APMA 1655 Honors Statistical Inference I February 26, 2023

Homework 2

Name: *Due: 11 pm, February 17*

Collaborators:

- You are strongly encouraged to work in groups, but solutions must be written independently.

Please feel free to use all the results in the Appendix of HW 2 without proving them.

1 Problem Set

1. (2 points) Suppose $(\Omega, \mathbb{P})$ is a probability space, and B is a event with $\mathbb{P}(B) > 0$. We define a function $\tilde{\mathbb{P}}$ of subsets of Ω by the following

$$\tilde{\mathbb{P}}(A) \stackrel{\text{def}}{=} \mathbb{P}(A \mid B), \quad \text{for all } A \subset \Omega.$$

Please prove that $\tilde{\mathbb{P}}$ is a probability, i.e., $(\Omega, \tilde{\mathbb{P}})$ is a probability space as well.

2. (1 point) Let $(\Omega, \mathbb{P})$ be a probability space and n be a positive integer. $B_1, B_2, \ldots, B_n$ are events and provide a partition of Ω, i.e.,

   - $\bigcup_{i=1}^{n} B_i = \Omega$,
   - $B_1, B_2, \ldots, B_n$ are mutually disjoint.

Let A be any event. **Please prove that $A \cap B_1, A \cap B_2, A \cap B_3, \ldots, A \cap B_n$ are mutually disjoint**, i.e.,

$$(A \cap B_i) \cap (A \cap B_j) = \emptyset, \quad \text{if } i \neq j.$$

3. (2 points) A box contains w white balls and b black balls. A ball is chosen at random.

   - If the chosen ball is white, we add d white balls to the box, that is, now there are $w + d$ white balls and b black balls.
   - If the chosen ball is black, we add d black balls to the box, that is, now there are w white balls and $b + d$ black balls.

After adding the d balls, another ball is drawn at random from the box. **Show that the probability that the second chosen ball is white does not depend on d.** Hint: Use the law of total probability (LTP).

4. (1 point) Suppose the underlying probability space is $(\Omega, \mathbb{P})$. Let G and H be events such that $0 < \mathbb{P}(G) < 1$ and $0 < \mathbb{P}(H) < 1$. **Give a formula for $\mathbb{P}(G|H^c)$ in terms of $\mathbb{P}(G)$, $\mathbb{P}(H)$ and $\mathbb{P}(G \cap H)$ only.**

5. (1 point) Suppose we have the following

$$\begin{aligned}
&\mathbb{P}(\text{“snow today”}) = 30\%,\\
&\mathbb{P}(\text{“snow tomorrow”}) = 60\%,\\
&\mathbb{P}(\text{“snow today and tomorrow”}) = 25\%.
\end{aligned}$$

Given that it snows today, what is the probability that it will snow tomorrow?

6. (3 points) Let $(\Omega, \mathbb{P})$ be a probability space. Suppose we have two events A and B such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. **Please prove that the following three equations are equivalent.**

   (a) $\mathbb{P}(A \mid B) = \mathbb{P}(A)$,

   (b) $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$,

   (c) $\mathbb{P}(B \mid A) = \mathbb{P}(B)$.

2 Appendix

Please feel free to use all the results in the appendix without proving them.

2.1 Appendix 1

Let A, B, and C be events. Then, we have

- (Commutative Law) $A \cup B = B \cup A$,
- (Commutative Law) $A \cap B = B \cap A$,
- (Associative Law) $(A \cup B) \cup B = A \cup (B \cup C)$,
- (Associative Law) $(A \cap B) \cap C = A \cap (B \cap C)$,
- (Distributive law) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$,
- (Distributive law) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$,
- $(A \cup B)^c = A^c \cap B^c$,
- $(A \cap B)^c = A^c \cup B^c$.

2.2 Appendix 2

Let $A_1, A_2, \ldots$ be any sequence of events and B be an event. We have the following

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c,$$
$$\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c,$$
$$B \cap \left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} (B \cap A_n),$$
$$B \cup \left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} (B \cup A_n).$$