APMA 1655 Honors Statistical Inference I March 2, 2023

Homework 1

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- You are strongly encouraged to work in groups, but solutions must be written independently.

Please feel free to use the following laws without proving them:
Let A, B, and C be events. Then, we have

- (Commutative Law) $A \cup B = B \cup A$,
- (Commutative Law) $A \cap B = B \cap A$,
- (Associative Law) $(A \cup B) \cup B = A \cup (B \cup C)$,
- (Associative Law) $(A \cap B) \cap C = A \cap (B \cap C)$,
- (Distributive law) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$,
- (Distributive law) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
- Let $\{A_1, A_2, \ldots, A_n, \ldots\}$ be a sequence of events, then we have

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c, \quad \left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c.$$

Problem 1 (Set theory)

Suppose we are interested in a sample space Ω. Please review the following definitions

$$\bigcup_{n=1}^{\infty} A_n = \left\{\omega \in \Omega : \text{ there exists at least one } n' \text{ such that } \omega \in A_{n'}\right\},$$
$$\bigcap_{n=1}^{\infty} A_n = \{\omega \in \Omega : \omega \in A_n \text{ for all } n = 1, 2, 3, \ldots\}$$

1. (0.5 points) We define a sequence $\{A_n\}_{n=1}^{\infty} = \{A_1, A_2, \ldots, A_n, \ldots\}$ of events as the following:

$$A_1 = \Omega,$$
$$A_n = \emptyset, \quad \text{for all } n = 2, 3, \ldots.$$

Please prove the following:

$$\Omega = \bigcup_{n=1}^{\infty} A_n. \tag{1}$$

Proof. We immediately have that $\Omega = A_1 \subset \bigcup_{n=1}^{\infty} A_n$. Let $x \in \bigcup_{n=1}^{\infty} A_n$. For some $A_i \in \{A_n\}_{n=1}^{\infty}$, we have that $x \in A_i \subset \Omega$. Thus, $x \in \Omega \implies \bigcup_{n=1}^{\infty} A_n \subset \Omega$ completing the double inclusion. Therefore, we have proven that $\Omega = \bigcup_{n=1}^{\infty} A_n$ as desired. □

2. Let E_1 and E_2 be two events with $E_1 \cap E_2 = \emptyset$. We define a sequence $\{A_n\}_{n=1}^{\infty} = \{A_1, A_2, \dots, A_n, \dots\}$ of events as the following:

$$\begin{aligned} A_1 &= E_1, \\ A_2 &= E_2, \\ A_n &= \emptyset, \quad \text{for all } n = 3, 4, \dots. \end{aligned} \tag{2}$$

Please prove the following:

(a) (0.5 points) The sequence $\{A_n\}_{n=1}^{\infty} = \{A_1, A_2, \dots, A_n, \dots\}$ defined in Eq. (2) is mutually disjoint.

Proof. Let $A_j, A_k \in \{A_n\}_{n=1}^{\infty}$ such that $j \neq k$. The possible combinations of A_j and A_k can be expressed with the following cases: (we have that $\cap$ is commutative, so the order of A_j and A_k does not matter)

- Case 1 ($A_j = E_1$ and $A_k = E_2$): $A_j \cap A_k = E_1 \cap E_2 = \emptyset$
- Case 2 ($A_j = E_1$ and $A_k = \emptyset$): $A_j \cap A_k = E_1 \cap \emptyset = \emptyset$
- Case 3 ($A_j = E_2$ and $A_k = \emptyset$): $A_j \cap A_k = E_2 \cap \emptyset = \emptyset$

Therefore, since $A_j, A_k \in \{A_n\}_{n=1}^{\infty} : j \neq k \implies j \cap k = \emptyset$, we have proven that $\{A_n\}_{n=1}^{\infty}$ is mutually disjoint. □

(b) (0.5 points) We have the following identity

$$E_1 \cup E_2 = \bigcup_{n=1}^{\infty} A_n,$$

where $A_1, A_2, \dots$ are defined in Eq. (2).

Proof. We have that $E_1 \cup E_2 = A_1 \cup A_2 \subset \bigcup_{n=1}^{\infty} A_n$. Let $x \in \bigcup_{n=1}^{\infty} A_n$. For some $A_i \in \{A_n\}_{n=1}^{\infty}$, we have that $x \in A_i \subset E_1 \cup E_2$. Thus, $x \in E_1 \cup E_2 \implies \bigcup_{n=1}^{\infty} A_n \subset E_1 \cup E_2$, completing the double inclusion. Therefore, we have proven that $E_1 \cup E_2 = \bigcup_{n=1}^{\infty} A_n$. □

3. (1 points) Let $\Omega = \mathbb{R}$ = the collection of all real numbers. We define a sequence of events as follows

$$A_n = \left[0, 1 + \frac{1}{n}\right), \quad \text{for all } n = 1, 2, 3, \dots. \tag{3}$$

Please prove the following identity

$$[0, 1] = \bigcap_{n=1}^{\infty} A_n,$$

where $A_1, A_2, A_3, \ldots$ are defined in Eq. (3).

Remark: Please read the following explanation for notations:

$$\begin{aligned}\left[0, 1+\frac{1}{n}\right) &= \left\{x : x \text{ is a real number such that } 0 \le x \text{ and } x < 1+\frac{1}{n}\right\} \\ &= \text{the collection of real numbers that are no less than 0 but smaller than } 1+\frac{1}{n};\end{aligned}$$

$$\begin{aligned}[0,1] = &\text{ the collection of real numbers that are no less than 0 but no higher than 1} \\ =&\{x : x \text{ is a real number such that } 0 \le x \text{ and } x \le 1\}.\end{aligned}$$

Proof. $A_{n+1} \subsetneq A_n$ since $[0, 1+\frac{1}{n+1}] \subsetneq [0, 1+\frac{1}{n}]$. Thus,

$$\bigcap_{n=1}^{\infty} A_n = \lim_{n\to\infty} A_n = \lim_{n\to\infty} [0, 1+\frac{1}{n}] = [0,1]$$

. □

Problem 2 (Definition of Probability Spaces)

(1 point) Suppose n is a fixed positive integer. We define the pair $(\Omega, \mathbb{P})$ as follows

- $\Omega = \{1, 2, \cdots, n\}$.
- For any $A \subset \Omega$, we define $\mathbb{P}(A) = \frac{\#A}{n}$, where $\#A$ denotes the number of elements in A.

Please prove that the pair $(\Omega, \mathbb{P})$ defined herein is a probability space.

Proof. We will show that $(\Omega, \mathbb{P})$ is a probability space by proving the following three axioms.

- $\underline{\mathbb{P}(A \subset \Omega) \ge 0}$: Let $A \subset \Omega$. We have that $\mathbb{P}(A) = \frac{\#A}{n} \ge 0$ since $\#A \ge 0$ and $n \ge 0$.
- $\underline{\mathbb{P}(\Omega) = 1}$: $\mathbb{P}(\Omega) = \frac{n}{n} = 1$
- Countable Additivity: Let $A_1, \ldots, A_m \subset \Omega$ be mutually disjoint events.

$$\begin{aligned}\mathbb{P}(A_1 \cup \cdots \cup A_m) &= \frac{\#A_1 + \cdots + \#A_m}{n} && \text{(definition of } \mathbb{P}) \\ &= \frac{\#A_1}{n} + \cdots + \frac{\#A_m}{n} && \text{(common denominator)} \\ &= \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_m) && \text{(definition of } \mathbb{P})\end{aligned}$$

□

Problem 3 (Properties of $\mathbb{P}$)

Let $(\Omega, \mathbb{P})$ be a probability space. Then, we have the following properties

1. (0 point) $\mathbb{P}(\emptyset) = 0$, i.e., the probability of the impossible event is zero;

2. (0 point) if two events E_1 and E_2 satisfy $E_1 \cap E_2 = \emptyset$, we have $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$;

3. (0.5 points) suppose $A, B \subset \Omega$. If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$;[1]

   Proof. Let $A \subset B$. We have that $B = (B \cap A^c) \cup A$ and $(B \cap A^c) \cap A = \emptyset$. Thus, $(B \cap A^c)$ and A are additive, which implies that $\mathbb{P}(B) = \mathbb{P}((B \cap A^c) \cup A) = \mathbb{P}(B \cap A^c) + \mathbb{P}(A)$. Because $\mathbb{P}(B \cap A^c) \geq 0$ by the definition of a probability space, it must be the case that $\mathbb{P}(B) \geq \mathbb{P}(A) \implies \mathbb{P}(A) \leq \mathbb{P}(B)$. □

4. (0.5 points) $0 \leq \mathbb{P}\{A\} \leq 1$ for any subsets $A \subset \Omega$;

   Proof. Let $A \subset \Omega$. Immediately, we have that $\mathbb{P}(A) \leq \mathbb{P}(\Omega) \implies \mathbb{P}(A) \leq 1$. Since $\emptyset \subset A$, we have that $\mathbb{P}(\emptyset) \leq \mathbb{P}(A) \implies 0 \leq \mathbb{P}(A)$. □

5. (0.5 points) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

   Proof. Let $A \subseteq \Omega$. By the definition of a complement, we have that $A \cup A^c = \Omega$ and $A \cap A^c = \emptyset$. Thus,

$$
\begin{aligned}
1 &= P(\Omega) && (P(\Omega) = 1) \\
&= P(A \cup A^c) && \text{(definition of complement)} \\
&= P(A) + P(A^c) && (A \text{ and } A^c \text{ disjoint})
\end{aligned}
$$

   Therefore, we can conclude that $P(A^c) = 1 - P(A)$ □

6. (1 point) for any $A, B \subset \Omega$, we have $\mathbb{P}\{A \cup B\} = \mathbb{P}\{A\} + \mathbb{P}\{B\} - \mathbb{P}\{A \cap B\}$;

   Proof. Let $A, B \in \Omega$.

$$
\begin{aligned}
A \cup B &= (A \cup B) \cap \Omega && (\text{definition of } \cap) \\
&= (A \cup B) \cap (A \cup A^c) && \text{(definition of complement)} \\
&= A \cup (B \cap A^c) && \text{(distributive law)}
\end{aligned}
$$

   Thus, $\mathbb{P}(A \cup B)$ can be expressed as the probability of the union of two disjoint events.

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \cup (B \cap A^c)) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c)$$

   Now, we can rewrite $\mathbb{P}(B \cap A^c)$ using a relation derived from $\mathbb{P}(B)$.

[1] Hint: If $A \subset B$, we have $B = (B \cap A^c) \cup A$; furthermore, $(B \cap A^c)$ and A are disjoint.

$$
\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}(B \cap \Omega) && \text{(definition of } \cap) \\
&= \mathbb{P}(B \cap (A \cup A^C)) && \text{(definition of complement)} \\
&= \mathbb{P}((B \cap A) \cup (B \cap A^c)) && \text{(distributive law)} \\
&= \mathbb{P}(A \cap B) + \mathbb{P}(B \cap A^c) && \text{(}p\text{ is additive))}
\end{aligned}
$$

Using this relation, we have that $\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$. By substituting this into the first relation, we get $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ as desired. □

7. (1 point) for any sequence of subsets $\{A_n\}_{n=1}^{\infty}$, we have $\mathbb{P}\{\bigcup_{n=1}^{\infty} A_n\} \leq \sum_{n=1}^{\infty} \mathbb{P}\{A_n\}$.[2]

Proof. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of events such that

$$
\begin{aligned}
B_1 &= A_1 \\
B_{n>1} &= A_n \setminus \left(\bigcup_{i=1}^{n-1} B_i \right) = A_n \cap \left(\bigcup_{i=1}^{n-1} B_i \right)^c
\end{aligned}
$$

We will first show that $\{B_n\}_{n=1}^{\infty}$ is mutually disjoint. Let $j, k \in \mathbb{Z}_{>0}$ such that $j < k$. By the definition of B_n, we have

$$B_k = A_k \setminus (B_1 \cup \cdots \cup B_j \cup \cdots \cup B_{k-1}) \implies B_j \cap B_k = \emptyset$$

Therefore, $\{B_n\}_{n=1}^{\infty}$ is mutually disjoint. Next, we will show that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$.

Let $x \in \bigcup_{n=1}^{\infty} A_n$. For some $A_k \in \{A_n\}_{n=1}^{\infty}$, we have that $x \in A_k$. We therefore have the following two cases:

- **Case 1** $(k = 1)$**:** $x \in A_1 = B_1 \implies x \in \bigcup_{n=1}^{\infty} B_n$
- **Case 2** $(k > 1)$**:** For contradiction, suppose $x \notin \bigcup_{n=1}^{k} B_n$. This implies that

$$x \notin \bigcup_{n=1}^{k-1} B_n \implies x \in \left(\bigcup_{n=1}^{k-1} B_n \right)^c$$

Since $B_k = A_k \cap \left(\bigcup_{n=1}^{k-1} B_n \right)^c$, we have that $x \in B_k$ which is a contradiction. Therefore, $x \in \bigcup_{n=1}^{k} B_n \implies x \in \bigcup_{n=1}^{\infty} B_n$.

[2]More precisely, we have the following:

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k \right) = \sum_{k=1}^{n} (-1)^{k-1} \left\{ \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \mathbb{P}\left(\bigcap_{l=1}^{k} A_{i_l} \right) \right\};$$

However, the identity above is not usually used in applications. You do not need to prove this precise identity in HW 1.

Both cases imply $x \in \bigcup_{n=1}^{\infty} B_n \implies \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n$.

Now let $x \in \bigcup_{n=1}^{\infty} B_n$. For some $B_k \in \{B_n\}_{n=1}^{\infty}$, we have that $x \in B_k \subset A_k \subset \bigcup_{n=1}^{\infty} A_n$ since $B_k = A_k \setminus (\cdots)$. Thus, $x \in \bigcup_{n=1}^{\infty} A_n \implies \bigcup_{n=1}^{\infty} B_n \subset \bigcup_{n=1}^{\infty} A_n$, completing the double inclusion.

Finally, we have that

$$\begin{aligned}
\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) &= \mathbb{P}(\bigcup_{n=1}^{\infty} B_n) && \text{(above)} \\
&= \sum_{n=1}^{\infty} \mathbb{P}(B_n) && (\{B_n\}_{n=1}^{\infty} \text{ mutually disjoint}) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}(A_n) && \text{since } \mathbb{P}(B_n) \leq \mathbb{P}(A_n)
\end{aligned}$$

□

Properties 1 and 2 were proved in class, and you do not need to prove them in HW 1.
Please prove Properties 3-7 above.

Problem 4 (Application of the Probability Properties)

Let $(\Omega, \mathbb{P})$ be a probability space.

1. (1 point) Let A and B are two events. Suppose $B \subset A$. Please prove the following:

$$\mathbb{P}(A^c) \leq \mathbb{P}(B^c).$$

Proof. Let $B \subset A$. We have that

$$\begin{aligned}
\mathbb{P}(B) &\leq \mathbb{P}(A) && \text{(definition of } \mathbb{P}) \\
1 - \mathbb{P}(B^c) &\leq 1 - \mathbb{P}(A^c) && \text{(definition of complement)} \\
-\mathbb{P}(B^c) &\leq -\mathbb{P}(A^c) && \text{(subtraction)} \\
\mathbb{P}(A^c) &\leq \mathbb{P}(B^c) && \text{(addition)}
\end{aligned}$$

□

2. (1 point) Let A and B are two events. If $\mathbb{P}(A) = 0.7$ and $\mathbb{P}(B) = 0.6$, what is the smallest possible value of $\mathbb{P}(A \cup B)$? What is the largest possible value of $\mathbb{P}(A \cup B)$?

The smallest possible value of $\mathbb{P}(A \cup B)$ is when $B \subset A \implies \mathbb{P}(A \cap B)$ is largest (*i.e.* when $\mathbb{P}(A \cap B) = \mathbb{P}(B) = 0.6$). In this case,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.7 + 0.6 - 0.6 = 0.7$$

The largest possibe value is attainable when $A \cup B = \Omega$:

$$\mathbb{P}(A \cup B) = \mathbb{P}(\Omega) = 1$$

3. (1 point) Let A and B are two events. If $\mathbb{P}(A) = 0.7$ and $\mathbb{P}(B) = 0.6$, what is the smallest possible value of $P(A \cap B)$? What is the largest possible value of $P(A \cap B)$?

   The smallest possible value of $\mathbb{P}(A \cap B)$ is attainable when the overlap between A and B is smallest. This occurs when $A \cup B = \Omega$. Thus, $\mathbb{P}(A \cap B) = 0.6 - 0.3 = 0.3$.

   The largest possible value of $\mathbb{P}(A \cap B)$ occurs when $B \subset A$. In this case, $\mathbb{P}(A \cap B) = \mathbb{P}(B) = 0.6$.