S), \ln(Z) \big\rangle(t)}{dt} \tag{4}$$

Proof. Once again we use $\mu_X(t, X(t)) = S(t)\,\mu_S(t)$ and $\Sigma_X(t, X(t))^\intercal = S(t)\,\Sigma_S(t)^\intercal$ leading to $\widetilde{\mu}_X(t, X(t)) = S(t)\,\widetilde{\mu}_S(t)$. Corollary 2.2 can be therefore written as follows:

$$\begin{aligned}
S(t)\,\widetilde{\mu}_S(t) &= S(t)\,\mu_S(t) + \frac{d\big\langle S\,, \ln(Z)\big\rangle(t)}{dt}\\
\widetilde{\mu}_S(t) &= \mu_S(t) + \frac{1}{S(t)}\,\frac{d\big\langle S\,, \ln(Z)\big\rangle(t)}{dt}\\
\widetilde{\mu}_S(t) &= \mu_S(t) + \frac{1}{S(t)}\,\frac{\big\langle dS(t)\,, d\ln(Z(t))\big\rangle}{dt}\\
\widetilde{\mu}_S(t) &= \mu_S(t) + \frac{\big\langle \frac{dS(t)}{S(t)}\,, d\ln(Z(t))\big\rangle}{dt}\\
\widetilde{\mu}_S(t) &= \mu_S(t) + \frac{\big\langle d\ln(S(t))\,, d\ln(Z(t))\big\rangle}{dt}\\
\widetilde{\mu}_S(t) &= \mu_S(t) + \frac{d\big\langle\, \ln(S)\,, d\ln(Z)\big\rangle(t)}{dt}
\end{aligned}$$

□

2.4 Three Step Process

As mentioned previously, and formula (2)

$$\widetilde{\mu}_X(t, X(t)) = \mu_X(t, X(t)) + \frac{d\big\langle X, \ln(Z)\big\rangle(t)}{dt}$$

and formula (4)

$$\widetilde{\mu}_S(t) = \mu_S(t) + \frac{d\big\langle\, \ln(S), \ln(Z)\big\rangle(t)}{dt}$$

are used in practice to calculate drifts under a new measure. This naturally leads to defining the following three step process:

1. First step consists in defining the Radon-Nikodym derivative Z linking $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$
2. Second step consists in calculating Z dynamics under $\mathbb{Q}$ allowing in turn $d\big\langle X, \ln(Z)\big\rangle(t)$ or $d\big\langle \ln(S), \ln(Z)\big\rangle(t)$ calculation
3. Third step consists simply in applying formula (2) or formula (4), depending on the dynamics considered

This process will be used in next sections to calculate quanto adjustment (section 3), LMM drifts (section 4) and foreign short rate Hull & White dynamics (section 5).

3 First Application: Quanto Adjustment

In order to price a foreign asset paid in domestic currency, one needs the foreign asset's dynamics under domestic measure. This leads to an drift adjustment in the foreign asset's dynamics known as quanto adjustment. We show hereafter how to apply the three step process defined in section 2.4 in order to calculate quanto adjustment. We start by expressing Radon-Nikodym derivative linking foreign and domestic measures (section 3.1) and then we calculate a foreign asset's (section 3.2) and forward rate's (section 3.3) dynamics under domestic measure. In the following, foreign currency shall denote the asset's currency and domestic the payment currency.

3.1 Linking Foreign and Domestic Measures

Since $\frac{S_f(t)}{X_{df}(t)}$ is the value of a tradable asset expressed in domestic currency d, $\frac{S_f(t)}{X_{df}(t)B_d(t)}$ is martingale under $\mathbb{Q}^d$.

Clearly $\frac{S_f(t)}{B_f(t)}$ is martingale under $\mathbb{Q}^f$.

Consequently the following equality holds:

$$S(t) = \mathbb{E}_t^{\mathbb{Q}^f}\left[B_f(t)\frac{S_f(T)}{B_f(T)}\right] = \mathbb{E}_t^{\mathbb{Q}^d}\left[X_{df}(t)B_d(t)\frac{S_f(T)}{X_{df}(T)B_d(T)}\right]$$

which leads to the following proposition linking $\mathbb{Q}^f$ and $\mathbb{Q}^d$:

Proposition 3.1. (Link Between Foreign and Domestic Risk Neutral Measures) $\mathbb{Q}^f$ *and* $\mathbb{Q}^d$ *are linked through Radon-Nikodym derivative:*

$$\mathbb{E}_t^{\mathbb{Q}^f}\left[\frac{d\mathbb{Q}^d}{d\mathbb{Q}^f}\right] = \frac{X_{df}(t)B_d(t)}{X_{df}(0)B_d(0)}\frac{B_f(0)}{B_f(t)}$$

Applying the exact same rationale to numeraires $P_f(\cdot, T)$ and $P_d(\cdot, T)$ leads to the proposition below, linking $\mathbb{Q}_T^f$ and $\mathbb{Q}_T^d$:

Proposition 3.2. (Link Between Foreign and Domestic Forward Measures) $\mathbb{Q}_T^f$ *and* $\mathbb{Q}_T^d$ *are linked through Radon-Nikodym derivative:*

$$\mathbb{E}_t^{\mathbb{Q}_T^f}\left[\frac{d\mathbb{Q}_T^d}{d\mathbb{Q}_T^f}\right] = \frac{X_{df}(t)P_d(t,T)}{X_{df}(0)P_d(0,T)}\frac{P_f(0,T)}{P_f(t,T)}$$

which can be reformulated using FX forward rate[3]*:*

$$\mathbb{E}_t^{\mathbb{Q}_T^f}\left[\frac{d\mathbb{Q}_T^d}{d\mathbb{Q}_T^f}\right] = \frac{X_{df}(t,T)}{X_{df}(0,T)}$$

3.2 Quanto Adjustment for Assets

Under foreign risk neutral measure $\mathbb{Q}^f$ we write S^f dynamics as follows:

$$\frac{dS_f(t)}{S_f(t)} = r_f(t)\,dt + \sigma_S(t)\,dW_S^f(t)$$

[3]This last equality comes from non arbitrage relation linking forward exchange rate to spot exchange rate and zero coupons in both currencies:

$$X_{df}(t,T) = X_{df}(t)\frac{P_d(t,T)}{P_f(t,T)}$$

A demonstration of this formula is given in appendix A.2.

where W_S^f is a brownian motion under $\mathbb{Q}^f$.

We want to write S_f dynamics under domestic measure $\mathbb{Q}^d$ so that we could price a payoff depending on asset S_f paid in domestic currency d. To do so, we go through the three step process defined in section 2.4:

1. First we define the Radon-Nikodym derivative Z linking $\mathbb{Q}^f$ and $\mathbb{Q}^d$
2. We then calculate Z dynamics under $\mathbb{Q}^f$ allowing $d\big\langle \ln(S^f), \ln(Z)\big\rangle(t)$ calculation
3. Finally we apply corollary 2.4 and rewrite S_f dynamics under domestic measure $\mathbb{Q}^d$

Step 1: Defining Radon-Nikodym Derivative Z

Proposition 3.1 leads to the following Radon-Nikodym derivative linking $\mathbb{Q}^f$ and $\mathbb{Q}^d$:

$$Z(t) = \frac{X_{df}(t)B_d(t)}{X_{df}(0)B_d(0)}\frac{B_f(0)}{B_f(t)}$$

Step 2: Z Dynamics under $\mathbb{Q}^f$

Let us start by writting B_d, B_f and X_{df} dynamics[4] separately:

$$\begin{aligned}
\frac{dB_d(t)}{B_d(t)} &= r_d(t)\,dt \\
\frac{dB_f(t)}{B_f(t)} &= r_f(t)\,dt \\
\frac{dX_{df}(t)}{X_{df}(t)} &= \left[r_f(t) - r_d(t)\right]\,dt + \sigma_X(t)\,dW_X^f(t)
\end{aligned}$$

where $W_X^f(t)$ is a brownian motion under $\mathbb{Q}^f$. This leads to Z dynamics:

$$\frac{dZ(t)}{Z(t)} = \sigma_X(t)\,dW_X^f(t)$$

Step 3: Quanto adjustment resulting from corollary 2.4

Let $\rho_{S,X}(t)$ denote instantaneous correlation between $S_f(t)$ and $X_{df}(t)$ brownian motions. In order to apply corollary 2.4, we need to calculate $d\big\langle \ln(S_f), \ln(Z)\,\big\rangle(t)$:

$$\begin{aligned}
d\big\langle \ln(S_f), \ln(Z)\,\big\rangle(t) &= \big\langle\, d\ln(S_f(t)), d\ln(Z(t))\,\big\rangle \\
&= \left\langle \frac{dS_f(t)}{S_f(t)}, \frac{dZ(t)}{Z(t)} \right\rangle \\
&= \big\langle\, \sigma_S(t)\,dW_S^f(t), \sigma_X(t)\,dW_X^f(t)\,\big\rangle \\
&= \sigma_S(t)\,\sigma_X(t)\,\big\langle\, dW_S^f(t), dW_X^f(t)\,\big\rangle \\
&= \rho_{S,X}(t)\,\sigma_S(t)\,\sigma_X(t)\,dt
\end{aligned}$$

We can now apply corollary 2.4 allowing to express S dynamics under domestic measure $\mathbb{Q}^d$:

$$\boxed{\frac{dS_f(t)}{S_f(t)} = \big[\,r_f(t) + \rho_{S,X}(t)\,\sigma_S(t)\,\sigma_X(t)\,\big]\,dt + \sigma_S(t)\,dW_S^d(t)}$$

where W_S^d is a standard brownian motion under $\mathbb{Q}^d$.

[4] A demonstration of how to obtain X_{df} drift under measure $\mathbb{Q}^f$ is given in appendix B.

3.3 Quanto Adjustment for IBOR Rates

Under foreign forward neutral measure $\mathbb{Q}^f_{T_2}$, $F_f(\cdot, T_1, T_2)$ is martingale. We therefore write its dynamics as follows:

$$\frac{dF_f(t, T_1, T_2)}{F_f(t, T_1, T_2)} = \sigma_F(t)\, dW^f_F(t)$$

where W^f_F is a brownian motion under $\mathbb{Q}^f_{T_2}$.

We want to write $F_f(\cdot, T_1, T_2)$ dynamics under domestic forward measure $\mathbb{Q}^d_{T_2}$ so that we could price a payoff depending on rate $F_f(\cdot, T_1, T_2)$ paid in domestic currency d. As in the quanto adjustment for assets, we go through the three step process defined in section 2.4:

1. First we define the Radon-Nikodym derivative Z linking $\mathbb{Q}^f_{T_2}$ and $\mathbb{Q}^d_{T_2}$
2. We then calculate Z dynamics under $\mathbb{Q}^f_{T_2}$ allowing $d\big\langle \ln(F_f(\cdot, T_1, T_2)), \ln(Z)\big\rangle(t)$ calculation
3. Finally we apply corollary 2.4 and rewrite $F_f(\cdot, T_1, T_2)$ dynamics under domestic measure $\mathbb{Q}^d_{T_2}$

Step 1: Defining Radon-Nikodym Derivative Z

Proposition 3.2 leads to the following Radon-Nikodym derivative linking $\mathbb{Q}^f_{T_2}$ and $\mathbb{Q}^d_{T_2}$:

$$Z(t) = \frac{X_{df}(t, T_2)}{X_{df}(0, T_2)}$$

Step 2: Z Dynamics Under $\mathbb{Q}^f_{T_2}$

Let us write now Z dynamics under $\mathbb{Q}^f_{T_2}$:

$$\frac{dZ(t)}{Z(t)} = \frac{dX_{df}(t, T_2)}{X_{df}(t, T_2)} = \sigma_X(t)\, dW^f_X(t)$$

where $W^f_X(t)$ is a brownian motion under $\mathbb{Q}^f_{T_2}$.

Step 3: Quanto adjustment resulting from corollary 2.4

Let $\rho_{F,X}(t)$ denote instantaneous correlation between foreign forward rate $F_f(t, T_1, T_2)$ and forward FX rate $X_{df}(t, T_2)$ brownian motions. In order to apply corollary 2.4, we need to calculate $d\big\langle \ln(F_f(\cdot, T_1, T_2)), \ln(Z) \big\rangle(t)$ where $Z(t) = X_{df}(t, T_2)$.

$$\begin{aligned}
d\big\langle \ln(F_f(\cdot, T_1, T_2)), \ln(Z) \big\rangle(t) &= \big\langle\, d\ln(F_f(t, T_1, T_2)),\, d\ln(Z(t))\, \big\rangle \\
&= \left\langle \frac{dF_f(t, T_1, T_2)}{F_f(t, T_1, T_2)}, \frac{dZ(t)}{Z(t)} \right\rangle \\
&= \big\langle\, \sigma_F(t)\, dW^f_F(t),\, \sigma_X(t)\, dW^f_X(t)\, \big\rangle \\
&= \sigma_F(t)\, \sigma_X(t) \big\langle\, dW^f_F(t),\, dW^f_X(t)\, \big\rangle \\
&= \rho_{F,X}(t)\, \sigma_F(t)\, \sigma_X(t)\, dt
\end{aligned}$$

We can now apply corollary 2.4 allowing to express $F_f(\cdot, T_1, T_2)$ dynamics under domestic forward measure $\mathbb{Q}^d_{T_2}$:

$$\boxed{\frac{dF_f(t, T_1, T_2)}{F_f(t, T_1, T_2)} = \big[\, r_f(t) + \rho_{F,X}(t)\, \sigma_F(t)\, \sigma_X(t)\, \big]\, dt + \sigma_F(t)\, dW^d_F(t)}$$

where $W^d_{T_2}$ is a brownian motion under $\mathbb{Q}^d_{T_2}$.

4 Second Application: LMM drifts

We show in this section how to calculate forward rates dynamics under LMM using a unique probability measure. The reader may refer to [BM2006] section 6.3.2 or [AP2010] section 14.2.2 for more details on the LMM model.

4.1 Notations and Framework

For the current section we consider only a single domestic measure. Therefore we do not use anymore superscripts d and f as in previous section.

- $(T_i)_{i \in [\![0,n]\!]}$ denotes a set of dates such as $T_i < T_j$ when $i < j$
- In order to lighten significantly notations, $F_i(\cdot)$ will denote forward rate $F(\cdot, T_{i-1}, T_i)$ and τ_i will denote day count fraction $\tau(T_{i-1}, T_i)$
- $W_i^{T_j}$ will denote the brownian motion driving F_i dynamics under $\mathbb{Q}_{T_j}$
- ρ_{ij} will denote correlation between $W_i^{T_j}$ and $W_j^{T_j}$

4.2 Linking Forward Measures

By definition, any asset divided by numeraire $P(\cdot, T_i)$, respectively $P(\cdot, T_j)$, is martingale under $\mathbb{Q}_{T_i}$, respectively $\mathbb{Q}_{T_j}$. By applying the same rationale as in section 3.1 we obtain the link between $\mathbb{Q}_{T_i}$ and $\mathbb{Q}_{T_j}$ through Radon-Nikodym derivative:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}_{T_i}}\left[\frac{d\mathbb{Q}_{T_j}}{d\mathbb{Q}_{T_i}}\right] &= \frac{P(t,T_j)}{P(0,T_j)}\frac{P(0,T_i)}{P(t,T_i)} \\
&= \frac{P(0,T_i)}{P(0,T_j)}\prod_{k=i+1}^{j}\frac{P(t,T_k)}{P(t,T_{k-1})}
\end{aligned}$$

Since, by non arbitrage[5]

$$F_k(t) = \frac{1}{\tau_k}\left(\frac{P(t,T_{k-1})}{P(t,T_k)} - 1\right)$$

we obtain the following proposition (with $j > i$)

Proposition 4.1. (Link Between Forward Measures) $\mathbb{Q}_{T_i}$ *and* $\mathbb{Q}_{T_j}$ *are linked through Radon-Nikodym derivative:*

$$\mathbb{E}_t^{\mathbb{Q}_{T_i}}\left[\frac{d\mathbb{Q}_{T_j}}{d\mathbb{Q}_{T_i}}\right] = \frac{P(0,T_i)}{P(0,T_j)}\frac{P(0,T_i)}{P(t,T_i)}$$

which can be reformulated using forward rates:

$$\mathbb{E}_t^{\mathbb{Q}_{T_i}}\left[\frac{d\mathbb{Q}_{T_j}}{d\mathbb{Q}_{T_i}}\right] = \frac{P(0,T_i)}{P(0,T_j)}\prod_{k=i+1}^{j}\frac{1}{1+\tau_k\,F_k(t)}$$

4.3 Calculating LMM Drifts

Each forward rate F_i is martingale under respective measure $\mathbb{Q}_{T_i}$, $i \in [\![1,n]\!]$. We therefore express F_i dynamics under measure $\mathbb{Q}_{T_i}$ as follows:

$$\frac{dF_i(t)}{F_i(t)} = \sigma_i(t)\,F_i(t)\,dW_i^{T_i}(t)$$

[5] See appendix A.1 for a proof of this equality.

where $W_i^{T_i}$ is a brownian motion under $\mathbb{Q}_{T_i}$.

Our aim here is to obtain F_i dynamics under $\mathbb{Q}_{T_j}$ with $j > i$. As we did for quanto adjustments, we follow the three step process defined in section 2.4.

Step 1: Defining Radon-Nikodym Derivative Z

Proposition 4.1, leads to the Radon-Nikodym derivative linking $\mathbb{Q}_{T_i}$ and $\mathbb{Q}_{T_j}$:

$$Z(t) = \frac{P(0,T_i)}{P(0,T_j)} \prod_{k=i+1}^{j} \frac{1}{1+\tau_k\, F_k(t)}$$

Step 2: Z Dynamics under $\mathbb{Q}_{T_j}$

We could calculate Z dynamics, but we will rather calculate directly the bracket $d\langle \ln(F_i),\, \ln(Z)\rangle(t)$ which is more simple[6].

Let us start by calculating $\ln(Z(t))$:

$$\ln(Z(t)) = \ln\left(\frac{P(0,T_i)}{P(0,T_j)}\right) - \sum_{k=i+1}^{j} \ln\left[1+\tau_k\, F_k(t)\right]$$

Therefore

$$\begin{aligned}
d\langle \ln(F_i),\, \ln(Z)\,\rangle(t) &= \langle\, d\ln(F_i(t)),\, d\ln(Z(t))\,\rangle \\
&= \left\langle \frac{dF_i(t)}{F_i(t)},\, d\left[\ln\left(\frac{P(0,T_i)}{P(0,T_j)}\right) - \sum_{k=i+1}^{j} \ln\left[1+\tau_k\, F_k(t)\right]\right]\right\rangle \\
&= \left\langle \sigma_i(t)\, dW_i^{T_i}(t),\, -\sum_{k=i+1}^{j} d\ln\left[1+\tau_k\, F_k(t)\right]\right\rangle \\
&= -\sigma_i(t) \sum_{k=i+1}^{j} \Big\langle dW_i^{T_i}(t),\, d\ln\left[1+\tau_k\, F_k(t)\right]\Big\rangle \\
&= -\sigma_i(t) \sum_{k=i+1}^{j} \left\langle dW_i^{T_i}(t),\, \frac{d\left[1+\tau_k\, F_k(t)\right]}{1+\tau_k\, F_k(t)}\right\rangle
\end{aligned}$$

Let us now write $1+\tau_k\, F_k(t)$ dynamics under $\mathbb{Q}^{T_j}$:

$$\begin{aligned}
\frac{d\left[1+\tau_k\, F_k(t)\right]}{1+\tau_k\, F_k(t)} &= \frac{\tau_k}{1+\tau_k\, F_k(t)}\, dF_k(t) \\
&= \frac{\tau_k\, \sigma_k(t)\, F_k(t)}{1+\tau_k\, F_k(t)}\, dW_k^{T_k}(t)
\end{aligned}$$

Therefore

$$\begin{aligned}
d\langle \ln(F_i),\, \ln(Z)\,\rangle(t) &= -\sigma_i(t) \sum_{k=i+1}^{j} \left\langle dW_i^{T_i}(t),\, \frac{\tau_k\, \sigma_k(t)\, F_k(t)}{1+\tau_k\, F_k(t)}\, dW_k^{T_k}(t)\right\rangle \\
&= -\sigma_i(t) \sum_{k=i+1}^{j} \frac{\tau_k\, \sigma_k(t)\, F_k(t)}{1+\tau_k\, F_k(t)} \Big\langle dW_i^{T_i}(t),\, dW_k^{T_k}(t)\Big\rangle
\end{aligned}$$

[6] In fact both calculations are closely related since we need volatility terms in both cases.

And since $\langle dW_i^{T_i}(t),\, dW_k^{T_k}(t)\rangle = \rho_{ik}(t)\, dt$, we can write:

$$d \left\langle \ln(F_i),\, \ln(Z) \right\rangle(t) = -\sigma_i(t) \sum_{k=i+1}^{j} \frac{\tau_k\, \rho_{ik}(t)\, \sigma_k(t)\, F_k(t)}{1 + \tau_k\, F_k(t)}\, dt$$

Step 3: Drift Resulting from corollary 2.4

We can now apply corollary 2.4 allowing to express F_i dynamics under measure $\mathbb{Q}_{T_j}$:

$$\boxed{\frac{dF_i(t)}{F_i(t)} = -\sigma_i(t) \sum_{k=i+1}^{j} \frac{\tau_k\, \rho_{ik}(t)\, \sigma_k(t)\, F_k(t)}{1 + \tau_k\, F_k(t)}\, dt + \sigma_i(t)\, dW_i^{T_j}(t)}$$

where $W_i^{T_j}$ is a brownian motion under $\mathbb{Q}_{T_j}$.

5 Third Application: Foreign Short Rate Hull & White Dynamics

When considering foreign short rate Hull & White dynamics, it is usefull to write these dynamics under domestic measure. It is the case for instance when working with a 3 factor cross currency model. Let us write the foreign short rate dynamics under foreign risk neutral measure $\mathbb{Q}^f$ as follows:

$$dr_f(t) = \left[\theta_f(t) - a_f r_f(t)\right] dt + \sigma_f(t)\, dW^f_{r_f}(t)$$

where $W^f_{r_f}$ is a brownian motion under $\mathbb{Q}^f$. We want to express r_f dynamics under domestic risk neutral measure $\mathbb{Q}^d$. Our task is simplified a lot by quanto adjustment application since steps 1 and 2 are exactly the same:

1. We already expressed Radon-Nikodym derivative linking domestic and foreign risk neutral measures in Proposition 3.1:
$$Z(t) = \mathbb{E}^{\mathbb{Q}^f}_t \left[\frac{d\mathbb{Q}^d}{d\mathbb{Q}^f}\right] = \frac{X_{df}(t) B_d(t)}{X_{df}(0) B_d(0)} \frac{B_f(0)}{B_f(t)}$$

2. We have expressed Radon-Nikodym derivative dynamics under foreign risk neutral measure $\mathbb{Q}^f$:
$$\frac{dZ(t)}{Z(t)} = \sigma_X(t)\, dW^f_X(t)$$

There is therefore only one step remaining which consists in applying corollary 2.2. We finally obtain the following dynamics under $\mathbb{Q}^d$:

$$dr_f(t) = \left[\theta_f(t) - a_f r_f(t) + \rho_{r,X}(t)\, \sigma_f(t)\, \sigma_X(t)\right] dt + \sigma_f(t)\, dW^d_{r_f}(t)$$

where $\rho_{r,X}(t)$ denotes instantaneous correlation between $r_f(t)$ and $X_{df}(t)$ brownian motions.

A Non Arbitrage Relations

We give hereafter a reminder of important relations resulting from non arbitrage.

A.1 Forward Rate

Non arbitrage implies the following relation linking forward rate to zero coupons:

$$F(t, T_1, T_2) = \frac{1}{\tau(T_1, T_2)} \left(\frac{P(t, T_1)}{P(t, T_2)} - 1 \right)$$

$F(t, T_1, T_2)$ is therefore a tradable asset (the tradable asset is $P(t, T_1)$) divided by numeraire $P(t, T_2)$ associated to measure $\mathbb{Q}_{T_2}$. Consequently, $F(\cdot, T_1, T_2)$ is martingale under forward measure $\mathbb{Q}_{T_2}$.

Let us now proove this non aribtrage relation. Consider at time t a Forward Rate Agreement (FRA) to invest one euro at time T_1 and giving back at maturity T_2 the amount of $1 + \tau(T_1, T_2)\, F(t, T_1, T_2)$ euros. The forward rate $F(t, T_1, T_2)$ is such as the FRA value at time t is null.

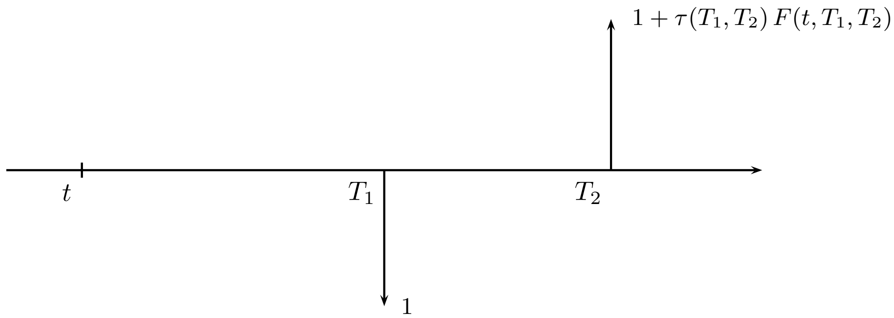

Figure 1: Forward Rate Agreement Cashflows

The FRA is perfectly replicated by buying an amount of $1 + \tau(T_1, T_2)\, F(t, T_1, T_2)$ zero coupons maturing at T_2 and selling a zero coupon maturing at T_1. Therefore the value at time t of the forward contract is $P(t, T_2)\, [1 + \tau(T_1, T_2)\, F(t, T_1, T_2)] - P(t, T_1)$.

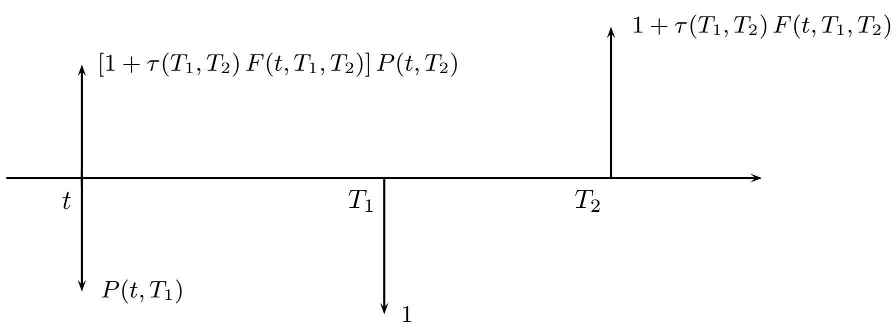

Figure 2: Forward Rate Agreement Replication

Since the value at time t of the FRA is null, we obtain $P(t, T_2)\, [1 + \tau(T_1, T_2)\, F(t, T_1, T_2)] - P(t, T_1) = 0$ and thus the result.

A.2 Forward Exchange Rate

Non arbitrage implies the following relation linking forward exchange rate to spot exchange rate and zero coupons in both currencies:

$$X_{df}(t,T) = X_{df}(t)\,\frac{P_d(t,T)}{P_f(t,T)}$$

Let us proove this result. Consider you buy at time t a forward contract allowing to receive at maturity T an amount $X_{df}(t,T)$ in foreign currency f in exchange of a unit of domestic currency d.

Figure 3: FX Forward

This payoff is perfectly replicated by buying at time t an amount $X_{df}(t,T)$ of foreign zero coupons and selling a domestic zero coupon, both maturing at time T.

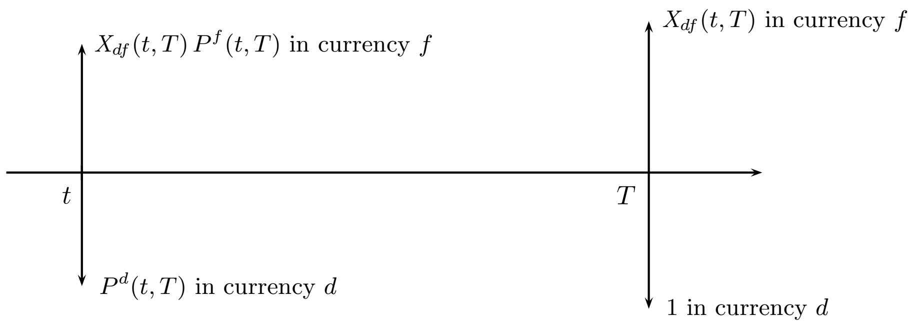

Figure 4: FX Forward Replication

Therefore, the value at time t of the forward contract in foreign currency is equal to $X_{df}(t,T)P_f(t,T) - X_{df}(t)P_d(t,T)$. Since a forward contract value is 0, the following relation holds:

$$X_{df}(t,T)P_f(t,T) = X_{df}(t)P_d(t,T)$$

which leads to the result.

B Spot FX Rate Dynamics

$\frac{B_d(t)X_{df}(t)}{B_f(t)}$ is martingale under foreign risk neutral measure $\mathbb{Q}^f$. B_d and B_f accounts evolve according to $dB_d(t) = r_d(t)B_d(t)dt$ and $dB_f(t) = r_f(t)B_f(t)dt$. Let us use the following notations for X_{df} dynamics under $\mathbb{Q}^f$:

$$\frac{dX_{df}(t)}{X_{df}(t)} = \mu_{FX}(t)dt + \sigma_{FX}(t)dW(t)$$

Since $\frac{B_d(t)X_{df}(t)}{B_f(t)}$ is martingale under foreign risk neutral measure $\mathbb{Q}^f$, its drift is null, implying the relation $\mu_X(t) + r_d - r_f = 0$. This leads to the expression of X_{df} drift under $\mathbb{Q}^f$:

$$\mu_X(t) = r_f(t) - r_d(t)$$

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