APMA 1655 Honors Statistical Inference I February 26, 2023

Homework 2

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Collaborators: N/A

- You are strongly encouraged to work in groups, but solutions must be written independently.

Please feel free to use all the results in the Appendix of HW 2 without proving them.

1 Problem Set

1. (2 points) Suppose $(\Omega, \mathbb{P})$ is a probability space, and B is a event with $\mathbb{P}(B) > 0$. We define a function $\tilde{\mathbb{P}}$ of subsets of Ω by the following

$$\tilde{\mathbb{P}}(A) \stackrel{\text{def}}{=} \mathbb{P}(A \mid B), \quad \text{for all } A \subset \Omega.$$

Please prove that $\tilde{\mathbb{P}}$ is a probability, i.e., $(\Omega, \tilde{\mathbb{P}})$ is a probability space as well.

Proof. We will prove that $(\Omega, \tilde{\mathbb{P}})$ is a probability space by proving the following three axioms:

- $\underline{\tilde{\mathbb{P}}(A \subset \Omega) \geq 0}$: $\tilde{\mathbb{P}}(A) = \mathbb{P}(A \mid B) \geq 0$
- $\underline{\tilde{\mathbb{P}}(\Omega) = 1}$: $\tilde{\mathbb{P}}(\Omega) = \mathbb{P}(\Omega \mid B) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$
- Countable Additivity: Let $A_1, \ldots, A_m \subset \Omega$ be mutually disjoint events.

$$\begin{aligned}
\tilde{\mathbb{P}}(A_1 \cup \cdots \cup A_m) &= \mathbb{P}((A_1 \cup \cdots \cup A_m) \mid B) \\
&= \frac{\mathbb{P}((A_1 \cup \cdots \cup A_m) \cap B)}{\mathbb{P}(B)} \\
&= \frac{\mathbb{P}((A_1 \cap B) \cup \cdots \cup (A_m \cap B))}{\mathbb{P}(B)} \\
&= \frac{\mathbb{P}(A_1 \cap B) + \cdots + \mathbb{P}(A_m \cap B)}{\mathbb{P}(B)} \\
&= \tilde{\mathbb{P}}(A_1) + \cdots + \tilde{\mathbb{P}}(A_m)
\end{aligned}$$

□

2. (1 point) Let $(\Omega, \mathbb{P})$ be a probability space and n be a positive integer. $B_1, B_2, \ldots, B_n$ are events and provide a partition of Ω, i.e.,

- $\bigcup_{i=1}^{n} B_i = \Omega$,
- $B_1, B_2, \ldots, B_n$ are mutually disjoint.

Let A be any event. **Please prove that $A \cap B_1, A \cap B_2, A \cap B_3, \ldots, A \cap B_n$ are mutually disjoint**, i.e.,

$$(A \cap B_i) \cap (A \cap B_j) = \emptyset, \quad \text{if } i \neq j.$$

Proof. Let $B_i, B_j \in \{B_k\}_{k=1}^n$ such that $i \neq j$. We have that $B_i \cap B_j = \emptyset$. It follows that $A \cap B_i \subset B_i$ and $A \cap B_j \subset B_j$. Thus, it must be the case that $(A \cap B_i) \cap (A \cap B_j) = \emptyset$. $\square$

3. (2 points) A box contains w white balls and b black balls. A ball is chosen at random.

   - If the chosen ball is white, we add d white balls to the box, that is, now there are $w + d$ white balls and b black balls.
   - If the chosen ball is black, we add d black balls to the box, that is, now there are w white balls and $b + d$ black balls.

   After adding the d balls, another ball is drawn at random from the box. **Show that the probability that the second chosen ball is white does not depend on d.** Hint: Use the law of total probability (LTP).

   Proof. Let W, B denote "white" and "black" respectively.

   We have that $\Omega = \{(W,W), (W,B), (B,W), (B,B)\}$. Let $A = \{(W,W), (B,W)\}$ represent the event that the second ball picked is white. We will now create a partition of Ω using the following elements:

$$B_1 = \{(W,W)\} \qquad B_2 = \{(W,B)\} \qquad B_3 = \{(B,W)\} \qquad B_4 = \{(B,B)\}$$

   Since $B_1, \ldots, B_4$ are mutually disjoint and $B_1 \cup \cdots \cup B_4 = \Omega$, we have that $\{B_i\}_{i=1}^4$ forms a partition of Ω.

   By the law of total probability,

$$\begin{aligned}
\mathbb{P}(A) &= \sum_{i=1}^{4} \mathbb{P}(A \mid B_i) \cdot \mathbb{P}(B_i) \\
&= \sum_{i=1}^{4} \mathbb{P}(A \cap B_i) \\
&= \mathbb{P}(W,W) + \mathbb{P}(B,B) + \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) \\
&= \frac{w}{w+b} \cdot \frac{w+d}{w+b+d} + \frac{b}{w+b} \cdot \frac{w}{w+b+d} \\
&= \frac{w^2 + dw + bw}{(w+b)(w+b+d)} \\
&= \frac{w(w+b+d)}{(w+b)(w+b+d)} \\
&= \frac{w}{w+b}
\end{aligned}$$

   Therefore, we have proven that $\mathbb{P}(A)$ does not depend on d. $\square$

4. (1 point) Suppose the underlying probability space is $(\Omega, \mathbb{P})$. Let G and H be events such that $0 < \mathbb{P}(\mathsf{G}) < 1$ and $0 < \mathbb{P}(\mathsf{H}) < 1$. **Give a formula for $\mathbb{P}(\mathsf{G}|\mathsf{H}^c)$ in terms of $\mathbb{P}(\mathsf{G})$, $\mathbb{P}(\mathsf{H})$ and $\mathbb{P}(\mathsf{G} \cap \mathsf{H})$ only.**

Proof. (a) We will first show that $\mathbb{P}(\mathsf{G} \cup \mathsf{H}^c) = \mathbb{P}(\mathsf{G}) + \mathbb{P}((\mathsf{H} \cup \mathsf{G})^c)$.

$$\begin{aligned}
\mathbb{P}(\mathsf{G} \cup \mathsf{H}^c) &= \mathbb{P}((\mathsf{G} \cup \mathsf{H}^c) \cap \Omega) && (\mathsf{A} \cap \Omega = \mathsf{A}) \\
&= \mathbb{P}((\mathsf{G} \cup \mathsf{H}^c) \cap (\mathsf{G} \cup \mathsf{G}^c)) && \text{(def of complement)} \\
&= \mathbb{P}([(\mathsf{G} \cup \mathsf{H}^c) \cap \mathsf{G}] \cup [(\mathsf{G} \cup \mathsf{H}^c) \cap \mathsf{G}^c]) && \text{(distributive law)} \\
&= \mathbb{P}(\mathsf{G} \cup (\mathsf{H}^c \cap \mathsf{G}^c)) && \text{(distributive law)} \\
&= \mathbb{P}(\mathsf{G}) + \mathbb{P}(\mathsf{H}^c \cap \mathsf{G}^c) && \text{(additivity)} \\
&= \mathbb{P}(\mathsf{G}) + \mathbb{P}((\mathsf{H} \cup \mathsf{G})^c) && \text{(De Morgan's law)}
\end{aligned}$$

(b) Next, we will show that $\mathbb{P}(\mathsf{G} \cap \mathsf{H}^c) = \mathbb{P}(\mathsf{G}) - \mathbb{P}(\mathsf{G} \cap \mathsf{H})$.

$$\begin{aligned}
\mathbb{P}(\mathsf{G} \cap \mathsf{H}^c) &= \mathbb{P}(\mathsf{G}) + \mathbb{P}(\mathsf{H}^c) - \mathbb{P}(\mathsf{G} \cup \mathsf{H}^c) && \text{(def of } \mathbb{P}) \\
&= \mathbb{P}(\mathsf{G}) + \mathbb{P}(\mathsf{H}^c) - \mathbb{P}(\mathsf{G}) - \mathbb{P}((\mathsf{H} \cup \mathsf{G})^c) && \text{(substitute (a))} \\
&= \mathbb{P}(\mathsf{G}) + 1 - \mathbb{P}(\mathsf{H}) - \mathbb{P}(\mathsf{G}) - 1 + \mathbb{P}(\mathsf{H} \cup \mathsf{G}) && \text{(def of complement)} \\
&= \mathbb{P}(\mathsf{H} \cup \mathsf{G}) - \mathbb{P}(\mathsf{H}) && \text{(subtraction)} \\
&= \mathbb{P}(\mathsf{H}) + \mathbb{P}(\mathsf{G}) - \mathbb{P}(\mathsf{G} \cap \mathsf{H}) - \mathbb{P}(\mathsf{H}) && \text{(def of } \mathbb{P}) \\
&= \mathbb{P}(\mathsf{G}) - \mathbb{P}(\mathsf{G} \cap \mathsf{H}) && \text{(subtraction)}
\end{aligned}$$

(c) Finally, we will express $\mathbb{P}(\mathsf{G} \,|\, \mathsf{H}^c)$ in terms of $\mathbb{P}(\mathsf{G})$, $\mathbb{P}(\mathsf{H})$, and $\mathbb{P}(\mathsf{G} \cap \mathsf{H})$.

$$\begin{aligned}
\mathbb{P}(\mathsf{G} \,|\, \mathsf{H}^c) &= \frac{\mathbb{P}(\mathsf{G} \cap \mathsf{H}^c)}{\mathbb{P}(\mathsf{H}^c)} && \text{(conditional probability)} \\
&= \frac{\mathbb{P}(\mathsf{G}) - \mathbb{P}(\mathsf{G} \cap \mathsf{H})}{\mathbb{P}(\mathsf{H}^c)} && \text{(substitute (b))} \\
&= \frac{\mathbb{P}(\mathsf{G}) - \mathbb{P}(\mathsf{G} \cap \mathsf{H})}{1 - \mathbb{P}(\mathsf{H})} && \text{(def of complement)}
\end{aligned}$$

□

5. (1 point) Suppose we have the following

$$\begin{aligned}
&\mathbb{P}(\text{“snow today”}) = 30\%, \\
&\mathbb{P}(\text{“snow tomorrow”}) = 60\%, \\
&\mathbb{P}(\text{“snow today and tomorrow”}) = 25\%.
\end{aligned}$$

Given that it snows today, what is the probability that it will snow tomorrow?

Proof.

$$\begin{aligned}\mathbb{P}(\text{“snow tomorrow”} \mid \text{“snow today”}) &= \frac{\mathbb{P}(\text{“snow tomorrow”} \cap \text{“snow today”})}{\mathbb{P}(\text{“snow today”})}\\ &= 0.25/3\\ &= 5/6\end{aligned}$$

□

6. (3 points) Let $(\Omega, \mathbb{P})$ be a probability space. Suppose we have two events A and B such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. **Please prove that the following three equations are equivalent.**

   (a) $\mathbb{P}(A \mid B) = \mathbb{P}(A)$,

   (b) $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$,

   (c) $\mathbb{P}(B \mid A) = \mathbb{P}(B)$.

   Proof. We will show that equations (b) and (c) are equivalent to (a).

   - (b) $\iff$ (a):

   $$\begin{aligned}\mathbb{P}(A \cap B) &= p(A) \cdot \mathbb{P}(B) && \text{(b)}\\ \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} &= \mathbb{P}(A) && \text{(division)}\\ \mathbb{P}(A \mid B) &= \mathbb{P}(A) && \text{(conditional probability)}\end{aligned}$$

   - (c) $\iff$ (a):

   $$\begin{aligned}\mathbb{P}(B \mid A) &= \mathbb{P}(B) && \text{(c)}\\ \frac{\mathbb{P}(A \mid B) \cdot \mathbb{P}(B)}{\mathbb{P}(A)} &= \mathbb{P}(B) && \text{(Bayes' Rule)}\\ \mathbb{P}(A \mid B) \cdot \mathbb{P}(B) &= \mathbb{P}(B) \cdot \mathbb{P}(A) && \text{(multiplication)}\\ \mathbb{P}(A \mid B) &= \mathbb{P}(A) && \text{(division)}\end{aligned}$$

   Therefore, we have that (a) $\iff$ (b) $\iff$ (c). □

2 Appendix

Please feel free to use all the results in the appendix without proving them.

2.1 Appendix 1

Let A, B, and C be events. Then, we have

- (Commutative Law) $A \cup B = B \cup A$,
- (Commutative Law) $A \cap B = B \cap A$,
- (Associative Law) $(A \cup B) \cup B = A \cup (B \cup C)$,
- (Associative Law) $(A \cap B) \cap C = A \cap (B \cap C)$,
- (Distributive law) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$,
- (Distributive law) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$,
- $(A \cup B)^c = A^c \cap B^c$,
- $(A \cap B)^c = A^c \cup B^c$.

2.2 Appendix 2

Let $A_1, A_2, \ldots$ be any sequence of events and B be an event. We have the following

$$\begin{aligned}
\left(\bigcup_{n=1}^{\infty} A_n\right)^c &= \bigcap_{n=1}^{\infty} A_n^c, \\
\left(\bigcap_{n=1}^{\infty} A_n\right)^c &= \bigcup_{n=1}^{\infty} A_n^c, \\
B \cap \left(\bigcup_{n=1}^{\infty} A_n\right) &= \bigcup_{n=1}^{\infty} (B \cap A_n), \\
B \cup \left(\bigcap_{n=1}^{\infty} A_n\right) &= \bigcap_{n=1}^{\infty} (B \cup A_n).
\end{aligned}$$