MATH 1530 Problem Set 5

Tanish Makadia

(Collaborated with Esmé and Kazuya)

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Problem 1. How many elements of order 6 are in S_7?

Proof. By (Gallian, 5.1), every permutation of a finite set can be expressed as a product of disjoint cycles. Additionally, by (Gallian, 5.3), the order of a permutation in disjoint cycle form is the lcm of lengths of the disjoint cycles.

Let $P = \{s \in S_7 \mid |s| = 6\}$. We must find the cardinality of P. Let $p \in P$. From above, p must have a disjoint cycle form in which the lcm of the disjoint cycle lengths equals 6. Therefore, the disjoint cycle form of p must fall under one of the following cases (note that the order of the disjoint cycles does not matter since they are commutative):

- **Case 1 (lengths: $2, 2, 3$):** $p = (a_1, a_2)(b_1, b_2)(c_1, c_2, c_3)$. In this case, the number of ways to construct p using elements of S_7 is:

$$\frac{1}{2}\left(\frac{7!}{5! \cdot 2} \cdot \frac{5!}{3! \cdot 2} \cdot \frac{3!}{3}\right) = 210$$

- **Case 2 (lengths: $3, 2, 1, 1$):** $p = (a_1, a_2, a_3)(b_1, b_2)(c_1)(d_1)$. In this case, the number of ways to construct p is:

$$\frac{7!}{4! \cdot 3} \cdot \frac{4!}{2! \cdot 2} = 420$$

- **Case 3 (lengths: $6, 1$):** $p = (a_1, a_2, a_3, a_4, a_5, a_6)(b_1)$. In this case, the number of ways to construct p is:

$$\frac{7!}{1! \cdot 6} = 840$$

Therefore, the number of elements of order 6 in S_7 is $\operatorname{card}(P) = 210 + 420 + 840 = 1470$. $\square$

Problem 2. Let D_4 denote the rigid operations on a square taking the square back to itself (i.e., the symmetries of the square). For example, rotating the square by π is a rigid operation taking the square back to itself. This is called the *dihedral group*, and it is a group under composition.

Label the vertices of the square from 1 to 4. Use this to represent the elements of D_4 a subgroup of S_4 (that is, list the elements of D_4 using cycle notation). What is the order of D_4? Is D_4 isomorphic to S_4?

Problem 3. Prove that a permutation with odd order must be an even permutation. Show that the converse is false.

Problem 4. Let $\mathbb{C}$ be the complex numbers and

$$M = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \middle| \ a, b \in \mathbb{R} \right\}.$$

prove that $\mathbb{C}^*$ and M^* (the nonzero elements of M), viewed as groups with multiplication, are isomorphic.

Problem 5. Let G be a group. An isomorphism from G to itself is called an *automorphism* of G. Let $\mathrm{Aut}(\mathsf{G})$ denote the set of all automorphisms of G. This is a group under the operation of function composition. Find two groups G and H such that $\mathsf{G} \not\approx \mathsf{H}$ but $\mathrm{Aut}(\mathsf{G}) \approx \mathrm{Aut}(\mathsf{H})$.