APMA 1655 Honors Statistical Inference I

Homework 4

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- You are strongly encouraged to work in groups, but solutions must be written independently.

1 Review

1.1 Definition of Discrete Random Variables

Let $(\Omega, \mathbb{P})$ be a probability space. Suppose X is a random variable defined on Ω, and F_X is the CDF of X.

1. We say X is a **discrete random variable** if its CDF F_X is of the following form

$$\boxed{F_X(x) = \sum_{k=0}^{K} p_k \cdot \mathbf{1}_{[x_k, +\infty)}(x),} \tag{1}$$

   where $p_k \geq 0$ for all $k = 1, 2, \ldots, K$ and $\sum_{k=0}^{K} p_k = 1$; the K is allowed to be ∞.

2. If X is a discrete random variable whose CDF is of the form in Eq. (1), we call the ordered sequence $\{p_k\}_{k=0}^{K}$ as the **probability mass function (PMF)**†[1] of X.

1.2 Independence between Events

Let $(\Omega, \mathbb{P})$ be a probability space. Suppose $\tilde{A}$ and $\tilde{B}$ are two events. We say $\tilde{A}$ and $\tilde{B}$ are **independent** if $\mathbb{P}(\tilde{A} \cap \tilde{B}) = \mathbb{P}(\tilde{A}) \cdot \mathbb{P}(\tilde{B})$.

1.3 Independence between Random Variables — Version I

Let Y and Z be two random variables defined on the probability space $(\Omega, \mathbb{P})$. We say that Y and Z are independent if they satisfy the following **for any** subsets $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$

$$\boxed{\mathbb{P}\left(\{\omega \in \Omega : Y(\omega) \in A \text{ and } Z(\omega) \in B\}\right) = \mathbb{P}\left(\{\omega \in \Omega : Y(\omega) \in A\}\right) \cdot \mathbb{P}\left(\{\omega \in \Omega : Z(\omega) \in B\}\right).}$$

[1] †: The ordered sequence $\{p_k\}_{k=0}^{K}$ is conventionally called as a function. You may view the map $k \mapsto p_k$ as a function. I think the reason $\{p_k\}_{k=0}^{K}$ is called a function is to make the names "PMF" and "PDF" look similar. In addition, if you are comfortable with the concept of vectors, you may view the ordered sequence $\{p_k\}_{k=0}^{K}$ as a vector $(p_0, p_1, \ldots, p_K)$; if $K = \infty$, this vector is infinitely long.

1.4 Independence between Random Variables — Version II

Let Y and Z be two random variables defined on the probability space $(\Omega, \mathbb{P})$. We say that Y and Z are independent if the following is true: **for any** subsets $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$, the following two events are independent

$$\tilde{A} = \{\omega \in \Omega : Y(\omega) \in A\}, \qquad \tilde{B} = \{\omega \in \Omega : Z(\omega) \in B\}.$$

2 Problem Set

1. (2 points) Let n be a positive integer, and $\Omega \overset{\text{def}}{=} \{1, 2, \dots, n\}$. Suppose $\mathbb{P}$ is a function of subsets of Ω defined as follows

$$\mathbb{P}(A) \overset{\text{def}}{=} \frac{\#A}{\#\Omega}, \quad \text{for all } A \subset \Omega.$$

We define a random variable X as follows

$$X(\omega) = \omega, \qquad \text{for all } \omega \in \Omega = \{1, 2, \dots, n\}.$$

Suppose you have done the following

- You have proved that $(\Omega, \mathbb{P})$ is a probability space (see HW 1).
- You have derived the CDF F_X of X (see HW 3).

Please represent the CDF F_X in the form in Eq. (1). Specifically, please show what the K, $\{p_k\}_{k=0}^K$, and $\{x_k\}_{k=0}^K$ in Eq. (1) should be.

$$\begin{aligned} F_X(x) &= \frac{1}{n}\mathbb{1}_{[1,+\infty)}(x) + \frac{1}{n}\mathbb{1}_{[2,+\infty)}(x) + \cdots + \frac{1}{n}\mathbb{1}_{[n,+\infty)}(x) \\ &= \sum_{k=0}^{n-1} \frac{1}{n} \cdot \mathbb{1}_{[k+1,+\infty)}(x) \end{aligned}$$

Therefore,

$$K = n - 1; \ \ \{p_k\}_{k=0}^K = \{\frac{1}{n}\}_{k=0}^{n-1}; \ \ \{x_k\}_{k=0}^{n-1} = \{k+1\}_{k=0}^{n-1}$$

2. (2 points) Let Y and Z be random variables defined on the probability space $(\Omega, \mathbb{P})$; the distribution of the random variable X defined as follows

$$\begin{aligned} & X(\omega) \overset{\text{def}}{=} Y(\omega) + (1 - Y(\omega)) \cdot Z(\omega), \qquad \text{for all } \omega \in \Omega, \quad \text{where} \\ & Y \sim \text{Bernoulli}\left(\frac{1}{2}\right), \qquad Z \sim N(0, 1), \\ & Y \text{ and } Z \text{ are independent.} \end{aligned} \tag{2}$$

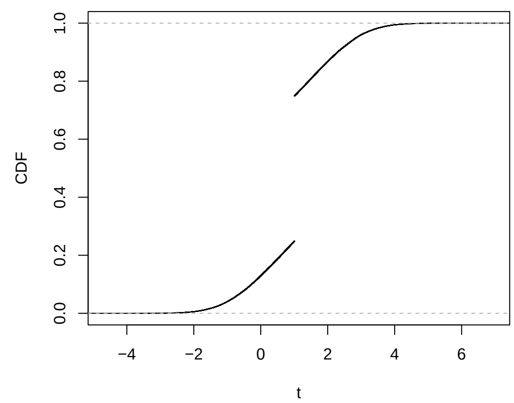


Figure 1: The CDF of the distribution of X defined in Eq. (2). This function is neither continuous nor piecewise constant/step-like.

Then, we claim that the CDF of the random variable X defined in Eq. (2) is the following

$$\begin{aligned} F_X(x) &= \frac{1}{2} \cdot \mathbf{1}_{[1,+\infty)}(x) + \frac{1}{2} \cdot F_Z(x) \\ &= \frac{1}{2} \cdot \mathbf{1}_{[1,+\infty)}(x) + \frac{1}{2} \cdot \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}} \, dt, \end{aligned} \tag{3}$$

where F_X denotes the CDF of X, and F_Z denotes the CDF of Z (i.e., the CDF of $N(0,1)$). The graph of the CDF in Eq. (3) is presented in Figure 1.

Please prove the formula in Eq. (3).

Proof. By the law of total probability,

$$F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(X \le x \mid Y = 1) \cdot \mathbb{P}(Y = 1) + \mathbb{P}(X \le x \mid Y = 0) \cdot \mathbb{P}(Y = 0)$$

We will now compute each half of the sum.

$$\begin{aligned} \mathbb{P}(X \le x \mid Y = 1) \cdot \mathbb{P}(Y = 1) &= \mathbb{P}(Y + (1 - Y)Z \le x \mid Y = 1) \cdot \mathbb{P}(Y = 1) \\ &= \mathbb{P}(1 + (1 - 1)Z \le x) \cdot \mathbb{P}(Y = 1) \\ &= \mathbb{P}(1 \le x) \cdot \mathbb{P}(Y = 1) \\ &= \frac{1}{2} \mathbb{1}_{[1,+\infty)}(x) \end{aligned}$$

$$\begin{aligned} \mathbb{P}(X \le x \mid Y = 0) \cdot \mathbb{P}(Y = 0) &= \mathbb{P}(Y + (1 - Y)Z \le x \mid Y = 0) \cdot \mathbb{P}(Y = 0) \\ &= \mathbb{P}(0 + (1 - 0)Z \le x) \cdot \mathbb{P}(Y = 0) \\ &= \mathbb{P}(Z \le x) \cdot \mathbb{P}(Y = 0) \\ &= \frac{1}{2} F_Z(x) \end{aligned}$$

Therefore, $F_X(x) = \frac{1}{2} \mathbb{1}_{[1,+\infty)}(x) + \frac{1}{2} F_Z(x)$. $\square$

3. (2 points) Let Y, Z, and W be random variables defined on the probability space $(\Omega, \mathbb{P})$. Suppose

   - $Y \sim$ Bernoulli(p);
   - the CDFs of Z and W are F_Z and F_W, respectively;
   - Y, Z, and W are mutually independent, i.e., Y and Z are independent, Y and W are independent, Z and W are independent.

   We define a new random variable X by $X(\omega) \stackrel{\text{def}}{=} Y(\omega) \cdot Z(\omega) + (1 - Y(\omega)) \cdot W(\omega)$ for all $\omega \in \Omega$. **Please prove that the CDF of X is the following**

$$F_X(x) = p \cdot F_Z(x) + (1 - p) \cdot F_W(x).$$

   Proof. By the law of total probability,

$$F_X(x) = \mathbb{P}(X \leq x \mid Y = 1) \cdot \mathbb{P}(Y = 1) + \mathbb{P}(X \leq x \mid Y = 0) \cdot \mathbb{P}(Y = 0)$$

   We will now compute each half of the sum.

$$\begin{aligned}
\mathbb{P}(X \leq x \mid Y = 1) \cdot \mathbb{P}(Y = 1) &= \mathbb{P}(YZ + (1 - Y)W \leq x \mid Y = 1) \cdot \mathbb{P}(Y = 1) \\
&= \mathbb{P}(1 \cdot Z + (1 - 1)W \leq x) \cdot \mathbb{P}(Y = 1) \\
&= \mathbb{P}(Z \leq x) \cdot \mathbb{P}(Y = 1) \\
&= p \cdot F_Z(x)
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(X \leq x \mid Y = 0) \cdot \mathbb{P}(Y = 0) &= \mathbb{P}(YZ + (1 - Y)W \leq x \mid Y = 0) \cdot \mathbb{P}(Y = 0) \\
&= \mathbb{P}(0 \cdot Z + (1 - 0)W \leq x) \cdot \mathbb{P}(Y = 0) \\
&= \mathbb{P}(W \leq x) \cdot \mathbb{P}(Y = 0) \\
&= (1 - p) \cdot F_W(x)
\end{aligned}$$

   Together, we have $F_X(x) = p \cdot F_Z(x) + (1 - p) \cdot F_W(x)$. □

4. (2 points) Let Y, Z, and W be random variables defined on the probability space $(\Omega, \mathbb{P})$. Suppose

   - $Y \sim$ Bernoulli$(1/3)$;
   - $Z \sim$ Pois(1);
   - $W \sim N(0, 1)$;
   - Y, Z, and W are mutually independent, i.e., Y and Z are independent, Y and W are independent, Z and W are independent.

   We define a new random variable X by $X(\omega) \stackrel{\text{def}}{=} Y(\omega) \cdot Z(\omega) + (1 - Y(\omega)) \cdot W(\omega)$ for all $\omega \in \Omega$. Let F_X denote the CDF of X. **Please draw the graph of $F_X(x)$ for $-1 \leq x \leq 5.5$, i.e.,**

$$\{(x, F_X(x)) : -1 \leq x \leq 5.5\}.$$

We have already shown that $F_X(x) = p \cdot F_Z(x) + (1-p) \cdot F_W(x)$.

Since $Y \sim \texttt{Bernoulli}(1/3)$, we have that

$$p = \mathbb{P}(Y=1) = 1/3 \text{ and } 1-p = \mathbb{P}(Y=0) = 2/3$$

Additionally, since $X \sim \texttt{Pois}(1)$, we have that

$$\begin{aligned} F_Z(x) &= \sum_{k=0}^{+\infty} \frac{1}{ek!} \cdot \mathbb{1}_{[k,+\infty)}(x) \\ &= \sum_{k=0}^{x} \frac{1}{ek!} \end{aligned}$$

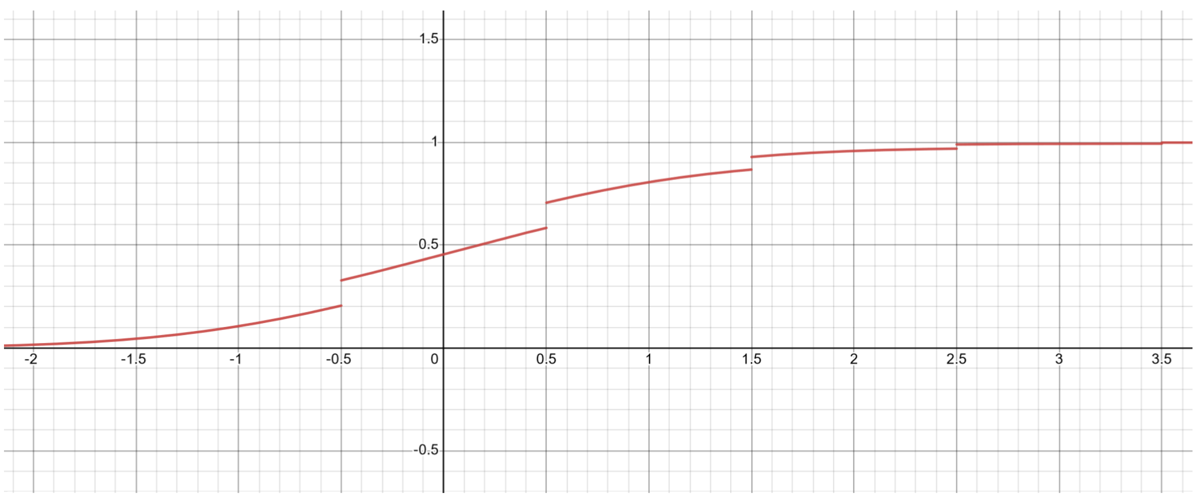
Finally, since $W \sim N(0,1)$, we have that

$$F_W(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}t^2} dt$$

Together,

$$F_X(x) = \frac{1}{3} \cdot \sum_{k=0}^{x} \frac{1}{ek!} + \frac{2}{3} \cdot \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}t^2} dt$$

The following is the graph of $F_X(x)$:



5. (2 points) Let $p_k = \frac{\lambda^k e^{-\lambda}}{k!}$ for all $k = 0, 1, 2, \ldots$, where $k!$ denotes the factorial of k; conventionally, $0! = 1$ (see Wikipedia). **Please prove the following identity**

$$\sum_{k=0}^{\infty} k \cdot p_k = \lambda. \tag{4}$$

Remark: Eq. (4) shows that the "expected value" of Pois(λ). We will discuss the concept of expected values in Chapter 3 of my lecture notes.

Proof. Since $p_k = \frac{\lambda^k e^{-\lambda}}{k!}$, we have that

$$\begin{aligned}
\sum_{k=0}^{\infty} k \cdot p_k &= \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!} \\
&= 0 + \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!} \\
&= e^{-\lambda} \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \\
&= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}\lambda}{(k-1)!} \\
&= \lambda \cdot e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda \cdot e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
&= \lambda \cdot e^{-\lambda} \cdot e^{\lambda} \\
&= \lambda
\end{aligned}$$

□