MATH 1530 Problem Set 6

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Problem 1. Let G be a finite Abelian group and let n be a positive integer that is relatively prime to $|G|$. Prove that the mapping $a \mapsto a^n$ is an automorphism of G.

Proof. Define $\alpha : G \to G$ such that $a \mapsto a^n$. Let $g, h \in G$.

1. **Injective:** Suppose $g^n = h^n$.

$$\begin{aligned} g^n = h^n &\implies e = g^n h^{-n} \\ &\implies e = (gh^{-1})^n \\ &\implies |gh^{-1}| \mid n \end{aligned}$$

   Additionally, $gh^{-1} \in G$. By *Lagrange's Theorem*, we have $|gh^{-1}| \mid |G|$. Since $|gh^{-1}|$ divides both n and $|G|$, and $\gcd(n, |G|) = 1$, we have that $|gh^{-1}| = 1$. Therefore, $gh^{-1} = e \implies g = eh \implies g = h$.

2. **Surjective:** Consider g^n. We have that $g \mapsto g^n$.

3. **Preserves Group Operation:** $\alpha(gh) = (gh)^n = g^n h^n = \alpha(g) \cdot \alpha(h)$.

□

Problem 2. Let G be a group of order pqr, where p, q, r are distinct primes. If H is a subgroup of G of order pq and K is a subgroup of G of order qr, prove that $|H \cap K| = q$.

Proof. We have already proven that $H \cap K$ is a subgroup of G. This implies that $H \cap K$ is also a subgroup of H and K. By *Lagrange's Theorem*, we have that

$$|H \cap K| \mid |H|, |K| \implies |H \cap K| \mid pq, qr$$

Therefore, $|H \cap K|$ is either 1 or q. Assume for contradiction that $|H \cap K| = 1$. By lemma 1, we have that

$$|HK| = \frac{pq \cdot qr}{1} = pq^2r$$

which is a contradiction since HK is a subset of G, which implies that $|HK| \leq |G|$. Therefore, we have shown that $|H \cap K| = q$ as desired. □

Lemma 1. *Let* H *and* K *be subgroups of a finite group* G*. Then,*

$$|HK| = \frac{|H||K|}{|H \cap K|} \text{ where } HK = \{hk \mid h \in H,\ k \in K\}$$

Proof. Consider the group $H \oplus K$ and the set HK.

We define a group action $H \oplus K \cdot HK \to HK$ such that $(h', k') \cdot hk \mapsto h'(hk)k'^{-1}$. This action is well-defined since

1. $(e, e) \cdot hk = e(hk)e^{-1} = e(hk)e = hk$.
2. $(h'', k'') \cdot ((h', k') \cdot hk) = (h'', k'') \cdot h'(hk)k'^{-1} = h''h'(hk)k'^{-1}k''^{-1} = (h''h', k''k') \cdot hk = ((h'', k'')(h', k')) \cdot hk$.

Consider $e \in HK$. By the *orbit-stabilizer theorem*,

$$|H \oplus K| = |H||K| = |\mathrm{orb}_{H\oplus K}(e)| \cdot |\mathrm{stab}_{H\oplus K}(e)|$$

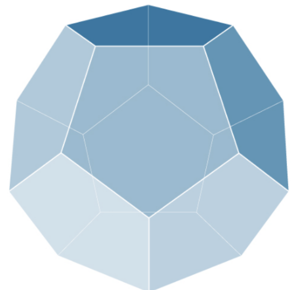
1. **Orbit:** $(h, k) \cdot e = hek^{-1} = hk^{-1}$ for all $(h, k) \in H \oplus K$. Therefore, $|\mathrm{orb}_{H\oplus K}(e)| = |\{hk^{-1} \mid h \in H,\ k \in K\}| = |HK|$.
2. **Stabilizer:** $(h, k) \cdot e = e \implies hk^{-1} = e \implies h = k$. Therefore, $|\mathrm{stab}_{H\oplus K}(e)| = |\{(h, k) \in H \oplus K \mid h = k\}| = |H \cap K|$.

Finally, we have that $|H||K| = |HK| \cdot |H \cap K|$,

$$\implies |HK| = \frac{|H||K|}{|H \cap K|}$$

□

Problem 3. Calculate the order of the group of rotations of a regular dodecahedron:



Proof. Let G be the rotation group of the dodecahedron. Assign each of the 12 faces of the dodecahedron a unique number $1-12$. Since every rotation must take each face to exactly one other face, G is a group of permutations on the set $\{1, \ldots, 12\}$.

Consider a single face, $\mathsf{f} \in \{1, \ldots, 12\}$, of the dodecahedron. By the *orbit-stabilizer theorem*, we have that

$$|\mathsf{G}| = |\mathsf{orb}_\mathsf{G}(\mathsf{f})| \cdot |\mathsf{stab}_\mathsf{G}(\mathsf{f})|$$

1. $|\mathbf{orb}_\mathbf{G}(\mathbf{f})|$: Picking an axis of rotation through the centers of any two parallel faces allows us to bring f to any other face $\mathsf{f}' \in \{1, \ldots, 12\}$. Therefore, $\mathsf{orb}_\mathsf{G}(\mathsf{f}) = \{1, \ldots, 12\}$ which implies that $|\mathsf{orb}_\mathsf{G}(\mathsf{f})| = 12$.

2. $|\mathbf{stab}_\mathbf{G}(\mathbf{f})|$: Let $\overline{\mathsf{f}} \in \{1, \ldots, 12\}$ be the face parallel to f. Picking an axis of rotation through the centers of f and $\overline{\mathsf{f}}$ allows us to rotate the dodecahedron in 5 distinct ways while fixing the position of f. This implies that $|\mathsf{stab}_\mathsf{G}(\mathsf{f})| = 5$.

Together, we have $|\mathsf{G}| = 12 \cdot 5 = 60$. $\square$

Problem 4. Determine the number of cyclic subgroups of order 15 in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$.

Proof. A cyclic subgroup of order 15 has $\phi(15) = 8$ distinct elements of order 15. We will now determine the number of distinct elements of order 15 in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$.

Let $(g_1, g_2) \in \mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$ such that $|(g_1, g_2)| = 15$. By *(Gallian, Theorem 8.1)*, we have that $\operatorname{lcm}(|g_1|, |g_2|) = 15$. For each of the resulting cases, we can use the *Euler phi function* since $\mathbb{Z}_{90}$ and $\mathbb{Z}_{36}$ are both cyclic.

1. **($|g_1| = 5,\ |g_2| = 3$):**
   - $\phi(5) = 4 \implies 4$ distinct elements of order 5 in $\mathbb{Z}_{90}$.
   - $\phi(3) = 2 \implies 2$ distinct elements of order 3 in $\mathbb{Z}_{36}$.

   Therefore, we have $4 \cdot 2 = 8$ ways to make (g_1, g_2) from this case.
2. **($|g_1| = 15,\ |g_2| = 1$):**
   - $\phi(15) = 8 \implies 8$ distinct elements of order 15 in $\mathbb{Z}_{90}$.
   - $\phi(1) = 1 \implies 1$ distinct element of order 1 in $\mathbb{Z}_{36}$.

   So there are $8 \cdot 1 = 8$ ways to make (g_1, g_2) from this case.
3. **($|g_1| = 15,\ |g_2| = 3$):** From above, we have 8 distinct elements of order 15 in $\mathbb{Z}_{90}$, and 2 distinct elements of order 3 in $\mathbb{Z}_{36}$. Hence, there are $8 \cdot 2 = 16$ ways to make (g_1, g_2) from this case.

In total, there are $8+8+16 = 32$ distinct elements of order 15 in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$. Since each cyclic subgroup of order 15 is disjoint and has 8 distinct elements of order 15 which can generate it, the number of cyclic subgroups of order 15 in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$ is $32/8 = 4$. $\square$

Problem 5. Let p and q be odd primes and let m and n be positive integers. Prove that $U(p^m) \oplus U(q^n)$ is not cyclic. [hint: read the book to find a useful result we didn't cover in class]

Proof. By *(Gallian, pg. 160)*, we have that $U(p^m) \approx \mathbb{Z}_{p^m - p^{m-1}}$ and $U(q^n) \approx \mathbb{Z}_{q^n - q^{n-1}}$. Because $\mathbb{Z}_{p^m - p^{m-1}}$ and $\mathbb{Z}_{q^n - q^{n-1}}$ are both cyclic, we have that $U(p^m)$ and $U(q^n)$ are cyclic as well. Therefore, by *(Gallian, Theorem 8.2)*, we must show that $|U(p^m)|$ and $|U(q^n)|$ are not relatively prime.

By lemma 3, we have that $|U(p^m)| = p^m - p^{m-1}$ and $|U(q^n)| = q^n - q^{n-1}$. Since the product of odds is odd, p^m, p^{m-1}, q^n, and q^{n-1} must all be odd. Since the difference of odds is even, we have that $2 \mid p^m - p^{m-1}, q^n - q^{n-1} \implies \gcd(p^m - p^{m-1}, q^n - q^{n-1}) \neq 1$. Therefore, $|U(p^m)|$ and $|U(q^n)|$ are not relatively prime, which means $|U(p^m)| \oplus |U(q^n)|$ is not cyclic. □

Lemma 2. *Let p be an odd prime. Then $U(p^n) \approx \mathbb{Z}_{p^n - p^{n-1}}$.*

Proof. By lemma 3, we have that $|U(p^n)| = p^n - p^{n-1}$. We can arrange the elements of $U(p^n)$ in ascending order so that $U(p^n) = \{u_1, \ldots, u_{p^n - p^{n-1}}\}$ where $j < k \implies u_j < u_k$. Similarly, we can arrange the elements of $\mathbb{Z}_{p^n - p^{n-1}}$ in ascending order so that $\mathbb{Z}_{p^n - p^{n-1}} = \{z_1, \ldots, z_{p^n - p^{n-1}}\}$ where $j < k \implies z_j < z_k$.

Define a mapping $\phi : U(p^n) \to \mathbb{Z}_{p^n - p^{n-1}}$ such that $u_i \mapsto z_i$. We will now show that ϕ is an isomorphism. Let $z_m, z_n \in \mathbb{Z}_{p^n - p^{n-1}}$.

1. **Injective:** $z_j = z_k \implies \phi(u_j) = \phi(u_k) \implies u_j = u_k$.
2. **Surjective:** For any $z_i \in \mathbb{Z}_{p^n - p^{n-1}}$, we have that $u_i \mapsto z_i$.
3. **Preserves Group Operation:** $\phi(u_m \cdot u_n) = \phi((u_m u_n)) = \ldots$ to be proved

□

Lemma 3. *Let p be an odd prime. Then $\phi(p^n) = p^n - p^{n-1}$.*

Proof. We will show $|U(p^n)| = p^n - p^{n-1}$. Of course, there are p^n integers up to p^n. Therefore, $|U(p^n)| = p^n - m$ where m is the number of integers in the set $\{1, \ldots, p^n\}$ that are not relatively prime with p^n. Evidently, the prime factorization of p^n only contains the prime p. This implies that p divides every integer that is not relatively prime with p^n. The number of such integers in the set $\{1, \ldots, p^n\}$ is p^n/p. Therefore,

$$|U(p^n)| = p^n - m = p^n - \frac{p^n}{p} = p^n - p^{n-1}$$

□