

Homework 2

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Due: 11 pm, February 17

Collaborators: N/A

- You are strongly encouraged to work in groups, but solutions must be written independently.

Please feel free to use all the results in the Appendix of HW 2 without proving them.

1 Problem Set

1. (2 points) Suppose (Ω, \mathbb{P}) is a probability space, and B is a event with $\mathbb{P}(B) > 0$. We define a function $\tilde{\mathbb{P}}$ of subsets of Ω by the following

$$\tilde{\mathbb{P}}(A) \stackrel{\text{def}}{=} \mathbb{P}(A | B), \quad \text{for all } A \subset \Omega.$$

Please prove that $\tilde{\mathbb{P}}$ is a probability, i.e., $(\Omega, \tilde{\mathbb{P}})$ is a probability space as well.

Proof. We will prove that $(\Omega, \tilde{\mathbb{P}})$ is a probability space by proving the following three axioms:

- $\tilde{\mathbb{P}}(A \subset \Omega) \geq 0$: $\tilde{\mathbb{P}}(A) = \mathbb{P}(A | B) \geq 0$
- $\tilde{\mathbb{P}}(\Omega) = 1$: $\tilde{\mathbb{P}}(\Omega) = \mathbb{P}(\Omega | B) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$
- Countable Additivity: Let $A_1, \dots, A_m \subset \Omega$ be mutually disjoint events.

$$\begin{aligned} \tilde{\mathbb{P}}(A_1 \cup \dots \cup A_m) &= \mathbb{P}((A_1 \cup \dots \cup A_m) | B) \\ &= \frac{\mathbb{P}((A_1 \cup \dots \cup A_m) \cap B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}((A_1 \cap B) \cup \dots \cup (A_m \cap B))}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(A_1 \cap B) + \dots + \mathbb{P}(A_m \cap B)}{\mathbb{P}(B)} \\ &= \tilde{\mathbb{P}}(A_1) + \dots + \tilde{\mathbb{P}}(A_m) \end{aligned}$$

□

2. (1 point) Let (Ω, \mathbb{P}) be a probability space and n be a positive integer. B_1, B_2, \dots, B_n are events and provide a partition of Ω , i.e.,

- $\bigcup_{i=1}^n B_i = \Omega$,
- B_1, B_2, \dots, B_n are mutually disjoint.

Let A be any event. **Please prove that $A \cap B_1, A \cap B_2, A \cap B_3, \dots, A \cap B_n$ are mutually disjoint, i.e.,**

$$(A \cap B_i) \cap (A \cap B_j) = \emptyset, \quad \text{if } i \neq j.$$

Proof. Let $B_i, B_j \in \{B_k\}_{k=1}^n$ such that $i \neq j$. We have that $B_i \cap B_j = \emptyset$. It follows that $A \cap B_i \subset B_i$ and $A \cap B_j \subset B_j$. Thus, it must be the case that $(A \cap B_i) \cap (A \cap B_j) = \emptyset$. \square

3. (2 points) A box contains w white balls and b black balls. A ball is chosen at random.
- If the chosen ball is white, we add d white balls to the box, that is, now there are $w + d$ white balls and b black balls.
 - If the chosen ball is black, we add d black balls to the box, that is, now there are w white balls and $b + d$ black balls.

After adding the d balls, another ball is drawn at random from the box. **Show that the probability that the second chosen ball is white does not depend on d .** Hint: Use the law of total probability (LTP).

4. (1 point) Suppose the underlying probability space is (Ω, \mathbb{P}) . Let G and H be events such that $0 < \mathbb{P}(G) < 1$ and $0 < \mathbb{P}(H) < 1$. **Give a formula for $\mathbb{P}(G|H^c)$ in terms of $\mathbb{P}(G)$, $\mathbb{P}(H)$ and $\mathbb{P}(G \cap H)$ only.**
5. (1 point) Suppose we have the following

$$\begin{aligned}\mathbb{P}(\text{"snow today"}) &= 30\%, \\ \mathbb{P}(\text{"snow tomorrow"}) &= 60\%, \\ \mathbb{P}(\text{"snow today and tomorrow"}) &= 25\%.\end{aligned}$$

Given that it snows today, what is the probability that it will snow tomorrow?

6. (3 points) Let (Ω, \mathbb{P}) be a probability space. Suppose we have two events A and B such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. **Please prove that the following three equations are equivalent.**
- $\mathbb{P}(A|B) = \mathbb{P}(A)$,
 - $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$,
 - $\mathbb{P}(B|A) = \mathbb{P}(B)$.

2 Appendix

Please feel free to use all the results in the appendix without proving them.

2.1 Appendix 1

Let A , B , and C be events. Then, we have

- (Commutative Law) $A \cup B = B \cup A$,
- (Commutative Law) $A \cap B = B \cap A$,
- (Associative Law) $(A \cup B) \cup C = A \cup (B \cup C)$,
- (Associative Law) $(A \cap B) \cap C = A \cap (B \cap C)$,
- (Distributive law) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$,
- (Distributive law) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$,
- $(A \cup B)^c = A^c \cap B^c$,
- $(A \cap B)^c = A^c \cup B^c$.

2.2 Appendix 2

Let A_1, A_2, \dots be any sequence of events and B be an event. We have the following

$$\begin{aligned}\left(\bigcup_{n=1}^{\infty} A_n\right)^c &= \bigcap_{n=1}^{\infty} A_n^c, \\ \left(\bigcap_{n=1}^{\infty} A_n\right)^c &= \bigcup_{n=1}^{\infty} A_n^c, \\ B \cap \left(\bigcup_{n=1}^{\infty} A_n\right) &= \bigcup_{n=1}^{\infty} (B \cap A_n), \\ B \cup \left(\bigcap_{n=1}^{\infty} A_n\right) &= \bigcap_{n=1}^{\infty} (B \cup A_n).\end{aligned}$$