

## Homework 1

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Due: 11 pm, February 10

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- You are strongly encouraged to work in groups, but solutions must be written independently.

Please feel free to use the following laws without proving them:

Let A, B, and C be events. Then, we have

- (Commutative Law )  $A \cup B = B \cup A$ ,
- (Commutative Law )  $A \cap B = B \cap A$ ,
- (Associative Law)  $(A \cup B) \cup C = A \cup (B \cup C)$ ,
- (Associative Law)  $(A \cap B) \cap C = A \cap (B \cap C)$ ,
- (Distributive law)  $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ ,
- (Distributive law)  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ .
- Let  $\{A_1, A_2, \dots, A_n, \dots\}$  be a sequence of events, then we have

$$\left( \bigcup_{n=1}^{\infty} A_n \right)^c = \bigcap_{n=1}^{\infty} A_n^c, \quad \left( \bigcap_{n=1}^{\infty} A_n \right)^c = \bigcup_{n=1}^{\infty} A_n^c.$$

## Problem 1 (Set theory)

Suppose we are interested in a sample space  $\Omega$ . Please review the following definitions

$$\bigcup_{n=1}^{\infty} A_n = \{ \omega \in \Omega : \text{there exists at least one } n' \text{ such that } \omega \in A_{n'} \},$$

$$\bigcap_{n=1}^{\infty} A_n = \{ \omega \in \Omega : \omega \in A_n \text{ for all } n = 1, 2, 3, \dots \}$$

1. (0.5 points) We define a sequence  $\{A_n\}_{n=1}^{\infty} = \{A_1, A_2, \dots, A_n, \dots\}$  of events as the following:

$$A_1 = \Omega,$$

$$A_n = \emptyset, \quad \text{for all } n = 2, 3, \dots$$

Please prove the following:

$$\Omega = \bigcup_{n=1}^{\infty} A_n. \tag{1}$$

*Proof.* Since  $A_1 = \Omega$ , we immediately have that  $\Omega \subset \bigcup_{n=1}^{\infty} A_n$ . Additionally, we have that for all  $A_i \in \{A_n\}_{n=1}^{\infty}$ ,  $A_i \subset \Omega$ . Thus,  $\bigcup_{n=1}^{\infty} A_n \subset \Omega$  completing the double inclusion. Therefore, we have proven that  $\Omega = \bigcup_{n=1}^{\infty} A_n$  as desired.  $\square$

2. Let  $E_1$  and  $E_2$  be two events with  $E_1 \cap E_2 = \emptyset$ . We define a sequence  $\{A_n\}_{n=1}^{\infty} = \{A_1, A_2, \dots, A_n, \dots\}$  of events as the following:

$$\begin{aligned} A_1 &= E_1, \\ A_2 &= E_2, \\ A_n &= \emptyset, \quad \text{for all } n = 3, 4, \dots \end{aligned} \tag{2}$$

**Please prove the following:**

- (a) (0.5 points) The sequence  $\{A_n\}_{n=1}^{\infty} = \{A_1, A_2, \dots, A_n, \dots\}$  defined in Eq. (2) is mutually disjoint.

*Proof.* Let  $x, y \in \{A_n\}_{n=1}^{\infty}$  such that  $x \neq y$ . The possible combinations of  $x$  and  $y$  can be expressed with the following cases: (we have that  $\cap$  is commutative, so the order of  $x$  and  $y$  does not matter)

- Case 1 ( $x = E_1$  and  $y = E_2$ ):  $x \cap y = E_1 \cap E_2 = \emptyset$
- Case 2 ( $x = E_1$  and  $y = \emptyset$ ):  $x \cap y = E_1 \cap \emptyset = \emptyset$
- Case 3 ( $x = E_2$  and  $y = \emptyset$ ):  $x \cap y = E_2 \cap \emptyset = \emptyset$

Therefore, since  $x, y \in \{A_n\}_{n=1}^{\infty} : x \neq y \implies x \cap y = \emptyset$ , we have proven that  $\{A_n\}_{n=1}^{\infty}$  is mutually disjoint.  $\square$

- (b) (0.5 points) We have the following identity

$$E_1 \cup E_2 = \bigcup_{n=1}^{\infty} A_n,$$

where  $A_1, A_2, \dots$  are defined in Eq. (2).

*Proof.* Since  $A_1 = E_1$  and  $A_2 = E_2$ , we have that  $E_1 \cup E_2 \subset \bigcup_{n=1}^{\infty} A_n$ . Additionally, for all  $A_i \in \{A_n\}_{n=1}^{\infty}$ , we have that  $A_i \subset E_1 \cup E_2$ . Thus,  $\bigcup_{n=1}^{\infty} A_n \subset E_1 \cup E_2$ , completing the double inclusion. Therefore, we have proven that  $E_1 \cup E_2 = \bigcup_{n=1}^{\infty} A_n$ .  $\square$

3. (1 points) Let  $\Omega = \mathbb{R}$  = the collection of all real numbers. We define a sequence of events as follows

$$A_n = \left[0, 1 + \frac{1}{n}\right), \quad \text{for all } n = 1, 2, 3, \dots \tag{3}$$

Please prove the following identity

$$[0, 1] = \bigcap_{n=1}^{\infty} A_n,$$

where  $A_1, A_2, A_3, \dots$  are defined in Eq. (3).

**Remark:** Please read the following explanation for notations:

$$\left[0, 1 + \frac{1}{n}\right) = \left\{x : x \text{ is a real number such that } 0 \leq x \text{ and } x < 1 + \frac{1}{n}\right\}$$

= the collection of real numbers that are no less than 0 but smaller than  $1 + \frac{1}{n}$ ;

$$[0, 1] = \text{the collection of real numbers that are no less than 0 but no higher than 1}$$

$$= \{x : x \text{ is a real number such that } 0 \leq x \text{ and } x \leq 1\}.$$

*Proof.*  $A_{n+1} \subsetneq A_n$  since  $[0, 1 + \frac{1}{n+1}] \subsetneq [0, 1 + \frac{1}{n}]$ . Thus,

$$\bigcap_{n=1}^{\infty} A_n = \lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} [0, 1 + \frac{1}{n}] = [0, 1]$$

□

## Problem 2 (Definition of Probability Spaces)

(1 point) Suppose  $n$  is a fixed positive integer. We define the pair  $(\Omega, \mathbb{P})$  as follows

- $\Omega = \{1, 2, \dots, n\}$ .
- For any  $A \subset \Omega$ , we define  $\mathbb{P}(A) = \frac{\#A}{n}$ , where  $\#A$  denotes the number of elements in  $A$ .

**Please prove that the pair  $(\Omega, \mathbb{P})$  defined herein is a probability space.**

*Proof.* We will show that  $(\Omega, \mathbb{P})$  is a probability space by proving the following three axioms.

- $\mathbb{P}(A \subset \Omega) \geq 0$ : Let  $A \subset \Omega$ . We have that  $\mathbb{P}(A) = \frac{\#A}{n} \geq 0$  since  $\#A \geq 0$  and  $n \geq 0$ .
- $\mathbb{P}(\Omega) = 1$ :  $\mathbb{P}(\Omega) = \frac{n}{n} = 1$
- Countable Additivity: Let  $A_1, \dots, A_m \subset \Omega$  be mutually disjoint events.

$$\begin{aligned} \mathbb{P}(A_1 \cup \dots \cup A_m) &= \frac{\#A_1 + \dots + \#A_m}{n} && \text{(definition of } \mathbb{P} \text{)} \\ &= \frac{\#A_1}{n} + \dots + \frac{\#A_m}{n} && \text{(common denominator)} \\ &= \mathbb{P}(A_1) + \dots + \mathbb{P}(A_m) && \text{(definition of } \mathbb{P} \text{)} \end{aligned}$$

□

### Problem 3 (Properties of $\mathbb{P}$ )

Let  $(\Omega, \mathbb{P})$  be a probability space. Then, we have the following properties

1. (0 point)  $\mathbb{P}(\emptyset) = 0$ , i.e., the probability of the impossible event is zero;
2. (0 point) if two events  $E_1$  and  $E_2$  satisfy  $E_1 \cap E_2 = \emptyset$ , we have  $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$ ;
3. (0.5 points) suppose  $A, B \subset \Omega$ . If  $A \subset B$ , then  $\mathbb{P}(A) \leq \mathbb{P}(B)$ ;<sup>1</sup>

*Proof.* Let  $A \subset B$ . We have that  $B = (B \cap A^c) \cup A$  and  $(B \cap A^c) \cap A = \emptyset$ . Thus,  $(B \cap A^c)$  and  $A$  are additive, which implies that  $\mathbb{P}(B) = \mathbb{P}((B \cap A^c) \cup A) = \mathbb{P}(B \cap A^c) + \mathbb{P}(A)$ . Because  $\mathbb{P}(B \cap A^c) \geq 0$  by the definition of a probability space, it must be the case that  $\mathbb{P}(B) \geq \mathbb{P}(A) \implies \mathbb{P}(A) \leq \mathbb{P}(B)$ .  $\square$

4. (0.5 points)  $0 \leq \mathbb{P}(A) \leq 1$  for any subsets  $A \subset \Omega$ ;

*Proof.* Let  $A \subset \Omega$ . Immediately, we have that  $\mathbb{P}(A) \leq \mathbb{P}(\Omega) \implies \mathbb{P}(A) \leq 1$ . Since  $\emptyset \subset A$ , we have that  $\mathbb{P}(\emptyset) \leq \mathbb{P}(A) \implies 0 \leq \mathbb{P}(A)$ .  $\square$

5. (0.5 points)  $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ .

*Proof.* Let  $A \subseteq \Omega$ . By the definition of a complement, we have that  $A \cup A^c = \Omega$  and  $A \cap A^c = \emptyset$ . Thus,

$$\begin{aligned} 1 &= \mathbb{P}(\Omega) && (\mathbb{P}(\Omega) = 1) \\ &= \mathbb{P}(A \cup A^c) && \text{(definition of complement)} \\ &= \mathbb{P}(A) + \mathbb{P}(A^c) && (A \text{ and } A^c \text{ disjoint}) \end{aligned}$$

Therefore, we can conclude that  $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$   $\square$

6. (1 point) for any  $A, B \subset \Omega$ , we have  $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ ;

*Proof.* Let  $A, B \in \Omega$ .

$$\begin{aligned} A \cup B &= (A \cup B) \cap \Omega && \text{(definition of } \cap) \\ &= (A \cup B) \cap (A \cup A^c) && \text{(definition of complement)} \\ &= A \cup (B \cap A^c) && \text{(distributive law)} \end{aligned}$$

Thus,  $\mathbb{P}(A \cup B)$  can be expressed as the probability of the union of two disjoint events.

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \cup (B \cap A^c)) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c)$$

Now, we can rewrite  $\mathbb{P}(B \cap A^c)$  using a relation derived from  $\mathbb{P}(B)$ .

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<sup>1</sup>Hint: If  $A \subset B$ , we have  $B = (B \cap A^c) \cup A$ ; furthermore,  $(B \cap A^c)$  and  $A$  are disjoint.

$$\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}(B \cap \Omega) && \text{(definition of } \cap \text{)} \\
&= \mathbb{P}(B \cap (A \cup A^c)) && \text{(definition of complement)} \\
&= \mathbb{P}((B \cap A) \cup (B \cap A^c)) && \text{(distributive law)} \\
&= \mathbb{P}(A \cap B) + \mathbb{P}(B \cap A^c) && (\mathbb{p} \text{ is additive})
\end{aligned}$$

Using this relation, we have that  $\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$ . □

7. (1 point) for any sequence of subsets  $\{A_n\}_{n=1}^{\infty}$ , we have  $\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$ .<sup>2</sup>

Properties 1 and 2 were proved in class, and you do not need to prove them in HW 1.  
**Please prove Properties 3-7 above.**

## Problem 4 (Application of the Probability Properties)

Let  $(\Omega, \mathbb{P})$  be a probability space.

1. (1 point) Let  $A$  and  $B$  are two events. Suppose  $B \subset A$ . Please prove the following:

$$\mathbb{P}(A^c) \leq \mathbb{P}(B^c).$$

2. (1 point) Let  $A$  and  $B$  are two events. If  $\mathbb{P}(A) = 0.7$  and  $\mathbb{P}(B) = 0.6$ , what is the smallest possible value of  $\mathbb{P}(A \cup B)$ ? What is the largest possible value of  $\mathbb{P}(A \cup B)$ ?
3. (1 point) Let  $A$  and  $B$  are two events. If  $\mathbb{P}(A) = 0.7$  and  $\mathbb{P}(B) = 0.6$ , what is the smallest possible value of  $\mathbb{P}(A \cap B)$ ? What is the largest possible value of  $\mathbb{P}(A \cap B)$ ?

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<sup>2</sup>More precisely, we have the following:

$$\mathbb{P}\left(\bigcup_{k=1}^n A_k\right) = \sum_{k=1}^n (-1)^{k-1} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \mathbb{P}\left(\bigcap_{l=1}^k A_{i_l}\right) \right\};$$

However, the identity above is not usually used in applications. You do not need to prove this precise identity in HW 1.