MATH 1530 Problem Set 5

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Problem 1. How many elements of order 6 are in S_7?

Proof. By (Gallian, 5.1), every permutation of a finite set can be expressed as a product of disjoint cycles. Additionally, by (Gallian, 5.3), the order of a permutation in disjoint cycle form is the lcm of lengths of the disjoint cycles.

Let $P = \{s \in S_7 \mid |s| = 6\}$. We must find the cardinality of P. Let $p \in P$. From above, p must have a disjoint cycle form in which the lcm of the disjoint cycle lengths equals 6. Therefore, the disjoint cycle form of p must fall under one of the following cases (note that the order of the disjoint cycles does not matter since they are commutative):

- **Case 1 (lengths: $2, 2, 3$):** $p = (a_1, a_2)(b_1, b_2)(c_1, c_2, c_3)$. In this case, the number of ways to construct p using elements of S_7 is:
$$\frac{1}{2}\left(\frac{7!}{5! \cdot 2} \cdot \frac{5!}{3! \cdot 2} \cdot \frac{3!}{3}\right) = 210$$
- **Case 2 (lengths: $3, 2, 1, 1$):** $p = (a_1, a_2, a_3)(b_1, b_2)(c_1)(d_1)$. In this case, the number of ways to construct p is:
$$\frac{7!}{4! \cdot 3} \cdot \frac{4!}{2! \cdot 2} = 420$$
- **Case 3 (lengths: $6, 1$):** $p = (a_1, a_2, a_3, a_4, a_5, a_6)(b_1)$. In this case, the number of ways to construct p is:
$$\frac{7!}{1! \cdot 6} = 840$$

Therefore, the number of elements of order 6 in S_7 is $\operatorname{card}(P) = 210 + 420 + 840 = 1470$. □

Problem 2. Let D_4 denote the rigid operations on a square taking the square back to itself (i.e., the symmetries of the square). For example, rotating the square by π is a rigid operation taking the square back to itself. This is called the *dihedral group*, and it is a group under composition.

Label the vertices of the square from 1 to 4. Use this to represent the elements of D_4 a subgroup of S_4 (that is, list the elements of D_4 using cycle notation). What is the order of D_4? Is D_4 isomorphic to S_4?

Proof. The elements of D_4 are the following permutations:

1. $\begin{array}{|cc|}\hline 1 & 2 \\ & \\ 4 & 3 \\ \hline\end{array} \xrightarrow[e]{\text{identity}} \begin{array}{|cc|}\hline 1 & 2 \\ & \\ 4 & 3 \\ \hline\end{array}$

2. $\begin{array}{|cc|}\hline 1 & 2 \\ \multicolumn{2}{|c|}{\longleftrightarrow} \\ 4 & 3 \\ \hline\end{array} \xrightarrow[(1,2)(4,3)]{\text{horizontal flip}} \begin{array}{|cc|}\hline 2 & 1 \\ & \\ 3 & 4 \\ \hline\end{array}$

3. $\begin{array}{|cc|}\hline 1 & 2 \\ \multicolumn{2}{|c|}{\updownarrow} \\ 4 & 3 \\ \hline\end{array} \xrightarrow[(1,4)(2,3)]{\text{vertical flip}} \begin{array}{|cc|}\hline 4 & 3 \\ & \\ 1 & 2 \\ \hline\end{array}$

4. $\begin{array}{|cc|}\hline 1 & 2 \\ \multicolumn{2}{|c|}{\nearrow} \\ 4 & 3 \\ \hline\end{array} \xrightarrow[(2,4)]{\text{left diagonal flip}} \begin{array}{|cc|}\hline 1 & 4 \\ & \\ 2 & 3 \\ \hline\end{array}$

5. $\begin{array}{|cc|}\hline 1 & 2 \\ \multicolumn{2}{|c|}{\nwarrow} \\ 4 & 3 \\ \hline\end{array} \xrightarrow[(1,3)]{\text{right diagonal flip}} \begin{array}{|cc|}\hline 3 & 2 \\ & \\ 4 & 1 \\ \hline\end{array}$

6. $\begin{array}{|cc|}\hline 1 & 2 \\ \multicolumn{2}{|c|}{\circlearrowright} \\ 4 & 3 \\ \hline\end{array} \xrightarrow[(1,2,3,4)]{\text{clockwise rotation}} \begin{array}{|cc|}\hline 4 & 1 \\ & \\ 3 & 2 \\ \hline\end{array}$

7. $\begin{array}{|cc|}\hline 1 & 2 \\ \multicolumn{2}{|c|}{2} \\ 4 & 3 \\ \hline\end{array} \xrightarrow[(1,3)(2,4)]{\text{clockwise rotation (x2)}} \begin{array}{|cc|}\hline 3 & 4 \\ & \\ 2 & 1 \\ \hline\end{array}$

8. $\begin{array}{|cc|}\hline 1 & 2 \\ \multicolumn{2}{|c|}{3} \\ 4 & 3 \\ \hline\end{array} \xrightarrow[(1,4,3,2)]{\text{clockwise rotation (x3)}} \begin{array}{|cc|}\hline 2 & 3 \\ & \\ 1 & 4 \\ \hline\end{array}$

Evidently, $|\mathrm{D}_4| = 8$. ... $\square$

Problem 3. Prove that a permutation with odd order must be an even permutation. Show that the converse is false.

Proof. Let p be a permutation such that $|p| = n$ where n is odd. We have that, $p^n = e$. By (Gallian, 5.4), $p = \beta_1 \cdots \beta_r$ where each β_i is a two-cycle. Combining these two equations, we obtain $(\beta_1 \cdots \beta_r)^n = e$. For contradiction, suppose r is odd. Thus, we have that

$$\begin{aligned} e &= (\beta_1 \cdots \beta_r)^n \\ &= (\beta_1 \cdots \beta_r) \overset{n \text{ times}}{\cdots} (\beta_1 \cdots \beta_r) \\ &= \beta_1 \cdots \beta_{nr} \end{aligned}$$

By lemma 1, nr is odd. Since e must equal the product of an even number of two cycles, this is a contradiction. Therefore, r must be even which implies that p is an even permutation. □

Lemma 1. *The product of two odd integers is odd*

Proof. Let $x, y \in \mathbb{Z}$ such that x and y are odd. By the division algorithm, we have that $x = 2b_x + 1$ and $y = 2b_y + 1$ where $b_x, b_y \in \mathbb{Z}$. Now consider the product of x and y:

$$\begin{aligned} x \cdot y &= (2b_x + 1) \cdot (2b_y + 1) \\ &= 4b_x b_y + 2b_x + 2b_y + 1 \\ &= 2(2b_x b_y + b_x + b_y) + 1 \end{aligned}$$

Therefore, $2 \nmid x \cdot y \implies x \cdot y$ is odd. □

Problem 4. Let $\mathbb{C}$ be the complex numbers and

$$\mathsf{M} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

prove that $\mathbb{C}^*$ and M^* (the nonzero elements of M), viewed as groups with multiplication, are isomorphic.

Problem 5. Let G be a group. An isomorphism from G to itself is called an *automorphism* of G. Let $\mathrm{Aut}(\mathsf{G})$ denote the set of all automorphisms of G. This is a group under the operation of function composition. Find two groups G and H such that $\mathsf{G} \not\approx \mathsf{H}$ but $\mathrm{Aut}(\mathsf{G}) \approx \mathrm{Aut}(\mathsf{H})$.