

## Homework 2

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Due: 11 pm, February 17

Collaborators: N/A

- You are strongly encouraged to work in groups, but solutions must be written independently.

Please feel free to use all the results in the Appendix of HW 2 without proving them.

## 1 Problem Set

1. (2 points) Suppose  $(\Omega, \mathbb{P})$  is a probability space, and  $B$  is a event with  $\mathbb{P}(B) > 0$ . We define a function  $\tilde{\mathbb{P}}$  of subsets of  $\Omega$  by the following

$$\tilde{\mathbb{P}}(A) \stackrel{\text{def}}{=} \mathbb{P}(A | B), \quad \text{for all } A \subset \Omega.$$

**Please prove that  $\tilde{\mathbb{P}}$  is a probability, i.e.,  $(\Omega, \tilde{\mathbb{P}})$  is a probability space as well.**

*Proof.* We will prove that  $(\Omega, \tilde{\mathbb{P}})$  is a probability space by proving the following three axioms:

- $\tilde{\mathbb{P}}(A \subset \Omega) \geq 0$ :  $\tilde{\mathbb{P}}(A) = \mathbb{P}(A | B) \geq 0$
- $\tilde{\mathbb{P}}(\Omega) = 1$ :  $\tilde{\mathbb{P}}(\Omega) = \mathbb{P}(\Omega | B) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$
- Countable Additivity: Let  $A_1, \dots, A_m \subset \Omega$  be mutually disjoint events.

$$\begin{aligned} \tilde{\mathbb{P}}(A_1 \cup \dots \cup A_m) &= \mathbb{P}((A_1 \cup \dots \cup A_m) | B) \\ &= \frac{\mathbb{P}((A_1 \cup \dots \cup A_m) \cap B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}((A_1 \cap B) \cup \dots \cup (A_m \cap B))}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(A_1 \cap B) + \dots + \mathbb{P}(A_m \cap B)}{\mathbb{P}(B)} \\ &= \tilde{\mathbb{P}}(A_1) + \dots + \tilde{\mathbb{P}}(A_m) \end{aligned}$$

□

2. (1 point) Let  $(\Omega, \mathbb{P})$  be a probability space and  $n$  be a positive integer.  $B_1, B_2, \dots, B_n$  are events and provide a partition of  $\Omega$ , i.e.,

- $\bigcup_{i=1}^n B_i = \Omega$ ,
- $B_1, B_2, \dots, B_n$  are mutually disjoint.

Let  $A$  be any event. **Please prove that  $A \cap B_1, A \cap B_2, A \cap B_3, \dots, A \cap B_n$  are mutually disjoint**, i.e.,

$$(A \cap B_i) \cap (A \cap B_j) = \emptyset, \quad \text{if } i \neq j.$$

*Proof.* Let  $B_i, B_j \in \{B_k\}_{k=1}^n$  such that  $i \neq j$ . We have that  $B_i \cap B_j = \emptyset$ . It follows that  $A \cap B_i \subset B_i$  and  $A \cap B_j \subset B_j$ . Thus, it must be the case that  $(A \cap B_i) \cap (A \cap B_j) = \emptyset$ .  $\square$

3. (2 points) A box contains  $w$  white balls and  $b$  black balls. A ball is chosen at random.

- If the chosen ball is white, we add  $d$  white balls to the box, that is, now there are  $w + d$  white balls and  $b$  black balls.
- If the chosen ball is black, we add  $d$  black balls to the box, that is, now there are  $w$  white balls and  $b + d$  black balls.

After adding the  $d$  balls, another ball is drawn at random from the box. **Show that the probability that the second chosen ball is white does not depend on  $d$ .** Hint: Use the law of total probability (LTP).

*Proof.* Let  $W, B$  denote "white" and "black" respectively.

We have that  $\Omega = \{(W, W), (W, B), (B, W), (B, B)\}$ . Let  $A = \{(W, W), (B, W)\}$  represent the event that the second ball picked is white. We will now create a partition of  $\Omega$  using the following elements:

$$B_1 = \{(W, W)\} \quad B_2 = \{(W, B)\} \quad B_3 = \{(B, W)\} \quad B_4 = \{(B, B)\}$$

Since  $B_1, \dots, B_4$  are mutually disjoint and  $B_1 \cup \dots \cup B_4 = \Omega$ , we have that  $\{B_i\}_{i=1}^4$  forms a partition of  $\Omega$ .

By the law of total probability,

$$\begin{aligned} \mathbb{P}(A) &= \sum_{i=1}^4 \mathbb{P}(A | B_i) \cdot \mathbb{P}(B_i) \\ &= \sum_{i=1}^4 \mathbb{P}(A \cap B_i) \\ &= \mathbb{P}(W, W) + \mathbb{P}(B, B) + \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) \\ &= \frac{w}{w+b} \cdot \frac{w+d}{w+b+d} + \frac{b}{w+b} \cdot \frac{w}{w+b+d} \\ &= \frac{w^2 + dw + bw}{(w+b)(w+b+d)} \\ &= \frac{w(w+b+d)}{(w+b)(w+b+d)} \\ &= \frac{w}{w+b} \end{aligned}$$

Therefore, we have proven that  $\mathbb{P}(A)$  does not depend on  $d$ .  $\square$

4. (1 point) Suppose the underlying probability space is  $(\Omega, \mathbb{P})$ . Let  $G$  and  $H$  be events such that  $0 < \mathbb{P}(G) < 1$  and  $0 < \mathbb{P}(H) < 1$ . **Give a formula for  $\mathbb{P}(G|H^c)$  in terms of  $\mathbb{P}(G)$ ,  $\mathbb{P}(H)$  and  $\mathbb{P}(G \cap H)$  only.**

*Proof.* (a) We will first show that  $\mathbb{P}(G \cup H^c) = \mathbb{P}(G) + \mathbb{P}((H \cup G)^c)$ .

$$\begin{aligned}
 \mathbb{P}(G \cup H^c) &= \mathbb{P}((G \cup H^c) \cap \Omega) && (A \cap \Omega = A) \\
 &= \mathbb{P}((G \cup H^c) \cap (G \cup G^c)) && (\text{def of complement}) \\
 &= \mathbb{P}([(G \cup H^c) \cap G] \cup [(G \cup H^c) \cap G^c]) && (\text{distributive law}) \\
 &= \mathbb{P}(G \cup (H^c \cap G^c)) && (\text{distributive law}) \\
 &= \mathbb{P}(G) + \mathbb{P}(H^c \cap G^c) && (\text{additivity}) \\
 &= \mathbb{P}(G) + \mathbb{P}((H \cup G)^c) && (\text{De Morgan's law})
 \end{aligned}$$

(b) Next, we will show that  $\mathbb{P}(G \cap H^c) = \mathbb{P}(G) - \mathbb{P}(G \cap H)$ .

$$\begin{aligned}
 \mathbb{P}(G \cap H^c) &= \mathbb{P}(G) + \mathbb{P}(H^c) - \mathbb{P}(G \cup H^c) && (\text{def of } \mathbb{P}) \\
 &= \mathbb{P}(G) + \mathbb{P}(H^c) - \mathbb{P}(G) - \mathbb{P}((H \cup G)^c) && (\text{substitute (a)}) \\
 &= \mathbb{P}(G) + 1 - \mathbb{P}(H) - \mathbb{P}(G) - 1 + \mathbb{P}(H \cup G) && (\text{def of complement}) \\
 &= \mathbb{P}(H \cup G) - \mathbb{P}(H) && (\text{subtraction}) \\
 &= \mathbb{P}(H) + \mathbb{P}(G) - \mathbb{P}(G \cap H) - \mathbb{P}(H) && (\text{def of } \mathbb{P}) \\
 &= \mathbb{P}(G) - \mathbb{P}(G \cap H) && (\text{subtraction})
 \end{aligned}$$

(c) Finally, we will express  $\mathbb{P}(G|H^c)$  in terms of  $\mathbb{P}(G)$ ,  $\mathbb{P}(H)$ , and  $\mathbb{P}(G \cap H)$ .

$$\begin{aligned}
 \mathbb{P}(G|H^c) &= \frac{\mathbb{P}(G \cap H^c)}{\mathbb{P}(H^c)} && (\text{conditional probability}) \\
 &= \frac{\mathbb{P}(G) - \mathbb{P}(G \cap H)}{\mathbb{P}(H^c)} && (\text{substitute (b)}) \\
 &= \frac{\mathbb{P}(G) - \mathbb{P}(G \cap H)}{1 - \mathbb{P}(H)} && (\text{def of complement})
 \end{aligned}$$

□

5. (1 point) Suppose we have the following

$$\begin{aligned}
 \mathbb{P}(\text{"snow today"}) &= 30\%, \\
 \mathbb{P}(\text{"snow tomorrow"}) &= 60\%, \\
 \mathbb{P}(\text{"snow today and tomorrow"}) &= 25\%.
 \end{aligned}$$

**Given that it snows today, what is the probability that it will snow tomorrow?**

*Proof.*

$$\begin{aligned}\mathbb{P}(\text{"snow tomorrow"} \mid \text{"snow today"}) &= \frac{\mathbb{P}(\text{"snow tomorrow"} \cap \text{"snow today"})}{\mathbb{P}(\text{"snow today"})} \\ &= 0.25/3 \\ &= 5/6\end{aligned}$$

□

6. (3 points) Let  $(\Omega, \mathbb{P})$  be a probability space. Suppose we have two events  $A$  and  $B$  such that  $\mathbb{P}(A) > 0$  and  $\mathbb{P}(B) > 0$ . **Please prove that the following three equations are equivalent.**

- (a)  $\mathbb{P}(A \mid B) = \mathbb{P}(A)$ ,
- (b)  $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$ ,
- (c)  $\mathbb{P}(B \mid A) = \mathbb{P}(B)$ .

*Proof.* We will show that equations (b) and (c) are equivalent to (a).

- (b)  $\iff$  (a):

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(A) \cdot \mathbb{P}(B) && \text{(b)} \\ \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} &= \mathbb{P}(A) && \text{(division)} \\ \mathbb{P}(A \mid B) &= \mathbb{P}(A) && \text{(conditional probability)}\end{aligned}$$

- (c)  $\iff$  (a):

$$\begin{aligned}\mathbb{P}(B \mid A) &= \mathbb{P}(B) && \text{(c)} \\ \frac{\mathbb{P}(A \mid B) \cdot \mathbb{P}(B)}{\mathbb{P}(A)} &= \mathbb{P}(B) && \text{(Bayes' Rule)} \\ \mathbb{P}(A \mid B) \cdot \mathbb{P}(B) &= \mathbb{P}(B) \cdot \mathbb{P}(A) && \text{(multiplication)} \\ \mathbb{P}(A \mid B) &= \mathbb{P}(A) && \text{(division)}\end{aligned}$$

Therefore, we have that (a)  $\iff$  (b)  $\iff$  (c).

□

## 2 Appendix

Please feel free to use all the results in the appendix without proving them.

### 2.1 Appendix 1

Let  $A$ ,  $B$ , and  $C$  be events. Then, we have

- (Commutative Law )  $A \cup B = B \cup A$ ,
- (Commutative Law )  $A \cap B = B \cap A$ ,
- (Associative Law)  $(A \cup B) \cup C = A \cup (B \cup C)$ ,
- (Associative Law)  $(A \cap B) \cap C = A \cap (B \cap C)$ ,
- (Distributive law)  $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ ,
- (Distributive law)  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ ,
- $(A \cup B)^c = A^c \cap B^c$ ,
- $(A \cap B)^c = A^c \cup B^c$ .

### 2.2 Appendix 2

Let  $A_1, A_2, \dots$  be any sequence of events and  $B$  be an event. We have the following

$$\begin{aligned}\left(\bigcup_{n=1}^{\infty} A_n\right)^c &= \bigcap_{n=1}^{\infty} A_n^c, \\ \left(\bigcap_{n=1}^{\infty} A_n\right)^c &= \bigcup_{n=1}^{\infty} A_n^c, \\ B \cap \left(\bigcup_{n=1}^{\infty} A_n\right) &= \bigcup_{n=1}^{\infty} (B \cap A_n), \\ B \cup \left(\bigcap_{n=1}^{\infty} A_n\right) &= \bigcap_{n=1}^{\infty} (B \cup A_n).\end{aligned}$$