MATH 1530 Problem Set 6

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Problem 1. Let G be a finite Abelian group and let n be a positive integer that is relatively prime to $|G|$. Prove that the mapping $a \mapsto a^n$ is an automorphism of G.

Proof. Define $\alpha : G \to G$ such that $a \mapsto a^n$. Let $g, h \in G$.

1. **Injective:** Suppose $g^n = h^n$.

$$\begin{aligned} g^n = h^n &\implies e = g^n h^{-n} \\ &\implies e = (gh^{-1})^n \\ &\implies |gh^{-1}| \mid n \end{aligned}$$

   Additionally, $gh^{-1} \in G$. By *Lagrange's Theorem*, we have $|gh^{-1}| \mid |G|$. Since $|gh^{-1}|$ divides both n and $|G|$, and $\gcd(n, |G|) = 1$, we have that $|gh^{-1}| = 1$. Therefore, $gh^{-1} = e \implies g = eh \implies g = h$.

2. **Surjective:** Consider g^n. We have that $g \mapsto g^n$.

3. **Preserves Group Operation:** $\alpha(gh) = (gh)^n = g^n h^n = \alpha(g) \cdot \alpha(h)$.

$\square$

Problem 2. Let G be a group of order pqr, where p, q, r are distinct primes. If H is a subgroup of G of order pq and K is a subgroup of G of order qr, prove that $|H \cap K| = q$.

Proof. We have already proven that $H \cap K$ is a subgroup of G. This implies that $H \cap K$ is also a subgroup of H and K. By *Lagrange's Theorem*, we have that

$$|H \cap K| \mid |H|, |K| \implies |H \cap K| \mid pq, qr$$

Therefore, $|H \cap K|$ is either 1 or q. Assume for contradiction that $|H \cap K| = 1$. By lemma 1, we have that

$$|HK| = \frac{pq \cdot qr}{1} = pq^2r$$

which is a contradiction since HK is a subset of G, which implies that $|HK| \leq |G|$. Therefore, we have shown that $|H \cap K| = q$ as desired. □

Lemma 1. *Let H and K be subgroups of a finite group G. Then,*

$$|HK| = \frac{|H||K|}{|H \cap K|} \text{ where } HK = \{hk \mid h \in H,\ k \in K\}$$

Proof. We can separate HK into a union of left cosets of K in G:

$$HK = \bigcup_{h \in H} hK$$

By the properties of cosets, we have that $hK = h'K$ or $hK \cap h'K = \emptyset$ for all $h, h' \in H$. We must now determine how many of these cosets are distinct.

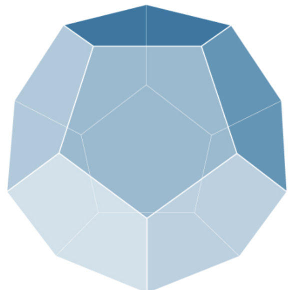
Suppose $hK = h'K$ for some $h, h' \in H$. Since $hK = h'K \Leftrightarrow h^{-1}h' \in K$, we have that $h^{-1}h' = k$ for some $k \in K$. This implies that $k \in H \implies k \in H \cap K$. Additionally, $h' = hk$. Thus, there are $|H \cap K|$ ways to create the same coset for each $h' \in H$ (by *Cayley's Theorem*, we know that each $k \in H \cap K$ has exactly one corresponding $h \in H$ such that $hk = h'$). Therefore, the number of distinct cosets hK where $h \in H$ is $|H|/|H \cap K|$.

Since $|hK| = |h'K|$ for all $h, h' \in H$, the number of elements in each coset is $|hK| = |K|$. Therefore, the cardinality of HK equals the number of distinct cosets times the number of distinct elements in each coset, giving us

$$|HK| = \frac{|H||K|}{|H \cap K|}$$

□

Problem 3. Calculate the order of the group of rotations of a regular dodecahedron:



Proof. Let G be the rotation group of the dodecahedron. Assign each of the 12 faces of the dodecahedron a unique number $1 - 12$. Since every rotation must take each face to exactly one other face, G is a group of permutations on the set $\{1, \ldots, 12\}$.

Consider a single face, $f \in \{1, \ldots, 12\}$, of the dodecahedron. By the *orbit-stabilizer theorem*, we have that

$$|G| = |\mathrm{orb}_G(f)| \cdot |\mathrm{stab}_G(f)|$$

1. $|\mathbf{orb_G(f)}|$: Picking an axis of rotation through the centers of any two parallel faces allows us to bring f to any other face $f' \in \{1, \ldots, 12\}$. Therefore, $\mathrm{orb}_G(f) = \{1, \ldots, 12\}$ which implies that $|\mathrm{orb}_G(f)| = 12$.

2. $|\mathbf{stab_G(f)}|$: Let $\bar{f} \in \{1, \ldots, 12\}$ be the face parallel to f. Picking an axis of rotation through the centers of f and $\bar{f}$ allows us to rotate the dodecahedron in 5 distinct ways while fixing the position of f. This implies that $|\mathrm{stab}_G(f)| = 5$.

Together, we have $|G| = 12 \cdot 5 = 60$. $\square$

Problem 4. Determine the number of cyclic subgroups of order 15 in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$.

Problem 5. Let p and q be odd primes and let m and n be positive integers. Prove that $U(p^m) \oplus U(q^n)$ is not cyclic. [hint: read the book to find a useful result we didn't cover in class]