MATH 1530 Problem Set 2

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Problem 1. Consider $\mathsf{U}(40)$. Find a subgroup which is cyclic of order 4. Find a subgroup which is noncyclic of order 4.

Proof. $\square$

Problem 2. If H and K are subgroups of a group G, prove that $\mathsf{H} \cap \mathsf{K}$ is a subgroup of G. If $\mathsf{H} \not\subset \mathsf{K}$ and $\mathsf{K} \not\subset \mathsf{H}$, prove that $\mathsf{H} \cup \mathsf{K}$ is never a subgroup of G.

Problem 3. Prove that a group G is Abelian if and only if $\mathsf{G} = \mathsf{Z}(\mathsf{G})$.

Problem 4. Suppose G is a group with exactly 8 elements of order 3. how many subgroups of order 3 does G have?

Proof. Let $H \subset G$ be a subgroup of order 3:

$$H = \{e, a, b\}$$

Since e is unique, we have that $ab \neq a$ and $ab \neq b$. In order for H to be closed, the only remaining choice is $ab = e$. Thus, for any subgroup of order 3, the two elements besides the identity must be each other's inverse.

Now, we will show that $|a| = |b| = 3$. Consider $a^2 \in H$. Since the identity is unique, $a^2 \neq a$, and since b is the unique inverse of a, the only remaining choice is $a^2 = b$. Therefore,

$$\begin{aligned} a^3 &= a \cdot a^2 & \qquad b^3 &= (a^2)^3 \\ &= a \cdot b & &= (a^3)^2 \\ &= e & &= e \end{aligned}$$

We have that there are exactly 8 elements of G of order 3. Because a subgroup of order 3 requires two distinct elements of order 3, we can conclude that the number of distinct subgroups of order 3 in G is $8/2 = 4$.

$\square$

Problem 5. Let G be a finite group with more than one element. Show that G has an element of prime order.