MATH 1530 Problem Set 3

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Problem 1. Please complete the mid-semester survey. Write "I have completed the mid-semester survey" and sign your name.

Problem 2. Let a be an element of a group G. Prove that $\langle a^m \rangle \cap \langle a^n \rangle$ is cyclic, where n, m are integers. What is its generator?

Proof. Let $a^k \in \langle a^m \rangle \cap \langle a^n \rangle$. We have that $a^k \in \langle a^m \rangle \implies a^k = a^{ms}$ where $s \in \mathbb{Z}$. We also have that $a^k \in \langle a^n \rangle \implies a^k = a^{nt}$ where $t \in \mathbb{Z}$. Together, we have

$$a^k = a^{ms} = a^{nt} \implies k = ms = nt$$

In other words, k must be a common multiple of both m and n. Since every common multiple of m and n is itself a multiple of $\operatorname{lcm}(m, n)$, we have that $\langle a^m \rangle \cap \langle a^n \rangle$ is equal to $\langle a^{\operatorname{lcm}(m,n)} \rangle$. $\square$

Problem 3. Let a and b belong to a group. If $|a|$ and $|b|$ are relatively prime, prove that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

Proof. Let G be a group containing elements a, b. Let $m = |a|$ and $n = |b|$. We can now express $\langle a \rangle$ and $\langle b \rangle$ as:

$$\langle a \rangle = \{e, a^1, \ldots, a^{m-1}\} \qquad \langle b \rangle = \{e, b^1, \ldots, a^{n-1}\}$$

Because the identity element of G is unique, we have that $e \in \langle a \rangle \cap \langle b \rangle$.

Next, we will show that for all $a^k \in \langle a \rangle$ such that $a^k \neq e$, we have that $a^k \notin \langle b \rangle$. By (Gallian, 4.2 Corollary 1), we know that if $a^k \in \langle a \rangle$, then $|a^k|$ divides m. Additionally, since $a^k \neq e$, we know $|a^k| > 1$. If $a^k \in \langle b \rangle$, $|a^k|$ must divide n. But since $|a|$ and $|b|$ are relatively prime, we have that $\gcd(m, n) = 1$. Because $|a^k| \neq 1$, we have shown that $a^k \notin \langle b \rangle$. The same process can be used to show that for all $b^k \in \langle b \rangle$ such that $b^k \neq e$, we have that $b^k \notin \langle a \rangle$.

Therefore, we have proven that $\langle a \rangle \cap \langle b \rangle = \{e\}$. $\square$