LMM support for RFR

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Abstract

This document describes the adaptation of the existing (Libor-based) production implementation of the Shifted-Lognormal Libor Market Model (SLMM) to support so-called RFR deals under Libor transition.

Keywords. Libor transition, alternative risk-free rates, RFR, ARRC, LMM, SLMM, FMM, SOFR, Term SOFR, SONIA, OAS1, DLIB.

DISCLAIMER

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1 Introduction

The market-wide transition from unsecured Libor benchmarks to new indices based on secured overnight risk-free rates (RFRs) has created the challenge of adapting the existing framework for pricing deals based on legacy Libors to support deals based on the new RFR rates.

Support is also required for the so-called *fallback protocol*, as described in [ISD], under which existing legacy Libor-based contracts are replaced by specified RFR-based contracts.

2 Modeling Issues Presented by the RFR Underlying

Daily compounded overnight rates typically have the form

$$R_{[t_0,t_n]} = \frac{1}{\Delta}\left[\prod_{i=1}^{n}(1+\delta_i R_{t_i}) - 1\right], \qquad \delta_i = \Delta(t_{i-1},t_i), \qquad \Delta = \Delta(t_0,t_n) = \sum_{i=1}^{n}\delta_i, \tag{2.1}$$

where each R_{t_i} is the daily value of the RFR to be compounded, such as SOFR, accruing over the overnight interval $[t_{i-1}, t_i]$, $i = 1, \dots, n$. Here $\Delta(T, T') > 0$ is the accrual fraction between any two dates $T < T'$, computed according to the chosen day-count convention. Typically,

$$\Delta(T,T') \approx T' - T \tag{2.2}$$

as a year fraction.

Our first modeling priority is to avoid the necessity to generate each individual daily value R_{t_i} separately, for model performance and efficiency reasons. This is accomplished by approximating the discrete product in (2.1) with a continuous-time integral[1],

$$\prod_{i=1}^{n}(1+\delta_i R_{t_i}) \approx \exp\left\{\int_{t_0}^{t_n} r_t dt\right\} = \frac{B_{t_n}}{B_{t_0}}, \tag{2.3}$$

so that, instead of a discrete time series of daily values R_{t_i}, the model uses a continuously defined *short rate* process r_t. The result of compounding is then expressed as a multiplicative increment, *i.e.* a ratio of the values of the *money market account*:

$$B_t = \exp\left\{\int_0^t r_s ds\right\}, \qquad t \geq 0. \tag{2.4}$$

The main requirement of the model, therefore, is the ability to generate the short rate and the money market account consistently with the discount factors and the term rates (Libors). In principle, any term structure model that admits a description in the HJM framework should be able to do this, but in practice most such known models have drawbacks and limitations. Simple models such as Hull-White have too few parameters so they cannot be calibrated well to swaption skews,

[1] Note that if $t_0 < 0$ as in the case of an *in-accrual* compounding period, then the integral is taken from $t = 0$ and the realized portion of the compounding is not handled by simulation.

implied rate correlations and other observed market information. More advanced HJM models such as general Cheyette or Quadratic Gaussian require non-trivial numerical computations. For all Markovian HJM models the complexity significantly increases with the number of risk factors.

By contrast, term structure models of the LMM type can handle multiple risk factors well, and have a rich set of parameters so that they can be calibrated to diverse market data. However, until now, they have been primarily used to generate discount factors and term rates only, and didn't have the short rate and the money market account processes. Our goal is to fill this gap, extending the current implementation of the SLMM so that both the *original Libor* observables (discount factors, on-grid and off-grid Libor rates) and the *new RFR* observables (short rate and money market account) can be generated together in a consistent, no-arbitrage manner.

3 SLMM for Risk-Free Term Rates

We assume that all randomness in the system comes from a d-dimensional vector of independent standard Brownian motions with respect to the risk-neutral probability measure $\mathbf{Q}$,

$$\vec{W}_t \in \mathbb{R}^d, \tag{3.1}$$

for some $d \in \mathbb{N}$. The exact value of $d > 0$ is immaterial and can always be assumed to be large enough so that as many individual Brownian motions, independent or correlated, as may be necessary could be constructed from (3.1).

We regard the valuation date as $t = 0$, and consider a fixed accrual grid

$$0 = T_0 < T_1 < \cdots < T_M, \qquad \tau_k = \Delta(T_{k-1}, T_k), \qquad k = 1, \ldots, M.$$

A single-period forward risk-free term rate with the accrual interval $[T, T']$ is defined in terms of the risk-free discount factors $P_t(T)$ as

$$F_t(T, T') = \frac{1}{\Delta(T, T')} \left[\frac{P_t(T)}{P_t(T')} - 1 \right], \qquad t \leq T < T'.$$

In particular, the forward risk-free on-grid Libors L_t^k are defined in terms of the grid discount factors $P_t(T_k)$ as

$$L_t^k := F_t(T_{k-1}, T_k) = \frac{1}{\tau_k} \left[\frac{P_t(T_{k-1})}{P_t(T_k)} - 1 \right], \qquad t \leq T_{k-1}, \tag{3.2}$$

for all $k = 1, \ldots, M$. From (3.2) we see that in a single-curve market environment there is no distinction between RFR forward term-rates and the forward Libors.

A Libor Market Model (LMM) specifies the dynamics of the forward risk-free on-grid Libors (3.2) in the form

$$dL_t^k = \vec{\sigma}_k^{\text{LMM}} \left(t, L_t^k \right) \cdot d\vec{W}_t^{T_k}, \qquad t < T_{k-1}. \tag{3.3}$$

Here each $\vec{\sigma}_k^{\text{LMM}}(t, L) \in \mathbb{R}^d$ is a given vector-valued deterministic function of the time t and the forward Libor variable L (and, possibly, other stochastic state variables of the system). Each $\vec{W}_t^{T_k} \in \mathbb{R}^d$ is a vector of independent standard Brownian motions with respect to the time-T_k

forward measure $\mathbf{Q}^{T_k}$ which corresponds to the choice of the discount factor $P_t(T_k)$ as the numeraire, and is equal to (3.1) plus an appropriate drift term.

The SLMM uses a *shifted lognormal* (SLN) dynamics with time-independent shifts,

$$\vec{\sigma}_k^{\text{LMM}}(t, L) = (L + \alpha_k)\vec{\sigma}_k(t), \qquad t \leq T_{k-1}, \tag{3.4}$$

with the shifts $\alpha_k \in \mathbb{R}$ that are deterministic constants, and vector-valued volatilities $\vec{\sigma}_k(t) \in \mathbb{R}^d$ that are deterministic functions of time. The LMM SDEs (3.3) with the SLN volatility coefficients (3.4) can be regarded as

$$dL_t^k = \left(L_t^k + \alpha_k\right)\vec{\sigma}_k(t) \cdot d\vec{W}_t^{T_k} = \left(L_t^k + \alpha_k\right) v_k(t) dZ_t^{T_k}, \qquad t < T_{k-1}, \tag{3.5}$$

with the scalar volatilities

$$v_k(t) = |\vec{\sigma}_k(t)| \geq 0,$$

and driven by the scalar Brownian motions $Z_t^{T_k} \in \mathbb{R}$ such that

$$dZ_t^{T_k} = \frac{\vec{\sigma}_k(t)}{v_k(t)} \cdot d\vec{W}_t^{T_k},$$

with pairwise instantaneous correlations

$$d\left\langle Z^{T_j}, Z^{T_k}\right\rangle_t = \rho_{j,k}(t)dt, \qquad \rho_{j,k}(t) = \frac{\vec{\sigma}_j(t) \cdot \vec{\sigma}_k(t)}{v_j(t)v_k(t)}, \qquad j, k = 1, \ldots, M.$$

Here each $Z_t^{T_k}$ is defined for $t \leq T_{k-1}$ only (because so are the volatilities $\vec{\sigma}_k(t)$ in (3.4), too), and is a standard, *i.e.* driftless, Brownian motion with respect to its own forward measure $\mathbf{Q}^{T_k}$. The set of model parameters for such a self-contained SLMM is completely described by the shifts α_k, the scalar volatilities $v_k(t)$, and the correlations $\rho_{j,k}(t)$.

4 Werpachowski Conditions for Stub Rates under SLMM

Werpachowski [Wer] aims to establish an arbitrage-free approach to simulating the discounts $P_t(T)$ at all times $t < T$. In order to be able to compute these discount factors, which can be obtained by performing off-grid interpolations in LMM, it is sufficient to know only the forward risk-free *stub rates* $F_t(T, T_k)$ for all $t \leq T$ and $T_{k-1} \leq T < T_k$. Indeed, if $T_{i-1} \leq t \leq T_i$ for some i, then on the date t all discount factors with maturities on the grid are

$$P_t(T_k) = \frac{1}{1 + \Delta(t, T_i)F_t(t, T_i)} \left(\prod_{j=i+1}^{k} \frac{1}{1 + \tau_j L_t^j}\right), \qquad k \geq i, \tag{4.1}$$

and, therefore, for any $T_{i-1} \leq t \leq T_i$ and $T_{k-1} \leq T \leq T_k$ with $i \leq k$, the corresponding off-grid discount factor is

$$P_t(T) = \frac{1}{1 + \Delta(t, T_i)F_t(t, T_i)} \left(\prod_{j=i+1}^{k} \frac{1}{1 + \tau_j L_t^j}\right) [1 + \Delta(T, T_k)F_t(T, T_k)]. \tag{4.2}$$

For the forward stub rates, [Wer] proposes a simple reduced model relating their values to their respective enclosing forward risk-free Libors. It then shows that this assumption satisfies no-arbitrage conditions.

Since [Wer] assumes a purely lognormal LMM while we are aiming at the SLN version as in (3.4), we have to generalize this prescription to accommodate non-zero SLN shifts for both the grid Libors and the stub rates. The generalized assumption is that, for any $T_{k-1} \leq T < T_k$, the ratio of the two appropriately shifted rates is constant as a function of time,

$$\frac{F_t(T, T_k) + \xi_k(T)}{\widehat{L}_t^k + \alpha_k} = \text{const}, \qquad 0 \leq t \leq T, \tag{4.3}$$

where $\xi_k(T)$ is some stub-specific but time-independent shift, and $\widehat{L}_t^k$ is a continuous extension of the k-th grid Libor (3.5) to the whole time interval $[0, T_k]$,

$$d\widehat{L}_t^k = \left(\widehat{L}_t^k + \alpha_k\right) \widehat{\sigma}_k(t) \cdot d\vec{W}_t^{T_k}, \qquad t < T_k,$$

where $\widehat{\sigma}_k(t) \in \mathbb{R}^d$ is some extension of the vector-valued volatility parameter $\vec{\sigma}_k(t)$ to $t \leq T_k$, such that

$$(t \leq T_{k-1} \quad \Longrightarrow \quad \widehat{\sigma}_k(t) \equiv \vec{\sigma}_k(t)) \qquad \Longrightarrow \qquad \left(t \leq T_{k-1} \quad \Longrightarrow \quad \widehat{L}_t^k \equiv L_t^k\right).$$

If we denote the stub-specific value of the time-independent ratio in (4.3) by $m_k(T)$. It can be computed using the deterministic data known at the time $t = 0$,

$$m_k(T) = \frac{F_0(T, T_k) + \xi_k(T)}{\widehat{L}_0^k + \alpha_k} = \frac{F_0(T, T_k) + \xi_k(T)}{L_0^k + \alpha_k},$$

so, in terms of it, the condition (4.3) can be expressed as

$$F_t(T, T_k) + \xi_k(T) = m_k(T) \left(\widehat{L}_t^k + \alpha_k\right), \qquad 0 \leq t \leq T.$$

The meaning of the assumption (4.3) is that, if the extended grid Libor $\widehat{L}_t^k$ has the SLN dynamics with a shift value α_k, then the dynamics of the forward stub rate $F_t(T, T_k)$ is also SLN with the same time-dependent volatility, but with a (possibly different) stub-specific shift $\xi_k(T)$. Indeed, then

$$dF_t(T, T_k) = m_k(T) d\widehat{L}_t^k = m_k(T) \left(\widehat{L}_t^k + \alpha_k\right) \widehat{\sigma}_k(t) \cdot d\vec{W}_t^{T_k} = [F_t(T, T_k) + \xi_k(T)]\, \widehat{\sigma}_k(t) \cdot d\vec{W}_t^{T_k}. \tag{4.4}$$

5 The Forward Market Model

In [LM1, LM2], Lyashenko and Mercurio introduced the Forward Market Model (FMM) as an extension of LMM that allows to model forward- and backward-looking risk-free rates simultaneously and consistently with each other. Similarly to the forward risk-free term rates (3.2), the *forward FMM rates* are defined as

$$R_t^k = \frac{1}{\tau_k} \left[\frac{\overline{P}_t(T_{k-1})}{\overline{P}_t(T_k)} - 1 \right], \qquad t \geq 0, \tag{5.1}$$

but, unlike the forward Libors, each FMM rate R_t^k is now defined at all times. Here the *extended* discount factors are given, for all times $t \geq 0$, by

$$\overline{P}_t(T) = \begin{cases} P_t(T), & t \leq T, \\ \frac{B_t}{B_T}, & t > T, \end{cases}$$

in terms of the ordinary discount factors $P_t(T)$ and the money market account (2.4), so that

$$R_t^k = \begin{cases} L_t^k, & t \leq T_{k-1}, \\ \frac{1}{\tau_k}\left[\frac{\frac{B_t}{B_{T_{k-1}}}}{P_t(T_k)} - 1\right], & T_{k-1} \leq t \leq T_k, \\ \frac{1}{\tau_k}\left(\frac{B_{T_k}}{B_{T_{k-1}}} - 1\right), & t \geq T_k \end{cases} \tag{5.2}$$

(note that the overlapping conditions agree at $t = T_{k-1}$ and $t = T_k$). The dynamics of the FMM rates (5.1) is assumed to be

$$dR_t^k = \gamma_k(t)\vec{\sigma}_k^{\mathrm{FMM}}(t, R_t^k) \cdot d\vec{W}_t^{T_k}, \qquad t > 0, \tag{5.3}$$

where the definitions of $\vec{W}_t^{T_k}$ and $\mathbf{Q}^{T_k}$ have been extended to all $t \geq 0$ by assuming that each time-T_k forward measure $\mathbf{Q}^{T_k}$ now corresponds to having the extended discount factor $\overline{P}_t(T_k)$ as its numeraire.

The vector-valued FMM volatility functions $\vec{\sigma}_k^{\mathrm{FMM}}(t, \cdot) \in \mathbb{R}^d$ are some extensions of the corresponding LMM volatility functions $\vec{\sigma}_k^{\mathrm{LMM}}(t, \cdot)$ to all $t \geq 0$. The deterministic scalar *decay coefficients* $\gamma_k(t) \in \mathbb{R}$ are defined for all $t \geq 0$, are globally continuous, and satisfy

$$\gamma_k(t) = \begin{cases} 1, & t \leq T_{k-1}, \\ \text{some smooth interpolation}, & T_{k-1} < t < T_k, \\ 0, & t \geq T_k. \end{cases} \tag{5.4}$$

Thus, by (5.3) and (5.4),

$$\big(t \leq T_{k-1} \implies \vec{\sigma}_k^{\mathrm{FMM}}(t, \cdot) \equiv \vec{\sigma}_k^{\mathrm{LMM}}(t, \cdot)\big) \implies \Big(t \leq T_{k-1} \implies R_t^k \equiv L_t^k\Big).$$

The particular choice of an extension of $\vec{\sigma}_k^{\mathrm{LMM}}(t, \cdot)$ to $\vec{\sigma}_k^{\mathrm{FMM}}(t, \cdot)$ may have an impact on the dynamics of the FMM rate R_t^k only for $T_{k-1} < t < T_k$, because any value of $\vec{\sigma}_k^{\mathrm{FMM}}(t, R)$ with $t \geq T_k$ will be multiplied by the zero value of $\gamma_k(t)$ in the SDE (5.3).

The introduction of the FMM rates (5.2) with their universal (*i.e.* for both $t < T_{k-1}$ and $t > T_{k-1}$) dynamics (5.3) makes it possible for the FMM to generate both the forward risk-free Libors L_t^k and the money market account B_t in the same model. In [LM3, LM4], Lyashenko and Mercurio define additional state variables that turn the FMM into a Markovian HJM model of the Cheyette type and allow exact closed-form calculations for discount factors $P_t(T)$ with arbitrary on- and off-grid maturities $T \geq t$. The values of the money market account can then be computed recursively over the accrual grid intervals by (5.2) as

$$B_t \;=\; B_{T_{k-1}} P_t(T_k) \left(1 + \tau_k R_t^k\right), \qquad T_{k-1} \leq t \leq T_k,$$

using the closed-form expressions for the discount factors in terms of the additional FMM state variables (see [LM3, LM4] for details).

6 SLMM methodology for RFR rates

While the FMM approach is an elegant formulation that extends the Libor Market model to handle RFR rates, the Werpachowski method provides some pragmatic advantages in terms of its integration into the current SLMM implementation, and at the same time respects the spirit of the FMM approach. Specifically, the SLMM with stub rates has been implemented as the term structure model for Libor-based interest rate derivatives such as European and Bermudan swaptions and CMS products, as documented in [Blo1] (with detail of the Werpachowski stub interpolation in [Blo1, App. E]). The necessary extensions of this implementation for supporting RFR overnight rates, and hence RFR payoffs derived as compounded rates, amounts to the accurate (or approximately accurate) simulation of the money market account. This is summarized in the following discussion.

The SLMM implementation employs time-independent shifts (3.4), and uses the additional assumption that the SLN shift for each forward risk-free stub rate in (4.3) is the same as for the enclosing forward risk-free *zombie* Libor,

$$\xi_k(T) = \alpha_k, \qquad T_{k-1} \leq T < T_k. \tag{6.1}$$

The SLN vol of the *zombie* rate uses a flat extrapolation[2], so that

$$d\widehat{L}_t^k = \left(\widehat{L}_t^k + \alpha_k\right) \widehat{\sigma}_k(t) \cdot d\vec{W}_t^{T_k}, \qquad t < T_k,$$

with

$$\widehat{\sigma}_k(t) \;=\; \vec{\sigma}_k(t \wedge T_{k-1}), \qquad t \leq T_k, \tag{6.2}$$

and, therefore, by (4.4),

$$dF_t(T, T_k) = \left[F_t(T, T_k) + \alpha_k\right] \widehat{\sigma}_k(t) \cdot d\vec{W}_t^{T_k}, \qquad T_{k-1} \leq T < T_k, \qquad t < T.$$

The simulations are performed in the *Libor spot* probability measure $\mathbf{Q}^*$, which has the effect of using a different forward measure $\mathbf{Q}^{T_k}$ during each simulation interval $T_{k-1} \leq t \leq T_k$.

[2] Strictly speaking, the flat extrapolation was mandated in the legacy Libor LMM, and persists in the RFR LMM when neither caps nor caplets are included among the calibration instruments. See §7.1 for the treatment of zombie vols when RFR-caps are included.

6.1 Modelling the money market account

The Werpachowski technique allows for calculation of any future discount factor $P(t,T)$ once we have simulated a Monte Carlo path up to sample date t (see (4.1) and (4.2)). In particular, we can sample the overnight discount factor seen at time t and thus incrementally build up the simulated money market account $B(t)$. Alternatively, one can derive the short-term rate $r(t)$ from the Werpachowski-based formula for the short-dated discount factor in terms of the zombie rate, and hence by sampling the short-term rate $r(t)$ we can calculate the money market account in terms of the approximation

$$\frac{B(T_{end})}{B(T_{start})} = \exp\left[\int_{T_{start}}^{T_{end}} r(t)dt\right] \approx \exp\left[\sum_k \tau_k r(t_k)\right]. \tag{6.3}$$

When pricing a given trade, for each pair of accrual-start and accrual-end dates T_{start} and T_{end} relevant to the trade's payoff, one needs to sample the money market account at T_{start} and T_{end} in order to obtain the compounded-in-arrears RFR rate between these dates:

$$R_{T_{end}}(T_{start}, T_{end}) = \frac{1}{\Delta}\left[\prod_{i=1}^{n}(1+\delta_i R_{t_i}) - 1\right] \approx \frac{1}{\Delta}\left[\frac{B(T_{end})}{B(T_{start})} - 1\right] \tag{6.4}$$

where $\Delta = \Delta(t_{start}, t_{end})$ is the day-count coverage of the accrual period.

In principle, one could simulate the money market account $B(t)$ daily over the period between T_{start} and T_{end}. In practice, however, this will be computationally slow. Instead, one simulates at a frequency much sparser than daily (such as monthly), thereby achieving much faster performance, while at the same time without incurring any significant impact on accuracy.

6.2 Multiple curves and basis

The Bloomberg implementation of the classical LMM supports dual curve calculations where there are separate discounting and floating rate projection curves. The details of the modelling approach can be found in [Blo1]. Without details, the idea is that both the spot Libor numeraire N_t and the on-grid forward rates L_t^k of (3.2)) which we model stochastically in the LMM are defined by reference to the *discount* curve[3]:

$$L_t^{k,disc} := F_t^{disc}(T_{k-1}, T_k) = \frac{1}{\tau_k}\left[\frac{P_t^{disc}(T_{k-1})}{P_t^{disc}(T_k)} - 1\right], \qquad t \le T_{k-1}. \tag{6.5}$$

The forward rates obtained from the projection curve are then modelled stochastically by applying a deterministic additive basis:

$$F_t^{proj}(T_{k-1}, T_k) = F_t^{disc}(T_{k-1}, T_k) + \beta(T_{k-1}, T_k) \tag{6.6}$$

where

$$\beta(T_{k-1}, T_k) = F_0^{proj}(T_{k-1}, T_k) - F_0^{disc}(T_{k-1}, T_k). \tag{6.7}$$

[3] The spot Libor numeraire is sometimes also referred to as the discrete Libor money market account.

Modelling the basis in this way aims to ensure that a forward contract which pays a floating rate defined by the projection curve is priced by the LMM consistently with the initial projection and discount curves:

$$E^N \left(F_t^{proj}(T_{k-1}, T_k)/N_{T_k} \right) = P^{disc}(0, T_k) F_0^{proj}(T_{k-1}, T_k). \tag{6.8}$$

In the RFR context there may be a requirement to support a specified discount curve and a different RFR projection curve. A typical example would be a trade where floating cashflows on funding and coupon legs are defined by reference to the SOFR compounded-in-arrears rate, but where the credit support annex (CSA) specifies that interest on collateral should be paid at the Fed Funds rate. In this case the projection curve should be the SOFR swap curve, while the discount curve should be the Fed Funds curve.

To support this basis between a discount curve and an RFR projection curve, we start by modelling the money market account $B^{disc}(t)$ defined by reference to the *discount* curve. We again use equations ((6.3)) and ((6.4)) to simulate the money market account $B^{disc}(t)$, but now all calculations are by reference to on-grid forward rates $L_t^{k,disc}$ inferred from the discount curve. Werpachowski interpolation is applied to these forward rates to sample a short rate $r^{disc}(t)$. We next define a *projection* money market account $B^{proj}(t)$ by applying a deterministic multiplicative factor to $B^{disc}(t)$:

$$B^{proj}(t) = \lambda_t B^{disc}(t), \tag{6.9}$$

where

$$\lambda_t := P^{disc}(0, t)/P^{proj}(0, t). \tag{6.10}$$

Finally, we simulate the realized value of the compounded-in-arrears RFR floating rate by reference to the sampled *projection* money market account:

$$R_{T_{end}}^{proj}(T_{start}, T_{end}) = \frac{1}{\Delta} \left[\frac{B^{proj}(T_{end})}{B^{proj}(T_{start})} - 1 \right]. \tag{6.11}$$

As with the classical case of forward-looking forward rates, this technique aims to give consistency of the pricing of floating RFR compounded-in-arrears contracts with the initial projection and discount curves:

$$E^N \left(R_{T_2}^{proj}(T_1, T_2)/N_{T_2} \right) = P^{disc}(0, T_2) R_0^{proj}(T_1, T_2) \tag{6.12}$$

where

$$R_0^{proj}(T_1, T_2) := \frac{1}{\Delta} \left(\frac{P^{proj}(0, T_1)}{P^{proj}(0, T_2)} - 1 \right). \tag{6.13}$$

More mathematical detail motivating the approach is presented in Appendix A.

7 Calibration to the RFR Market

The set of RFR market instruments used to calibrate the LMM contains both RFR swaptions and RFR caps. Because there is no material difference in the pricing methodology for RFR swaptions compared to classical Libor-based swaptions, there is consequently no significant change required to the calibration methodology to RFR swaptions compared to the classical case. However, given that the payoff and pricing methodology for RFR caps is different to the Libor-based case, some change is needed to the RFR cap calibration algorithm. This is described in this section.

For convenience, we recall that the calibration to swaptions in the classical LMM is based on approximating the terminal LMM distribution of the forward swap rate as shifted lognormal. The relevant *swaption approximation formula* is given in detail in the Bloomberg LMM documentation (see [Blo1, §D.2]) for pricing a Payer Swaption of expiry T_a, tenor $(T_b - T_a)$, and strike K:

$$\mathbf{PS}(0; a, b, K) \quad \approx \quad A^{ab}(0) \cdot \mathrm{Black}\left(S^{ab}(0) + \alpha^{ab}, K + \alpha^{ab}, \sqrt{V^{ab}}\right), \tag{7.1}$$

where V^{ab} is the terminal variance defined by

$$V^{ab} \quad := \quad \int_0^{T_a} \left[\sigma^{ab}(s)\right]^2 ds = \sum_{i,j=a}^{b-1} \rho_{i,j} \int_0^{T_a} {\gamma^{ab}}_i(s){\gamma^{ab}}_j(s) ds, \tag{7.2}$$

and we have used expressions $A^{ab}(0)$ for the swap annuity and α^{ab} for the effective shift derived in [Blo1, §C.4].

Note that when specializing (7.1) to the case of a forward caplet where $b = a + 1$ (which in the classical non-RFR case is just a 1-period swaption), the formula becomes the exact pricing formula for forward caplets:

$$\mathbf{Cpl}(0, T_{k-1}, T_k, \tau_k, K) \quad = \quad \tau_k P_k \,\mathrm{Black}\left(L_k(0) + \alpha_k, K + \alpha_k, \sqrt{V_k}\right), \tag{7.3}$$

where the terminal variance up to time T_k is given in terms of LMM model parameters by

$$V_k \quad := \quad \int_0^{T_k} \sigma_k^2(s)\ ds = \sum_{j=0}^{k-1} \tau_j \sigma_{j,k}^2. \tag{7.4}$$

Note also that in the classical case, the zombie rate is not required in the calibration. Instead, the zombie rate and its instantaneous volatility value are only used for evaluating *stub-rates* when pricing an instrument by Monte Carlo simulation (per [Blo1, §E]). In the classical LMM configuration, a modelling assumption is that the zombie rate instantaneous volatility parameter is assigned using flat-extrapolation: $\sigma_{k,k} := \sigma_{k,k-1}$.

By contrast to the classical case, when considering the calibration of the LMM to the RFR market which includes *backward* caps, the following points should be considered:

1. The swaption approximation formula continues to be directly applicable when calibrating to RFR swaptions.
2. The exact caplet pricing formula is still applicable to *forward-looking* RFR caplets, but it is not applicable to the market-standard *backward* RFR caplets which will be new calibration instruments.
3. The zombie rate $\widehat{L_k(t)}$ and its volatility $\widehat{\sigma}_k(t) = \sigma_{k,k}$ for $t \in [T_{k-1}, T_k]$, will now be required in the instrument pricing simulation for evaluating the RFR short-rate $r(t)$ (and thereby the realized compounded-in-arrears RFR rate). In this case, instead of flat-extrapolating the zombie rate instantaneous volatility value $\sigma_{k,k}$ we need to calibrate a value for this parameter in order to reprice backward caplets/floorlets (at least approximately).

7.1 Description of RFR Caplet Calibration Methodology

In this section we describe in more detail the extension of the LMM calibration methodology required to calibrate to RFR compounded-in-arrears caps and caplets. We will in fact only describe the calibration to RFR caplets. That is, we will assume that we have available a calibrated RFR cap volatility cube, which can be queried to return the undiscounted PV of a standard in-arrears RFR caplet for any expiry date, compounding tenor and strike (see also Remark 7.1 below). The methodology used by Bloomberg to construct and calibrate such an RFR cap volatility cube is summarized in [Blo2, §3.2.2].

The RFR caplet calibration methodology in fact reuses much of the existing mathematical methodology already present in the classical LMM calibration to standard forward-looking Libor-based caplets. The first step in the methodology is to use a simple modeling assumption to transform a set of PVs of target RFR-in-arrears caplet calibration instruments into a corresponding set of PVs for synthetic RFR forward-looking caplets. The forward-looking caplet expiries are the start dates of the corresponding compounding periods. We then simply apply the classical calibration methodology to these synthetic forward-looking RFR caplets (see (7.6) below). The technique above requires the choice of a modelling assumption to infer the PV of the synthetic forward-looking caplet premium from the corresponding backward-looking premium. The assumption is that the forward-looking premium is calculated using the *linear vol decay during compounding period* version of the Bachelier option pricing formula as a tool to translate between implied volatility quotes and caplet premiums. Equations explaining the key points are given below.

One additional complication in the RFR case is that, as noted in the previous section, there is an additional volatility parameter for each RFR forward rate which is not explicitly present in the classical case. This is the volatility parameter $\widehat{\sigma}_k(t) = \sigma_{k,k}$ controlling the instantaneous volatility of the zombie rate $\widehat{L_k(t)}$ over the final time period $[T_{k-1}, T_k]$ of its existence, namely the compounding period of the corresponding RFR-in-arrears caplet calibration instrument. These additional volatility parameters are calibrated in the new RFR cap calibration instead of simply being flat-extrapolated. The idea, at least intuitively, is that the additional zombie rate volatility parameter $\sigma_{k,k}$ will be determined by requiring the total variance of the backward-looking RFR forward rate to be consistent with the PV of the calibrating RFR caplet under Bachelier model assumptions, where the portion of this total variance accruing up to compounding-start date is determined by calibrating to the synthetic forward-looking RFR caplet.

We now present selected mathematical detail in support of the overview above. Suppose we have been provided the undiscounted PV $c^k_{back}(K)$ of an RFR-in-arrears caplet to which we wish to calibrate. Here the caplet has strike K and compounding period start date T_{k-1} and end date T_k which are aligned with LMM grid dates. The compounding period is τ_k. We solve for an implied *forward vol* $\sigma^k_{fwd}(K)$ corresponding to the given in-arrears premium by using the following *linear decay* version of the Bachelier option pricing formula[4]:

$$c^k_{back}(K) \quad = \quad \tau_k \,\mathrm{Bachelier}(L_k(0), K, \sigma^k_{fwd}(K)\sqrt{T_{k-1} + \tau_k/3}). \tag{7.5}$$

Here $L_k(0)$ is the RFR forward rate observed on the calibration date for the period from T_{k-1} to

[4] The linear decay version of the Bachelier formula assumes that the underlying forward rate follows a normal process with constant volatility up to T_{k-1}, after which the instantaneous normal volatility decays linearly over time to zero at T_k.

T_k. Once this step is complete we compute the price of the corresponding synthetic forward-looking RFR caplet with expiry T_{k-1} as

$$c^k_{fwd}(K) \quad = \quad \tau_k \,\text{Bachelier}(L_k(0), K, \sigma^k_{fwd}(K)\sqrt{T_{k-1}}). \tag{7.6}$$

We next explain how the zombie rate volatilities are specified. Consider first an idealized case where the LMM follows normal rate dynamics. In this case, we would simply set the zombie rate volatility parameter $\sigma_{k,k}$ for the calibrating RFR-in-arrears caplet equal to the implied forward volatility $\sigma^k_{fwd}(K)$ calculated as in (7.5) above. In reality, the LMM dynamics are shifted lognormal, and so we need to convert the parameter $\sigma_{k,k}$ by reference to this convention. This is done using the following equation, which implicitly defines the value of the parameter σ_{kk}:

$$c^k_{fwd}(K) \quad = \quad \tau_k \,\text{Black}(L_k(0) + \alpha_k, K + \alpha_k, \sqrt{T_{k-1}}\sigma_{k,k}). \tag{7.7}$$

Remark 7.1. As suggested earlier, caps can be selected in the UI as calibration instruments. Moreover, in terms of the LMM implementation, caps are first decomposed into their constituent caplets. Because caplets are also selectable as instruments in the UI, it may be found useful to illustrate some of the consequences of the cap decomposition into constituent caplets:

1. A single USD cap of maturity M will introduce $4M$ OTM caplets of the same strike.
2. A strip of OTM caps of the same fixed strike will introduce the same set of caplets as the single cap of maximal maturity.
3. A strip of ATM caps (each of its own strike) will introduce multiple caplets of multiple OTM strikes. For example, a strip 1Y to 3Y ATM caps will introduce a three-strike smile on the 9M-1Y caplet, and a two-point smile on the 1Y-1Y3M caplets, and many others targets.
4. An RFR caplet of maturity 3M is a valid calibration instrument.

8 Term RFR Rates

8.1 Regulatory background

The present document does not aim to give an overview of the regulatory background for Libor Transition. However, there are specific regulatory considerations regarding Term RFR Rates which are relevant to modelling choices, and which we discuss in this section.

As a general principle, global regulators overseeing Libor Transition recommend the use of specified daily risk-free-rates (RFRs) such as SOFR (and similar RFRs in other currencies) as a basis for replacing Libor. Regulators recommend that the significant majority of new contracts, and also the majority of fallback contracts, directly references averages of these daily-fixing benchmark RFR rates. These averages (used as reference rates within the calculation of trade cashflow amounts) can be based on geometric compounding, simple averaging, or in some cases, published indices (such as the SOFR Index published by the NY Fed, see [Fed]) from which a reference average can be calculated in a simple way[5].

[5] The calculation of a reference compounded average rate by means of a published daily-compounded RFR index is essentially ReferenceRate = (Index(End Date)/Index(StartDate)-1)/DayCountFraction, where the dates are the start and end of the reference interest period.

Notwithstanding the background above, regulators are prepared to countenance the use of so-called "Term Risk-Free Rates" in certain specific situations. In such cases, the reference rate is no longer an average of daily RFR rates over a specified interest period. Instead, the reference rate is a single fixing, typically made at the start of the reference interest period of a contract, of a Term Rate published by an institution which has been authorized by regulators to publish this Term Rate.

In particular, the US Alternative Reference Rates Committee (ARRC) has set out recommendations regarding the use of Term SOFR rates (see [ARR2] and [ARR1]). Term SOFR rates are countenanced only in a limited set of use cases, such as fallbacks for certain legacy cash products referencing Libor. The methodology for calculating Term SOFR must meet certain regulatory criteria for an alternative reference rate, similar to the criteria which SOFR itself is required to meet (see [ARR2, ¶2]). The use of derivative transactions based on Term SOFR is discouraged by regulators, and is only envisaged by them to be acceptable in strictly limited situations, such as end-user hedging of cash products referencing Term SOFR (see [ARR2, §1]).

As of first-quarter 2023, Term SOFR rates are published by CME for use as reference rates subject to the regulatory guidelines above (see [CME1]). Term rates are published daily for 1M, 3M, 6M and 12M tenors. The methodology implemented by CME to calculate these rates has been designed with the intent to follow regulatory guidelines for alternative reference rates and for financial benchmarks. The methodology estimates the value of each published rate using an algorithm developed by CME which is based on observing transaction prices from 13 consecutive 1M CME SOFR futures contracts and five consecutive 3M CME SOFR futures contracts. More information can be found at the CME Term SOFR FAQ page [CME2].

8.2 Support for Term SOFR

Bloomberg has added modelling support for Term SOFR to the RFR extension of the Libor Market Model in DLIB. This section summarizes the modelling approach.

We first consider the nature of Term SOFR trade payoffs and how these can be represented in the BLAN scripting language used by DLIB. Trade coupons which reference floating Term SOFR rates are defined in essentially the same way as the classical forward-looking Libor case. In more detail, a fixing value $R^{term}_{t_{fix}}$ is observed of a Term SOFR rate published on the coupon fixing date t_{fix}. The coupon is then calculated by reference to this fixing value. For example, the coupon of a capped and floored Term SOFR floater would be of the form

$$\text{Coupon} = \max(\,\text{Floor},\ \min(\,\text{Cap},\ R^{term}_{t_{fix}}\,)\,).$$

As described in §8.1 above, CME currently publishes the Term SOFR rate fixing each day for 1M, 3M, 6M and 12M tenors. Bloomberg has implemented market data tickers these published rates. For example, "TSFR3M Index", "TSFR12M Index" are the Bloomberg tickers for the published 3M and 12M Term SOFR rates, respectively. Term SOFR trades can then be created straightforwardly in DLIB by referencing the Term SOFR tickers above in the BLAN script or template which defines the coupon formula.

We next consider the modelling assumptions made for the Term SOFR rate within the LMM. The technique is similar to the classical LMM approach for forward-looking rates which reference a specified projection curve (as described in §6.2 above). To be more definite, suppose that T_1, T_2 are the start and end dates of the interest period corresponding to a Term SOFR contract with fixing date t_{fix}[6]. Then we model the stochastic forward Term SOFR rate by applying a deterministic additive basis to the SOFR curve forward rate:

$$F_t^{\text{Term SOFR}}(T_1, T_2) = F_t^{\text{SOFR}}(T_1, T_2) + \beta_{\text{SOFR}}^{\text{Term SOFR}}(T_1, T_2), \tag{8.1}$$

where

$$\beta_{\text{SOFR}}^{\text{Term SOFR}}(T_1, T_2) = F_0^{\text{Term SOFR}}(T_1, T_2) - F_0^{\text{SOFR}}(T_1, T_2). \tag{8.2}$$

The technique is consistent with the classical LMM approach given by equations (6.6) and (6.7) in §6.2. That is, the LMM stochastically models the evolution of on-grid forward rates derived from a specified *discount* curve. If we specify a different *projection* curve, then forward-looking floating rates referencing this curve are modelled stochastically by applying a deterministic additive basis. Indeed, equations (8.1) and (8.2) could equivalently be reformulated in terms of this additive basis to the discount curve. For example, if the discount curve were chosen to be the Fed Funds curve, then we have the equivalent formulation

$$F_t^{\text{Term SOFR}}(T_1, T_2) = F_t^{\text{Fed Funds}}(T_1, T_2) + \beta_{\text{Fed Funds}}^{\text{Term SOFR}}(T_1, T_2), \tag{8.3}$$

where

$$\beta_{\text{Fed Funds}}^{\text{Term SOFR}}(T_1, T_2) := F_0^{\text{Term SOFR}}(T_1, T_2) - F_0^{\text{Fed Funds}}(T_1, T_2). \tag{8.4}$$

The Bloomberg market data infrastructure includes provision for a wide variety of basis curves, including basis curves for Term SOFR against the standard SOFR swap curve. Specifically, within the Bloomberg curves function ICVS we have basis curves

- S558 (USD 1M CME Term SOFR)
- S559 (USD 3M CME Term SOFR)
- S560 (USD 6M CME Term SOFR)
- S561 (USD 12M CME Term SOFR)

When pricing a Term SOFR trade, the Term SOFR basis curve of the required tenor is passed to DLIB, and this enables the Term SOFR basis adjustment of (8.2) to be calculated in a straightforward way.

It should be noted that, while the Bloomberg infrastructure exists to support all required Term SOFR basis curves, nonetheless, as of first-half 2023, there is only very limited availability of market data sources from which to populate these curves. This is because, as noted in section §8.1 above, regulators at this time strongly discourage inter-bank Term SOFR derivative transactions (see [ARR2, §1]). Currently each of the Bloomberg Term SOFR basis curves S558-S561 is built using only a single front-end basis swap instrument, whose market price is inferred from the most recent

[6] For example, for a 3M contract, T_1 would be two business days after t_{fix} and T_2 would be three months after T_1 (taking into account appropriate date rolling conventions).

Term SOFR fixing published by CME[7]. Despite these observations, client end-users of Bloomberg's DLIB and MARS functions who wish to price and risk manage Term SOFR transactions can, in principle, create their own custom Term SOFR basis curves within the Bloomberg infrastructure, and populate such custom curves with indicative market levels chosen at the client's discretion.

When pricing a Term SOFR trade, the LMM can be calibrated by reference to vanilla SOFR option instruments, as described in §7. These calibration instruments can include vanilla SOFR swaptions and vanilla SOFR compounded-in-arrears caplets.

We make a few further remarks concerning LMM calibration for Term SOFR trades. Currently DLIB does not support direct calibration to Term SOFR instruments such as Term SOFR caplets. The main practical reason for this is that, as noted above, inter-bank Term SOFR derivative transactions are, at the time of writing, strongly discouraged by regulators. Therefore there is little or no transparent Term SOFR option market data available from which Bloomberg might attempt to populate a Term SOFR volatility cube, and to which one might try to calibrate the LMM. Despite these practical considerations, from a purely theoretical modelling perspective at least, it is in principle possible to define the volatility parametrization of the LMM in such a way as to facilitate a simultaneous calibration both to market-standard backward-looking SOFR caplets and/or SOFR swaptions and at the same time to forward-looking caplets such as Term SOFR caplets. This could be done, for example, by allowing the instantaneous volatility parameter $\sigma_{k,k}$ controlling the instantaneous volatility of the so-called *zombie rate* $\widehat{L_k(t)}$ over the RFR compounding period $[T_{k-1}, T_k]$ to be an additional free parameter in the model calibration (see paragraph 3 of §7.1 to recall the notation and the background). The use of this additional free volatility parameter $\sigma_{k,k}$ applicable over the in-arrears compounding period could (in principle) allow joint calibration to forward and backward-looking caplets. This is because this parameter could be calibrated by reference to the additional time value observed in the market price of a backward-looking caplet compared to its forward-looking equivalent. However, once again, there are practical considerations. Even if market prices for forward-looking Term SOFR caplets could be inferred, at least approximately (for example informed by reference to a market consensus pricing service) it is highly probable that these prices would be skewed by structural and liquidity effects arising from the regulatory restrictions on inter-bank transactions. In this context, a joint calibration of the model to standard SOFR options (which are liquid) and also to Term SOFR options (which are illiquid and where prices are likely to be skewed) may well lead to unrealistic calibrated model volatility parameters (through over-fitting of the model to inconsistent input data). Given the considerations discussed above, when calibrating to SOFR option instruments, currently DLIB only supports calibration to the more liquid standard SOFR swaptions and standard backwards-looking SOFR caplets[8].

To conclude this section on Term SOFR, we note that Bloomberg at present does not support automatic generation of Term SOFR fallback trades within DLIB. That is, if an end-user has a

[7] Notwithstanding the limited availability of data sources for populating the Term SOFR curve, the use of the synthetic front-end basis swap in this way when building the Term SOFR basis curve should have the result that the Term SOFR forward rate modelled by (8.2) converges to the published Term SOFR rate as a trade seasons towards the rate fixing date.

[8] When calibrating the LMM by reference to backward-looking SOFR caplets $c^k_{back}(K)$, one might ask what would be the normal term volatility of the corresponding Term SOFR caplets implied by this calibration. In this case, one would expect the normal term volatility of the Term SOFR caplet to be given approximately by the "forward volatility" σ^k_{fwd} defined by equations (7.5) and (7.6) in §7.1.

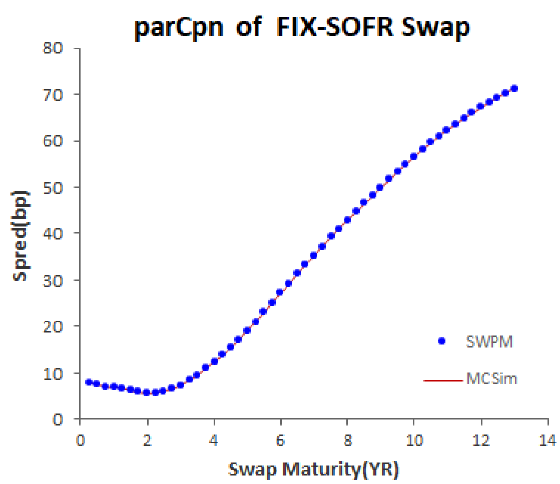
DLIB position booked within MARS which is subject to Term SOFR fallback (such as a legacy cash product referencing Libor) then the end-user will need to manually re-book a separate fallback trade which references Term SOFR appropriately, following the Term SOFR fallback protocol.

9 Suporting Tests

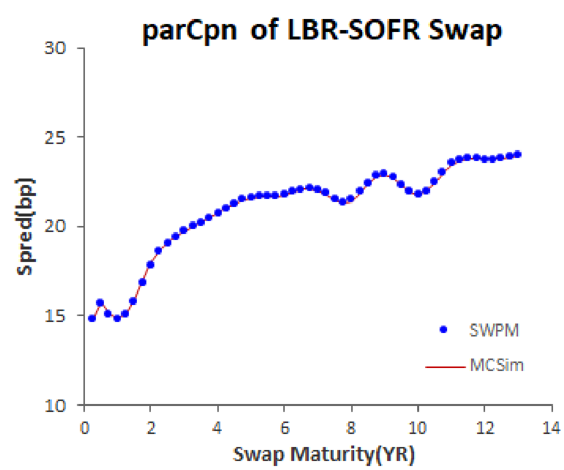
In this section we present testing results in support of several key validation points.

9.1 Correct pricing of linear RFR deals

A basic test is to confirm that the RFR extension of the SLMM correctly prices linear RFR instruments, and in particular RFR swaps. The tests in this section compare prices of SOFR swaps computed using (in the DLIB application) the SLMM with prices computed from SWPM. The SWPM calculations use simple analytic formulas depending only on discount factors and RFR forward rates interpolated from the SOFR curve. On the other hand, the SLMM calculations involve Monte Carlo simulation of the daily money market account, as described in the sections above.



(a) Par fixed rate of USD swap paying fixed vs. 90D SOFR (deal SL5157MZ) valued on October 12, 2020.



(b) Par margin of USD swap paying 3M Libor vs. 90D SOFR (deal SL9K0HRK) valued on October 12, 2020.

Figure 9.1: SOFR based Fix-Float and Float-Float swap comparisons with SWPM.

In this context, we consider comparisons between DLIB and SWPM in the case of standard SOFR swap and also a SOFR vs Libor basis swap. Simulated values of par rates for SOFR swaps paying quarterly fixed coupons can be compared against the values generated in equivalent deals constructed in SWPM. For this purpose a Fix-Float (SOFR) swap was priced with two fixed coupons from which a par-coupon could be backed out. The comparison results are presented in Figure 9.1a.

Similarly, simulated values of par spreads for SOFR swaps paying quarterly floating coupons can be compared against their values to equivalent deals constructed in SWPM. For this purpose a Float (LIBOR)-Float (SOFR) swap was priced with two fixed LIBOR-spreads from which a par-spread could be backed out. The comparison results are presented in Figure 9.1b.

9.2 Qualitative tests of RFR option pricing

In the following test we confirm the expected qualitative relationship between RFR compounded-in-arrears caplets and the corresponding classical forward-looking caplets. Assuming a single-curve setting, we would expect an RFR caplet to be worth more than its forward-looking equivalent, owing to the additional volatility experienced by the underlying RFR forward rate once the compounding period has started. We also expect the difference in prices between RFR and forward-looking caplets to decrease as time to maturity increases (because the compounding period, over which additional volatility is experienced, becomes a smaller proportion of total maturity).

When this test is performed, the resulting comparison of implied volatilities, as shown in Figure 9.2, displays the expected behavior. Note that, for purposes of this test only, the volatility profiles are implied from applying the Bachelier formula with the expiry date taken to be the payment date, which differs from the normal caplet pricing convention.

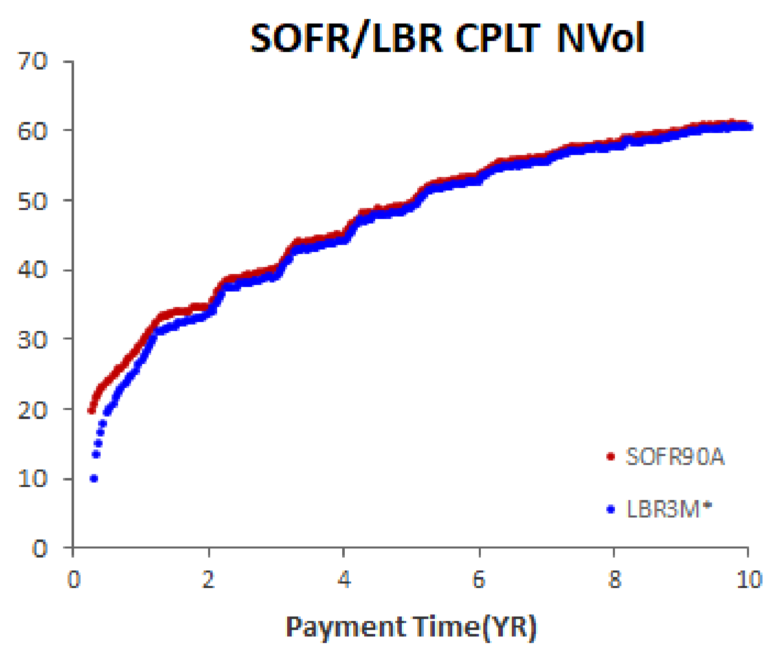
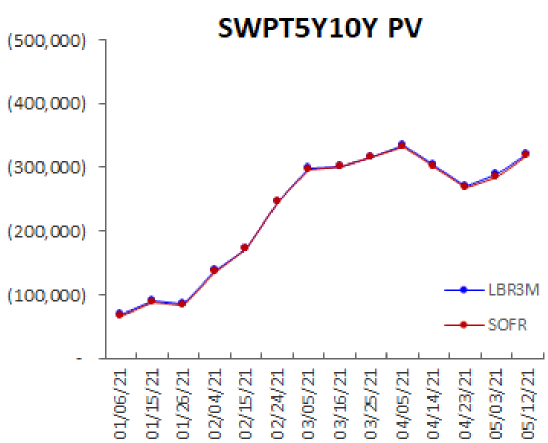


Figure 9.2: Normal implied volatilities of ATM caplets on 3M Libor and 90-day compounded SOFR with aligned payment dates, plotted as functions of the time to the payment date. Note that volatilites are implied from the Bachelier formula in which the expiry date is taken to be the payment date.

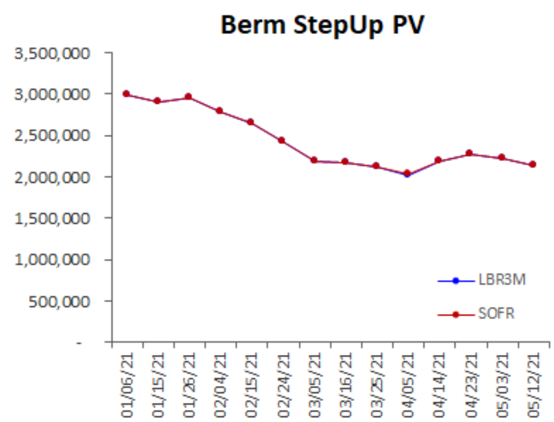
9.3 Consistency tests for European and Bermudan swaptions

In this test we check an expected edge case. Assuming a single-curve setting, the price of the RFR version of a European or Bermudan swaption should agree with the price of the classical forward-looking counterpart in the case where the pricing date is before the compounding period starts. Mathematically, this is simply because, in this case, on the pricing date the underlying RFR compounded-in-arrears forward rates coincide with the classical forward-looking rates; and moreover, all future option-exercise decisions of these trades depend only on forward rate values observed prior to a compounding period start. As the deals are priced using the same calibrated SLMM, these RFR and classical forward rates also have the same volatility up to the compounding start date. Although mathematically we would naturally expect the prices to coincide, the implementation of the pricing in these two cases is significantly different. In the RFR case, pricing the deal involves simulation of the daily-accrued money market account over compounding periods; while in the Libor-style case, all that is needed is to simulate a single Libor-style fixing for each forward rate. So the test is a good practical evaluation of whether the pricing algorithm is robust.

When this test is performed, the resulting comparison of simulated prices, as shown in Figure 9.3, displays the expected behavior.



(a) Price of European swaption 5Yx10Y.

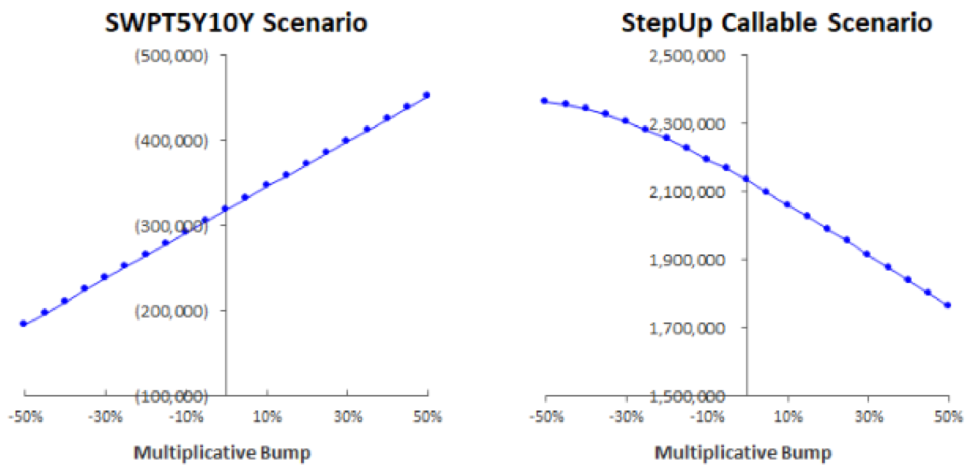


(b) Price of Bermudan Step-Up.

Figure 9.3: Comparing swaptions with compounded SOFRRATE versus LIBOR3m as floating index.

9.4 Stress test

As a follow up to the two swaptions considered in Figure 9.3, we examine their respective behaviors when undergoing a stressed volatility market. The resulting volatility ladders, or *vega scenario profiles*, are observed to be smooth with no kinks throughout the range of multiplicative bumps, ranging in size from -50% to $+50\%$.



(a) Volatility ladder for European 5Yx10Y swaption.

(b) Volatility ladder for Bermudan Step-Up.

Figure 9.4: Shock scenarios applied to the European and Bermudan trades showing smooth profiles, where multiplicative bumps ranging from −50% to +50% have been applied.

9.5 IBOR Fallback

Booking an RFR deal in DLIB which uses the RFR compounded rate as an underlying is straightforward to the extent that the underlying is explicitly identified by the RFR overnight curve[9] In terms of market data, the single RFR curve will be specified for both discounting[10]and forward projection. What is perhaps less straightforward is the specification of the Volatility Cube, where the swaption volatilities to which the SLMM is calibrated can be taken from either a LIBOR or RFR market.

However, additional consideration must be given to deals already in play that have been booked using IBOR underlyings. Following each currency's cessation date when IBOR *goes away*, not only will IBOR underlyings be inaccessible to freshly booked trades, but they will also need to be *proxied* with an underlying that makes reference to the prevailing RFR compounded rate. The applicable methodology is referred to as *IBOR Fallback*, and its implementation in DLIB is illustrated below.

For each IBOR rate, per term and currency, ISDA specifies fallback rates that are defined by adding a published spread to the compounded RFR rate with the corresponding accrual period, and are tickerized with the prefix V (such as VUS0003M for USD Libor 3M). After the IBOR cessation in a given currency, then under fallback, the IBOR indices used in legacy deal bookings must be mapped to the corresponding fallback rate.

The impact of fallback to a non-floater deal, taken here to be a European 5Yx10Y swaption in USD, is evaluated by comparing its price using the IBOR fallback ticker VUS0003M, against its price using the compounded SOFR rate plus ISDA spread of 26.16 bps. The comparison presented in Figure 9.5 shows good agreement.

[9]In the Bloomberg system these curves have unique identifiers such as SOFRRATE in USD, or SONIO/N in GBP.
[10]This applies as well to deals not depending on an RFR underlying.

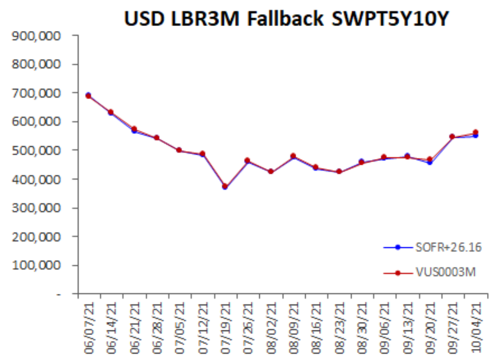
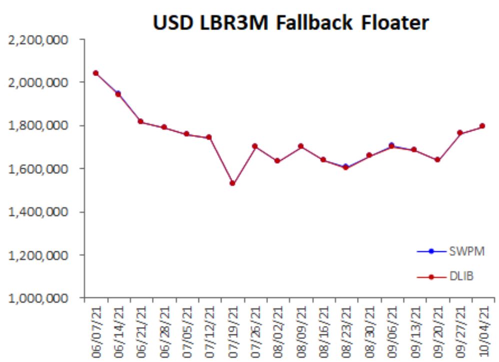
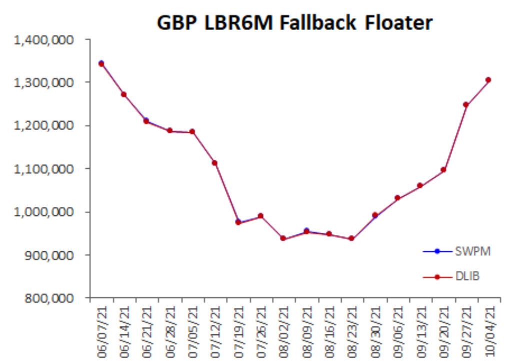


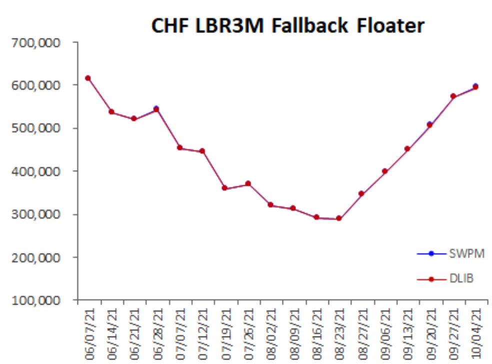
Figure 9.5: Historical trend of European 5Yx10Y swaption in USD using SOFR+26.16bp, and also the IBOR fallback ticker VUS0003M.



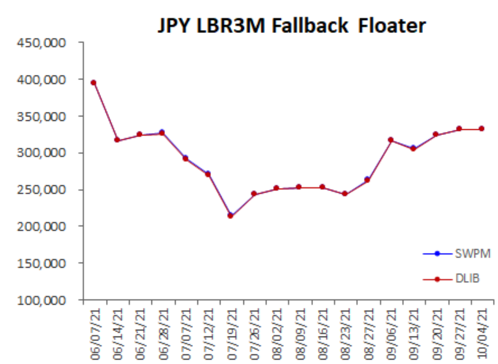
(a) Historical trend of USD fallback floater (SOFRATE+29.16bp vs VUS0003M).



(b) Historical trend of GBP fallback floater (SONIO/N+27.66 vs VBP0006M).



(c) Historical trend of CHF fallback floater (SRFXON3+7.41 vs VSF0006M).



(d) Historical trend of JPY fallback floater (MUTKCALM+5.81 vs VJY0006M).

Figure 9.6: Comparing historical trends of fallback floater booked in SWPM and DLIB. The SWPM floater is RFR compounded rate+ISDA-Spread, and DLIB floater uses the IBOR fallback ticker.

As shown above in Figure 9.6, the prices for a non-callable fallback floater in several currencies are compared between SWPM (based on the compounded RFR plus ISDA spread) and DLIB (based on the fallback IBOR index) with a good agreement.

APPENDICES

A Consistency of basis modelling for RFR-in-arrears rates

In this appendix we give mathematical motivation for the technique described in §6.2 for modelling the basis between a discount curve and an RFR compounded-in-arrears projection curve.

We show that, assuming that the spot Libor numeraire N_t and the money market account $B^{disc}(t)$, both of which are constructed by reference to the *discount* curve, satisfy all the usual martingale conditions required for absence of arbitrage, then floating RFR compounded-in-arrears contracts will theoretically be priced consistently with the initial projection and discount curves. We will use the notation of §6.2 without further explanation.

$$\begin{aligned}
E^N\left(R^{proj}_{T_2}(T_1,T_2)/N_{T_2}\right) &\overset{(6.9)}{=} E^N\left(\frac{1}{\Delta}\left(\frac{B^{proj}(T_2)}{B^{proj}(T_1)}-1\right)/N_{T_2}\right)\\
&\overset{(6.10)}{=} \frac{1}{\Delta}\left(E^N\left(\frac{\lambda_{T_2}B^{disc}(T_2)}{\lambda_{T_1}B^{disc}(T_1)}/N_{T_2}\right)-E^N\left(\frac{1}{N_{T_2}}\right)\right)\\
&= \frac{1}{\Delta}\left(\frac{\lambda_{T_2}}{\lambda_{T_1}}E^N\left(\frac{B^{disc}(T_2)}{B^{disc}(T_1)}/N_{T_2}\right)-P^{disc}(0,T_2)\right). \qquad (A.1)\\
E^N\left(\frac{B^{disc}(T_2)}{B^{disc}(T_1)}/N_{T_2}\right) &= E^N\left(\frac{1}{B^{disc}(T_1)}E^N\left(\frac{B^{disc}(T_2)}{N_{T_2}}|\mathcal{F}_{T_1}\right)\right)\\
&= E^N\left(\frac{1}{B^{disc}(T_1)}\frac{B^{disc}(T_1)}{N_{T_1}}\right)\\
&= P^{disc}(0,T_1); \qquad (A.2)\\
\frac{\lambda_{T_2}}{\lambda_{T_1}} &= \frac{P^{disc}(0,T_2)/P^{proj}(0,T_2)}{P^{disc}(0,T_1)/P^{proj}(0,T_1)}; \qquad (A.3)\\
\frac{\lambda_{T_2}}{\lambda_{T_1}}E^N\left(\frac{B^{disc}(T_2)}{B^{disc}(T_1)}/N_{T_2}\right) &\overset{(A.2),(A.3)}{=} P^{disc}(0,T_2)\frac{P^{proj}(0,T_1)}{P^{proj}(0,T_2)}. \qquad (A.4)\\
E^N\left(R^{proj}_{T_2}(T_1,T_2)/N_{T_2}\right) &\overset{(A.1),(A.4)}{=} \frac{1}{\Delta}\left(P^{disc}(0,T_2)\frac{P^{proj}(0,T_1)}{P^{proj}(0,T_2)}-P^{disc}(0,T_2)\right)\\
&= P^{disc}(0,T_2)\frac{1}{\Delta}\left(\frac{P^{proj}(0,T_1)}{P^{proj}(0,T_2)}-1\right)\\
&\overset{(6.13)}{=} P^{disc}(0,T_2)R^{proj}_0(T_1,T_2). \qquad (A.5)
\end{aligned}$$

This completes the theoretical derivation. Note that the equality

$$\frac{B^{disc}(T_1)}{N_{T_1}} = E^N\left(\frac{B^{disc}(T_2)}{N_{T_2}}|\mathcal{F}_{T_1}\right)$$

used above in (A.2) simply expresses the theoretical observation that $B^{disc}(t)$ is the price of a tradable security. One perspective on this is that a contract paying off $B^{disc}(T_2)$ units of cash at T_2 can be risklessly replicated by holding $B^{disc}(T_1)$ units of cash at T_1 and continuously rolling this cash up at the instantaneous rate $r^{disc}(t)$ until time T_2.

As a closing remark, we note that the basis modelling assumption for RFR-in-arrears floating rates as set out in §6.2 (in particular (6.9) and (6.10)) can also be regarded from the Heath-Jarrow-Morton (HJM) perspective as assuming that the *projection* curve instantaneous forward rate $f^{proj}(t, T)$ equals the *discount* curve instantaneous forward rate plus a deterministic additive basis $b(t, T)$:

$$f^{proj}(t, T) = f^{disc}(t, T) + b(t, T),$$

where $b(t, T)$ can be inferred from the initial discounting and projection curves.

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