MATH 1530 Problem Set 5

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Problem 1. Let G be a finite Abelian group and let n be a positive integer that is relatively prime to $|G|$. Prove that the mapping $a \mapsto a^n$ is an automorphism of G.

Problem 2. Let G be a group of order pqr, where p, q, r are distinct primes. If H is a subgroup of G of order pq and K is a subgroup of G of order qr, prove that $|H \cap K| = q$.

Proof. We have already proven that $H \cap K$ is a subgroup of G. This implies that $H \cap K$ is also a subgroup of H and K. By *Lagrange's Theorem*, we have that

$$|H \cap K| \mid |H|, |K| \implies |H \cap K| \mid pq, qr$$

Therefore, $|H \cap K|$ is either 1 or q. Assume for contradiction that $|H \cap K| = 1$. By lemma 1, we have that

$$|HK| = \frac{pq \cdot qr}{1} = pq^2r$$

which is a contradiction since HK is a subset of G, which implies that $|HK| \leq |G|$. Therefore, we have shown that $|H \cap K| = q$ as desired. $\square$

Lemma 1. *Let* H *and* K *be subgroups of a finite group* G*. Then,*

$$|HK| = \frac{|H||K|}{|H \cap K|} \text{ where } HK = \{hk \mid h \in H,\ k \in K\}$$

Proof. We can separate HK into a union of left cosets of K in G:

$$HK = \bigcup_{h \in H} hK$$

By the properties of cosets, we have that $hK = h'K$ or $hK \cap h'K = \emptyset$ for all $h, h' \in H$. We must now determine how many of these cosets are distinct.

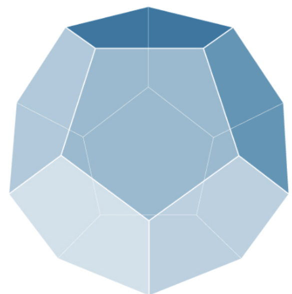
Suppose $hK = h'K$ for some $h, h' \in H$. Since $hK = h'K \Leftrightarrow h^{-1}h' \in K$, we have that $h^{-1}h' = k$ for some $k \in K$. This implies that $k \in H \implies k \in H \cap K$. Additionally, $h' = hk$. Thus, there are $|H \cap K|$ ways to create the same coset for each $h' \in H$ (by *Cayley's Theorem*, we know that each $k \in H \cap K$ has exactly one corresponding $h \in H$ such that $hk = h'$). Therefore, the number of distinct cosets hK where $h \in H$ is $|H|/|H \cap K|$.

Since $|hK| = |h'K|$ for all $h, h' \in H$, the number of elements in each coset is $|hK| = |K|$. Therefore, the cardinality of HK equals the number of distinct cosets times the number of distinct elements in each coset, giving us

$$|HK| = \frac{|H||K|}{|H \cap K|}$$

$\square$

Problem 3. Calculate the order of the group of rotations of a regular dodecahedron:



Problem 4. Determine the number of cyclic subgroups of order 15 in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$.

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Problem 5. Let p and q be odd primes and let m and n be positive integers. Prove that $U(p^m) \oplus U(q^n)$ is not cyclic. [hint: read the book to find a useful result we didn't cover in class]