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Shifted-Lognormal LIBOR Market Model

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Abstract

This document provides a client overview of the Bloomberg implementation of the shifted-lognormal LIBOR market model using several volatility and correlation models.

Keywords. LIBOR Market Model, Monte Carlo simulation, shifted lognormal, correlation model, cap calibration, swaption calibration, CMS-spread calibration, DLIB, SWPM, YASN.

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1 Executive Summary

The Bloomberg Shifted-lognormal LIBOR Market Model (SLMM) has been implemented with the objective of providing accurate pricing of both vanilla and exotic interest rate derivative products. The simplest examples include key-rate projections such as a 10 year swap rate, as required for certain mortgage applications. The more exotic examples include those with complex payoffs such as range accruals, complex optionality such as Bermudan callables, or complex underlyings such as CMS spreads. In practice, only a sufficiently flexible model can accommodate negative rates, or reproduce swaption skews at different expiries and tenor, or capture correlation between the two year and ten year swap rates. The SLMM, when properly calibrated, can achieve all of these objectives.

In addition to the prevailing interest rate curves used for discounting and LIBOR projections, the inputs to the SLMM model include the ATM and OTM swaption market, cap and caplet markets, and single and multi-look CMS spread option premiums.

The key mathematical elements of the model are the specification of a coupled system of SDE giving the dynamics of the forward LIBOR rates, according to which each forward LIBOR rate exhibits shifted-lognormal behavior controlled by certain *model parameters*. Each forward rate has a time-dependent volatility and a constant shift coefficient. Each pair of forward rates has a constant correlation coefficient obtained from a simple[1] two-parameter functional form. In sum, an N-LIBOR market is modeled by SLMM with N volatility functions, and (N+2) constants.

The Bloomberg SLMM model calibrates its model parameters to a user-definable set of calibration instruments selected from the swaption, cap, caplet, and cms spread option markets. The calibrated SLMM model can subsequently be used by DLIB's Monte Carlo simulator to price any genuine[2] interest rate deal supported by DLIB.

As with any model, the Bloomberg SLMM has a number of limitations. As a variant of the lognormal model, one might expect limitations regarding negative rates and volatility skew. However, the shifted lognormal does accommodate negative rates, and the proper calibration of the shift parameters can largely accommodate volatility skew, though not reverse skew, nor smile convexity. More precisely, shifted-lognormal Libors implies (approximately) shifted-lognormal Swap Rates, where the shift (an almost linear function of the Libor shifts) can modulate between lognormal and normal skew, but cannot reproduce smile curvature. This is an advantage over Hull-White models which imply a shifted-lognormal Swap Rate with a large fixed shift, which is to say *normal skew*. On the other hand, realistically modeling a Swap Rate smile requires a more general model, one in which the Libors follow SABR or other stochastic volatility dynamics. The correlation parameters can accommodate correlation dependent products to a degree, but will have difficulty depending on the OTM strike coverage. The possibility of lognormal blow-up during path generation does exist in theory, but in practice this is suppressed by imposing an upper bound on evolved states.

A notable limitation of the SLMM is its performance with long-dated complex deals. For example, a 30 year deal depending on a 30 year USD CMS will necessitate calibrating an exceedingly large number of model parameters to accomodate 60 years of Libors, as well as the simulation of 240

[1] The constant will be less straightforward in the case when factor-reduction is applied to the correlations.
[2] The SLMM is not available for pricing *hybrid* interest rate products in DLIB, such as Quanto Cap.

states over an average lifetime of 15 years. With a simulation granularity of three months, this implies the generation of roughly 15,000 states *per path*. On the other, factor reduction achieved by applying PCA in the calibration process, and the application of variance reduction techniques (such as Sobol quasi-random numbers) in the simulation phase, are among the several techniques employed by the SLMM implementation to overcome the numerical complexity associated with its high-dimensionality.

The model has been tested to confirm that its implementation meets its key objectives. The testing plan includes verifying that repricing vanillas matches the market quotes to which the model has been calibrated, and in particular that positive volatility skew observed in the market is reproduced by accurate calibration of the shift parameters; verifying that interpolation methods accurately price off-grid *stub-rates* associated with seasoned deals; verifying calibration stability as reflected in the bump-reprice scenarios, such as bucketed-vega and other Greeks; quantifying the limits of accuracy for approximation formulas, including CMS convexity adjustments; and finally, validating performance criteria when pricing computationally demanding products such as callable range accruals. All elements of the testing plan have demonstrated satisfactory results, several of which are presented in the final chapter.

2 Introduction and Context

Interest rate models

No-arbitrage modeling of the term structure of interest rates has a long history. The original attempts focused on modeling the dynamical (stochastic) evolution of the (unobservable) instantaneous risk-free interest rate, the so-called *short rate*. The short rate models of Hull-White and Ho-Lee are some popular examples of models in this category. The difficulty of calibrating these models to market instruments, which are based on actual observed rates with a finite tenor such as the LIBOR rates, led to the development of so called "Market Models". These market models directly postulate the dynamic evolution of observable rates which allows for a much easier interpretation of their parameters, as well as their calibration to observable market instruments.

Rates related products are ubiquitous, ranging from exotic derivative products to the projected level of interest rates required by mortgage products. A prominent example of the role of the interest rate model is its rate projections for the two year and ten year swap rates, whose correlation should be realistically modeled. Whereas the short rate models are ineffective, if not altogether incapable, at modeling the correlation between two year and ten year swap rates, the Libor Market Model, with its rich model parameter structure, has no such limitation.

A robust and consistent modeling of the interest rate process is critical for properly pricing these securities, with Market Models having become the de facto industry standard model, being used by most top financial institutions and market practitioners.

In what follows we describe Bloomberg's implementation of the LIBOR Market Model (LMM).

LIBOR Market Models

The LMM, in contrast to those models which simulate short-rate or instantaneous forward rates, simulates a set of forward LIBOR rates. LIBOR Market Models come in many flavors, the most popular of which are the Lognormal, Shifted-Lognormal, and CEV.

Short-rate models, such as Hull-White, may have been favored by many market practitioners in the past, but the sizable distribution of negative rates produced by such models in a very low interest rate environment makes them unacceptable. Moreover, the one and two factor Hull-White models cannot capture the volatility smile found in the swaption market, nor can they realistically model the correlation between the key two year and ten year swap rate projections.

The Shifted-Lognormal and CEV are essentially hybrids of the Lognormal and Normal models, modulating between these two extremes; in the former case by introducing an "additive shift parameter" ($\alpha \geq 0$), and in the latter case by introducing a "blending exponent parameter" ($0 < \beta \leq 1$). Unlike the short-rate models and the Lognormal LMM, the CEV and Shifted-Lognormal LMM can model the volatility skew found in the OTM swaption market.

Among the Lognormal, Shifted-Lognormal, and CEV LMMs (see ([SG]), only the Shifted-Lognormal LMM allows for negative rates, while at the same time preventing rates from becoming too negative.

We summarize in Table 2.1 the salient comparisons between the Shifted-Lognormal LMM and the Hull-White two-factor short-rate model.

	Hull-White Two-Factor	Shifted-Lognormal LMM
Underlying States	instantaneous short rate	directly observable LIBOR rates
Model Dynamics	Normal model	shifted-lognormal model
Swap Rate Dynamics	shifted-lognormal with fixed shift	shifted-lognormal (approximate)
Model Parameters	non-observable	realistic correlation structure
		observable volatility structure (ATM level)
		observable shift structure (OTM skew)
Calibration Features	caplets, ATM swaptions	full Swaption Matrix
	one additional OTM swaption	CMS-spread options
Negative Rates	prevalent in low-rate environment	low probability if properly calibrated
	no lower bound on negativity	negativity of rates bounded below
Flexibility	limited interest rate environments	highly adaptable due to rich parameter structure
Index Projections	unrealistically correlated swap rates	realistically correlated swap rates

Table 2.1: Contrasting Interest Rate Models

Shifted-Lognormal LMM (SLMM)

The Shifted-Lognormal LMM ameliorates the difficulties described above. For example, the model itself imposes no prohibition on negative forward rates, but the *shifted* forward rates are guaranteed to remain positive, while the distribution of *negative* forward rates turns out to be quite small in practice.

As explained above, all LMM models are based on simulating the evolution of a set of forward LIBOR rates, and thereby benefit from being directly observable in the market. In this spirit, the Shifted-Lognormal LMM has the additional advantage that the forward volatilities are intuitively associated with the volatility levels quoted in the ATM swaption market, and likewise the forward shifts are intuitively associated with the volatility skews quoted in the OTM swaption market. Finally, the forward-forward correlations are not only associated with directly observable data, but they may even be specified based on historical market data. These benefits may be contrasted with the less transparent association of CEV model parameters to market observables. Consequently, the Shifted-Lognormal LMM addresses certain shortcomings associated with competing LMM models.

For all of these reasons the Shifted-Lognormal LMM has been developed as a superior interest rate model for pricing exotic interest rate derivative products, as well as MBS pricing.

Document coverage

In this document we describe in detail the Shifted-Lognormal LIBOR Market Model (SLMM) as implemented in Bloomberg's DLIB terminal function. We present in the following sections a description of our SLMM model, providing mathematical details and motivation for certain design choices that have been made, for example regarding different flavors of volatility, correlation, and shift parameterizations. Finally, we discuss calibration quality using the Caplet, Swaption and CMS-spread option markets, and also pricing comparisons with real-world data including vanillas, exotics, and callable deals, as well as comparisons with SWPM [Blo4] which uses the LGM short-rate model and the CMS-replication model.

3 Model Overview

This chapter will summarize the key modeling components and their interaction.

1. Specification of deal characterizing interest rate derivative being priced (see BLAN)
2. Specification of LIBOR Market (schedule of reset dates, accrual dates, and accrual coverages) is set out in §3.1.
3. Specification of LMM Model Parameters (shifts, volatility and correlation models, including factor reduction) is set out in §3.2.
4. Specification of calibration market data (discount curve, swaption volatilities, and/or cap and cms spread option prices) is set out in §3.3.
5. Specification of calibration model data (initial values and bounds of calibrated model parameters, fixed values of non-calibrated model parameters).
6. Specification of calibration configuration settings (price *vs.*Black volatility *vs.*Normal volatility), while noting the non-configurable settings (*e.g.* the maximum number of optimizer iterations).
7. Specification of Monte Carlo configuration settings (number of simulation paths), while noting the non-configurable settings (*e.g.* choice of numeraire, seed, and sampling time interval).

3.1 LIBOR Market Specification

The starting point for all pricing is the specification of a deal, *i.e.* translating a deal's term-sheet into parameters that can be specified to the model . Although the details of specifying a deal are not discussed in this document, certain elements of the deal's structure are necessary to pricing with the LIBOR Market Model. To the extent that specific underlying indexes comprising a deal are related to the LIBOR rates of some tenor and in some currency (*e.g.* US0003M or EUR006M), for example CMS rates or the LIBOR rates themselves, special attention must be given to the schedules implicit in the deal structure. In particular, if the deal requires LIBOR rates $\{F_1, \ldots, F_n\}$ which are fixed at fixing dates[3] $\{T_1^-, \ldots, T_n^-\}$ and are applied to accrual periods $\{T_1, \ldots, T_n, T_{n+1}\}$, then the following LIBOR Market would be diagrammatically specified as follows:

$$\text{today} = T_0 \longrightarrow T_1 \xrightarrow[\tau_1]{F_1} T_2 \xrightarrow[\tau_2]{F_2} \cdots \xrightarrow[\tau_{n-1}]{F_{n-1}} T_n \xrightarrow[\tau_n]{F_n} T_{n+1}$$

The time T_0 indicates the evaluation[4] date of the deal, and the coverages $\{\tau_1, \ldots, \tau_n\}$ are the accrual times given in units of year-fractions, whose exact values are deduced from the day-count conventions specified by the deal. For example, the following data are associated to the US0003M index, using the ACT/360 day-count convention and an effective date of December 11, 2012:

k	Reset Date T_k^-	Accrual Start T_k	Accrual End T_{k+1}	Days -	Coverage τ_k	Reset Rate F_k
1	03/11/2013	03/13/2013	06/13/2013	92	0.2555	0.2801
2	06/11/2013	06/13/2013	09/13/2013	92	0.2555	0.2865
3	09/11/2013	09/13/2013	12/13/2013	91	0.2527	0.3113

Table 3.1: US0003M (ACT/360) on 12/10/2012 (First Three LIBOR Periods)

In general, every deal will determine a Libor schedule identical to that of a swap which starts on the settlement date (normally today's date, unless it is an aged deal), and which extends to include the deal's horizon date. Note that if calibration instruments are later selected (see §4.3) whose maturities necessitate the inclusion of additional Libors, then the Libor schedule's end date is correspondingly extended beyond the horizon date for purposes of calibration (but not pricing).

3.2 Model Specification and Model Parameters

The LiborMarketSpec described in §3.1 only partially characterizes the LIBOR Market Model. In fact, it is only the "LIBOR Market" part of the "LIBOR Market Model". Specifying the "Model" part of LMM requires the specification of the main model (the shifted-lognormal model in the present case) and its three additional sub-models (shifts, volatility and correlation).

The shifted-lognormal LMM supports three flavors of correlation sub-model (Rebonato two-parameter full-factor, Rebonato two-parameter reduced-factor, and of course a fully specified correlation matrix), two flavors of the volatility sub-model (constant, piecewise-constant), and one flavor of shift model (alpha-shift).

While we provide technical details in the appendices, for purposes of this section we merely state that the shifted-lognormal LMM models the dynamics of the LIBOR rates $F_k(t)$ using the following

[3]Fixing dates and forward rates are often referred to as reset dates and reset rates, respectively.
[4]The evaluation date is sometimes referred to as the settlement date, the pricing date, or the as-of date.

coupled system of stochastic differential equations[5]:

$$dF_k(t) = \mu_k^Q(t)dt + \sigma_k(t)(F_k(t)+\alpha_k)dW_k^Q(t), \tag{3.1a}$$
$$F_k(0) = L_k, \tag{3.1b}$$

where the various Brownian motions $dW_k^Q(t)$ have the "forward-forward" correlation structure given by ρ:

$$dW^Q(t)' \cdot dW^Q(t) = \begin{bmatrix} dW_1^Q \\ \vdots \\ dW_n^Q \end{bmatrix} \cdot [dW_1^Q, \ldots, dW_n^Q] = \Big(\{\rho_{i,j}dt\}\Big) =: \rho dt. \tag{3.2}$$

These SDE's contain both diffusion terms $\sigma_k(t)F_k(t)dW_k^Q(t)$ and drift terms $\mu_k^Q(t)dt$ which are described in Appendix C. What concerns us in this section is identifying the model parameters $\sigma_k(t), \alpha_k(t)$ and $\rho_{i,j}(t)$, and explaining how they are specified by the client user.

Strictly speaking, the model parameters are the functions $\sigma_k(t), \alpha_k(t)$ and $\rho_{i,j}(t)$. However, inasmuch as we mandate parametric forms corresponding to a "volatility-model" and a "correlation-model" for these functions, the model parameters are identified with the associated functional parameters. For example, when using the piecewise-constant volatility-model we have $\sigma_i(t) = \sigma_{i,j}$ for $t \in [T_j, T_{j+1})$, and so $n(n+1)/2$ model parameters must be specified. Similarly, when using the Rebonato (two-parameter, full-factor) correlation-model[6] we have:

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)e^{-\rho_d|T_i - T_j|}, \tag{3.3}$$

and so only the two parameters ρ_d and ρ_∞ must be specified.

We summarize in tables Table 3.3, Table 3.4, and Table 3.2 the model parameters associated with the different flavors of volatility structure and forward-forward correlation structure, as well as the shift parameters specific to the shifted-lognormal base model:

Shifts	Constant
$\alpha_i(t)$	α_i
Parameters	α_i (= 0 for lognormal)
Number Parameters	n

Table 3.2: Shift-Model Parameters

Volatilities	Constant	Piecewise Constant
$\sigma_i(t)$	σ_i	$\sigma_{i,j}, \quad t \in [T_{j-1}, T_j)$
Parameters	σ_i	$\sigma_{i,j}$
Number Parameters	n	$n(n+1)/2$

Table 3.3: Volatility-Model Parameters

[5]The superscript Q here indicates the Q-measure associated with a choice of Numeraire.
[6]None of the correlation models considered here are time-dependent; $\rho_{ij}(t)$ depends only on the indices (i, j).

Correlations	Rebonato Full-Factor	Rebonato Reduced-Factor	Explicit Cholesky $A \cdot A'$
$\rho_{i,j}$	$\rho_\infty + (1-\rho_\infty)e^{-\rho_d \lvert T_i - T_j \rvert}$	f-factor PCA	$\sum_k a_{i,k} a_{j,k}$
Parameters	ρ_∞, ρ_d	ρ_∞, ρ_d, f	$\rho_{i,j} \quad j>i$
Number Parameters	2	2	$n(n-1)/2$

Table 3.4: Correlation-Model Parameters.

The starting point for constructing a LIBOR market model will therefore consist of specifying the choice of each sub-model, as well as the initial values (or default values if uncalibrated) for all of the corresponding model parameters.

To illustrate how a LIBOR market model can be constructed in practice, let's assume that in a three-period LIBOR market the lognormal model is chosen with a volatility model of constant type $(\sigma_1, \sigma_2, \sigma_3)$, and a correlation model of Rebonato full-factor type (ρ_d, ρ_∞). The selection of a lognormal model implies $\{\alpha_1 = \alpha_2 = \alpha_3 = 0\}$, and we arbitrarily assign $\{\sigma_1 = 0.11, \sigma_2 = 0.12, \sigma_3 = 0.13\}$ to the constant vols associated with the forwards F_1, F_2, F_3, and set $\{\rho_\infty = 0.35, \rho_d = 0.15\}$ in the Rebonato full-factor correlation parameters.

Figure 3.1: Volatility and Shift model parameters

Note that, as displayed in Figure 3.1, the collection of $\sigma_{i,j}$, each of which is the volatility of the i'th LIBOR $F_i(t)$ during the time interval when $T_j < t < T_{j+1}$, has been flattened accordingly:

$$\begin{bmatrix} \sigma_{1,1} & 0 & 0 \\ \sigma_{2,1} & \sigma_{2,2} & 0 \\ \sigma_{3,1} & \sigma_{3,2} & \sigma_{3,3} \end{bmatrix} \longmapsto [\sigma_{1,1}, \sigma_{2,1}, \sigma_{2,2}, \sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}];$$

$$\begin{bmatrix} 0.11 & 0 & 0 \\ 0.21 & 0.22 & 0 \\ 0.31 & 0.32 & 0.33 \end{bmatrix} \longmapsto [0.11, 0.12, 0.21, 0.22, 0.23, 0.31, 0.32, 0.33].$$

Observe that $\sigma_{i,j} = 0$ when $j > i$ because $F_i(t > T_i)$ has no volatility after its fixing date, and is therefore omitted from the parametrization.

3.3 Model Inputs and Outputs

The market inputs to the SLMM model are the discount curve, forward curve, volatility cube, and CMS-SO premiums. The market inputs are snapshotted as of a particular Market Data date specified in the DLIB screen. The fundamental configuration inputs to the SLMM model are the number of factors (normally four), and the number of Monte Carlo paths. The discount curve is used for discounting of cashflows, while the forward curve, represented as a *basis spread* over the *OIS Libors* backed out from the discount curve, determines the intial values of the Libor forward rates. Swaption, cap, and caplet calibration instruments are obtained as either Black or Normal volatilities from VCUB. See §4.3 for details on selection of calibration instruments.

The calibration process will apply numerical algorithms (root-finding, approximation formulas, optimization methods) to determine the model parameters $\alpha_k, \sigma_{i,j}, \rho_\infty, \rho_{decay}$, and return this *calibrated* model in its response to a calibration request. A subsequent pricing request will be passed the calibrated model, along with other configuration parameters such as the number of paths, and will perform a Monte Carlo simulation which first and foremost will return a price and a standard-error. In addition to the price and standard-error, a particular BLAN script may trigger cashflow data to be returned in the pricing response.

In the case of Greeks, additional market data will be passed in the amended calibration requests, namely the *SHOC'ed* market data, and additional simulation results will be returned in the pricing response, namely the requested Greeks such as DV01.

3.4 Summary of Inputs and Outputs

In Table 3.5 we present a high-level summary of model inputs, including key market-observable rates, non-market-observable parameters, model settings, data used to build the model.

Model Input	**Input type**	**Remarks**
Forward LIBOR rates: $F_i(0)$	Market-observable	Implied from par swap rates.
Discount curve: $D(t)$	Market-observable	Implied from OIS rates.
Interest Rate Volatility cube: $\hat{\sigma}(e,t,k)$	Data stripped from market-observables	Implied from swaption and cap ATM and OTM markets.
CMS spread option premiums	Market-observable	Contributor quotes or implied from historical correlations.
$\hat{\sigma}_{i,j}, \hat{\alpha}_k, \hat{\rho}_d, \hat{\rho}_\infty$	Model setting	Initial guess (or, if not calibrated, fixed value) for optimizer.
Number of reduced-factors	Control parameter	Controls the number of independent Brownian drivers for the SDE system.
Number of Monte Carlo paths	Control parameter	Controls the number of simulations when deals are priced by Monte Carlo. DLIB uses 20,000 paths as default.

Table 3.5: Summary of Model Inputs

In Table 3.6 we present a high-level summary of model outputs.

Model Output	Output type	Remarks
Shifted Libor volatilities: $\sigma_i(t)$	Calibrated parameter	Piecewise constant term structure of shifted-lognormal volatility for each forward Libor F_i.
Libor shifts: α_i	Calibrated parameter	Constant shift associated with each F_i.
Libor-Libor correlation: $\rho_{i,j}$	Calibrated parameter	Constant correlation between F_i and F_j (after possible factor-reduction).
Price and Std-Error	Simulation results	For a calibrated model, price and std-error depend only on the number of paths.
MC Projections	Cashflow payments and BLAN notifications	BLAN script can *notify* on any function of underlying states.

Table 3.6: Summary of Model Outputs

3.5 Model Components and Design

This section summarizes the key modeling components and their interaction.

- **Configuration of calibrator**
  This includes specifying the number of factors in the factor-reduction, and which model parameters will be *fixed* versus *calibrated*. This component is described in §4.5 - §4.7.

- **Calibration of shift parameters** (α_k)
  This is the first stage, called the Decoupled-Shift calibration, in the calibration process. This is described in §4.9.

- **Calibration of volatility and correlation parameters** ($\sigma_{i,j}, \rho_{i,j}$)
  This process is set out in §4.8.

- **Deal Pricing**
  Once the model is fully calibrated it can be used to price trades by Monte Carlo simulation. These model components are discussed in §5.

3.6 Conceptual soundness of the model

The Shifted-lognormal Libor Market Model is a robust and arbitrage-free model for modeling Libor rates, and consequently for pricing (using Monte Carlo simulation) any deal whose payoff is a function of Libor rates. Because of the robust and arbitrage-free interpolation methods employed, payoffs that are functions of *off-grid* Libors can equally well be priced. By virtue of the shifted-lognormal modeling of swap rates, volatility skews (between lognormal and normal) can be reproduced in the swaptions market. Furthermore, by virtue of the model's Libor-Libor correlation structure calibrated to CMS spread option premiums, one can price CMS spread products. As already discussed in §1 and tabulated in Table 3.8, modeling limitations include extreme swaption volatility skew, swaption volatility smiles and correlation smiles.

3.7 Summary of Assumptions and Approximations

In Table 3.7 we present a high-level summary of key modeling assumptions and approximations.

Assumption or Approximation	**Description**	**Assessment**
Dynamics of forward LIBOR rates $F_i(t)$: (3.1a)	Each forward LIBOR rate follows a shifted-lognormal process, where the Brownian drivers $dW_i(t)$ for each $dF_i(t)$ are correlated.	This specification of the underlying dynamics has all the advantages of a LIBOR Market Model, while enjoying additional benefits afforded by the shifted-forward, namely allowing negative rates, and reproducing volatility skew.
Form of volatility functions: $\sigma_i(t)$	Piecewise-constant term-structure: $\sigma_i(t) = \sigma_{i,j}$	Because of the large number of parameters in comparison with the number of calibration instruments, the assumption does not impose any real limitation. In fact, see section on *linking* which explains how additional identifications are sometimes desirable to limit over-parameterization.
Form of LIBOR shifts: $\alpha_k(t)$	Constant term-structure: $\alpha_k(t) = \alpha_k$	Constant shifts are adequate for calibrating to swaption volatility skew, as described in section on Shift-Calibration. On the other hand, it may represent a limitation to the simultaneous calibration to OTM CMS spread options.
Forward-Forward Correlation:$\rho_{i,j}(t)$	Correlations are time-independent: $\rho_{i,j}(t) = \rho_{i,j}$	There is no clear advantage to supporting time-dependent correlations, since only the *terminal* correlations are observable, and needless over-parameterization is undesirable.
Forward-Forward Correlation:$\rho_{i,j}$ (3.3)	Full correlation matrix has Rebonato (2-parameter) functional form. In the case of factor-reduction, PCA is performed on Rebonato functional form, thereby producing a reduced-rank f-factor correlation matrix	Guarantees positive-definiteness which is requirement of any correlation matrix. Functional form reflects (F_i, F_j) de-correlation over $\lvert i-j \rvert$ to asymptotic value. In practice it is found that the principal eigenvalue is dominant, and all but a few eigenvalues are negligibly small, thereby justifying the application of PCA.
Swaption approximation formula: (D.6)	Swap rate dynamics modeled as shifted-lognormal by matching moments	Accuracy of approximation degrades with increasing maturity and increasing moneyness.
Convexity-adjusted CMS rate: (D.25)	Model-dependent approximation, bivariate Copula, or replication integral	All approaches are susceptible to limits of numerical approximation.

Table 3.7: Summary of Model Input Assumptions and Approximations

3.8 Summary of Limitations

A consequence of the shifted-lognormal dynamics used in the SLMM is that volatility skew must lie between lognormal (shift $\alpha_k = 0$) and normal (shift $\alpha_k > 20\%$). Super-lognormal (negative shift) calibration to match positive skew is not supported, as well as sub-normal (unachievable shift) calibration to match excessive negative skew. In Table 3.8 we present a summary of certain limitations of the model. See also Appendix I for more detail on the 20% upper-bound to the shifts α_k.

Limitation	Description	Assessment
Calibration to positive volatility skews	Super-lognormal dynamics, which correspond to negative shifts ($\alpha_k < 0$), is not supported by SLMM.	This extreme case has been observed in some non-USD currencies, and can result in poor OTM calibration.
Calibration to extreme negative volatility skews	Sub-normal dynamics, which correspond to unachievable shifts, cannot be supported.	This extreme occurrence will result in poor OTM calibration.
Calibration to flat Black volatilities in combination with large negative rates	Inconsistent or competing constraints may impose imperfect shift-calibrations.	Extreme negative rates will impose a lower-bound on the shift-calibration, which bound may constrain even a flat OTM calibration.
Calibration to swaption volatility smile	Shifted-lognormal dynamics cannot model swaption convexity.	More general models, such as stochastic volatility, are required to model swaption smile.
Calibration to correlation smile	Simultaneous calibration to multiple OTM CMS-SO is problematic.	More general models, such as stochastic volatility and/or local correlation, are required to model correlation smile.
Monte Carlo performance	Deals whose payoffs depend on low probability events will require many paths to be priced. Deals whose payoffs depend on frequent monitoring will perform slowly.	Deep OTM deals will be slow to price, although SOBOL random number generation offers variance-reduction. Daily range accruals with long maturity will be slow to price, although Brownian bridging technique offers a performance benefit.
Stability and accuracy Greeks	If Monte Carlo prices are unstable, then Greeks cannot be accurate.	Freezing technique for American options offers a stability benefit for accurate Greeks, although non-differentiable payoffs will always present difficulties, especially for higher-order Greeks.

Table 3.8: Summary of Model Limitations

3.9 Calculation of Greeks

There are two varieties of Greeks; the so-called *pricer-Greeks* generated by DLIB, and the *GE-Greeks* generated by the Greeks Engine.

The GE Greeks are computed using the Greeks Engine, which implements a bump-reprice methodology in response to a SHOC scenario. A description of the Theta, KRR, and Vega bucketing (tenting) methodologies for these Greeks are described in [Blo3].

The pricer Greeks for Interest Rate deals, in particular those using the LMM model, include Theta, DV01, (parallel) Gamma, and (parallel) Vega. There is no bucketing for either KRR or Vega. Details on the pricer Greeks for LMM, and DLIB generally, can be found in [Blo2].

GE Greeks: Stickiness

For GE Greeks, stickiness assumptions are controlled by the SHOC API, in particular, the Interest Rate SHOC setting called *Shift Cube Underlying Curves*, where No indicates Sticky Strike, and Yes indicates Sticky Moneyness. In other words, a bump to the *spot*, which in interest rates means either a *parallel* bump to the interest rate curve, or a bucketed bump to a *key-rate* corresponding to a par swap rate from which the curve is stripped (or a tenor point in the case of a synthesized curve such as the Forward curve or Zero curve). In any case, after bumping the curve the Vol-Cube is either regenerated (*i.e.* sticky moneyness), or not (*i.e.* sticky strike). In the former case, the ATM vol is still associated with the ATM swaption after the bump, whereas in the latter the ATM vol becomes associated with what was formerly an OTM strike.

Pricer Greeks: DV01 and Vega

Pricer Greeks refer to the Greeks computed in DLIB and displayed in the Pricing:Greeks screen shown in Figure 3.2.

The DV01 computed by DLIB is similar to a *model Greek*, as no re-calibration will occur. While it is similar to a model Greek in that no re-calibration is performed, it is unlike most model Greeks which will generally bump internal model parameters as proxies for market bumps. In the case of DLIB however, not even model parameters are bumped. Up and down shifted curves (swap curves, forward curves, and basis curves) are passed for the given currency for the *pricing request* only. In other words, the base calibrated model is reused for pricing, in which the Monte Carlo simulation will evolve the SDE system (3.1a) using *bumped initial values* for the forward Libors $F_k(0)$ in (3.1b), effectively[7] using bumped curves for discounting cashflows. From this point of view, they are not genuine Greeks and will differ from the GE Greeks.

In the case of Vega, the calculation of Pricer Greek involves a parallel shift to the Black (or Normal) volatilities of all calibration instruments obtained from the Vol-Cube. In particular, all swaption and caps volatilities will be shifted. This new collection of swaption and cap calibration instruments are submitted to the LMM for calibration, where volatilities have been bumped by the bump size specified in Figure 3.2. This Vega should agree with GE Vega.

In principle, any CMS-SO instruments whose premiums are used in the calibration must be recalculated.

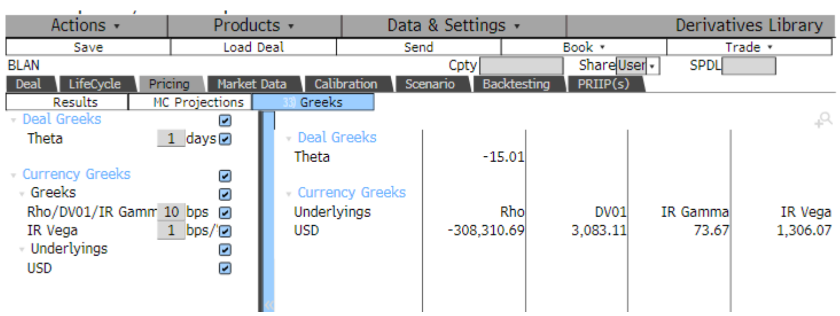


Figure 3.2: DLIB Pricer Greeks showing Theta, DV01, Gamma, and Vega

4 Calibration

Calibration refers to the method of finding values of model-parameters which produce observable values (e.g. swaption volatilities or cap premiums or cms-spread correlations) of (vanilla) market data. In general, it may not be possible to find such values of the model-parameters, and so one constructs a nonnegative-valued "objective function" of the model-parameters which will be zero

[7] For non interest rate deals, it is accurate to say the effect of the bumped curves is expressed only in the discounting of cashflows. For interest rate deals, however, discounting is expressed through the stochastic Numeraire, whose evolution depends on the bumped curves through its initial values.

precisely when the market data is perfectly reproduced, and whose minimization is taken as a "best-fit" of the model-parameters to market data.

4.1 Objective Function

Writing the vector of model-parameters as $\mathbf{x}$, we can describe the objective function

$$\mathbf{F} : \mathbb{R}^{dim(\mathbf{x})} \to \mathbb{R}^1$$

as follows: Assume there are M calibration targets, each with a market value (whether a swaption volatility, undiscounted swaption price, cap premium, etc.) represented by C_k $(1 \le k \le M)$. Assume further that the model predicts (based upon its vector of model-parameters $\mathbf{x}$) a value of $\hat{C}_k(\mathbf{x})$ for each of the M calibration instruments. Then the objective function can be described as taking the length of the vector $\mathbf{e}(\mathbf{x})$ of errors:

$$\mathbf{x} \in \mathbb{R}^{dim(\mathbf{x})} \longrightarrow \|\mathbf{e}(\mathbf{x})\| = \sqrt{\sum_{1 \le k \le M} (\hat{C}_k(\mathbf{x}) - C_k)^2}.$$

We emphasize here and throughout all discussion of calibration that the targets $\hat{C}_k(\mathbf{x})$ appearing in the objective function above are *Analytic Approximation Formulas* as described in Appendix D, particularly (D.2), (D.5), (D.6), (D.8), (D.25) and (D.26).

Note that the k'th component of $\mathbf{e}(\mathbf{x})$ is the signed-error in the k'th calibration instrument, and so the vector $\mathbf{e}(\mathbf{x})$ records all errors on an equal footing. If a weighting of the individual errors is desired (e.g. by the inverse of a bid-ask spread), one may replace the Euclidean norm with a weighted norm with weights $\omega_k > 0$:

$$\mathbf{F}(\mathbf{x}) = \sqrt{\sum_{1 \le k \le M} \omega_k (\hat{C}_k(\mathbf{x}) - C_k)^2}. \tag{4.1}$$

Note further that market instruments quoted in different units will create an objective function that combines terms of incompatible units. From this point of view, the weights may also be employed as unit conversions, for example multiplying terms quoted in Black volatility by a vega factor, effectively converting to a common unit of price which can be meaningfully combined with other terms quoted in price. The use of the ω_k to accommodate different calibration target units is depicted in Table 4.1, and described more fully in §4.7.

Target Type	Instrument Weight
Price	1.0
Black Volatility	Inverse Black Vega
Normal Volatility	Inverse Normal Vega

Table 4.1: Target types and their associated ω_k

4.2 Least Squares

The least-squares method addresses the minimization of the scalar objective function $\mathbf{F}(\mathbf{x})$ by applying specialized linear algebra algorithms to the geometry of the vector of signed errors $\mathbf{e}(\mathbf{x})$.

The collection of all $\mathbf{e}(\mathbf{x})$ forms a hypersurface in Euclidean space of m dimensions and we seek an $\mathbf{x}$ for which $\mathbf{e}(\mathbf{x})$ is closest to the origin. The two basic algorithms used in this context are Steepest-Descent, which is stable but converges slowly, and Gauss-Newton, which is fast but need not converge. The Levenberg-Marquadt algorithm is a hybrid of Steepest-Descent and Gauss-Newton, and can be proved to converge in the presence of constraints. Note that unconstrained optimization is not applicable to our calibrations, since, for example, volatility and shift model parameters must be non-negative.

Unlike many other optimization approaches which employ a *bootstrapping* methodology, the LMM optimizer calibrates all model parameters in one step. An exception to this which relates to skew calibration is described in §4.9.

4.3 Selection of Calibration Instruments

Calibration will require having market prices (or volatilities, or correlations, etc.) of a set of calibration instruments (*i.e.* the quantities $C_k(\mathbf{x})$ appearing in (4.1)). In general, the choice of these instruments will be tuned to the term-sheet being priced.

The following instrument types are supported which can be quoted as Black Volatility or Normal Volatility for ATM strike or Absolute Strike (OTM):

- Swaptions (Receiver)
- Caps
- Caplets

As an example, we show in Figure 4.1 the instruments associated with an *Upper Triangle* selection of ATM swaptions.

Figure 4.1: ATM Swaption Matrix

Additionally, the following CMS-Spread instrument types are supported which can be quoted as Normal Volatility, Spot Premium, Forward Premium for ATM strike, or Absolute Strike (OTM):

- CMS-Spread Options (Straddles, Calls, Puts)

Finally, the following CMS-Spread instrument type is supported which is a quoted Spread Correlation for ATM strike or Absolute Strike (OTM):

- CMS-Spread Correlations

Market quotes for each of these instrument types are retrievable by invoking the appropriate Terminal function, or by using Bloomberg's `blph()` function in Excel with the appropriate ticker symbol. DLIB retrieves volatility quotes from the same source as the VCUB function. Some examples are provided in Table 4.2.

Instrument	Function	Ticker example
Discount Factors	SWDF EU	'S0045D 18M BLC2 Curncy'
ATM Swaption Volatility	GVSP 295	'EUSV0A1 BVOL Curncy'
OTM Swaption Volatility	IAEP 2 1	'EUPA0C02 BVOL Curncy'
ATM Cap Premium	ICAE 17	'EUCPAM CMPN Curncy'
OTM Cap Premium		'EUCF201 CMPN Curncy'
CMS Premium over 3M LIBOR	ICAE 20	'EUCM025 CMPN Curncy'
CMS 10/2 Spread Options	GDCO 3449 3	'EUSO2023 Curncy'

Table 4.2: Example Instrument Tickers for the EUR markets

CMS-specific issues

Note that when calibrating to CMS spread options, it is important to include among the calibration instruments the swaption instruments which correspond to the reference swaps of each leg of the CMS spread. For example, calibrating to a CMS spread option 10/2 with expiry 3 years should include the 3x10 and the 3x2 swaptions. Also note that the ATM of the CMS spread straddle is a (shifted-lognormal) model-dependent convexity-corrected strike. Unless an absolute strike for the spread is supplied, the ATM strike is computed internally. One expedient pricing formula for the CMS spread premium uses a Gaussian bivariate model whose ingredients are the normal volatilities (possibly converted from quoted ATM Black volatilities) of the reference swaps comprising the legs of the spread, and also their terminal correlation (which may be historically determined). The well known formula for the resulting normal volatility of the spread

$$\sigma_{spread} = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \tag{4.2}$$

can be used (if not supplied directly) in a Bachelier pricing formula to price the straddle.

When internally calibrating to the CMS spread option price, whether provided directly as a Spot Premium, or Forward Premium, or indirectly as the implied normal volatility σ_{spread} or implied terminal correlation of the spread (see Table 4.3 below), the calibrator uses an approximation formula (see § D.3) for the price as a function of model parameters, specifically the volatility and shift parameters of the constituent Libors belonging to the legs. As already stated, without calibration instruments which influence the model parameters associated with the CMS legs, such as swaptions on their underlying reference swaps, the leg volatilities will be indeterminate.

Quotation Type	Quotation Meaning	Conversion to Forward Premium
Forward Premium	Forward Premium	None
Spot Premium	Discounted Forward Premium	Un-Discount to Forward Premium
SpreadVol	Normal Volatility σ_{spread} of Spread	Apply Bachelier Pricing Formula
IndexCorr	Correlation ρ between CMS legs	Apply (4.2) and Bachelier Pricing Formula

Table 4.3: Quotation alternatives for CMS Spread Options.

Figure 4.2: CMS Spread Options.

4.4 Calibration Procedure

Having described the specification of calibration instruments, we proceed to describe the details of performing a calibration with the shifted-lognormal LMM.

Recalling that the calibration process is a matter of tuning model parameters through an optimization algorithm, we must consider the question of the "initial values" of the model parameters, and also whether certain model parameters will be "non-calibrated".

4.5 Initial and Default Parameter Values

If a model parameter is a *calibrated parameter*, then its initial value plays the role of the "initial guess" used by the optimizer. If one views optimization as the process of incrementally improving the fit to market data by tuning model parameters, then the initial guess is simply the starting point for this process. In this context, the user need not (and typically would not) be burdened with specifying an initial guess, as default values are internally provided. Of course, users can coerce their own choice of an initial guess by overriding the internal defaults.[8]

[8] If the user specifies a value outside of the range of acceptable values for a given model parameter, for example a negative Black volatility, then an error will be thrown.

4.6 Fixed (non-calibrated) Parameter Values

If a model parameter is a *non-calibrated parameter*, then its initial value will be held fixed during the optimization process (and will thereby imply a limitation on how good of a fit the optimizer can achieve), and consequently this initial value will also be its final value.

More generally, users may provide their own choice of parameter values by explicitly overriding the internal defaults. Non-calibrated parameters typically arise when a user has already performed a calibration (in which all shifts, volatilities, and correlations are calibrated parameters as in §4.5), and subsequently wants to perform a quick recalibration of *only the volatilities* to a subset of market data. To explain this more fully, consider the following loosely accurate identifications:

- calibrate to ATM swaptions $\longleftrightarrow$ calibration of volatilities
- calibrate to OTM swaptions $\longleftrightarrow$ calibration of shifts
- calibrate to CMS-spread options $\longleftrightarrow$ calibration of correlations

A concrete example would be to perform a calibration to the ATM and OTM swaption markets and also the CMS-spread option market, which will result in all model parameters being calibrated. Alternatively, one could choose to calibrate the volatility parameters to the ATM swaption market only, while specifying the shifts and correlations as fixed non-calibrated parameters whose initial values would be specified under DLIB's Correlation tab. Table 4.4 summarizes the relationship between selected calibration instruments and the calibrated model parameters.

Calibration Instruments	Volatility Parameters	Shift Parameters	Correlation Parameters
ATM Swaption/Caplets only	calibrated	fixed	fixed
ATM + OTM Swaptions/Caplets	calibrated	calibrated	fixed
ATM Swaption/Caplets + CMS-SO	calibrated	fixed	calibrated
ATM + OTM + CMS-SO	calibrated	calibrated	calibrated

Table 4.4: Effect of selected calibration instruments on calibrated model parameters.

4.7 Internal Optimizer Settings

There are several calibration settings which are in effect when performing a calibration. The most important setting is the specification of the *target type* of the calibration, which is to say the *units* used by the optimizer. Calibrating to the volatilities would be a natural choice in the case of a swaption-only market, whereas calibrating to the price is suggested when caps or cms-spread option premiums are quoted. The target type used in LMM is chosen to be "price using vega-weighting", which will respect the instrument quotation, whether price, Black volatility, or Normal volatility. Calibrating to quoted volatility will effectively change the units of price to units of volatility[9] by applying an appropriate vega-weighting to the instrument's price. This vega-weight adjustment can be thought of as a conversion factor from units of price to units of volatility based on the market sensitivity. Note that it is desirable to avoid a mix of Black and Normal volatilities by ensuring that all instruments have consistent *vol-type* settings. For example, calibrating to a mix of swaptions

[9] Whether the volatility is Black or Normal is determined by the *vol-type* associated with the instrument.

and cms-spread options should use Normal Volatilities exclusively, since the cms-spread options do not support Black Volatility.

Another calibration setting is the specification of the *error type* of the calibration, which in the LMM implementation is always Absolute Error, as given in (4.1).

Smoothing of the shift and volatility model parameters through the use of a penalty function appended to the objection function is another feature of LMM calibration. See Appendix G for a description of how this feature is implemented.

The remaining internal configuration settings relate to the optimizer, and have been tuned to enhance performance.

4.8 Performing the Calibration

We have already described the minimal specification of the LMM model parameters, which will be supplemented with the selection of calibration instruments, LiborMarketSpec, and YieldCurve[10]. Performing a calibration will result in a *calibrated model* whose model parameters have been *fitted* to the market instruments.

Some remarks about the methodology for calibrating correlation model parameters. In the full-factor calibration, the Rebonato two-parameter full-factor correlation matrix

$$\boldsymbol{\rho}^N := (\rho_{i,j}) = \left(\rho_\infty + (1-\rho_\infty)e^{-\rho_d|T_i - T_j|}\right)$$

is generated as the (ρ_∞, ρ_d) are adjusted in each optimizer iteration. In the case of the reduced-factor correlation model of F factors, a rank F approximation $\boldsymbol{\rho}^F$ is obtained from $\boldsymbol{\rho}^N$ by applying a PCA (Principal Component Analysis) algorithm each optimizer iteration. It may happen that $\boldsymbol{\rho}^F$ exhibits features not present in $\boldsymbol{\rho}^N$, such as positivity or monotonicity in the columns (for example, `DLIB RACL` using 3-factor with initial values $\rho_\infty = 0.0, \rho_d = 0.6$). On the other hand, while producing *instantaneous* correlations which could possibly possess unexpected characteristics (such as non-monotonicity or non-positivity), the *terminal* correlations between the forward Libors, which are ultimately the correlations of interest, do not inherit any of these non-intuitive features.

Parameter reduction

Regarding the volatility model parameters, $\sigma_{i,j} = \sigma(F_i, t), t \in [T_{j-1}, T_j)$, one can observe the extreme over-parameterization of the $\sigma(F_i, t)$ with respect to quarterly T_j whereby the expiries and maturities of the swaption instruments become annual after the first year (see Figure 4.1). To address this over-parameterization, the calibrator will equate the $\sigma_{i,j}$ for $i = \{4m, 4m+1, 4m+2, 4m+3\}$ and $j = \{4n, 4n+1, 4n+2, 4n+3\}$ (where $i \geq j$) to a single $\sigma_{4m,4n}$. For example:

$$\begin{bmatrix} \sigma_{4,4} & 0 & 0 & 0 \\ \sigma_{5,4} & \sigma_{5,5} & 0 & 0 \\ \sigma_{6,4} & \sigma_{6,5} & \sigma_{6,6} & 0 \\ \sigma_{7,4} & \sigma_{7,5} & \sigma_{7,6} & \sigma_{7,7} \end{bmatrix} \longmapsto \sigma_{4,4}; \qquad \begin{bmatrix} \sigma_{8,4} & \sigma_{8,5} & \sigma_{8,6} & \sigma_{8,7} \\ \sigma_{9,4} & \sigma_{9,5} & \sigma_{9,6} & \sigma_{9,7} \\ \sigma_{10,4} & \sigma_{10,5} & \sigma_{10,6} & \sigma_{10,7} \\ \sigma_{11,4} & \sigma_{11,5} & \sigma_{11,6} & \sigma_{11,7} \end{bmatrix} \longmapsto \sigma_{8,4}.$$

[10] The YieldCurve data structure is described elsewhere.

In other words, the $\sigma_{i,j}$ are *linked* to a single $\sigma_{\hat{\imath},\hat{\jmath}}$ according to the mapping:

$$\sigma_{i,j} = \sigma_{\hat{\imath},\hat{\jmath}}, \qquad \text{where } \hat{\imath} = 4 \cdot \left\lfloor \frac{i}{4} \right\rfloor, \text{ and } \hat{\jmath} = 4 \cdot \left\lfloor \frac{j}{4} \right\rfloor.$$

Evidently, in the EUR market which has 6-month Libors and semi-annual swap schedules, the corresponding rule will be that $\hat{\imath} = 2 \cdot \left\lfloor \frac{i}{2} \right\rfloor$, and $\hat{\jmath} = 2 \cdot \left\lfloor \frac{j}{2} \right\rfloor$.

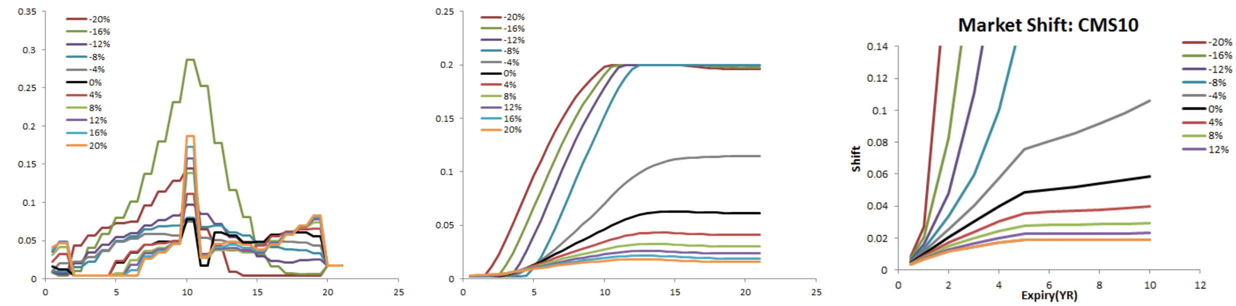
4.9 Decoupled-Shift Calibration

The *Decoupled-Shift* calibration is a methodology which was developed to address the regularity and robustness of the calibration of the shift parameters α_k. Although there are techniques that are available to impose regularity on the model parameters (see Appendix G), they are inadequate for producing a smooth behavior visible in DLIB's Vega Scenario Analysis[11].

Results from volatility scenarios when calibrating the shifts to a collared floater, which have been calibrated to 10Y swaptions (ATM & OTM) ranging from 6M to 10Y expiries, are shown in Figure 4.3(a) and Figure 4.3(b). Assuming a shifted-lognormally distributed swap rate, one can back out from each pair of swaption quotes at ATM & OTM strike an *implied shift* α_{SR} for each swap rate, and these can be systematically obtained from a range of bumps to the market volatilities ranging from -20% to 20%. These market α_{SR} are displayed in Figure 4.3(c). The smooth structure of the calibrated shifts α_k across the range of volatility bumps, achieved in Figure 4.3(b), is necessary to reproduce the market-implied shifts α_{SR}.

The Decoupled-Shift calibration method achieves the desired stability and regularity shown in Figure 4.3(b) by separating the calibration of the shifts from the calibration of the volatilities. The key idea is to notice that the shift of the swap rate is, to a good approximation, a linear combination of LMM shifts and the swap rate shift (see (C.16)), which can be directly computed by referring to the market swaption quotes at different strikes. Essentially, one collects all the equations (C.16) from the pairs of swaption quotes at different strikes, and then analytically solves the system of equations by using a constrained least-squares method. After mapping the market skew into LMM shifts by solving this system of equations, the standard calibration process is performed by fixing the shifts and proceeding to calibrate the volatility and correlation parameters.

[11] The Vega Scenario Analysis in DLIB involves multiple calibrations to swaption skews by parallel shifting the market volatility data.



(a) Term structure (to 21 years) of calibrated model shifts α_k for a ladder of vol-cube bumps without DSC

(b) Term structure (to 21 years) of calibrated model shifts α_k for a ladder of vol-cube bumps with DSC

(c) Market implied shifts α_{SR} for shifted-lognormally distributed swap rates, over a ladder of vol-cube bumps.

Figure 4.3: Typical market-implied shifts α_{SR} of CMS10Y to expiry 10Y, obtained directly from swaption volatility skew data for a range of volatility bumps, and their calibrated model shifts α_k. The trend is increasing α_{SR} with respect to expiry, and decreasing with respect to volatility.

It should be noted that the Decoupled-Shift calibration cannot be applied to the cap instruments because of the numerical complexity. Therefore, a second phase of calibration needs to be applied with the result of the first phase as the initial guess.

The Decoupled-Shift approach yields a very good result as shown in Figure 4.3(b), and produces a systematic trend of very smooth shift profiles per varying bump levels as shown in Figure 4.3(c). More detailed illustrations of the calibration quality associated with the Decoupled-Shift algorithm are given in §6.2, where in particular the excellent fitting quality is demonstrated in Figure 6.6.

4.10 Quality of Fit

After performing a calibration, it is of interest to determine the *quality of fit*, which is to say how well the optimizer was able to achieve fitting the model parameters to the set of calibration instruments. Moreover, in addition to determining the fit of the "in-sample" instruments to which the model parameters were calibrated, it is often desirable to determine the prices of "out-of-sample" instruments predicted by the calibrated model parameters (either by analytic formula prices in the case of caps and caplets, or approximation pricing formulas in the case of swaptions and cms-spread options).

5 Monte Carlo Pricing

Once model parameters have been determined by the calibration phase, the Monte-Carlo simulation will be invoked by the Pricer. Specifically, the Pricer will require an MCSimulator to generate simulated LIBOR rates which will be used in determining the price of the deal. The specifications required when constructing the MCSimulator, albeit only a subset of which are user-configurable, are summarized in Table 5.1 below:

Feature	Setting	Default
Number of Paths	$< 100,000$	$20,000$
Numeraire	Internal	Spot Libor
Variance Reduction	Internal	Moment Matching
Variance Reduction	Internal	Enable SOBOL Random Numbers
Seed	Internal	N/A
Sampling Interval	Internal	N/A

Table 5.1: Simulator Configuration Settings

5.1 Numeraires

Recall that a numeraire is a choice of asset price used to normalize other derivative asset prices. The numeraires described below derive from purchasing bonds at time T_0 of certain maturity T_m, and then, at some intermediate time $t \leq T_m$ (typically $t = T_m$), cashing them out and using the proceeds to purchase new bonds of maturity $T_{m'}$, and continuing in this fashion for future times $T_{m''}, T_{m'''}, \ldots$.

$$\begin{aligned}
\text{Spot-LIBOR} &: \quad m = 0,\ m' = 1,\ m'' = 2, \ldots \\
\text{Terminal} &: \quad m = N \\
T_k\text{-Forward} &: \quad m = k,\ m' = m + 1,\ m'' = m' + 1, \ldots
\end{aligned}$$

In general, a choice of numeraire must be made before any Monte-Carlo simulation is performed, as different choices imply that different evolution SDEs will be implemented. However, in the present case of the Shifted-Lognormal Libor Market Model, only the Spot-Libor numeraire is supported.

Although Monte Carlo simulations using the Spot-Libor numeraire may exhibit blow-up by virtue of its drift being comprised of many positive terms, the small number of offending paths can be controlled by capping the increments to the state variables. In all other respects, the Spot Libor numeraire is found to be superior to alternative numeraires. For example, the Terminal numeraire, whose evaluation of discount bonds may even fail monotonicity with respect to maturity, may also exhibit blow-down from a large negative drift. Additionally, the Spot Libor numeraire has the important feature that *unbiasing* the discount bonds will automatically unbias the forward Libor rates (see §A.1), thereby enforcing two important no-arbitrage properties.

5.2 Internal Simulator Settings

In addition to the specification of the number of paths, there are several internal configuration settings applied to the LMM simulator. These include the integer *seed* used for random number generation, and the *sampling interval*, which controls the time step used by the simulator during SDE simulation. In practice, a smaller amount of time might be used between two evolution dates if either of those dates correspond to a LIBOR fixing time or payment time or observation time. In no case, however, will the time between evolution dates exceed the sampling interval, as it provides a maximum granularity to the discretization of the time grid.

Variance reduction is an often used technique in Monte Carlo simulations, which is employed to achieve, using a small number of paths, the same simulation variance that would otherwise be obtainable only by using a large number of paths. The LMM implementation offers SOBL random number generation as a variance reduction methodology, but does not employ other techniques such as control-variates or antithetic random number generation.

5.3 LMM as a Continuous Model

As described in §B.1, and in more detail in Appendix E, the evolution of off-grid Libor rates $L(t, T_a, T_b)$ for T_a, T_b arbitrary start and end times, requires a special interpolation algorithm. This evalution of $L(t, T_a, T_b)$ is easily reduced to the evaluation of the so-called *stub-rates* $L(t, T, T_{k+1})$ where $T_k < T < T_{k+1}$, and T_k and T_{k+1} are consecutive grid dates. When the evaluation time $t < T_k$ predates the Libor reset time, then the fundamental formula defining the stub-rate in terms of known quantities $L(t, T_k, T_{k+1}), L(0, T_k, T_{k+1})$, and $L(0, T, T_{k+1})$ is given by

$$\begin{aligned}
[L(t, T, T_{k+1}) + \alpha_k] &:= f_T \cdot [L(t, T_k, T_{k+1}) + \alpha_k], \\
f_T &:= \frac{L(0, T, T_{k+1}) + \alpha_k}{L(0, T_k, T_{k+1}) + \alpha_k}.
\end{aligned} \tag{5.1}$$

For those cases when $T_k < t < T < T_{k+1}$, the premature freezing of the rate $L(t, T, T_{k+1})$ at $t = T_k$ is problematic, and is overcome by letting $L(t, T, T_{k+1})$ become a *zombie rate* which continues to evolve during the time $T_k < t < T$. Using this technique to compute arbitrary off-grid rates $L(t, T_a, T_b)$ effectively renders the LMM implementation a *continuous* model.

5.4 Brownian Bridge

When pricing a deal which requires many (perhaps daily) observation dates, such as a Range Accrual, the LMM simulation of Libor rates can become prohibitively expensive. In such cases, the LMM Monte Carlo simulation will apply the technique of Brownian Bridging, which allows Libor rates snapshotted at high-frequency observation dates (such as daily), intermediate between low-frequency dates (such as quarterly Libor reset dates), to be obtained from the low-frequency states by a form of "stochastic interpolation". In the absence of stochasticity, the Brownian Bridge algorithm amounts to a linear interpolation of rates snapshotted at low-frequency dates. On the other hand, in the presence of stochasticity, this interpolation generates random numbers to mimic the conditional distribution consistent with the end-point states (see [Gla, §3.1]), the distribution mean agreeing with the linear interpolant. Consequently, the Brownian Bridge approximation provides an alternative to the SDE simulator evolving many intermediate states using a costly high-frequency step size.

5.5 Pricing in a Dual-Curve Environment

Note that when pricing in a *dual-curve* setting, the simulation is performed by evolving OIS Forward rates only, and does not directly simulate market Libor rates. Any Libor rate required by the pricer is obtained from the simulated OIS Forward rate by adding a *basis*, which is obtained by retrieving the time-zero difference between OIS Forward rates and Libor Forward rates. This methodology is similarly applied to other Libor rates; the six month Libor rates are obtained from simulated three month OIS Forward rates by the addition of a basis obtained from the time-zero OIS and Forward curves. More details are provided in Appendix F.

6 Testing

6.1 Repricing calibration analysis

The validation of any model should include repricing the vanilla market to which the model has been calibrated. In this section we consider some examples of calibrating the LMM to an entire market (OTM caplets, ATM swaptions, OTM swaptions) of instruments, and then repricing all of these instruments in a single Monte Carlo simulation.

OTM Caplets

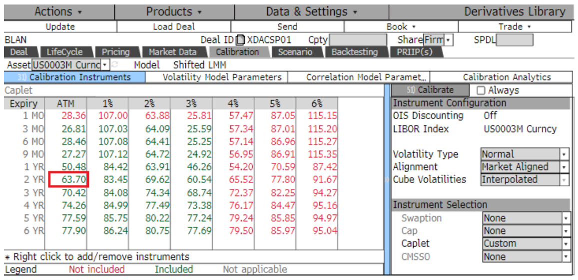
In this test with the USD caplet market on 03/04/2019, the ATM ($\sim$ 2.5%) and several OTM caplets up to five year expiries have been selected. The results for repricing of ATM caplets is shown in Table 6.1.

Expiry	Instrument Normal Vol	Repriced Normal Vol	Relative Error (%)
3M	26.8126	26.6352	0.6615
6M	28.4615	28.2721	0.6653
9M	27.2657	27.0836	0.6675
1Y	50.4796	50.6558	0.3490
2Y	63.7036	63.0604	1.0097
3Y	70.4187	69.8628	0.7894
4Y	74.2573	74.2859	0.0385
5Y	77.5854	77.5006	0.1093

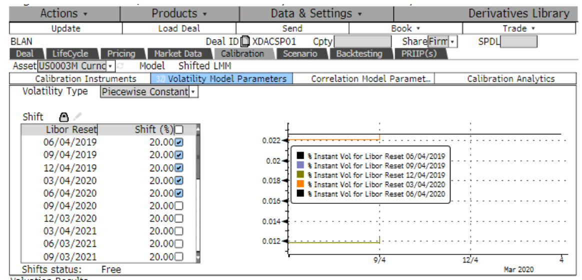
Table 6.1: ATM caplet calibration and repricing by Monte Carlo. Relative errors are less than 1%.

It turns out that this particular market (USD 03/04/2019) exhibits excessive (downward-sloping) skew at each maturity, and is therefore outside the modeling capability of the SLMM. More specifically, the SLMM can modulate between the two extremes of pure lognormal (flat slope in lognormal space, shifts = 0) and pure normal (negative slope in lognormal space, shifts = ∞). Any skew between these extremes, and only such skews, can be modeled by the shifted-lognormal model. The market data shown below in Figure 6.1(a) is effectively *sub-normal*.

We display in Figure 6.1(b) the calibrated shift-parameters α_k, and notice that they are pinned at their maximum allowed values of 20%, an increase from the 5% shift used as an initial value input to the calibrator. A value of 20% is effectively infinity; increasing the shift further in the calibration will not achieve sub-normal behavior. In Figure 6.2 we show plots of smile fitting below and above ATM.



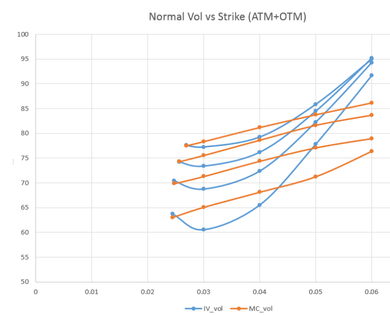
(a) Selection of caplets (ATM and OTM) as calibration instruments.



(b) Calibrated shift parameters in USD market of 20%, consistent with sub-normal skew.

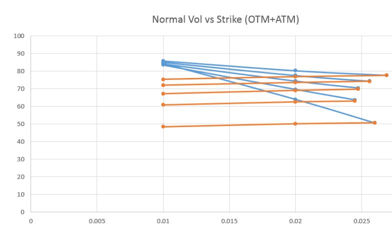
Figure 6.1: Calibrating to caplet skews.

(a) USD caplet OTM pricing from ATM ($\sim 2.5\%$) to 6% strike.



(b) USD caplet normal volatility from ATM ($\sim 2.5\%$) to 6% strike.

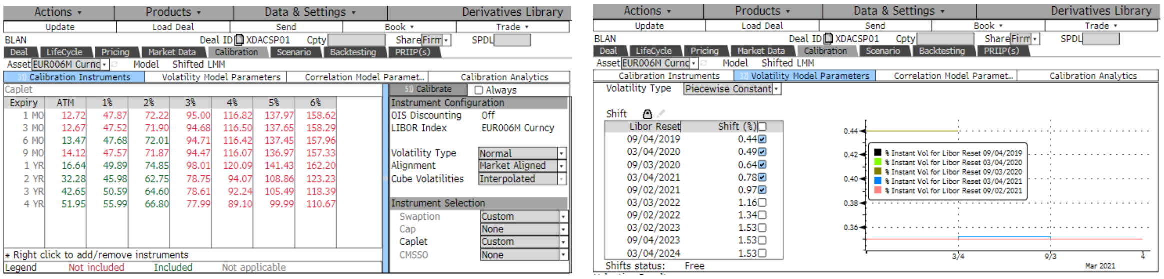
(c) USD caplet OTM pricing from 0.01 strike to ATM ($\sim 2.5\%$).



(d) USD caplet normal volatility from 0.01 strike to ATM ($\sim 2.5\%$).

Figure 6.2: OTM strikes below and above ATM of the caplet smile for USD market on 03/04/2019. ATM strike has a negative slope in normal vol-space showing sub-normal skew, after which it is positive skew.

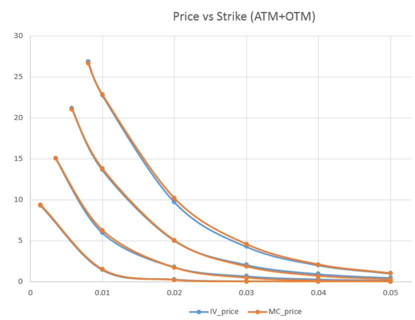
In contrast to the above, we observe the caplet matrix for the EUR market exhibits a reproducible skew on 03/04/2019 as shown in Figure 6.3(a), which is to say *sub-lognormal*. Correspondingly, the calibrated shift parameters displayed in Figure 6.4 are close to zero, consistent with the sub-lognormal volatility skew. As this is within the capability of SLMM calibration, this accounts for the much improved OTM calibration and matched re-pricing.



(a) Caplet normal volatilities in EUR market showing sub-lognormal skew.

(b) Calibrated shift parameters in EUR market close to 0%, consistent with super-lognormal skew.

Figure 6.3: Calibrating to caplet skews.



(a) EUR caplet OTM pricing from ATM (< 1%) to 5% strike.

Figure 6.4: Calibrating to caplet skews.

ATM Swaptions

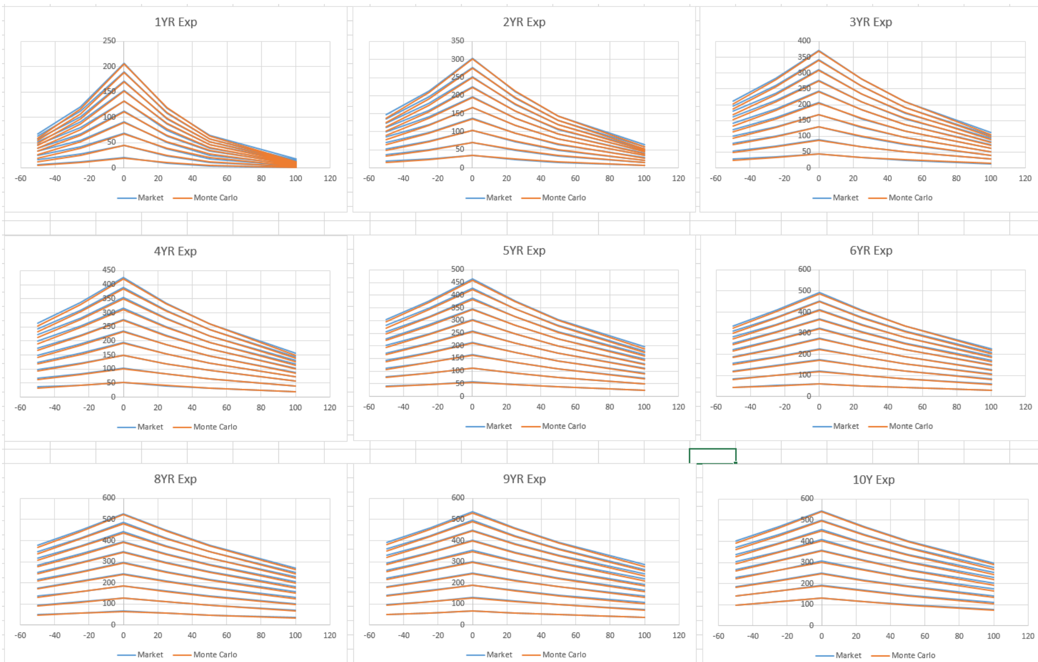
The ability to reprice the vanilla ATM swaption market after calibrating to that same market is a natural validation checkpoint. In Table 6.2 we observe the excellent quality of reproducing the ATM swaption market in a single Monte Carlo simulation.

Expiry	1YR	2YR	3YR	4YR	5YR	6YR	7YR	8YR	9YR	10YR
1YR	1.063	0.546	0.411	0.403	0.393	0.409	0.490	0.447	0.461	0.480
2YR	0.438	0.498	0.535	0.688	0.641	0.605	0.615	0.601	0.595	0.628
3YR	0.742	0.804	1.035	0.906	0.851	0.878	0.874	0.867	0.926	0.850
4YR	1.287	1.596	1.406	1.273	1.273	1.242	1.209	1.255	1.164	1.141
5YR	1.134	1.125	1.280	1.118	1.109	1.100	1.093	1.134	1.083	1.030
6YR	1.203	1.445	1.195	1.197	1.210	1.219	1.288	1.244	1.200	1.182
7YR	1.694	1.188	1.188	1.199	1.211	1.293	1.234	1.183	1.157	1.134
8YR	1.313	1.293	1.291	1.417	1.285	1.259	1.219	1.174	1.194	1.137
9YR	1.502	1.528	1.676	1.477	1.444	1.404	1.356	1.372	1.303	1.239
10YR	1.560	1.476	1.504	1.372	1.298	1.259	1.232	1.247	1.175	1.113

Table 6.2: Relative errors (%) when comparing 10Yx10Y ATM swaption matrix of normal volatilities of market instruments to Monte Carlo repricings. The market is USD observed on 03/04/2019.

OTM Swaptions

The ability to reprice the vanilla OTM swaption market after calibrating to that same market is another natural validation checkpoint. The matching of the volatility cube of OTM swaptions against Monte Carlo repricing is excellent, as shown in Figure 6.5. In Table 6.3 we illustrate the quality of reproducing the swaption smiles for ATM-relative strikes of -50bp to 100bp for a selection of expiry-tenor pairs, where the quality is expressed as a relative error between the market normal volatility and the implied volatility of the Monte Carlo repriced swaption using 200k paths.



(a) Matching of repriced Volatility Cube instruments.

Figure 6.5: Comparison of OTM Swaption Calibration and Monte Carlo Repricing against Market Prices using 200k paths. Each plot shows prices for 10 tenors across ATM-relative strikes from -50bp to +100bp. The market is USD observed on 03/04/2019.

Expiry-Tenor	-50	-25	0	25	50	100
1Yx5Y	7.503	3.273	0.626	1.724	2.082	2.015
2Yx6Y	4.623	1.991	0.574	0.698	0.774	2.133
3Yx3Y	4.081	1.855	0.976	0.357	0.865	0.520
4Yx9Y	2.822	1.374	1.061	0.373	0.280	1.660
6Yx8Y	2.294	1.121	1.127	0.574	0.495	1.722
7Yx3Y	2.202	0.921	1.109	0.501	0.410	1.692
8Yx4Y	2.137	1.008	1.312	0.786	0.743	1.983
9Yx2Y	1.609	0.676	1.387	1.095	1.220	2.671
10Yx8Y	2.100	0.943	1.179	0.721	0.759	2.183

Table 6.3: Relative errors (%) of market quoted normal volatility of OTM swaptions compared to implied volatility of Monte Carlo repricings of selected expiry-tenor instruments. Data is across ATM-relative strikes from -50bp to +100bp. The market is USD observed on 03/04/2019.

6.2 QC of Calibration

SLMM supports a wide range of calibration instruments including swaption smiles and CMS spread options. LMM has three key sub-models which are the shift, correlation and volatility. The shift parameter controls the skew behavior of instruments, *i.e.* the volatility trend across the strikes. The correlation model employs the reduced factor technique which is more practical than the full factor correlation in terms of simplicity and pricing speed, and also conforms to the general market concept that the curve movement is driven by a handful of factors. The volatility is the essential parameter controlling the degree of stochasticity in time evolution of the underlying rates.

Shift calibration quality

This section gives additional background to the discussion of the Decoupled-shift calibration described in §4.9. A "EUR collared floater with CMS10Y underlying" deal is tested for a *volatility scenario analysis* using a range of parallel bumps to the vol-cube, and which involves as calibration instruments 10Y swaptions struck at two strikes (ATM and OTM 3.9%) and whose expiries range from 6M to 10Y. We assume the 10Y swap rate at given expiry follows a shifted-lognormal distribution parameterized by an α_{SR} and σ_{SR}. The swaption volatility quotes at the two strikes will determine the two model parameters $\{\alpha_{SR}, \sigma_{SR}\}$, namely the swap-rate shift and the swap-rate (shifted-lognormal) volatility. Figure 6.3(c) from §4.9 shows the actual profile of implied swap-rate shifts across expiries, reproduced for six specific bumps to the volatility level.

The market trend shown in Figure 6.6 below is associated with the monotone increasing swaption shift α_{SR} with respect to increasing expiries as described in Figure 6.3(c), and can be interpreted as a transition from a lognormal ($\alpha_{SR} \sim 0$) to a shifted-lognormal ($\alpha_{SR} \gg 0$) distribution of 10Y swap rate (CMS10Y), as well as a monotone decreasing shift profile for increasing volatility bump levels. These features are consistent with one's intuition. The calibrated shift profiles already shown in Figure 6.3(b) confirm the consistent smoothness guaranteed by the decoupled-shift algorithm. The fitting quality for the ATM/OTM swaptions is excellent across all the volatility bump levels as shown in Figure 6.6 demonstrating the robustness of the shift calibration.

When the same test is conducted using other currency markets, the shift calibration method shows a consistently good fitting quality and also a smooth shift profile. As shown in Figure 6.7 where we consider the currencies USD, JPY, CAD and GBP, we observe excellent matching with the market quotes for each of the four currency regions.

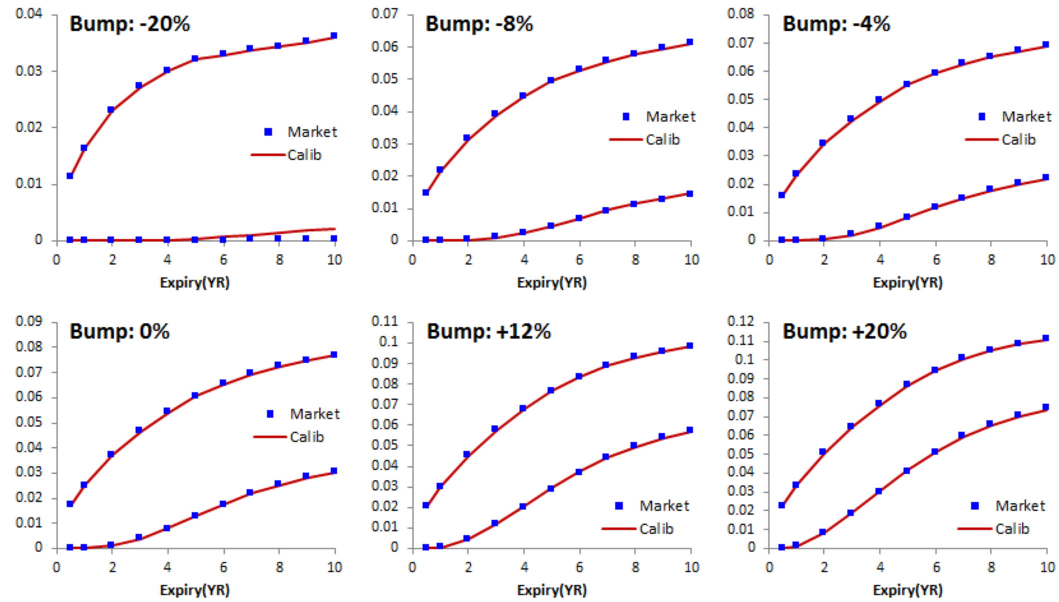
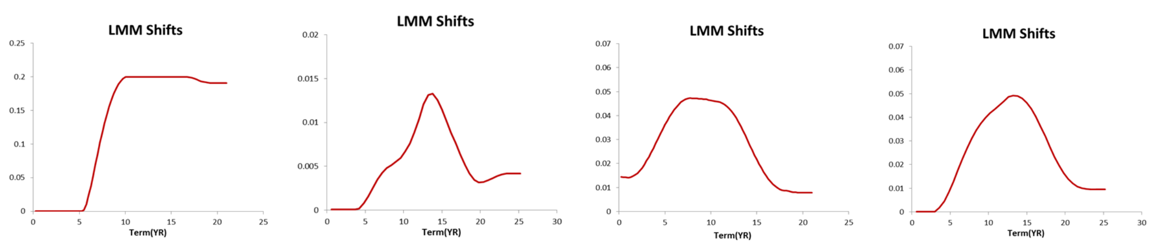
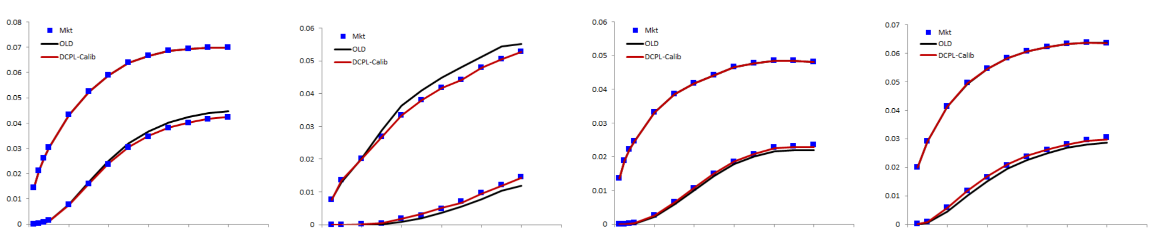


Figure 6.6: Repricing of CMS10Y ATM & OTM swaptions for multiple expiries over a range of bumped volatility shifts (-20% to 20%), showing excellent fitting quality.



(a) USD: shift profile (b) JPY: shift profile (c) CAD: shift profile (d) GBP: shift profile



(e) USD: calib quality (f) JPY: calib quality (g) CAD: calib quality (h) GBP: calib quality

Figure 6.7: Calibration tests across multiple currency markets. The shift profile is very smooth and the fitting quality is remarkably good.

Finally, using the *all in one* deal described in §6.4 which has been calibrated to a collection of instruments consisting of skews of caplets, 10Y and 2Y swaptions, caps and CMS (10Y-2Y) spread options, we observe the extremely smooth shift profile and smooth repricing of the instruments.

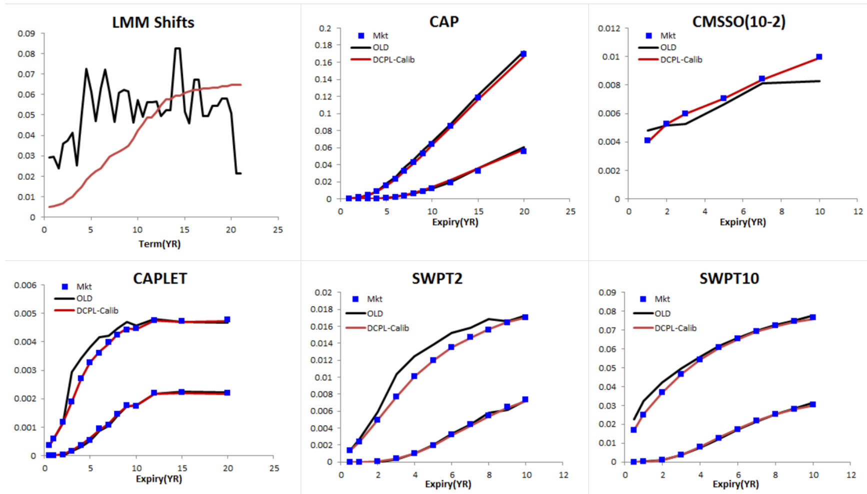


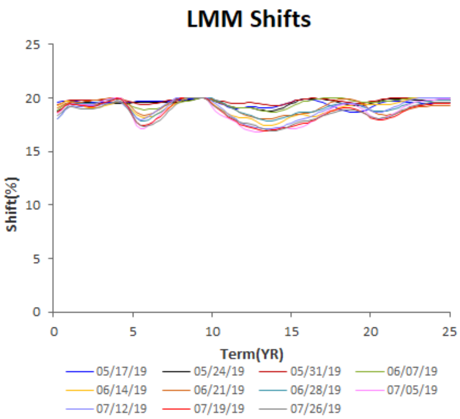
Figure 6.8: Calibrating to skews of caplets, 10Y and 2Y swaptions, caps and (10Y-2Y) CMS spread options. Comparison without the Decoupled-shift algorithm is indicated in plots with black curves.

Historical shift calibration stability

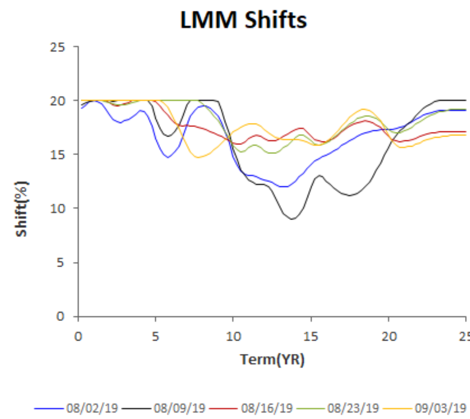
Historical analysis, as displayed in Figure 6.9(a) for a two month period in 2019, shows very stable shift calibration results. On the other hand, in the month following one can observe oscillations developing as displayed in Figure 6.9(b). As is well understood, the LMM shift calibration depends directly upon, and indeed should be reflective of, the swaption market skew/smile. It is therefore worthwhile to verify that the observed oscillations in the August LMM shift profile mirror a corresponding market trend in swaption skew. In the present example, the calibration has been performed using the swaption quotes at the three strikes ATM, ATM + 50bp, and ATM - 50bp. By computing the *slopes of the secants* to the (normal) volatility smile $\sigma_N(\cdot)$

$$\frac{\sigma_N(+50bp) - \sigma_N(-50bp)}{100bp}$$

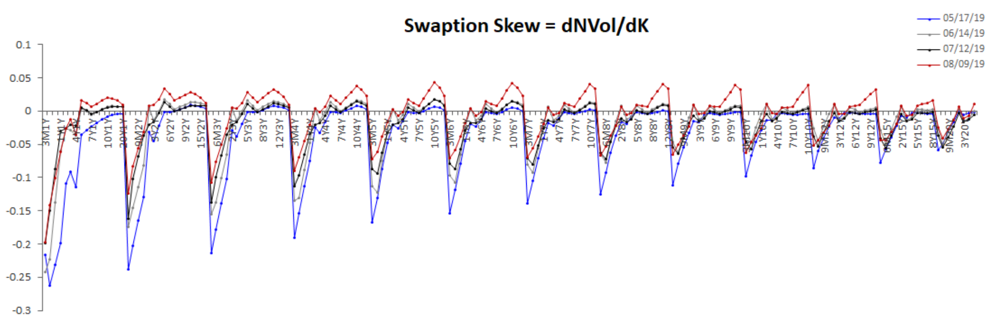
for swaption quotes of all (expiry, tenor)-pairs, one can observe in Figure 6.9(c) the strong increase in smile/skew in August, 2019. This increase in slope implies a trend toward more lognormal-like market skew, and hence is consistent with the decrease in the shift profile.



(a) Shift profile over two month period in 2019.



(b) Shift profile over month of August 2019.



(c) Slopes of normal volatility smiles for swaption matrix over the period 05/17/19 to 08/09/19.

Figure 6.9: Calibrated shift profiles over three month period showing two months of stability followed by the appearance of oscillations. The oscillations are consistent with market conditions.

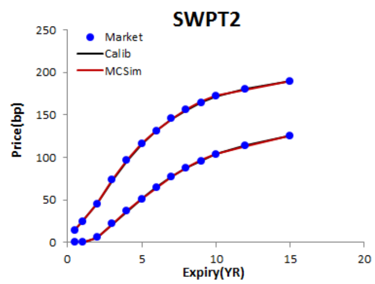
Correlation calibration quality

The SLMM is the unique Interest Rate model in DLIB which can be calibrated to the correlation of CMS rates, and thus many booked deals are priced using LMM due to their payoffs relying on this correlation structure. The CMS spread option is the de facto market instrument which manifests the implied correlation of two CMS rates. The proper calibration to CMS spread options is indeed quite demanding, since two CMS rates cannot be martingales simultaneously, and the convexity correction[12] is necessary to determine the model-dependent ATM strike of the underlying CMS Spread of the CMS spread option.

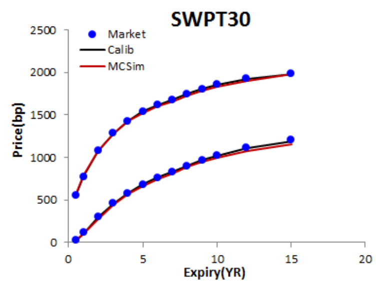
A deal in the EUR market is tested which involves swaption smiles (ATM, ATM+100bp) and CMS spread options (30Y-2Y) as shown in Figure 6.10. The quality of the swaption smile calibration and MC repricing are excellent, as shown in Figure 6.10(a) and Figure 6.10(b). The CMS spread option (straddle) prices from the calibration approximation formula (§D.3) agree well with Monte Carlo repricing, and are also in agreement with the market quotes shown in Figure 6.10(c). Even

[12] It should be noted that the accurate skew calibration also plays a key role for this purpose as discussed in [Hag].

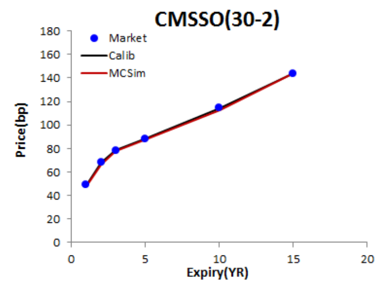
though the ATM strikes are not calibrated, the ATM strikes by MC simulation shown in Figure 6.10(d) are in good agreement with contributor quotes.



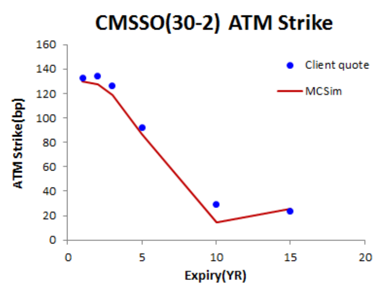
(a) Swaption 2Y smile calibration (ATM, ATM+100bp) and MC repricing.



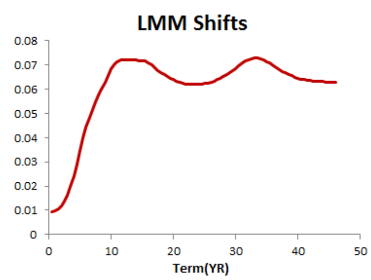
(b) Swaption 30Y smile calibration (ATM, ATM+100bp).



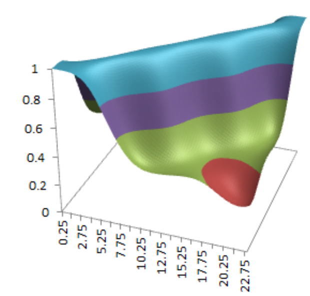
(c) ATM CMS20Y2Y SO calibration and MC repricing.



(d) ATM strike of CMS-30Y2Y SO's by MC simulation in good agreement with contributor quotes.



(e) Calibrated LMM shift profile.



(f) Calibrated correlation matrix $\rho(T_i, T_j)$ reduced from model parameters $\rho_{inf} = 0.26, \rho_{decay} = 0.08$.

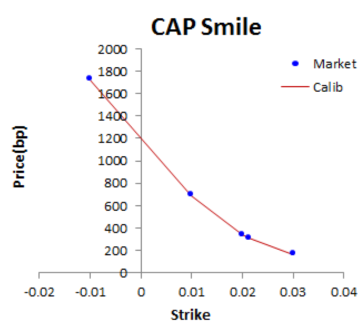
Figure 6.10: Calibration results and MC repricing of the swaption 2Y/30Y smiles and CMS spread options (30Y-2Y). The MC repricing of the ATM strike of the CMS spread option agrees well with contributor quotes. The calibrated shift profile (e), and the correlation structure with 4-factors (f), are also demonstrated.

The profile of the calibrated shift parameters shown in Figure 6.10(e) is very smooth, and the parameters for the 4-factor correlation structure $\rho(T_i, T_j)$ shown in Figure 6.10(f) are $\rho_{inf} = 0.26$ and $\rho_{decay} = 0.08$. It should be noted that the non-monotone profile of the correlation structure $\{\rho_{i,j}\}$ can be interpreted as a natural artifact of the PCA (Principal Component Analysis)-based factor reduction of the full-factor Rebonato correlation matrix during the calibration stage. Indeed[13], even for a full-factor 2×2 correlation matrix $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ with $\rho < 1$, which has been reduced by PCA to the one-factor matrix with $\rho = 1$, one can observe the loss of monotonicity in ρ increasing to 1.

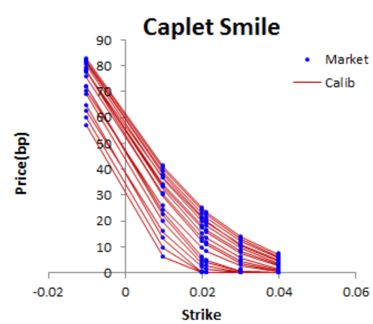
[13] If the PCA is viewed as a low-pass filter, then one can apply the intuition of subtracting from the original monotone $\rho(x, y)$ surface its *high-frequency* (small eigenvalue) components, resulting in a slightly oscillatory surface.

Cap calibration quality

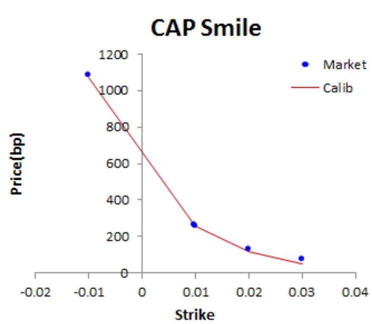
At first glance, the Cap calibration may be considered a subtle task, since it requires the stripping of Caps into Caplet volatilities, and the stripping methods of the vol cube (VCUB) and the LMM model differ. In order to maintain consistency with the Caplet volatilities obtained in VCUB, the Cap vols are therefore not directly used when Cap instruments are selected. Specifically, DLIB processes a Cap instrument by passing it to LMM *as the collection of its constituent Caplets*, all struck at the Cap strike. The excellent performance of LMM's Cap calibration is a direct consequence of its excellent performance with respect to OTM Caplet calibration. Indeed, the QC result of the Cap calibration looks remarkably good as shown in Figure 6.11. It also shows that the calibration performs very well for the negative strikes of Cap and Caplets.



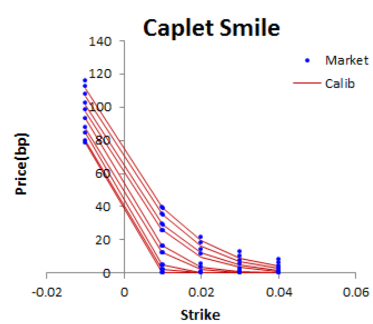
(a) USD: caplet smiles inside Cap with expiries 3M∼6Y in 3M interval



(b) USD: caplet smiles inside Cap with expiries 3M∼6Y in 3M interval



(c) GBP: 6Y CAP smile



(d) GBP: caplet smiles inside CAP with expiries 6M∼6Y in 6M interval

Figure 6.11: Calibration results for 6Y Cap smile for USD and GBP markets. The Cap smile calibration is very good as a consequence of the caplet smile calibration. Each caplet smile in the plot corresponds to a specific caplet maturity belonging to the Cap instrument.

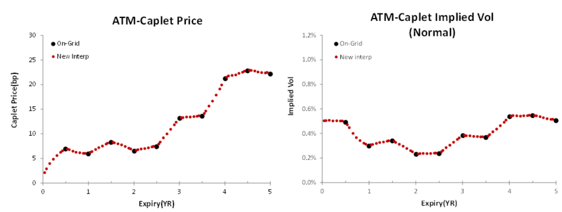
6.3 QC of Monte Carlo Simulation

Interpolation algorithm

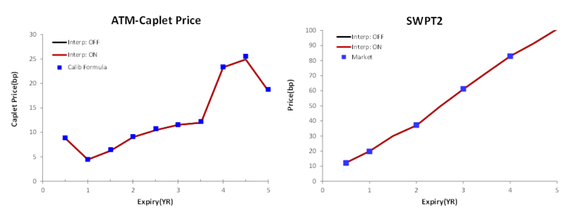
LMM is intrinsically a discrete model which defines the dynamics only for those Libor rates on the grid points, *e.g.* 3M Libor grid for USD, or 6M Libor grid for EUR markets. Many deals, however, need to evaluate the Libor rates with off-grid schedules and non-standard tenors. Therefore, it has been criticized by market practitioners that LMM model is only applicable to a limited class of deals. This is a striking difference from the Hull-White model which is a continuous model, and therefore has no restriction on the tenor and schedule of the Libor rates.

The SLMM implements the interpolation method first proposed by [Wer], appropriately adapted to shifted-lognormal dynamics. The computational overhead is very minor and the interpolated rates show a very smooth and continuous trend between the grid points. This technique plays a crucial role in pricing deals with high-frequency observation dates, such as daily range accruals.

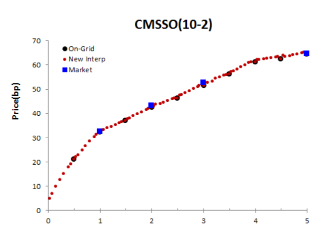
As shown in Figure 6.12(a) and Figure 6.12(b), the ATM caplet price and volatilities for the off-grid dates are very smooth as a function of expiry. The MC repricing of the CMS spread options (10Y-2Y) and the model-dependent ATM strike on off-grid dates confirms a smooth interpolation trend as shown in Figure 6.12(c) and Figure 6.12(d).



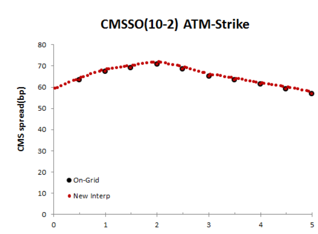
(a) Trend of ATM caplet price and normal volatility with increasing off-grid expiry.



(b) ATM caplet and swaption prices showing consistency of the off-grid algorithms with respect to on-grid pricing.



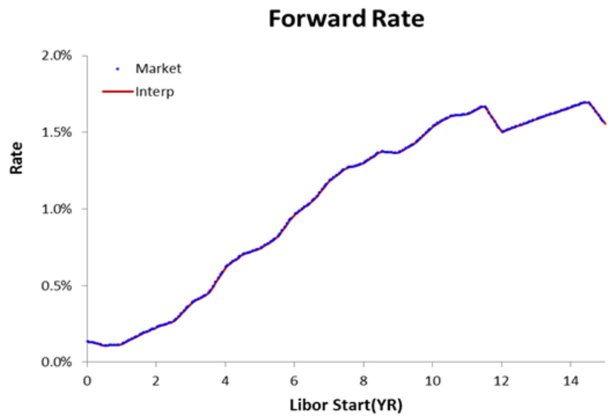
(c) CMS spread option (10Y-2Y) repriced by MC simulation for the off-grid dates.



(d) Model-dependent ATM strike of CMS spread option(10Y-2Y) by MC simulation for the off-grid dates.

Figure 6.12: Off-grid interpolation for the ATM Caplet vols and prices, Swaption prices, ATM CMS rates, and ATM CMSSO prices. In all plots the method shows a very smooth interpolation between grid points.

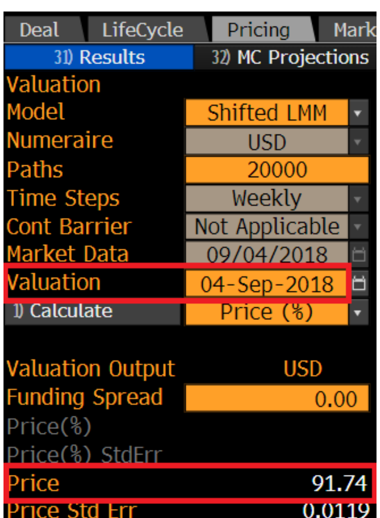
Finally, another important feature of the interpolation methodology is the off-grid forward matching of the Libor rates. As shown in Figure 6.13, the algorithm produces perfect agreement with the market implied forwards from the discount curve.



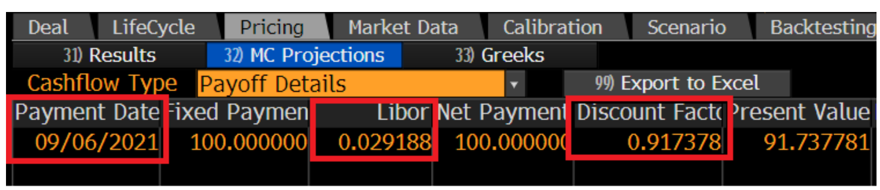
(a) Matching of forward Libors from discount curve and Monte Carlo projections.

Figure 6.13: Perfect forward Libor matching is obtained with unbiasing algorithm (§A.1).

An additional validation of the discount-matching and forward-matching can be observed directly from the DLIB terminal function, as shown in Figure 6.14.



(a) Performing a Monte Carlo simulation of 33 month ZCB using LMM.



(b) Payoff Details shows a simulated Libor rate fixed on 06/04/21.

(c) Market Data shows a market Libor rate fixed on 06/04/21.

Figure 6.14: ZCB of maturity 09/06/21 priced on 09/04/18 gives value 91.74 consistent with discount factor 0.917378. Additionally, the notified Libor rate of 0.029188 exactly matches the market Libor rate fixed on 06/04/21.

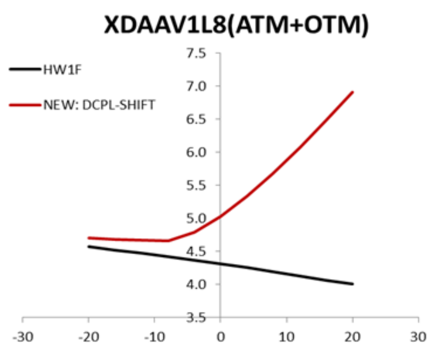
American Monte Carlo

LMM pricing of deal with American optionality are not discussed here, however Monte Carlo pricing of American deals in DLIB generally is discussed in [Blo1].

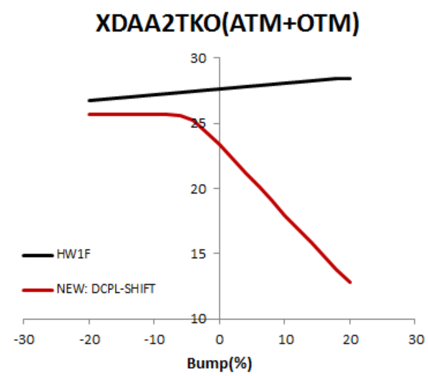
Vega stress test

A vega scenario test using a volatility ladder is conducted for a "EUR collared floater with CMS10Y coupon" deal as discussed in §6.2, and also a "USD collared floater with CMS5Y coupon" discussed below in §6.4, in order to stress test the calibration/MC-pricing stability in the high volatility regime. Whereas an unstable shift calibration will often result in a non-smooth and nearly flat vega profile, the LMM calibration shows a very smooth profile and a robust positive/negative slope in the region of extreme volatility bump.

The Hull-White model, also plotted, shows an opposite vega trend to that of the LMM due to its intrinsic Normal modeling of the volatility skew, which further confirms the strong *skew sensitivity* of these collared floater deals.



(a) EUR collared floater with CMS10Y. HW1F shows a downward trend.



(b) EUR collared floater with CMS5Y. HW1F shows an upward trend.

Figure 6.15: Vega scenario analysis of two EUR collared floaters with CMS underlying, for additive volatility bumps in Black space ranging from −20% to +20%. The Decoupled-shift calibration yields a very smooth vega profile showing a strong positive trend consistent with the calibrated shift profiles, while the Hull-White model yields an opposite trend (negative vega) due to its intrinsic Normal skew.

6.4 Comparisons with other Pricers

DLIB deals can replicate existing deals typically priced in YASN and SWPM. In this section we consider some comparisons against those other pricers as an independent verification, as YASN and SWPM do not use Monte Carlo simulation and therefore present independent checkpoints for model validation.

Price Comparisons with SWPM

For purposes of this section, we consider a USD CMS Collared Floater as a very challenging deal to validate the swaption skew and convexity correction of CMS rates in comparison with SWPM. As shown in Figure 6.17, the deal is best described by the details of the payment legs in the SWPM terminal. It can be regarded as a comprehensive deal which verifies the discount-matching, Libor-matching, the CMS convexity correction as well as the swaption volatility skew.

(a) SWPM deal showing the details of multiple legs.

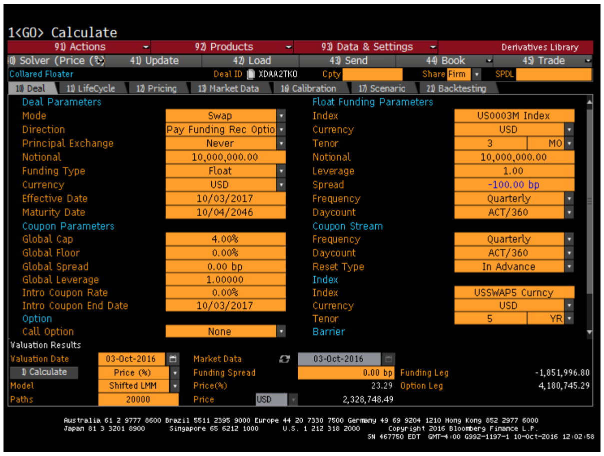
Legs	Type	Validation points
leg 1	Fixed coupon	Discount matching
leg 2	Libor float	Libor matching
leg 3	CMS float	CMS Convexity Correction
leg 4	CMS cap	CMS Convexity Correction + volatility skew
leg 5	CMS floor	CMS Convexity Correction + volatility skew

Figure 6.16: Detailed description of the legs and the validation points

Figure 6.17: SWPM deal showing the detailed payment legs and validation points. It can be regarded as a comprehensive deal which checks almost every aspect of model calibration and MC repricing.

A DLIB equivalent to a SWPM deal is shown in Figure 6.18. It should be noted that the DLIB coupon stream is actually the sum of legs 3, 4 and 5 of the SWPM deal.

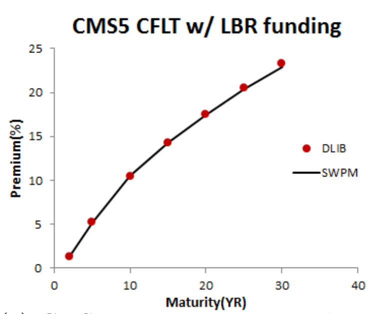
The deal is tested for a wide range of deal maturities and CMS tenors for coupon index as shown in Figure 6.19. The DLIB price shows an excellent agreement with SWPM across maturities up to 30Y and different CMS tenors. In other words, the SLMM accurately reproduces the market discounts, Libor forwards, CMS convexity correction and the swaption skew, all vital requirements for reliable DLIB pricing for most structured deals. The stability of DLIB pricing across monthly valuation dates is tested for one year time span for the deal shown in Figure 6.19(d), yielding a smooth trend without abrupt jumps.



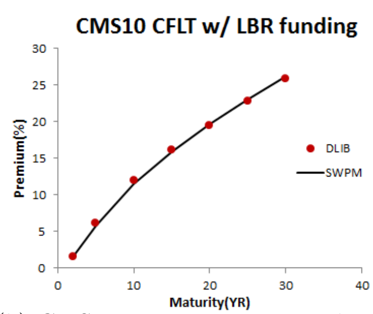
(a) DLIB deal showing the payment details based on Libor- and CMS5Y rate.

(b) Calibration instruments: ATM and OTM 5Y-swaptions.

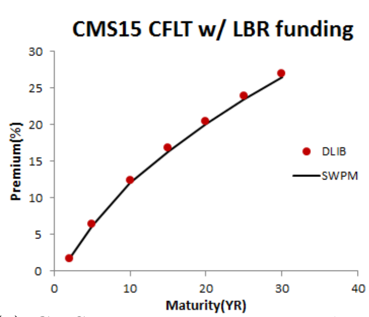
Figure 6.18: DLIB deal XDAA2TKO showing the payment details and the calibration instruments. This deal is priced by adjusting the deal maturity and the tenor of CMS coupon index to compare with the SWPM deal SLQG117Q in this analysis.



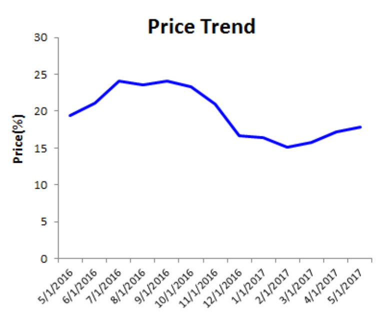
(a) CMS5Y rate as coupon index.



(b) CMS10Y rate as coupon index.



(c) CMS15Y rate as coupon index.



(d) Monthly price trend across one year time span for CMS5Y rate (XDAA2TKO).

Figure 6.19: DLIB prices across maturities up to 30 year with varying CMS tenors showing an excellent agreement with SWPM. Monthly price trend for CMS5Y for one year time span shows a continuous profile without abrupt jumps.

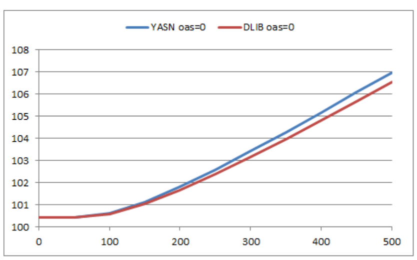
Price Comparisons with YASN

For purposes of this section, we will consider three representative structured products: Capped (Floored) Floater, European Cancellable Bond, and a Bermudan Cancellable Bond. Three YASN deals of the above type have been replicated in DLIB in an effort to verify pricing consistency between YASN and DLIB. For each deal, there is a remarkable price consistency for volatilities in the typical market range of under 100 bps. When priced with VCUB, the price differences between YASN and DLIB are within 0.01 when varying OAS from 0 to 500 bps.

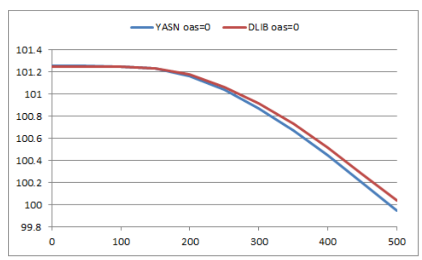
When volatility becomes unrealistically large, there are noticeable price differences between YASN and DLIB. The maximum price difference (at vol=500) are 0.447, −0.087, and −0.027, respectively.

The observed price divergence for the floored floater observed in Figure 6.20(a) is due to the fact that YASN pricing of the no-option bond utilizes the replication method instead of using the LGM model. The observed divergence simply reflects the inability of the HW1F model to fully replicate the market. At the same time, the confirmation of highly accurate prices from LGM/HW1F model at the market vol, and their graceful degradation shown in Figure 6.20, actually adds confidence in the quality of both model implementations.

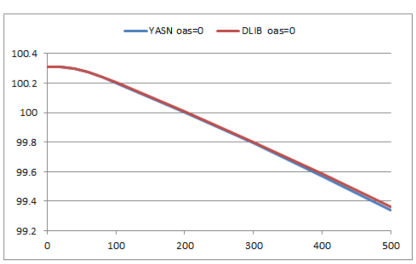
The price divergence for the other two bonds is quite small, even for very large volatilities, as shown in Figure 6.20(b) and Figure 6.20(c). That some divergence is present can be attributed to an increasingly greater numerical sensitivity, as volatility increases, to minor differences in their choice of calibration instruments.



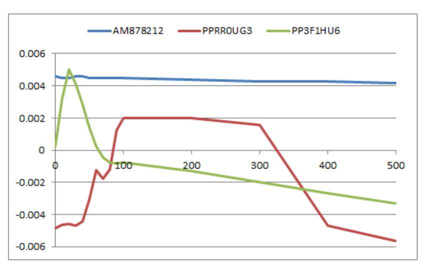
(a) Floored floater (AM878212 oas=0)



(b) European fixed (PPRR0UG3 OAS=0)



(c) Bemudan capped floater (PP3F1HU6 oas=0)



(d) Price difference vs OAS (market vol)

Figure 6.20: Price comparison between YASN and DLIB with respect to increasing volatility.

6.5 Conclusions from QC

The capability of CMS correlation is critical for many client deals which is unique to LMM compared to the Hull-White model. Testing results show excellent calibration quality to the CMS spread option and swaption smiles simultaneously with a reduced correlation structure. The interpolation technique of Monte Carlo simulation is crucial for general deal pricing to evaluate the underlying/numeraire on off-grid dates. The outcome analysis yields a very smooth interpolation for the forward rates, swaption volatilities, and correlation of CMS rates.

APPENDICES

A Spot-LIBOR Numeraire

In this section we describe the numeraire used in the LMM pricing. Notationally, $P_T(t)$ denotes the time t value of a zero-coupon bond of maturity T (necessarily a pathwise quantity for $t > 0$), so $P_T := P_T(0)$ is today's value of the bond maturing at time T which is determined from the initial discount curve; and $P_n := P_{T_n}(0)$ is today's value of the bond maturing at the Libor date T_n.

The Spot-Libor numeraire, also called the "discretely rebalanced bank account" numeraire, is a discrete version of the risk-neutral numeraire associated with the continuously compounded money market account. Specifically, its value at time T_0 is \$1. ; at time T_1 has increased in value to P_1^{-1} at which time the LIBOR from T_1 to T_2 has been fixed at $F_1(T_1)$; at time T_2 the numeraire has increased in value to $P_1^{-1}(1 + \tau_1 F_1(T_1))$; and so on, until Libor date T_k at which time its value becomes

$$\begin{aligned} \mathcal{N}(T_k) &:= P_1^{-1} \cdot \prod_{j=1}^{k-1} (1 + \tau F_j(T_k)) \\ &= P_1^{-1} \cdot \prod_{j=1}^{k-1} (1 + \tau_j F_j(T_j)) \qquad \text{(since } F_j \text{ freezes at } T_j\text{)}. \end{aligned}$$

In other words, starting with a \$1. purchase of bonds paying P_1^{-1} at maturity T_1, one cashes out at each LIBOR date T_k and uses the proceeds to purchase new bonds paying a return of $(1+\tau_k F_k(T_k))$ at maturity T_{k+1}.

A more complete description would account for cashing out the bonds at an intermediate time $T_k \le t < T_{k+1}$. Using the γ-notation to indicate, at an arbitrary time, the Libor index of the first rate "not yet frozen":

$$\gamma(t) = k + 1 \Longleftrightarrow T_k \le t < T_{k+1}^-,$$

we write more generally (by discounting the overshoot from $(1 + F_k(T_k))$ by $P_{k+1}(t)$):

$$\begin{aligned} \mathcal{N}(t) &:= \mathcal{N}(T_{k+1}) \cdot P_{k+1}(t) \qquad (T_k \le t < T_{k+1}) \\ &= P_1^{-1} \cdot \prod_{j=1}^{\gamma(t)-1} (1 + \tau_j F_j(T_j)) \cdot P_{\gamma(t)}(t). \end{aligned} \tag{A.1}$$

Strictly speaking $P_{\gamma(t)}(t)$, the time t value of the bond maturing at time T_{k+1} whose rate was reset at $F_k(T_k)$, is not mandated by the discrete rates $F_k(t)$ modeled in the Libor Market Model. A fuller discussion of these off-grid "stub discounts", and their related "stub rates"

$$F(t; t, T_{\gamma(t)}) := \frac{P_{\gamma(t)}(t)^{-1} - 1}{\tau_k}$$

is given in Appendix B.

A.1 Unbiasing Algorithm

Unbiasing, also called moment-matching, is a technique used to enforce agreement between simulated quantities, known to be martingales, with their initial values. For example, zero-coupon bonds are tradable assets, paying \$1 at maturity, and when discounted by (any) numeraire their expected values (with respect to the numeraire's associated measure) will, in theory, agree with their initial values. In practice, however, when using a reasonable number of Monte Carlo paths, one finds

$$P_T \;\neq\; \left\langle \mathcal{N}(T)^{-1} \right\rangle ,$$

where we use the notation $\langle \cdot \rangle$ to indicate the sample mean over all paths.

The unbiasing methodology replaces $F_k(T) \to \hat{F}_k(T)$ so that

$$\hat{\mathcal{N}}(T) \;:=\; P_1^{-1} \prod_{k=1}^{\gamma(T)-1} (1+\tau_k \hat{F}_k) \cdot P_{\gamma(T)}(T), \tag{A.2a}$$

$$P_T \;=\; \left\langle \hat{\mathcal{N}}(T)^{-1} \right\rangle . \tag{A.2b}$$

Unbiasing formulas are developed so that all fixed-coupon bonds (with arbitrary coupon schedules) and on-grid float-coupon bonds are exactly priced, while the off-grid float-coupon bonds are priced very nearly exactly.

B Determining the LIBOR Grid $\{T_k\}$

There are certain scheduling dates that are naturally included when determining the Libor grid. One perspective is that T_0 is the deal's evaluation date, and T_n is the deal's horizon date. Another view is that T_0 is today's date, and T_n is the maturity of the furthest dated calibration instrument. These views are equally valid, but have different implications for calibration and simulation.

B.1 Market Aligned

The "market aligned" approach is more natural for the purist, whose view is that the unadulterated quotes of the liquid instruments in today's market are the most reliable inputs to which model parameters should be calibrated. In this view, the Libor grid should agree with that of the prevailing swap whose tenor encompasses the market instruments relevant to the deal. This choice will, for the most part, give consistency between the Libor grid and the calibration instruments, and hence will be straightforward for the calibration phase. On the other hand, if the deal dates, which will include fixing and accrual schedules of Libor and CMS underlyings relevant to the deal, is misaligned with today's market (for example, when pricing an aged deal), then some interpolation methodology at the pricing phase will have to account for the mismatch between those underlying Libors being priced and the internally simulated Libor states.

The interpolation scheme used in adapting a market-aligned Libor grid to the pricing of off-grid Libor rates amounts to the evaluation of "short-dated bonds" $P_{\gamma(t)}(t)$ and their associated "stub rates" $L(t;T,T_{\gamma(T)})$, namely the forward rate from $T \to T_{\gamma(T)}$ evaluated at time t. This approach suffers from some performance degradation during the pricing phase, but is completely flexible in that it does not impose any constraint on the set of pricing dates. It should be remarked that when

$$t < T_k < T < T_{k+1} \quad \gamma(T) = k+1,$$

the evaluation of the interpolated stub rate $L(t;T,T_{k+1})$ is determined by the arbitrage-free requirement to be a multiple (whose value depends only on the initial discount curve) of the simulated state $[L(t;T_k,T_{k+1}) + \alpha_k]$:

$$[L(t;T,T_{k+1}) + \alpha_k] \quad = \quad c \cdot [L(t;T_k,T_{k+1}) + \alpha_k]\,. \tag{B.1}$$

On the other hand, when $T_k \le t \le T < T_{k+1}$, which includes the special case when $t = T$ relevant to evaluating $P_{\gamma(T)}(T)$, the right hand side of (B.1) has been frozen at L_k's fixing time T_k^-, and so the stub rate will suffer from the *premature freezing* of $L(T_k;T_k,T_{k+1})$ which has no evolution between T_k and t. The method of [Wer] was adopted to overcome this premature freezing by allowing L_k to continue evolving for $T_k \le t < T_{k+1}$, and was found to have many advantages over competing algorithms. An additional section Appendix E presents many more details related to pricing short-dated bonds using this technique of *zombie-rates*.

Finally, it should be noted that the evaluation of stub rates described here is critical to the time t evaluation of discount factors, in particular the numeraire $\mathcal{N}^S(t)$ discussed in Appendix A, when t is not a grid point T_k.

B.2 Deal Aligned

The "deal aligned" approach locks in the Libor grid to be compatible with the pricing dates relevant to the deal. Of course, this may not be possible, as in the case of a range accrual with daily observations and monthly coupons based on quarterly Libors, in which case the methods of §B.1 will be necessary. However, if the deal dates are a subset of a natural Libor schedule, but T_0 does not agree with today, then another approach is available. Since the calibration phase cannot handle calibrating model parameters associated with a given Libor grid to calibration instruments associated with a different Libor grid, a new set of calibration instruments whose expiries (but not tenors) must be translated to be relative to T_0 instead of today.

For this purpose, one can invoke the functionality of the terminal's `VCUB` function to generate quotes for *synthetic* caps and swaptions whose expiries are aligned with the deal's T_0 (hence the description "deal-aligned"), but misaligned with today's market. Other than any possible overhead arising from `VCUB` generating the synthetic quotes, the calibration and simulation phases are unaffected. As mentioned above, this option is appropriate only when all of the dates relevant to pricing belong to a single Libor schedule. Furthermore, one should note the following additional drawbacks to this approach. The interpolation algorithms employed by VCUB make certain model-dependent assumptions[14], and may not be consistent with the shifted-lognormal LMM model specification.

[14] See the VCUB Model Parameters option (38).

C Model Dynamics

C.1 Conventions and Notations

With regard to mathematical notation, we will generally denote scalar quantities in lower-case, vector quantities in lower-case bold-face, and matrix quantities in upper-case. Array representations of vectors will use row-vector notation, while transposes are indicated using an apostrophe. Thus

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v}' &= \sum_k u_k\cdot v_k =:\ \langle\mathbf{u},\mathbf{v}\rangle,\\ \mathbf{u}'\cdot\mathbf{v} &= \big(\{u_i v_j\}\big) =:\ \mathbf{u}\wedge\mathbf{v},\\ \langle\mathbf{u}A,\mathbf{v}\rangle &= \sum_{i,j} u_i a_{i,j} v_j = \langle\mathbf{u},\mathbf{v}A'\rangle.\end{aligned}$$

We will also need to consider interest-rate swaps, forward starting at time T_a, and where, at each time T_j ($j = a+1,\ldots,b$), a fixed rate K is exchanged for the Libor rate $F_{j-1}(T_j)$. The corresponding forward swap rate at time t is denoted by $S^{ab}(t)$. We have:

$$S^{ab}(t) \quad := \quad \frac{P_a(t)-P_b(t)}{\sum_{i=a}^{b-1}\tau_i P_{i+1}(t)} = \sum_{k=a}^{b-1}\omega_k^{ab}(t)F_k(t), \qquad \omega_k^{ab}(t) := \frac{\tau_k P_{k+1}(t)}{\sum_{i=a}^{b-1}\tau_i P_{i+1}(t)}. \tag{C.1}$$

Here, the denominator $\sum_{i=a}^{b-1}\tau_i P_{i+1}(t)$ is referred to as the "level" or "annuity", and in cases when the floating-leg and fixed-leg schedules are misaligned, is more properly written:

$$A^{ab}(t) := \sum_{i=\widetilde{a}}^{\widetilde{b-1}}\widetilde{\tau}_i P(t,\widetilde{T}_i). \tag{C.2}$$

For example, in the US swaption market the payment schedule for the floating leg is quarterly ($\tau_i \approx 0.25$), whereas the payment schedule for the fixed leg payments is semi-annual ($\widetilde{\tau}_i \approx 0.5$). In any case, the accrual start and end dates of the entire swap must agree, as must the overall coverage:

$$\begin{aligned}\widetilde{T}_{\widetilde{a}} &= T_a\\ \widetilde{T}_{\widetilde{b}} &= T_b\\ \sum_{i=\widetilde{a}}^{\widetilde{b-1}}\widetilde{\tau}_i &= \sum_{i=a}^{b-1}\tau_i.\end{aligned}$$

Again, for simplicity of presentation we may suppress this notational detail unless it is essential for a mathematical calculation.

When considering CMS options, we will require use of the Annuity numeraire, also called the Forward Swap numeraire, which is defined as the following bond portfolio

$$\mathcal{N}^{ab}(t) \quad := \quad \sum_{k=a}^{b-1}\tau_k P(t,T_{k+1}),. \tag{C.3}$$

The Annuity numeraire is associated with the Forward Swap measure Q^{ab} in which the forward swap rate $S^{ab}(t)$ becomes a martingale. Indeed, denoting by $\mathbf{E}^{ab}$ the expectation corresponding to Q^{ab}, we have

$$S^{ab}(0) \quad := \quad \frac{P_a - P_b}{\sum_{k=a}^{b-1} \tau_k P_{k+1}} = \frac{P_a(0) - P_b(0)}{\mathcal{N}^{ab}(0)} = \mathbf{E}^{ab}\left(\frac{P_a(t) - P_b(t)}{\mathcal{N}^{ab}(t)}\right), \tag{C.4}$$

where the latter equality reflects the pricing of a difference of bonds in units of the Annuity numeraire.

C.2 Diffusion Terms

The rank of ρ is the number of independent Brownian drivers for the correlated Brownian motions dW_j^Q, and is called the number of "factors" of our model. Denoting the rank of ρ by r, we have $r \geq 1$ since $\rho_{0,0} = 1$, and when $r = N$ we say that our model is "full-factor". In the particular case when $\rho = \mathbf{I}$, the SDE system $\{dF_j(t)\}$ will decouple into N independent shifted-lognormal evolutions. When ρ is rank-deficient, we can nonetheless find (statistically) independent Brownian drivers $dZ_i^Q(t)$ for which

$$\mathbb{E}\left[dZ_i^Q(t) \cdot dZ_j^Q(t)\right] = \delta_{i,j} dt, \qquad 0 \leq i, j < r, \tag{C.5}$$

and for which

$$[dW_0^Q, \ldots, dW_{N-1}^Q] \quad = \quad [dZ_0^Q, \ldots, dZ_{r-1}^Q] \cdot \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,N-1} \\ \vdots & \ddots & \vdots \\ \lambda_{r-1,0} & \cdots & \lambda_{r-1,N-1} \end{pmatrix} =: dZ^Q \cdot \Lambda. \tag{C.6}$$

Furthermore, combining (3.2), (C.5), and (C.6) gives

$$\begin{aligned} \rho dt &= \mathbb{E}\left[dW^{Q'} \cdot dW^Q\right] \\ &= \Lambda' \cdot \mathbb{E}\left[dZ^{Q'} \cdot dZ^Q\right] \cdot \Lambda \\ &= \Lambda' \cdot \mathbf{I} \cdot \Lambda dt, \end{aligned}$$

and so

$$\rho \quad = \quad \Lambda' \cdot \Lambda. \tag{C.7}$$

When $r = N$ and Λ is upper-triangular, (C.7) is referred to as the Cholesky factorization of ρ. When $r < N$, Λ is often referred to as the "pseudo square root" of ρ.

C.3 Drift Terms

By definition, in the shifted-lognormal LMM each forward rate F_j evolves under the corresponding forward measure $Q^{T_{j+1}}$ as a shifted geometric Brownian motion. Precisely, the instantaneous volatility of F_j is assumed to be given by

$$\sigma_j(t) \cdot (F_j(t) + \alpha_j),$$

where α_j's are real constants and σ_j's are deterministic functions of time, so that the dynamics of F_j under $Q^{T_{j+1}}$ is

$$dF_j(t) \quad = \quad \sigma_j(t) \cdot (F_j(t) + \alpha_j)\; dW_j^j(t), \tag{C.8}$$

where W_j^j is a standard Brownian motion under $Q^{T_{j+1}}$.

As (C.8) implies

$$d(F_j(t) + \alpha_j) \quad = \quad \sigma_j(t)(F_j(t) + \alpha_j)\; dW_j^j(t),$$

it follows that the forward rate F_j can be explicitly written as

$$\boxed{F_j(T) \quad = \quad -\alpha_j + (F_j(t) + \alpha_j)e^{-\frac{1}{2}\int_t^T \sigma_j^2(u)\,du + \int_t^T \sigma_j(u)\,dW_j^j(u)} \qquad t < T \leq T_j. \quad \text{(C.9)}}$$

The distribution of $F_j(T)$, conditional on $F_j(t)$, $t < T \leq T_{j-1}$, is then shifted-lognormal with density

$$p_{F_j(T)|F_j(t)}(x) \quad = \quad \frac{1}{(x+\alpha_j)U_j(t,T)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{\ln \frac{x+\alpha_j}{F_j(t)+\alpha_j} + \frac{1}{2}U_j^2(t,T)}{U_j(t,T)} \right)^2 \right\}, \tag{C.10}$$

for $x > -\alpha_j$, where $U_j(t,T)$ is the cumulative volatility defined by

$$U_j(t,T) \quad := \quad \sqrt{\int_t^T \sigma_j^2(u)du}\;.$$

We give below the dynamics of the forwards $F_j(t)$ in the Spot LIBOR measure Q_S associated with the Spot LIBOR numeraire (A.1).

Recalling (3.1a), we write

$$dX_j(t) \quad = \quad \mu_j^Q(t)dt + \sigma_j(t)X_j(t)dW_j^Q(t), \tag{C.11}$$

$$X_j(t) \quad := \quad F_j(t) + \alpha_j. \tag{C.12}$$

The drifts μ_j^Q of the forwards F_j are computed by requiring that the forwards be martingales in their own measure, *i.e.* by applying the change-of-numeraire technique to (C.8). Derivations of the explicit expression for μ_j^Q are provided in [AP, §15.2], [BM, §6.3], and give the following:

$$\mu_j^{Q_S}(t) \quad = \quad \sigma_j(t)X_j(t) \sum_{i=\gamma(t)}^{j} \frac{\rho_{i,j}\sigma_i(t)\tau_i X_i(t)}{1+\tau_i F_i(t)}. \tag{C.13}$$

C.4 Swap Rate Dynamics

It is not possible to model the dynamics of a swap rate exactly since the sum of shifted-lognormal processes is not itself a shifted-lognormal process. On the other hand, one may strive to model the swap rate S^{ab} *approximately* as a shifted-lognormal process in the forward swap measure Q^{ab} associated with the annuity numeraire A^{ab}. One seeks, therefore, to determine scalar parameters σ^{ab}, α^{ab}, and a single Q^{ab}-Brownian motion $W^{ab}(t)$, such that

$$\begin{aligned} X^{ab} &:= S^{ab} + \alpha^{ab}, \\ dX^{ab} &\approx \sigma^{ab}(t) \cdot X^{ab}(t) \cdot dW^{ab}(t). \end{aligned}$$

Specifically, consider the following quantities from which the swap rate from T_a to T_b is derived:

$$X_k(t) := F_k(t) + \alpha_k, \tag{C.14a}$$

$$A^{ab}(t) := \sum_{k=a}^{b-1} \tau_k P_t(T_{k+1}), \qquad \text{(annuity)} \tag{C.14b}$$

$$\omega_k^{ab}(t) := \frac{\tau_k P_t(T_{k+1})}{A_{ab}(t)} \qquad k = a, \dots, b-1, \tag{C.14c}$$

$$S^{ab}(t) := \sum_{k=a}^{b-1} \omega_k^{ab}(t) F_k(t). \qquad \text{(swap rate)} \tag{C.14d}$$

Following [BM, §6.15], one derives the following approximate shifted-lognormal dynamics of S^{ab}:

$$dS^{ab}(t) \approx \left(S^{ab}(t) + \alpha^{ab}\right) \cdot \langle \gamma_{ab}, dW_0^{ab} \rangle, \tag{C.15}$$

where α^{ab} is the "shift of the swaption"

$$\alpha^{ab} := \sum_{k=a}^{b-1} \omega_k^{ab} \alpha_k, \tag{C.16}$$

and dW_{ab} is the "row vector of correlated Brownian motions"

$$dW_{ab}(t) := \left[dW_a(t), \dots, dW_{b-1}(t)\right],$$

and γ_{ab} is the "vectorized volatility of the swaption"

$$\gamma_{ab}(t) := \frac{\left[\tau_a P_{a+1} X_a \sigma_a(t), \dots, \tau_{b-1} P_b X_{b-1} \sigma_{b-1}(t)\right]}{\sum_{k=a}^{b-1} \tau_k P_{k+1} X_k}, \tag{C.17}$$

and the constants $X_k = X_k(0)$ and $P_k = P_0(T_k)$ are values frozen at time T_0. Furthermore, by giving the explicit expression for the variance of the one dimensional Brownian motion $\langle \gamma_{ab}, dW_0^{ab} \rangle$ in terms of a quadratic form in the model parameters:

$$\begin{aligned} \rho_{ab} &:= \left(\rho_{i,j}\right)_{i=a,\dots,b-1; j=a,\dots,b-1} \\ \langle \gamma_{ab}, dW_0^{ab} \rangle^2 &= \langle \gamma_{ab}, dW_0^{ab} \rangle \cdot \langle dW_0^{ab}, \gamma_{ab}{}' \rangle = \gamma_{ab} \cdot dW_{ab}' \cdot dW_{ab} \cdot \gamma_{ab}{}', \\ \mathbb{E}\left[\langle \gamma_{ab}, dW_0^{ab} \rangle^2\right] &\overset{(3.2)}{=} \left(\gamma_{ab} \cdot \rho_{ab} \cdot \gamma_{ab}{}'\right) dt = \langle \gamma_{ab} \cdot \rho_{ab}, \gamma_{ab} \rangle dt, \end{aligned}$$

we can simplify (C.15) by introducing the "volatility of the swaption"

$$\sigma^{ab}(t) \quad := \quad \langle \gamma_{ab} \cdot \rho_{ab}, \gamma_{ab} \rangle^{\frac{1}{2}}. \tag{C.18}$$

Using the above we can write the dynamics of $S^{ab}(t)$ in terms of (approximate) shifted-lognormal parameters α^{ab} and $\sigma^{ab}(t)$ and Q^{ab}-Brownian motion $dW^{ab}(t)$:

$$\begin{aligned} X^{ab}(t) \quad &:= \quad S^{ab}(t) + \alpha^{ab}, \\ dW^{ab}(t) \quad &:= \quad \langle \gamma_{ab}, dW_0^{ab} \rangle / \sigma^{ab}, \\ dX^{ab}(t) \quad &\approx \quad \sigma^{ab}(t) \cdot X^{ab}(t) \cdot dW^{ab}(t). \end{aligned} \tag{C.19}$$

D Pricing Formulas

D.1 Exact Caplet Pricing Formula

In this section we show that caplet and floorlet prices in the shifted-lognormal LMM can be calculated in closed form. Consider a caplet whose payoff, fixed at T_j and paid at T_{j+1}, is given by

$$\tau_j [F_j(T_j) - K]^+$$

where K is its strike price. Then, assuming unit notional, the price of the caplet can be calculated as follows:

$$\begin{aligned} \mathbf{Cpl}(0, T_j, T_{j+1}, \tau_j, K) \quad &= \quad \tau_j P(0, T_{j+1}) E^{Q_{j+1}} \left[(F_j(T_j) - K)^+ \right] \\ &= \quad \tau_j P_{j+1} E^{Q_{j+1}} \left[\big(F_j\left(T_j\right) + \alpha_j - \left(K + \alpha_j\right) \big)^+ \right]. \end{aligned} \tag{D.1}$$

Since, according to (3.1a), $X_j(T_j) = F_j(T_j) + \alpha_j$ is a lognormal random variable in the Q_j measure, the last expectation in (D.1) yields the adjusted Black caplet price that corresponds to a shifted geometric Brownian motion:

$$\boxed{\mathbf{Cpl}(0, T_j, T_{j+1}, \tau_j, K) \quad = \quad \tau_j P_{j+1} \mathrm{Bl}\big(F_j + \alpha_j, K + \alpha_j, V_j\big) \qquad \text{(D.2)}}$$

where the terminal volatility up to time T_j is given by

$$V_j \quad := \quad U_j(0, T_j) = \sqrt{\int_0^{T_j} \sigma_j^2(s)\ ds}\ , \tag{D.3}$$

and

$$\boxed{\mathrm{Bl}(F, K, v) := F \cdot \Phi\left(\frac{\ln(F/K) + v^2/2}{v} \right) - K \cdot \Phi\left(\frac{\ln(F/K) - v^2/2}{v} \right) \text{(D.4)}}$$

is the Black-Scholes formula with Φ denoting the standard normal distribution function.

Likewise, the shifted-lognormal LMM price of the floorlet paying out at time T_{j+1} the quantity

$$\tau_j[K - F_j(T_j)]^+$$

is given by

$$\boxed{\mathbf{Flt}(0, T_j, T_{j+1}, \tau_j, K) = \tau_j P_{j+1}\ \mathrm{Bl}\big(K + \alpha_j, F_j + \alpha_j, V_j\big). \qquad \text{(D.5)}}$$

D.2 Approximate Swaption Pricing Formula

A European payer (receiver) swaption is an option giving its owner the right to enter, at a given maturity T_a, an interest-rate swap where a fixed rate K is paid (received) on dates $T_a, \ldots, T_{b-1}$. The swaption payoff at time T_a is given by

$$\Big[\omega(S^{ab}(T_a) - K)\Big]^+ \sum_{i=a}^{b-1} \tau_i P(T_a, T_{i+1})$$

where $\omega = 1$ for a payer and $\omega = -1$ for a receiver, and $S^{ab}(t)$ denotes the forward swap rate (C.14) at time t for the set of times $T_a, \ldots, T_b$:

$$S^{ab}(t) \quad = \quad \frac{P(t, T_a) - P(t, T_b)}{\sum_{k=a}^{b-1} \tau_k P(t, T_{k+1})}.$$

Using the swap rate dynamics developed in §C.4, we can price swaptions by following the same procedure as in the caplet case discussed in §D.1. To this end, we first consider a European payer swaption with maturity T_α and strike K, whose underlying swap pays on times $T_{a+1}, \ldots, T_b$. Assuming unit notional, the swaption price at time zero can be calculated as follows:

$$\begin{aligned}
\mathbf{PS}(0; a, b, K) \quad &= \quad \sum_{h=a}^{b-1} \tau_h P\left(0, T_{h+1}\right) \mathbf{E}^{ab}\left[\left(S^{ab}\left(T_a\right) - K\right)^+\right] \\
&= \quad A^{ab}(0) \cdot \mathbf{E}^{ab}\left[\left(X^{ab}\left(T_\alpha\right) - \left(K + \alpha^{ab}\right)\right)^+\right].
\end{aligned}$$

Since, according to (C.19), $X^{ab}(T_j) = S^{ab}(T_j) + \alpha^{ab}$ is (approximately) a lognormal random variable in the Q^{ab} measure, the last expectation in (D.1) yields the adjusted Black swaption price that corresponds to a shifted geometric Brownian motion. We thus obtain:

$$\boxed{\mathbf{PS}(0; a, b, K) \quad \approx \quad A^{ab}(0) \cdot \mathrm{Bl}\big(S^{ab}(0) + \alpha^{ab}, K + \alpha^{ab}, \Gamma^{ab}\big) \qquad \text{(D.6)}}$$

where Γ^{ab} is the terminal volatility defined by

$$\Gamma^{ab} \quad := \quad \sqrt{\int_0^{T_\alpha} \left[\gamma^{ab}(s)\right]^2 ds} = \sqrt{\sum_{i,j=a}^{b-1} \rho_{i,j} \int_0^{T_a} \gamma_i(s)\gamma_j(s) ds} \qquad \text{(D.7)}$$

and we have used expressions (C.16) and (C.18) derived in §C.4.

Likewise, the price of a receiver swaption is given by

$$\boxed{\mathbf{RS}(0;a,b,K) \approx A^{ab}(0)\cdot \mathrm{Bl}\big(K+\alpha^{ab}, S^{ab}(0)+\alpha^{ab}, \Gamma^{ab}\big) \qquad \text{(D.8)}}$$

D.3 Approximate CMS Spread Option Pricing Formula

In this section we provide an approximation formula for the CMS spread option price, which gives straddle, call, and put prices as a function of the strike and the model parameters $\alpha_k, \sigma_k, \rho_{i,j}$.

The basic tool for pricing the difference of two instruments is the Margrabe spread option pricing formula [Mar]. Now the Margrabe formula applies to the difference between two processes which are lognormal in a common measure. The first issue to confront is adapting the Margrabe formula to the difference between two shifted-lognormal processes, which is easily addressed. Less straightforward is the second issue, namely that the formulas for the CMS rates are naturally derived in their respective annuity measures Q^{ab} and Q^{ac}, and therefore these formulas must be adapted to a common measure which we choose to be that of the T_a-Forward. Thirdly, it will turn out that the CMS rates are no longer shifted-lognormal processes once adapted to this shared Q^{T_a} measure, and so we will need to invoke the technique of moment-matching to determine genuine shifted-lognormal process which best approximate (in the sense of agreement with their first two moments) the CMS rates.

Summarizing, in order to price CMS spread options we need to consider the two forward rates $S^{ab}(t)$ and $S^{ac}(t)$, $a < b < c$, at time T_a under the same measure Q^{T_a}. Having derived the dynamics of the forward swap rate processes in their natural annuity measures, the adapted formulas giving their moments in the common Q^{T_a} measure can be deduced by applying the change-of-measure technique. Next, one performs moment-matching to determine a new shifted-lognormal processes under the Q^{T_a} measure with suitable first and second moments. Finally, one invokes a variant of the Margrabe spread option pricing formula adapted to two shifted-lognormal underlying processes.

In more detail, we approximate the distribution of the forward CMS rate under the shifted Libor market model (for a given set of Libor market model parameter values) as a shifted lognormal distribution:

$$\hat{S}^{ab}(T_a) = -\hat{\alpha}^{ab} + (\hat{S}^{ab}(0)+\hat{\alpha}^{ab})e^{-\frac{1}{2}T_a(\hat{\sigma}^{ab})^2+\hat{\sigma}^{ab}W_{cms1}(T_a)}, \qquad \text{(D.9)}$$

where $W_{cms1}(T_a) \sim \mathcal{N}(0, T_a)$ under the T_a-forward measure. Here we use the notation $\hat{S}^{ab}(t)$ for the forward CMS rate

$$\hat{S}^{ab}(t) = \mathbf{E}_t^{T_a}(S^{ab}(T_a)), \qquad t < T_a,$$

where $\mathbf{E}_t^{T_a}(\cdot)$ denotes time t conditional expectation under the T_a-forward measure.

We need to obtain fitted values for $\hat{\alpha}^{ab}, \hat{S}^{ab}(0), \hat{\sigma}^{ab}$ in terms of the Libor market model parameters. First of all, we will use the same shift value $\hat{\alpha}^{ab} := \alpha^{ab}$ as that of $S^{ab}(T_a)$ in the Q^{ab} measure, as defined in (C.16).

We next need to estimate the value of the initial forward CMS rate $\hat{S}^{ab}(0)$. We do this using a closed-form approximation for the CMS convexity adjustment. There are a variety of standard practitioner techniques for deriving such approximations, but typically the starting point is to note that, by change of measure, we have

$$\hat{S}^{ab}(0) = \mathbf{E}^{T_a}[S^{ab}(T_a)] = \mathbf{E}^{ab}[\eta^{ab}(T_a)S^{ab}(T_a)], \tag{D.10}$$

where $\eta^{ab}(T_a)$ is the Radon-Nikodym derivative

$$\eta^{ab}(T_a) = \frac{B(T_a,T)}{A^{ab}(T_a)} / \mathbf{E}^{ab}\left[\frac{P(T_a,T)}{A^{ab}(T_a)}\right], \qquad T_a \le T \le T_b, \tag{D.11}$$

and where T denotes the payment time of the forward CMS rate, and $B(T_a,T)$ is the discount factor at time T evaluated at time T_a.

The next step is to approximate the Radon-Nikodym derivative using a functional form which allows the expectation (D.10) to be computed in closed form. For simplicity and practicality we will use a linear approximation of the form

$$\frac{B(T_a,T)}{A^{ab}(T_a)} \approx \Theta(T) + \Psi(T)\left(S^{ab}(T_a) - S^{ab}(0)\right), \tag{D.12}$$

where $\Theta(T), \Psi(T)$ are deterministic functions of the initial market data. We will set out below two methods, well-known in the financial modelling literature, which Bloomberg has implemented to determine $\Theta(T), \Psi(T)$. Assuming for the time being that these functions are known, substituting (D.12) into (D.10) gives the CMS approximation formula

$$\hat{S}^{ab}(0) = S^{ab}(0) + \frac{\Psi(T)}{\Theta(T)}\Big[\mathbf{E}^{ab}[S^{ab}(T_a)^2] - S^{ab}(0)^2\Big]. \tag{D.13}$$

To complete the calculation we need to compute the expectation $\mathbf{E}^{ab}[S^{ab}(T_a)^2]$ in (D.16) in terms of the Libor market model parameters. We recall that in §C.4 we fitted the dynamics of $S^{ab}(t)$ under the Q^{ab} annuity measure to a shifted lognormal process, where the time-dependent shifted lognormal parameters are obtained as functions of the Libor market model parameters[15]. Under this shifted lognormal approximation we can determine a closed-form expression for $\mathbf{E}^{ab}[S^{ab}(T_a)^2]$. To be explicit, from the shifted lognormal dynamics described in (C.19) and the definition of the shifted lognormal swaption terminal volatility Γ^{ab} given by (D.7)we obtain

$$\mathbf{E}^{ab}[S^{ab}(T_a)^2] = (\alpha^{ab})^2 - 2\alpha^{ab}(S^{ab}(0) + \alpha^{ab}) + (S^{ab}(0) + \alpha^{ab})^2 e^{(\Gamma^{ab})^2 T_a}. \tag{D.14}$$

Combining this with (D.16) gives the required closed-form approximation for the forward CMS rate $\hat{S}^{ab}(0)$.

It remains to determine the functions $\Theta(T), \Psi(T)$. As noted above, Bloomberg has implemented two methods for this, namely the *cash annuity formula* and *linear swap rate model* approximations.

[15] The results of the shifted lognormal fitting of the swap rate process are summarised in (C.19) above.

Cash annuity formula approximation

This method follows the well-known CMS modelling approach set out in [Hag]. In this case we approximate the annuity ratio using a linearized cash annuity formula[16]

$$\frac{B(T_a,T)}{A^{ab}(T_a)} \approx G(S^{ab}(0)) + G'(S^{ab}(0))\left(S^{ab}(T_a) - S^{ab}(0)\right), \tag{D.15}$$

where

$$G(S) := \frac{S}{(1+S/q)^{\Delta}} \frac{1}{1-\frac{1}{(1+S/q)^n}}.$$

From (D.15) one easily identifies $\Theta = G$ and $\Psi = G'$, and so the CMS formula (D.13) becomes

$$\hat{S}^{ab}(0) = S^{ab}(0) + \frac{G'(S^{ab}(0))}{G(S^{ab}(0))}\Big[\mathbf{E}^{ab}[S^{ab}(T_a)^2] - S^{ab}(0)^2\Big]. \tag{D.16}$$

Linear swap rate model approximation

This method uses a version of the well-known linear swap rate model described in the references [HK, Ch. 8] and [Pel]. We first prescribe the following functional form using the notation $B_k(T_a) := B(T_a, T_k)$:

$$\frac{B_k(T_a)}{A^{ab}(T_a)} = \hat{\alpha} + \hat{\beta}_k\left(\frac{S^{ab}(T_a)+\alpha^{ab}}{S^{ab}(0)+\alpha^{ab}}\right), \qquad k = a,\dots,b, \tag{D.17}$$

and then apply no-arbitrage constraints to deduce values for the coefficients $\hat{\alpha}, \hat{\beta}_k$. The first no-arbitrage constraints are obtained by taking expectation of (D.17) under the Q^{ab} annuity measure, which gives

$$\frac{B_k(0)}{A^{ab}(0)} = \hat{\alpha} + \hat{\beta}_k, \qquad k = a,\dots,b. \tag{D.18}$$

To derive the final constraint we use the definition of the annuity

$$A^{ab}(T_a) = \sum_{k=a}^{b-1} \widetilde{\tau}_k B_{k+1}(T_a),$$

to obtain

$$1 = \frac{\sum_{k=a}^{b-1}\widetilde{\tau}_k B_{k+1}(T_a)}{A^{ab}(T_a)} \overset{(D.18)}{=} \hat{\alpha}\left(\sum_{k=a}^{b-1}\widetilde{\tau}_k\right) + \left(\frac{S^{ab}(T_a)+\alpha^{ab}}{S^{ab}(0)+\alpha^{ab}}\right)\left(\sum_{k=a}^{b-1}\widetilde{\tau}_k\hat{\beta}_{k+1}\right). \tag{D.19}$$

[16] In the cash annuity formula n is the number of fixed coupon payments, q is the fixed leg swap frequency, and Δ is the fraction of a swap period between the swap start date and the CMS pay date.

This equation must be satisfied for every value of the random variable $S^{ab}(T_a)$, which requires $\sum_{k=a}^{b-1} \widetilde{\tau}_k \hat{\beta}_{k+1} = 0$ and $\hat{\alpha} \sum_{k=a}^{b-1} \widetilde{\tau}_k = 1$. We have therefore obtained the following explicit expressions for $\hat{\alpha}$ and $\hat{\beta}_k$:

$$\hat{\alpha} = \Big(\sum_{k=a}^{b-1} \widetilde{\tau}_k \Big)^{-1} \tag{D.20}$$

$$\hat{\beta}_k = \Big(\frac{B_k(0)}{A^{ab}(0)} - \hat{\alpha} \Big). \tag{D.21}$$

To complete the calculation, note that by comparing (D.17) and (D.12) we have for any choice of $T_k = T_a, T_{a+1}, \ldots$ as a payment time[17]

$$\Theta(T_k) = \hat{\alpha} + \hat{\beta}_k,$$

$$\Psi(T_k) = \frac{\hat{\beta}_k}{S^{ab}(0) + \alpha^{ab}},$$

which combined with (D.13) gives the closed-form CMS approximation formula.

Estimation of the second moment

At this point we have calculated the values of the first two parameters $\hat{\alpha}^{ab}, \hat{S}^{ab}(0)$ in the shifted lognormal fitting (D.9) of the forward CMS rate $\hat{S}^{ab}(T_a)$. It remains to consider the third parameter $\hat{\sigma}^{ab}$. We can extend the technique employed above to calculate a closed-form approximation formula for $\hat{\sigma}^{ab}$. We first note that (D.9) implies that

$$\mathbf{E}^{T_a}[\hat{S}^{ab}(T_a)^2] = (\hat{\alpha}^{ab})^2 - 2\hat{\alpha}^{ab}(\hat{S}^{ab}(0) + \hat{\alpha}^{ab}) + (\hat{S}^{ab}(0) + \hat{\alpha}^{ab})^2 e^{(\hat{\sigma}^{ab})^2 T_a}. \tag{D.22}$$

Therefore, given the already-computed values of $\hat{\alpha}^{ab}$ and $\hat{S}^{ab}(0)$, the value of $\hat{\sigma}^{ab}$ is determined by the second moment $\mathbf{E}^{T_a}[\hat{S}^{ab}(T_a)^2]$. Extending the approach presented above, we derive a closed-form approximation to this second moment from the change of measure formula

$$\mathbf{E}^{T_a}[S^{ab}(T_a)^2] = \mathbf{E}^{ab}[\eta^{ab}(T_a) S^{ab}(T_a)^2], \tag{D.23}$$

together with the linear approximation (D.12) to the Radon-Nikodym derivative $\eta^{ab}(T_a)$. The approximation formula is

$$\mathbf{E}^{T_a}[S^{ab}(T_a)^2] = \mathbf{E}^{ab}\left[\Big(\Theta(T) + \Psi(T) \Big(S^{ab}(T_a) - S^{ab}(0) \Big) \Big) S^{ab}(T_a)^2 \right].$$

The calculation of the right-hand-side requires computation of the third moment $\mathbf{E}^{ab}[S^{ab}(T_a)^3]$ under the Q^{ab} annuity measure. This is straightforward to derive in closed form, given the shifted lognormal fit (C.19) of the forward swap rate $S^{ab}(T_a)$ under the natural annuity measure Q^{ab}.

[17] Typically $T_k = T_a$ for single-looks, and $T_k = T_{a+1}$ for multi-looks.

Further remarks on moment matching

The Bloomberg implementation of the Libor market model may be configured additionally to allow improved matching to CMS forward rates by the use of an "adjuster" technique. When adjusters are used for CMS forward matching, then when pricing a CMS-linked instrument, the theoretical CMS forward rate $\hat{S}^{ab}(0)$ implied by the Libor market model and its parameter values is replaced by the adjusted rate

$$\hat{S}^{ab}(0) + \lambda^{ab},$$

where

$$\lambda^{ab} = \hat{S}^{ab}_{Market} - \hat{S}^{ab}(0).$$

Here $\hat{S}^{ab}_{Market}$ is the market CMS forward rate, extracted from market quotes for CMS swaps.

If the model is configured by the user not to use CMS forward adjusters, then the fitted parameter $\hat{S}^{ab}(0)$ is determined through the cash annuity formula approximation. The parameter $\hat{\sigma}^{ab}$ is always fitted using the linear swap rate model, irrespective of whether or not CMS forward adjusters are used.

The final step before applying the Margrabe spread option formula is to establish the correlation between $W_{cms1}(T_a)$ and $W_{cms2}(T_a)$:

$$\begin{aligned} \rho &:= Correl\left(\ln\left(\frac{\hat{S}^{ab}(T_a)+\alpha^{ab}}{\hat{S}^{ab}(0)+\alpha^{ab}}\right), \ln\left(\frac{\hat{S}^{ac}(T_a)+\alpha^{ac}}{\hat{S}^{ac}(0)+\alpha^{ac}}\right)\right) \\ &\approx \frac{\int_0^{T_a}\langle\gamma_{ab}, \gamma_{ac}\rho^{ab}_{ac}\rangle dt}{\sqrt{\int_0^{T_a}\langle\gamma_{ab},\gamma_{ab}\rho_{ab}\rangle dt \ \int_0^{T_a}\langle\gamma_{ac},\gamma_{ac}\rho_{ac}\rangle dt}}, \end{aligned} \tag{D.24}$$

where γ_{ab}, γ_{ac} are given in (C.17), and ρ^{ab}_{ac} is the rectangular sub-matrix of ρ given by

$$\begin{pmatrix} \rho_{a,a} & \cdots & \rho_{a,b} \\ \vdots & \rho_{i,j} & \vdots \\ \rho_{c,a} & \cdots & \rho_{c,b} \end{pmatrix}.$$

By making the additional definitions where the symbol x indicates an integration parameter:

$$\begin{aligned} \widetilde{K} &= K + \alpha^{ab} - \alpha^{ac}, \\ X_1(x) &= \hat{S}^{ab}(0)\exp\left(\hat{\sigma}^{ab}x\sqrt{T_a} - \frac{1}{2}T_a(\hat{\sigma}^{ab})^2\right), \\ X_2(x) &= \hat{S}^{ac}(0)\exp\left(\hat{\sigma}^{ac}x\rho\sqrt{T_a} - \frac{1}{2}\rho^2 T_a(\hat{\sigma}^{ac})^2\right), \end{aligned}$$

we can express the price of a call option on a CMS spread of strike K using the Margrabe spread option formula:

$$\boxed{\begin{aligned} \mathbf{CMSSC}(0; T_a, K, \hat{S}^{ab}, \hat{\sigma}^{ab}, \alpha^{ab}, \hat{S}^{ac}, \hat{\sigma}^{ac}, \alpha^{ac}, \rho) &= B_a(0)\cdot\mathbf{E}^{T_a}\left[\left(\hat{S}^{ac}(T_a) - \hat{S}^{ab}(T_a) - K\right)^+\right] \\ &\approx B_a(0)\cdot\int_{-\infty}^{\infty}\mathrm{Bl}(X_2(x), X_1(x)+\widetilde{K}, \sqrt{T_a(1-\rho^2)}\hat{\sigma}^{ac})\frac{e^{-x^2/2}dx}{\sqrt{2\pi}}. \qquad \text{(D.25)} \end{aligned}}$$

The price of a put option on a CMS spread of strike K is derived similarly to (D.25):

$$\begin{aligned}\mathbf{CMSSP}(0;T_a,K,\hat{S}^{ab},\sigma^{ab},\alpha^{ab},\hat{S}^{ac},\sigma^{ac},\alpha^{ac},\rho) &= B_a(0)\cdot\mathbf{E}^{T_a}\Big[\big(-\hat{S}^{ac}(T_a)+\hat{S}^{ab}(T_a)+K\big)^+\Big] \\ &\approx B_a(0)\cdot\int_{-\infty}^{\infty}\mathrm{Bl}(X_1(x)+\widetilde{K},X_2(x),\sqrt{T_a(1-\rho^2)}\hat{\sigma}^{ac})\frac{e^{-x^2/2}dx}{\sqrt{2\pi}}.\end{aligned} \tag{D.26}$$

E Interpolation

For purposes of this section we also make the following useful definition for identifying the Libor index of the first rate "not yet frozen" and that of the last rate "already frozen" corresponding to an arbitrary time t, *i.e.* whenever $T^-_{i-1}\le t<T^-_i$:

$$\begin{aligned} q(t) &:= i, \\ q'(t) &:= (i-1). \end{aligned}$$

In other words, $T^-_{q'(t)}\le t<T^-_{q(t)}$ for all t, and this condition defines q and q'.

The rates $L_k(t)$ are determined by the Monte Carlo simulation by snapshotting the evolution of each simulated state variable L_k at time t. In terms of these forward Libor rates, the Zero Coupon Bond with maturity T_k, denoted $P_k(t)$, is determined recursively for $t<T_{q(t)}<T_k$ according to:

$$P_k(t) := P_{k-1}(t)\cdot[1+\tau_{k-1}L_{k-1}(t)]^{-1}, \quad \text{(inductive step)} \tag{E.1a}$$

$$P_{q(t)}(t) = \text{to be determined} \quad \text{(initial condition).} \tag{E.1b}$$

More generally, for $t\le T_a<T_b$ (a and b are *not* indices), we denote by

$$P(t,T_a,T_b)$$

the forward discount from time T_a to T_b, evaluated at time t. Thus, $P_k(t)=P(t,t,T_k)$, the Zero Coupon Bond with arbitrary maturity T is given by $P(t,t,T)$, and from (E.1b) we have $P(t,t,T_{q(t)})=P_{q(t)}(t)$ is yet to be determined. Intrinsically, the Libor Market Model provides only $P(t,T_i,T_j)$ for grid points T_i and T_j.

In terms of the simulated Libor rates $L_k(t)$, the complete set of forward discounts at *grid dates* $(T_a,T_b)=(T_i,T_j)$ (for grid indices $i<j$) generated by the simulation is determined by

$$P(t,T_i,T_j) := P_j(t)/P_i(t) \overset{\text{(E.1a)}}{=} \prod_{k=i}^{j-1}[1+\tau_kL_k(t)]^{-1}. \tag{E.2}$$

A natural non-arbitrage condition imposed on the discounts for *arbitrary* times $t\le T_a\le T_b$ requires

$$P(t,T_a,T_c)=P(t,T_a,T_b)\cdot P(t,T_b,T_c). \tag{E.3}$$

Using (E.3), we write for *non-grid dates* T_a and T_b, and $q(a) := q(T_a)$,

$$\begin{aligned} P(t,T_a,T_b) &= P(t,T_a,T_{q(a)})\cdot P(t,T_{q(a)},T_{q'(b)})\cdot P(t,T_{q'(b)},T_b) \\ &=: P_{front}(t,T_a)\cdot P(t,T_{q(a)},T_{q'(b)})\cdot P_{back}(t,T_b) \qquad \text{(E.4a)} \\ &\overset{(E.3)}{=} P_{front}(t,T_a)\cdot P(t,T_{q(a)},T_{q(b)})/P(t,T_b,T_{q(b)}) \qquad \text{(E.4b)} \\ &= P_{front}(t,T_a)\cdot P(t,T_{q(a)},T_{q(b)})/P_{front}(t,T_b). \qquad \text{(E.4c)} \end{aligned}$$

We have denoted by P_{front}[18] the discount $P(t,T_a,T_{q(a)})$, which is variously referred to as the "short-dated bond", or the "front stub" [AP]. The "back stub" $P(t,T_{q'(b)},T_b)$ has been denoted by P_{back}[19] in (E.4a). From (E.4c) it is clear that it suffices to define the front stubs $P_{front}(t,T)$ for all T in order to determine the left hand side $P(t,T_a,T_b)$ generally.

Note that in (E.4b) we have applied (E.3) in the form

$$P_{back}(t,T_b) = P(t,T_{q'(b)},T_{q(b)})/P(t,T_b,T_{q(b)}). \qquad \text{(E.5)}$$

When $t=T_b$, and after inverting all terms, (E.5) states that \$1 invested at the fixed Libor rate at the accrual start date $T_{q'(b)}$ will be worth

$$1+\tau_{q'(b)}L_{q'(T_b)} = P(T_{q'(b)},T_{q'(b)},T_{q(b)})^{-1} = P(T_b,T_{q'(b)},T_{q(b)})^{-1}$$

at the accrual end date $T_{q(b)}$, and will have corresponding value at the pre-maturity date T_b obtained by applying the front stub discount $P(T_b,T_b,T_{q(b)})$. The assumption that $P(T_{q'(b)},T_{q'(b)},T_{q(b)}) = P(T_b,T_{q'(b)},T_{q(b)})$ used above is based on the freezing of the Libor rate $L_{q'(b)}$ at $t=T_{q'(b)}$, and will be revisited in §E where we discuss the so-called "zombie rates".

In addition to the non-arbitrage condition (E.3), we impose the additional non-arbitrage condition, which we formulate for the short-dated bond with $t\le T_a<T_q$, as

$$P_{front}(t,T_a) = \mathbb{E}_t^{T_a}\left[P(T_a,T_a,T_q)\right], \qquad (q=q(T_a)). \qquad \text{(E.6)}$$

Finally, we impose the condition that our interpolation formula for $P_{front}(t,T_a)$ agrees with the initial discount curve:

$$P_{front}(0,T_a) = \frac{P_{market}(T_q)}{P_{market}(T_a)}. \qquad \text{(E.7)}$$

We will see below how (E.3), (E.6), and (E.7) determine the ultimate scheme described in [Wer] and [Jäc, p24-26].

Calculation of the Short-Dated Bond using Zombie-Rates

In this section we will abbreviate $q(T)$ with simply q, and also $q'(T)$ with q'.

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[18] $P_{front}(t,T_a)$ can be identified with the discount $P(t,T_a,T_b)$ when T_b is the "next" grid point $T_{q(a)}$.

[19] $P_{back}(t,T_b)$ can be identified with the discount $P(t,T_a,T_b)$ when T_a is the "previous" grid point $T_{q'(b)}$.

As opposed to other approaches which introduce stochasticity into $L_{q'}(t > T_{q'})$ by mandating a dependence on the next (living) rate $L_q(t)$, the approach of [Wer] introduces stochasticity by allowing $L_{q'}(t)$ to evolve past its fixing $T_{q'}$, namely for $t \in [T_{q'}, T_q]$. As $L_{q'}(t)$ becomes officially "dead" when $t = T_{q'}$, [Wer] refers to $L_{q'}(t)$ at later times as a "zombie rate". The basic idea is to retain the shape of the initial curve of forward Libor rates $L(0, T, T_q)$ ($T_{q'} < T < T_q$) by mandating

$$L(t, T, T_q) := \left[L(0, T, T_q)/L(0, T_{q'}, T_q)\right] L(t, T_{q'}, T_q).$$

To ease the notation in what follows, L_k indicates the Libor $L_k(T_k)$ evaluated at its fixing. Also, P_k indicates the time-0 value of the discount bond maturing at T_k. Finally, P_T indicates the time-0 value of the discount bond maturing at time T.

We may express the interpolated rate in terms of the time-0 discounts:

$$\begin{aligned} f_T &:= L(0, T, T_q)/L(0, T_{q'}, T_q) && \text{(E.8)} \\ &= \left(\frac{\tau}{\tau_T}\right)\left(\frac{P_T/P_q - 1}{P_{q'}/P_q - 1}\right) \\ &= \left(\frac{\tau}{\tau_T}\right)\left(\frac{P_T - P_q}{P_{q'} - P_q}\right), && \text{(E.9)} \\ L(t, T, T_q) &:=^{20} f_T L_{q'}(t). && \text{(E.10)} \end{aligned}$$

In the case of shifted-lognormal, we look for a multiplier

$$\left[L(t, T, T_q) + \alpha_{q'}\right] = f_T \left[L_{q'}(t) + \alpha_{q'}\right],$$

so the corresponding formula is

$$\begin{aligned} f_T &:= \frac{L(0, T, T_q) + \alpha_{q'}}{L(0, T_{q'}, T_q) + \alpha_{q'}} && \text{(E.11)} \\ &= \left(\frac{\tau}{\tau_T}\right)\left(\frac{P_T - P_q(1 - \alpha_{q'}\tau_T)}{P_{q'} - P_q(1 - \alpha_{q'}\tau)}\right), && \text{(E.12)} \\ L(t, T, T_q) &= f_T\left(L_{q'}(t) + \alpha_{q'}\right) - \alpha_{q'} = f_T L_{q'}(t) + (f_T - 1)\alpha_{q'}. && \text{(E.13)} \end{aligned}$$

We may also express the interpolated front-stub discount in terms of f_T:

$$\begin{aligned} P_{front}(t, T) &:= P(t, T, T_{q(T)}) \\ &= \left[1 + \tau_T L(t, T, T_q)\right]^{-1} \\ &= \left[1 + \tau_T f_T L_{q'}(t))\right]^{-1}. && \text{(E.14)} \end{aligned}$$

In the shifted-lognormal case,

$$P_{front}(t, T) := \left[1 + \tau_T f_T L_{q'}(t) + \tau_T(f_T - 1)\alpha_{q'}\right]^{-1}. \qquad \text{(E.15)}$$

[20] An equivalent formulation is $[L(t, T, T_q) - L(0, T, T_q)] := f_T \left[L_{q'}(t) - L_{q'}(0)\right]$.

Consequently, the interpolated back-stub discount can be expressed as:

$$\begin{aligned} P_{back}(t,T) &:= P(t, T_{q'(T)}, T) \\ &= P(t, T_{q'(T)}, T_{q(T)})/P_{front}(t,T) \\ &= \frac{1+\tau_T f_T L_{q'}(T)}{1+\tau L_{q'}(T)}. \end{aligned} \tag{E.16}$$

If $t > T_{q'}$ then $L_{q'}(t)$ may be regarded to be a zombie rate in both numerator and denominator. On the other hand, for reasons of continuity ($T \to T_q, \tau_T \to 0$), it is better to use the frozen rate in the denominator and the zombie rate only in the numerator:

$$\begin{aligned} P_{back}(t,T) &:= \frac{1+\tau_T f_T L_{q'}(t)}{1+\tau L_{q'}(T_{q'})} \\ P_{back}(t, T \to T_q) &\longrightarrow \left[1+\tau L_{q'}(T_{q'})\right]^{-1} \quad \text{(and not } \left[1+\tau L_{q'}(T_q)\right]^{-1}\text{)}. \end{aligned}$$

In this way, when the front stub disappears in the limit ($\tau_T \to 0$), we are left with a back stub which is consistent with the native Libor Market calculation of the discount to T_q. Moreover, recalling the discussion of (E.5), the (inverse) back stub should be viewed as the time-T value of a \$1 Libor deposit made at time $T_{q'}$, which is to say the time-T value of $1 + \tau L_{q'}(T_{q'})$ paid at T_q.

Dynamics of the Zombie Rate

The dynamics of the zombie rate is given by

$$\begin{aligned} dL_n(t) &= (L_n(t) + \alpha_n)\sigma_n(T_n^-)dW_n(t), \qquad (n \geq 0) \\ dL_{-1}(t) &= (L_{-1}(t) + \alpha_0)\sigma_0(t)dW_0(t). \end{aligned}$$

In other words, $L_n(t)$ is a martingale in the spot Libor numeraire for $t \in [T_n, T_{n+1}]$, and its volatility is obtained by flat extrapolation of the volatility $\sigma_n(T_n^-)$. The martingale property is consistent with both $L_n(t)$ and $\mathcal{N}(t)$ being martingales with respect to the T_{n+1}-Forward measure when $t \in [T_n, T_{n+1}]$. Note that this flat extrapolation is consistent with the dynamics of the non-zombie Libor $L_n(t)$:

$$\begin{aligned} dL_n(t) &= (L_n(t) + \alpha_n)\left[\mu_n(t)dt + \sigma_n(t)dW_n(t)\right], \\ \mu_n(t) &= \sigma_n(t) \sum_{k=q(t)}^{n} \frac{\tau_k(t)\sigma_k(t)L_k(t)\rho_{k,n}(t)}{1+\tau_k L_k(t)}, \end{aligned}$$

where $q(t) = n+1$ implies the summation on the right hand side $\sum_{k=n+1}^{n}$ becomes vacuous. Additionally, the generation of $dW_n(t)$ is obtained just as it would be if $t < T_n$, giving $\langle L_n, L_i \rangle = \rho_{n,i}$ for $i > n$, and retaining $\langle L_i, L_j \rangle = \rho_{i,j}$ for $n < i, j < N$.

Additionally, to allow for stub rates when $t < T_0$, one evolves an artificial $L_{-1}(t)$ for $0 < t < T_0$. As there are no model parameters corresponding to L_{-1}, in this case both the volatility and shift are borrowed from L_0, as is its diffusion term.

The figures displayed in the following sections support the characterization of the LMM implementation as a *continuous* model which consistently generalizes its discrete on-grid pricing.

F Dual Curve Calculations

When calculating in a dual-curve setup, one makes the distinction between the OIS forward rates derived from the OIS discount curve $P_t(T)$

$$F_k(t) := \frac{1}{\tau_k}\left[\frac{P_t(T_k)}{P_t(T_{k+1})} - 1\right], \tag{F.1}$$

and the LIBOR rates $L_k(t)$ derived from the quoted LIBOR index. In particular, the implied LIBOR spread over OIS, associated with each LIBOR rate $L_k(t)$, is given by

$$\beta_k(t) \quad := \quad L_k(t) - F_k(t). \tag{F.2}$$

We make the simplifying assumption that $\beta_k(t)$ is time independent, hence $L_k(t) = F_k(t) + \beta_k$ where $\beta_k := L_k(0) - F_k(0)$. As only the OIS forward rates $F_k(t)$ are evolved in the Monte Carlo simulation, we need only explore the implications of the basis adjustment to calibration and deal pricing. Consider the following expression for the swaption price $\mathbf{PS}(t; a, b, K)$ on a payer swap from T_a to T_b struck at K:

$$\begin{aligned}
S^{ab}(t) \quad &:= \quad \sum_{k=a}^{b-1} \omega_k^{ab}(t) L_k(t) \\
&\overset{(F.2)}{=} \quad \sum_{k=a}^{b-1} \omega_k^{ab}(t)(F_k(t) + \beta_k), \qquad \text{(swap rate}^{21}\text{)} \\
\mathbf{PS}(t; a, b, K) \quad &= \quad N_{\mathbb{Q}}(t)\ \mathbb{E}_t^{\mathbb{Q}}\left[\frac{A^{ab}(T)\big(S^{ab}(T) - K\big)^+}{N_{\mathbb{Q}}(T)}\right],
\end{aligned}$$

which, when priced with respect to the forward swap measure $\mathbb{Q} = Q^{ab}$ at $t = 0, T = T_a$, gives

$$\begin{aligned}
\mathbf{PS}(0; a, b, K) \quad &= \quad A^{ab}(0)\ \mathbb{E}^{Q^{ab}}\left[\left(\sum_{k=a}^{b-1} \omega_k^{ab}(T_a)\big[F_k(T_a) - (K - \beta_k)\big]\right)^+\right] \\
&= \quad A^{ab}(0)\ \mathbb{E}^{Q^{ab}}\left[\left(\sum_{k=a}^{b-1} \omega_k^{ab}(T_a)\big[(X_k(T_a) - (K + \alpha_k - \beta_k)\big]\right)^+\right] \\
&= \quad A^{ab}(0)\ \mathbb{E}^{Q^{ab}}\left[\left(X^{ab}(T_a) - \sum_{k=a}^{b-1} \omega_k^{ab}(T_a)(K + \alpha_k - \beta_k)\right)^+\right].
\end{aligned} \tag{F.3}$$

Comparing (F.3) with the single-curve pricing of a swaption with respect to the shifted lognormal LMM as in (D.6), we obtain its analog in the dual-curve setup:

$$\boxed{\begin{aligned}
\mathbf{PS}(0; a, b, K) \quad &\approx \quad A^{ab}(0)\ \mathrm{Bl}\big(K + \alpha^{ab} - \beta^{ab}, X^{ab}(0), \Gamma^{ab}(T_a)\big) \\
&= \quad A^{ab}(0)\ \mathrm{Bl}\big(K + \alpha^{ab} - \beta^{ab}, S^{ab}(0) + \alpha^{ab} - \beta^{ab}, \Gamma^{ab}(T_a)\big) \qquad \text{(F.4)}
\end{aligned}}$$

[21] This expression differs from the single-curve formula by the quantity $\sum_{k=a}^{b-1} \omega_k^{ab} \beta_k$.

where

$$\beta^{ab} := \sum_{k=a}^{b-1} \omega^{ab}(0)\beta_k(0).$$

Setting $b = a + 1$ in (F.4) gives the analog of (D.1) for dual-curve caplet pricing. In other words, the dual-curve calibrated shifts are essentially the single-curve shifts $(\alpha_k - \beta_k)$, calibrated to the dual-curve quoted swaptions using the dual-curve ATM, and then adjusted by the LIBOR spreads β_k. As explained earlier, the Monte Carlo simulation evolves the OIS forward rates $F_k(t)$, or more precisely the $\log(F_k(t) + \alpha_k)$. Therefore, with respect to pricing, when a pathwise LIBOR underlying is required for evaluating a payoff, the returned rate is obtained from the OIS forward by the addition of the basis adjustment $F_k \to F_k + \beta_k$.

G Shift and Volatility Regularization

The regularization technique employed in the calibration process utilizes a penalty function which imposes smoothness by penalizing spikes in "adjacent" model parameter values. In the one dimensional case (shift parameters or constant volatility parameters), the penalty on variations with respect to the *forward index* takes the form:

$$\begin{aligned} p(\mathbf{x}; \omega_1, \omega_2) &= \sqrt{\omega_1 f_1(\mathbf{x}) + \omega_2 f_2(\mathbf{x})}, \\ f_1(\mathbf{x}) &:= \sum_{i=1}^{N-1} (x_{i+1} - x_i)^2, \\ f_2(\mathbf{x}) &:= \sum_{i=2}^{N-1} (x_{i+1} - 2x_i + x_{i-1})^2, \end{aligned} \tag{G.1}$$

where $\mathbf{x}$ is the vector of model parameters and ω_1, ω_2 are weights.

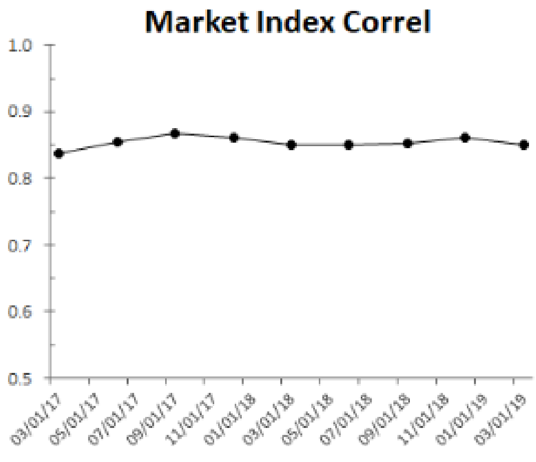
In the two dimensional case (piecewise constant volatility parameters), we have a similar formulation obtained by taking the rms of the $p(\cdot)$ evaluated over all rows and columns of X:

$$\begin{aligned} q^2(X; \omega_1, \omega_2, \eta_1, \eta_2) &= \omega_1 \sum_{j=1}^{N-1} f_1(X_{\cdot,j}) + \omega_2 \sum_{j=1}^{N-2} f_2(X_{\cdot,j}) + \eta_1 \sum_{i=2}^{N} f_1(X_{i,\cdot}) + \eta_2 \sum_{i=3}^{N} f_2(X_{i,\cdot}) \\ &= \sum_k p^2(X_{\cdot,k}; \omega_1, \omega_2) + \sum_k p^2(X_{k,\cdot}; \eta_1, \eta_2), \end{aligned} \tag{G.2}$$

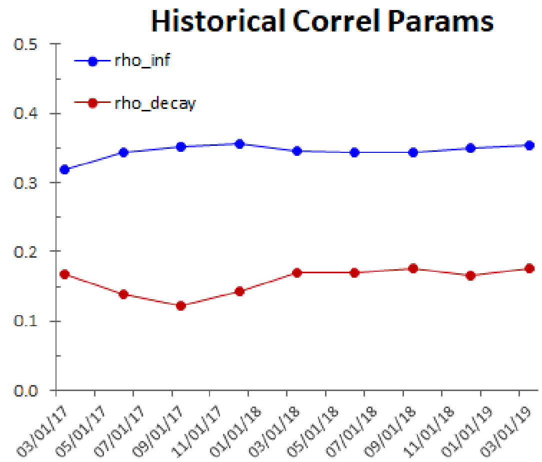
where ω_1, ω_2 are as above, and η_1, η_2 are independent weights applied to variations with respect to the time dimension.

H Correlation Default Values

The default values for the correlation parameter pair $(\rho_{\inf}, \rho_d)$ have been chosen to be $(0.35, 0.15)$. This choice is supported by USD historical data taken from March 2017 through March 2019, during which time one observes (see Figure H.1) that using (0.35, 0.15) as an initial guess consistently results in a *nearby* calibrated value. Note that the calibration has been performed using the CMSSO 10Y-2Y implied correlations as calibration instruments, in addition to the 10Y and 2Y ATM swaptions.



(a) Historical CMSSO 10Y-2Y Index Correlations.



(b) Calibrated (ρ_∞, ρ_d).

Figure H.1: Calibrating to the USD CMSSO 10Y-2Y implied correlation using an initial guess of $(\rho_\infty, \rho_d)_{init} = (0.35, 0.15)$. The calibrated (ρ_∞, ρ_d) shows a stable trend indicating the initial choice is robustly close to a stable equilibrium in the context of calibrating to historical correlation data.

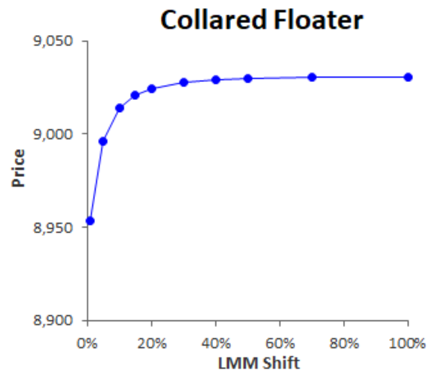
Remark H.1. It is worth emphasizing that the (0.35, 0.15) defaults, while appropriate for USD markets, could require modification when adapted to non-USD markets.

I Practical Upper Bound to Allowable Shift Parameters

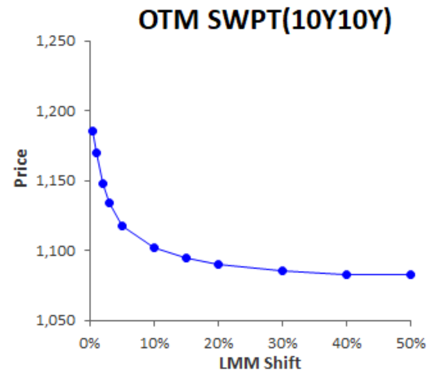
The fact that a shift value of $\alpha_k = 0.20$ effectively represents normal dynamics for F_k, and that further increasing the shift value within the range $\alpha_k \in (0.20, \infty)$ does not materially change this dynamics, has already been alluded to in previous sections. In particular, one cannot expect to achieve sub-normal dynamics by increasing α_k beyond 0.20 any more than one can achieve sub-normal dynamics for *any* value of the shift parameter.

In Figure I.1 we illustrate this behavior by considering two particular deals, a collared floater and an OTM swaption, chosen for their strong sensitivity to the skew of the underlying rates. By varying the shift levels in the range $\alpha_k \in [0.0, 1.0]$, one verifies the expected sensitivity to α_k values in the range $\alpha_k \in [0.0, 0.2]$, as well as the loss of sensitivity beyond $\alpha_k = 0.2$.

Moreover, referring to §6.1, one can see the 20% shift values shown in Figure 6.1 already achieve the nearly flat skew in Normal volatility space observed in Figure 6.2, known to be the theoretical minimum. Any shift values will necessarily fall short of producing the negative skew found in the corresponding USD caplet market.



(a) Pricing of a Collared Floater in the USD market on 08/05/2019.



(b) Pricing of a OTM 10Yx10Y swaption in the EUR market on 08/05/2019.

Figure I.1: By varying the LMM shift levels from $0.0 < \alpha_k < 1.0$, pricing of skew-sensitive deals shows insensitivity beyond a 20% level,

J FAQs

1. How does the LMM calibration handle volatility skew? As described in §3.2, LMM is a shifted-lognormal model where the lognormal dynamics of the *shifted rates* allows the model to reproduce the market skew. In order to calibrate the model to the volatility skew, one chooses both ATM *and* OTM swaptions when selecting instruments in the DLIB `Calibration` screen. The LMM calibrator will trigger the calibration of the shifts to reflect the market skew.

2. How are the LMM model parameters calibrated? LMM calibration applies a closed-form formula for caplet (§D.1), swaption (§D.2), and CMS spread options (§D.3), and applies an optimizer to match the market quotes. See §4.6 where Table 4.4 shows the relationship between selected calibration instruments and calibrated model parameters is indicated. With specific regard for calibrating the correlation parameters, note that the user can also provide equivalent quotes `SpreadVol` or `IndexCorrelation` as CMS spread options, where `SpreadVol` is the Normal volatility of the CMS spread, and `IndexCorrelation` is the Pearson's correlation of two CMS indices (*i.e.* swap rates). The default correlation model for LMM is the reduced-factor correlation which has four factors by default, but can freely configured to any desired number of factors in the `Calibration Model Parameters` screen.

3. Regarding the default Rebonato correlation parameter settings ρ_{Inf} and ρ_{Decay}, when should these fixed Model Values be overwritten with different fixed values? Moreover, when should they be disregarded in favor of specifying a user's choice for an Initial Guess used in calibrating them? Swaption, cap, and caplet instruments are all very insensitive to the correlation, and therefore it is always recommended, for reasons of stability, that these correlation parameters be held fixed in the calibration. On the other hand, when including CMSSO instruments in the calibration, whether as a Premium, or Spread Volatility, or Index Correlation, the correlation parameters should be calibrated by using the Initial Guess. The default values provided, as described in Appendix H, should be satisfactory in both cases.

4. What simulation method is employed in LMM pricing? As described in § 5, LMM uses Monte-Carlo simulation to generate all forward rates F_i to be snapshotted at grid times T_j for each path ω_n, resulting in (#Libors $\times$ #ObservationTimes $\times$ #Paths) internal states. Although LMM is generally understood to be a discrete model and therefore subject to the practical limitations implied by all discrete models, the Bloomberg LMM implementation supports rates with any tenor and at any observation time in an arbitrage-free way due to the interpolation technique described in §B.1, and thus behaves like a continuous model from the user perspective. It has been optimized for accurate moment matching (§A.1), speed and memory consumption.

5. When the Brownian bridge feature is applied (*e.g.* when pricing range accrual deals), are all observation dates respected, and what specific quantities are being *bridged*? As described in §5.4, the Brownian bridge technique is a numerical shortcut to obtaining internal states at every observation date (without evolving the internal states with excessively small time step). Thus, it is the internal states (forward Libor rates) that are bridged, and all observation dates are respected when the Brownian bridge methodology is applied.

6. What is the formula to obtain CMS fixings from Forward Libors, and is there any convexity adjustment? See (D.25) for the CMS-SO approximation formula, which does indeed incorporate the convexity adjustment obtained from the Linear Swap Model [Pel]. Note that when calibrating to swaptions, there is no convexity adjustment since the option is to enter the swap, rather than receiving a (single-look) payment.

7. Does the model generate Forward Libors only, and not longer tenor rates? Could calibrating to longer tenored or expiry swaptions create grid accrual periods longer than the deal's inherent 3-month or 6-month Libors? The Libor grid is determined by the granularity of the deal's discount curve, and then compatible (equivalent tenor) instruments are made available in the `Calibration Tab` (based on the Volatility source), where expiries are sufficiently long-dated relative to the deal's horizon.

8. How does the sparseness of long-dated swaptions influence the stability of calibrated model parameters? Stability is enforced using the Parameter Reduction method described in §4.8, while smoothness is enforced using a Regularization method described in Appendix G. Note that interpolation of market quotes to artificially reduce the sparseness of swaption instruments is not employed.

9. What is an appropriate choice of calibration instruments for an ATM Bermudan swaption? A reasonable default choice is the "Upper Triangle" method, which calibrates the model to ATM swaptions having total maturity (expiry plus underlying swap rate tenor) less than or equal to the maturity of the Bermudan. With this choice of calibration instruments, the shift parameters are not calibrated, but are instead fixed to values specified by the user. Equally important to the pricing of Bermudan swaptions is the choice of regression variables, for which one should consult [Blo1, §3].

10. What is an appropriate choice of calibration instruments for a long dated steepener range-accrual on the 20Y-2Y EUR CMS spread? The selected calibration instruments should naturally include CMS Spread Options (20y-2y) with expiries within the time horizon (6m, 1y-10y, 12y, 15y, 20y, 25y), and necessarily the associated swaptions corresponding to the 2y and 20y legs of each of these CMS-SO's. The Correlation Parameters must be non-fixed (Initial Guess only), and in order to account for relevant skew the OTM swaptions (*e.g.* $\pm 25bp$) should be included as well.

11. How are Greeks calculated when pricing with LMM? DLIB provides the `Scenario` screen for more customized analysis of price sensitivity to either IR-curve shifts (rho) or Vol-cube shifts (vega). The two supported methods are `Shoc Shift` and `Range Shift`. In Shoc shift, the user can define multiple scenarios of combined/individual shifts in IR-curve and Vol-cube. In Range shift, the user can set a range of shifts to investigate the valuation trend for the range. Additional information on Theta, Rho, IR-Gamma, and IR-Vega is provided in [Blo2, §8].

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