APMA 1655 Honors Statistical Inference I March 5, 2023

Homework 3

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- You are strongly encouraged to work in groups, but solutions must be written independently.

1 Review

To help you better answer the questions in HW 3, we review the example of Bernoulli distributions as follows:

- The experiment of interest is flipping a fair coin;
- the sample space corresponding to this experiment is $\Omega = \{\texttt{heads}, \texttt{tails}\}$;
- the probability $\mathbb{P}$ is defined by $\mathbb{P}(A) = \frac{\#A}{\#\Omega}$, i.e., $\mathbb{P}(\{\texttt{heads}\}) = \mathbb{P}(\{\texttt{tails}\}) = \frac{1}{2}$;
- the random variable X is defined by

$$X(\texttt{heads}) = 1, \quad X(\texttt{tails}) = 0.$$

The CDF of X is

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ \dfrac{1}{2}, & \text{if } 0 \le x < 1, \\ 1, & \text{if } x \ge 1. \end{cases} \tag{1}$$

Proof:

1. When $x < 0$, we have $A_x = \{\omega \in \Omega : X(\omega) \le x\} = \emptyset$; then, $F_X(x) = \mathbb{P}(A_x) = \mathbb{P}(\emptyset) = 0$.
2. When $0 \le x < 1$, we have $A_x = \{\omega \in \Omega : X(\omega) \le x\} = \{\texttt{tails}\}$; then, $F_X(x) = \mathbb{P}(A_x) = \mathbb{P}(\{\texttt{tails}\}) = \frac{1}{2}$.
3. When $x \ge 1$, we have $A_x = \{\omega \in \Omega : X(\omega) \le x\} = \Omega$; then, $F_X(x) = \mathbb{P}(A_x) = \mathbb{P}(\Omega) = 1$.

The proof is completed. $\square$

In addition, the Wikipedia page on random variables is nice material for learning the concept of random variables.

2 Problem Set

1. Let $(\Omega, \mathbb{P})$ be a probability space. Suppose B is an event and $0 < \mathbb{P}(B) < 1$. **Please prove the following:**

   (a) (1 point) If A and B are independent, then A and B^c are also independent.

   Proof. Using (b), we have that $\mathbb{P}(B^c \mid A) = 1 - \mathbb{P}(B \mid A) = 1 - \mathbb{P}(B) = \mathbb{P}(B^c)$. □

   (b) (1 point) $\mathbb{P}(A|B) + \mathbb{P}(A^c|B) = 1$.

   Proof. We will first prove that $\mathbb{P}(A \cap B) = \mathbb{P}(B) - \mathbb{P}(A^c \cap B)$.

$$
\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}(B \cap A) && \text{(commutativity)}\\
&= 1 - \mathbb{P}((B \cap A)^c) && \text{(def of complement)}\\
&= 1 - \mathbb{P}(B^c \cup A^c) && \text{(De Morgan's Law)}\\
&= 1 - \mathbb{P}(B^c \cup A^c \cap \Omega) && (E \cap \Omega = E)\\
&= 1 - \mathbb{P}(B^c \cup A^c \cap (B \cup B^c)) && \text{(def of complement)}\\
&= 1 - \mathbb{P}(B^c \cup (A^c \cap B) \cup (A^c \cap B^c)) && \text{(distributive law)}\\
&= 1 - \mathbb{P}(B^c \cup (A^c \cap B^c) \cup (A^c \cap B)) && \text{(commutativity)}\\
&= 1 - \mathbb{P}(B^c \cup (A^c \cap B)) && \text{(def of } \cup)\\
&= 1 - \mathbb{P}(B^c) - \mathbb{P}(A^c \cap B) && \text{(additivity)}\\
&= \mathbb{P}(B) - \mathbb{P}(A^c \cap B)
\end{aligned}
$$

   Now, we can use this relation to show that $\mathbb{P}(A|B) + \mathbb{P}(A^c|B) = 1$.

$$
\begin{aligned}
\mathbb{P}(A \mid B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} && \text{(conditional probability)}\\
&= \frac{\mathbb{P}(B) - \mathbb{P}(A^c \cap B)}{\mathbb{P}(B)} && \text{(substitute from above)}\\
&= \frac{\mathbb{P}(B)}{\mathbb{P}(B)} - \frac{\mathbb{P}(A^c \cap B)}{\mathbb{P}(B)} && \text{(distributive prop.)}\\
&= 1 - \mathbb{P}(A^c \mid B) && \text{(conditional probability)}
\end{aligned}
$$

   □

2. (2 points) Let n be a positive integer, and $\Omega \overset{\text{def}}{=} \{1, 2, \ldots, n\}$. Suppose $\mathbb{P}$ is a function of subsets of Ω defined as follows

$$\mathbb{P}(A) \overset{\text{def}}{=} \frac{\#A}{\#\Omega}, \quad \text{for all } A \subset \Omega.$$

   You have proved in HW 1 that $(\Omega, \mathbb{P})$ is a probability space.

   We define a random variable X as follows

$$X(\omega) = \omega, \qquad \text{for all } \omega \in \Omega = \{1, 2, \ldots, n\}.$$

   Please derive the CDF of the random variable X defined above. Please present your answer using a formula like the one in Eq. (1).

Proof. Consider the following cases for the CDF of the random variable X:

- When $x < 1$, we have that $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(\emptyset) = 0$.
- When $1 \le x < n$, we have that $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(\{1, \dots, \lfloor x \rfloor\}) = \frac{\lfloor x \rfloor}{n}$.
- When $x \ge n$, we have that $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(\Omega) = 1$.

Therefore, $F_X(x) = \begin{cases} 0 & \text{if } x < 1, \\ \frac{\lfloor x \rfloor}{n} & \text{if } 1 \le x < n, \\ 1 & \text{if } x \ge n. \end{cases}$ □

3. (2 points) Let X be a random variable defined on the probability space $(\Omega, \mathbb{P})$. Suppose X satisfies the following

x	1	2	3	4	5
$\mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$	1/2	1/4	1/8	1/16	1/16

Please derive the CDF of the random variable X. Please present your answer using a formula like the one in Eq. (1)**.**

Proof. Consider the following cases for $F_X(x)$:

- When $x < 1$, we have that $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(\emptyset) = 0$.
- When $1 \le x < 2$, we have that $F_X(x) = \mathbb{P}(X \le x) = 0 + 1/2 = 1/2$.
- When $2 \le x < 3$, we have that $F_X(x) = \mathbb{P}(X \le x) = 1/2 + 1/4 = 3/4$.
- When $3 \le x < 4$, we have that $F_X(x) = \mathbb{P}(X \le x) = 3/4 + 1/8 = 7/8$.
- When $4 \le x < 5$, we have that $F_X(x) = \mathbb{P}(X \le x) = 7/8 + 1/16 = 15/16$.
- When $x \ge 5$, we have that $F_X(x) = \mathbb{P}(X \le x) = 15/16 + 1/16 = 1$.

Therefore, $F_X(x) = \begin{cases} 0 & \text{if } x < 1, \\ 1/2 & \text{if } 1 \le x < 2, \\ 3/4 & \text{if } 2 \le x < 3, \\ 7/8 & \text{if } 3 \le x < 4, \\ 15/16 & \text{if } 4 \le x < 5, \\ 1 & \text{if } x \ge 5. \end{cases}$ □

4. (2 points) Let X be a random variable defined on the probability space $(\Omega, \mathbb{P})$. Suppose X satisfies the following

$$\mathbb{P}(\{\omega \in \Omega : X(\omega) = 0\}) = 1.$$

Please derive the CDF of the random variable X. Please present your answer using a formula like the one in Eq. (1)**.**

Proof. We have that $\mathbb{P}(X = 0) = 1 = \mathbb{P}(\Omega) \implies \{\omega \in \Omega \mid X(\omega) = 0\} = \Omega$. Thus,

- When $x < 0$, we have that $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(\emptyset) = 0$.
- When $x \ge 0$, we have that $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(\Omega) = 1$.

Therefore, $F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \ge 1. \end{cases}$ □

5. (2 points) Let X be a random variable defined on the probability space $(\Omega, \mathbb{P})$. Suppose the CDF of X is the following

$$F_X(x) = \begin{cases} 0, & \text{if } x < 1; \\ \log x, & \text{if } 1 \le x < e; \\ 1, & \text{if } e \le x. \end{cases}$$

Please compute the values of the following:

(a) $\mathbb{P}(\{\omega \in \Omega : X(\omega) < 2\})$;

Proof.

$$\begin{aligned} \mathbb{P}(X < 2) &= \mathbb{P}(X \le 2) - \mathbb{P}(X = 2) && \text{(additivity)} \\ &= \mathbb{P}(X \le 2) && (\mathbb{P}(X = 2) = 0) \\ &= F_X(2) && \text{(def of CDF)} \\ &= \log(2) \end{aligned}$$

□

(b) $\mathbb{P}(\{\omega \in \Omega : 0 < X(\omega) \le 3\})$;

Proof.

$$\begin{aligned} \mathbb{P}(0 < X \le 3) &= \mathbb{P}(X \le 3) - \mathbb{P}(X \le 0) && \text{(additivity)} \\ &= F_X(3) - F_X(0) && \text{(def of CDF)} \\ &= 1 - 0 && \text{(def of } F_X(x)\text{)} \\ &= 1 && \text{(subtraction)} \end{aligned}$$

□

(c) $\mathbb{P}(\{\omega \in \Omega : 2 < X(\omega) < 2.5\})$.

Proof.

$$\begin{aligned} \mathbb{P}(2 < X < 2.5) &= \mathbb{P}(X \le 2.5) - \mathbb{P}(X = 2.5) - \mathbb{P}(X \le 2) && \text{(additivity)} \\ &= \mathbb{P}(X \le 2.5) - \mathbb{P}(X \le 2) && (\mathbb{P}(X = 2.5) = 0) \\ &= F_X(2.5) - F_X(2) && \text{(def of CDF)} \\ &= \log(2.5) - \log(2) && \text{(def of } F_X(x)\text{)} \\ &= \log(1.25) && \text{(quotient rule)} \end{aligned}$$

□

Remark: For simplicity, many textbooks suppress the ω and represent $\mathbb{P}(\{\omega \in \Omega : X(\omega) < 2\})$, $\mathbb{P}(\{\omega \in \Omega : 0 < X(\omega) \leq 3\})$, and $\mathbb{P}(\{\omega \in \Omega : 2 < X(\omega) < 2.5\})$ as $\mathbb{P}(X < 2)$, $\mathbb{P}(0 < X < 3)$, and $\mathbb{P}(2 < X < 2.5)$, respectively. When you read those textbooks, this remark helps you understand what they mean.