

Homework 3

Name: *Tanish Makadia*Due: *11 pm, February 24*Collaborators: *Taj Gillin*

- You are strongly encouraged to work in groups, but solutions must be written independently.

1 Review

To help you better answer the questions in HW 3, we review the example of Bernoulli distributions as follows:

- The experiment of interest is flipping a fair coin;
- the sample space corresponding to this experiment is $\Omega = \{\mathbf{heads}, \mathbf{tails}\}$;
- the probability \mathbb{P} is defined by $\mathbb{P}(A) = \frac{\#A}{\#\Omega}$, i.e., $\mathbb{P}(\{\mathbf{heads}\}) = \mathbb{P}(\{\mathbf{tails}\}) = \frac{1}{2}$;
- the random variable X is defined by

$$X(\mathbf{heads}) = 1, \quad X(\mathbf{tails}) = 0.$$

The CDF of X is

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{2}, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x \geq 1. \end{cases} \quad (1)$$

Proof:

1. When $x < 0$, we have $A_x = \{\omega \in \Omega : X(\omega) \leq x\} = \emptyset$; then, $F_X(x) = \mathbb{P}(A_x) = \mathbb{P}(\emptyset) = 0$.
2. When $0 \leq x < 1$, we have $A_x = \{\omega \in \Omega : X(\omega) \leq x\} = \{\mathbf{tails}\}$; then, $F_X(x) = \mathbb{P}(A_x) = \mathbb{P}(\{\mathbf{tails}\}) = \frac{1}{2}$.
3. When $x \geq 1$, we have $A_x = \{\omega \in \Omega : X(\omega) \leq x\} = \Omega$; then, $F_X(x) = \mathbb{P}(A_x) = \mathbb{P}(\Omega) = 1$.

The proof is completed. □

In addition, the Wikipedia page on random variables is nice material for learning the concept of random variables.

2 Problem Set

1. Let (Ω, \mathbb{P}) be a probability space. Suppose B is an event and $0 < \mathbb{P}(B) < 1$. **Please prove the following:**

(a) (1 point) If A and B are independent, then A and B^c are also independent.

Proof. Using (b), we have that $\mathbb{P}(B^c | A) = 1 - \mathbb{P}(B | A) = 1 - \mathbb{P}(B) = \mathbb{P}(B^c)$. □

(b) (1 point) $\mathbb{P}(A|B) + \mathbb{P}(A^c|B) = 1$.

Proof. We will first prove that $\mathbb{P}(A \cap B) = \mathbb{P}(B) - \mathbb{P}(A^c \cap B)$.

$$\begin{aligned}
 \mathbb{P}(A \cap B) &= \mathbb{P}(B \cap A) && \text{(commutativity)} \\
 &= 1 - \mathbb{P}((B \cap A)^c) && \text{(def of complement)} \\
 &= 1 - \mathbb{P}(B^c \cup A^c) && \text{(De Morgan's Law)} \\
 &= 1 - \mathbb{P}(B^c \cup A^c \cap \Omega) && (E \cap \Omega = E) \\
 &= 1 - \mathbb{P}(B^c \cup A^c \cap (B \cup B^c)) && \text{(def of complement)} \\
 &= 1 - \mathbb{P}(B^c \cup (A^c \cap B) \cup (A^c \cap B^c)) && \text{(distributive law)} \\
 &= 1 - \mathbb{P}(B^c \cup (A^c \cap B^c) \cup (A^c \cap B)) && \text{(commutativity)} \\
 &= 1 - \mathbb{P}(B^c \cup (A^c \cap B)) && \text{(def of } \cup) \\
 &= 1 - \mathbb{P}(B^c) - \mathbb{P}(A^c \cap B) && \text{(additivity)} \\
 &= \mathbb{P}(B) - \mathbb{P}(A^c \cap B)
 \end{aligned}$$

Now, we can use this relation to show that $\mathbb{P}(A|B) + \mathbb{P}(A^c|B) = 1$.

$$\begin{aligned}
 \mathbb{P}(A | B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} && \text{(conditional probability)} \\
 &= \frac{\mathbb{P}(B) - \mathbb{P}(A^c \cap B)}{\mathbb{P}(B)} && \text{(substitute from above)} \\
 &= \frac{\mathbb{P}(B)}{\mathbb{P}(B)} - \frac{\mathbb{P}(A^c \cap B)}{\mathbb{P}(B)} && \text{(distributive prop.)} \\
 &= 1 - \mathbb{P}(A^c | B) && \text{(conditional probability)}
 \end{aligned}$$

□

2. (2 points) Let n be a positive integer, and $\Omega \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$. Suppose \mathbb{P} is a function of subsets of Ω defined as follows

$$\mathbb{P}(A) \stackrel{\text{def}}{=} \frac{\#A}{\#\Omega}, \quad \text{for all } A \subset \Omega.$$

You have proved in HW 1 that (Ω, \mathbb{P}) is a probability space.

We define a random variable X as follows

$$X(\omega) = \omega, \quad \text{for all } \omega \in \Omega = \{1, 2, \dots, n\}.$$

Please derive the CDF of the random variable X defined above. Please present your answer using a formula like the one in Eq. (1).

Proof. Consider the following cases for the CDF of the random variable X :

- When $x < 1$, we have that $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\emptyset) = 0$.
- When $1 \leq x < n$, we have that $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\{1, \dots, \lfloor x \rfloor\}) = \frac{\lfloor x \rfloor}{n}$.
- When $x \geq n$, we have that $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\Omega) = 1$.

Therefore, $F_X(x) = \begin{cases} 0 & \text{if } x < 1, \\ \frac{\lfloor x \rfloor}{n} & \text{if } 1 \leq x < n, \\ 1 & \text{if } x \geq n. \end{cases}$ □

3. (2 points) Let X be a random variable defined on the probability space (Ω, \mathbb{P}) . Suppose X satisfies the following

x	1	2	3	4	5
$\mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$	1/2	1/4	1/8	1/16	1/16

Please derive the CDF of the random variable X . Please present your answer using a formula like the one in Eq. (1).

Proof. Consider the following cases for $F_X(x)$:

- When $x < 1$, we have that $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\emptyset) = 0$.
- When $1 \leq x < 2$, we have that $F_X(x) = \mathbb{P}(X \leq x) = 0 + 1/2 = 1/2$.
- When $2 \leq x < 3$, we have that $F_X(x) = \mathbb{P}(X \leq x) = 1/2 + 1/4 = 3/4$.
- When $3 \leq x < 4$, we have that $F_X(x) = \mathbb{P}(X \leq x) = 3/4 + 1/8 = 7/8$.
- When $4 \leq x < 5$, we have that $F_X(x) = \mathbb{P}(X \leq x) = 7/8 + 1/16 = 15/16$.
- When $x \geq 5$, we have that $F_X(x) = \mathbb{P}(X \leq x) = 15/16 + 1/16 = 1$.

Therefore, $F_X(x) = \begin{cases} 0 & \text{if } x < 1, \\ 1/2 & \text{if } 1 \leq x < 2, \\ 3/4 & \text{if } 2 \leq x < 3, \\ 7/8 & \text{if } 3 \leq x < 4, \\ 15/16 & \text{if } 4 \leq x < 5, \\ 1 & \text{if } x \geq 5. \end{cases}$ □

4. (2 points) Let X be a random variable defined on the probability space (Ω, \mathbb{P}) . Suppose X satisfies the following

$$\mathbb{P}(\{\omega \in \Omega : X(\omega) = 0\}) = 1.$$

Please derive the CDF of the random variable X . Please present your answer using a formula like the one in Eq. (1).

Proof. We have that $\mathbb{P}(X = 0) = 1 = \mathbb{P}(\Omega) \implies \{\omega \in \Omega \mid X(\omega) = 0\} = \Omega$. Thus,

- When $x < 0$, we have that $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\emptyset) = 0$.
- When $x \geq 0$, we have that $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\Omega) = 1$.

Therefore, $F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$ □

5. (2 points) Let X be a random variable defined on the probability space (Ω, \mathbb{P}) . Suppose the CDF of X is the following

$$F_X(x) = \begin{cases} 0, & \text{if } x < 1; \\ \log x, & \text{if } 1 \leq x < e; \\ 1, & \text{if } e \leq x. \end{cases}$$

Please compute the values of the following:

- (a) $\mathbb{P}(\{\omega \in \Omega : X(\omega) < 2\})$;

Proof.

$$\begin{aligned} \mathbb{P}(X < 2) &= \mathbb{P}(X \leq 2) - \mathbb{P}(X = 2) && \text{(additivity)} \\ &= \mathbb{P}(X \leq 2) && (\mathbb{P}(X = 2) = 0) \\ &= F_X(2) && \text{(def of CDF)} \\ &= \log(2) \end{aligned}$$

□

- (b) $\mathbb{P}(\{\omega \in \Omega : 0 < X(\omega) \leq 3\})$;

Proof.

$$\begin{aligned} \mathbb{P}(0 < X \leq 3) &= \mathbb{P}(X \leq 3) - \mathbb{P}(X \leq 0) && \text{(additivity)} \\ &= F_X(3) - F_X(0) && \text{(def of CDF)} \\ &= 1 - 0 && \text{(def of } F_X(x)) \\ &= 1 && \text{(subtraction)} \end{aligned}$$

□

- (c) $\mathbb{P}(\{\omega \in \Omega : 2 < X(\omega) < 2.5\})$.

Proof.

$$\begin{aligned} \mathbb{P}(2 < X < 2.5) &= \mathbb{P}(X \leq 2.5) - \mathbb{P}(X = 2.5) - \mathbb{P}(X \leq 2) && \text{(additivity)} \\ &= \mathbb{P}(X \leq 2.5) - \mathbb{P}(X \leq 2) && (\mathbb{P}(X = 2.5) = 0) \\ &= F_X(2.5) - F_X(2) && \text{(def of CDF)} \\ &= \log(2.5) - \log(2) && \text{(def of } F_X(x)) \\ &= \log(1.25) && \text{(quotient rule)} \end{aligned}$$

□

Remark: For simplicity, many textbooks suppress the ω and represent $\mathbb{P}(\{\omega \in \Omega : X(\omega) < 2\})$, $\mathbb{P}(\{\omega \in \Omega : 0 < X(\omega) \leq 3\})$, and $\mathbb{P}(\{\omega \in \Omega : 2 < X(\omega) < 2.5\})$ as $\mathbb{P}(X < 2)$, $\mathbb{P}(0 < X \leq 3)$, and $\mathbb{P}(2 < X < 2.5)$, respectively. When you read those textbooks, this remark helps you understand what they mean.