MATH 0540 Final Review

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Field Axioms

1. Commutativity

$$\alpha + \beta = \beta + \alpha$$
$$\alpha\beta = \beta\alpha$$

2. Associativity

$$(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$$
$$(\alpha\beta)\lambda = \alpha(\beta\lambda)$$

3. Additive and Multiplicative Identities

$$\alpha + 0 = \alpha$$
$$\alpha \cdot 1 = \alpha$$

4. Unique Additive Inverse

$$\forall \alpha \in \mathbb{F},\ \exists \beta \in \mathbb{F} \text{ such that } \alpha + \beta = 0$$

5. Unique Multiplicative Inverse

$$\forall \alpha \in \mathbb{F},\ \exists \beta \in \mathbb{F} \text{ such that } \alpha\beta = 1$$

6. Distributive Property

$$\alpha(\beta + \lambda) = \alpha\beta + \alpha\lambda$$

Vector Spaces

1. A vector space is a set V with an addition on V and scalar multiplication on V such that the following properties hold:

a) Commutativity
b) Associativity
c) Unique Additive Identity
d) Multiplicative Identity
e) Unique Additive Inverse
f) Distributive Properties:

$$v(a + b) = av + bv$$
$$a(u + v) = au + av$$

2. Let p be a prime number. Then $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ is a vector space with addition and scalar multiplication $\mod p$.

$\mathbb{F}^S$ Notation

If S is a set, $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$. For any $f, g \in \mathbb{F}^S$ and $\lambda \in \mathbb{F}$

$$f + g \in \mathbb{F}^S \text{ is defined by } (f+g)(x) = f(x) + g(x) \qquad \text{for all } x \in S$$
$$\lambda f \in \mathbb{F}^S \text{ is defined by } (\lambda f)(x) = \lambda f(x) \qquad \text{for all } x \in S$$

Subspaces

U is a subspace of a vector space V if it is also a vector space under the same addition and scalar multiplication on V.

1. Unique Additive Identity $0 \in U$
2. Closed Under Addition and Scalar Multiplication

Subsets

1. Sum of subsets:

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_i \in U_i\}$$

2. If $U_1, \ldots, U_m$ are subspaces of V, then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$. 3. Every subspace of a finite-dimensional vector space is also finite-dimensional

Direct Sum

Let $U_1, \ldots, U_m$ be subspaces of V.

1. $U_1 + \cdots + U_m$ is called a direct sum if each of its elements can only be written in one way as a sum $u_1 + \cdots + u_m$ where each $u_i \in U_i$.

2. $U_1 \oplus \cdots \oplus U_m$ denotes that $U_1 + \cdots + U_m$ is a direct sum.

3. $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to express $0 = u_1 + \cdots + u_m$ is by taking each $u_i = 0$.

4. $U_1 + U_2$ is a direct sum if and only if $U_1 \cap U_2 = \{0\}$.

5. There exists a direct sum $U_1 \oplus W = V$ where W is some other subspace of V; *i.e.* every subspace is part of a direct sum equaling its parent space.

Span

1. The span of a list of vectors is the set of all possible linear combinations of them

$$span(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_i \in \mathbb{F}\}$$

2. The span of an empty list of vectors ( ) is defined to be $\{0\}$.

3. The span of a list of vectors in V is the smallest subspace of V containing all of the vectors.

4. Every spanning list contains a basis

Dimension

1. A vector space is finite-dimensional if some finite list of vectors spans the space.

2. Every finite dimensional vector space has a basis

4. If U is a subspace of V, $\dim U \leq \dim V$.

5. Dimension of a sum of two subspaces

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Linear Independence

1. The length of a linearly independent list of vectors is less than or equal to the length of a spanning list of vectors

2. The only way to express $0 = a_1 v_1 + \cdots + a_m v_m$ is by fixing each $a_1, \ldots, a_m$ to zero.

Linear Dependence

Let $v_1, \ldots, v_m$ be a linearly dependent list of vectors in V. The following must be true:

1. There exists $v_j \in \mathrm{span}(v_1, \ldots, v_{j-1})$

2. If this v_j is removed, the span of the list does not change

Bases

1. Linearly Independent

2. Spanning

3. All bases of the same space have the same length

4. Every linearly independent or spanning list of vectors of length $\dim V$ is a basis of V.

Linear Maps

1. Additivity

$$T(u + v) = T(u) + T(v)$$

2. Homogeneity

$$T(\lambda u) = \lambda T(u)$$

3. Suppose $v_1, \ldots, v_n$ are a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T \in \mathcal{L}(V, W)$ such that $T(v_j) = w_j$ for each $j = 1, \ldots, n$

4. Suppose $S, T \in \mathcal{L}(V, W)$, and $\lambda \in \mathbb{F}$. Then for all $v \in V$,

$$(S + T)(v) = Sv + Tv$$
$$(\lambda T)(v) = \lambda(Tv)$$

5. $\mathcal{L}(V, W)$ is a vector space with the addition and scalar multiplication defined in (4).

6. The product of linear maps is their composition such that if $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$

$$(ST)(u) = S(Tu)$$

7. Whenever the products make sense (i.e. the codomain of one map is the domain of the next), linear maps follow associativity, identity, and distributive properties as follows

$$(T_1T_2)T_3 = T_1(T_2T_3)$$
$$TI = IT = T$$
$$(S_1 + S_2)T = S_1T + S_2T$$

8. All linear maps take 0 to 0.

Range and Null Space

Consider a linear map $T \in \mathcal{L}(V, W)$:

1. $\operatorname{null} T = \{v \in V \mid T(v) = 0\}$

2. $\operatorname{null} T$ is a subspace of V

3. $\operatorname{range} T = \{T(v) \mid v \in V\}$

4. $\operatorname{range} T$ is a subspace of T

5. $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$

Injective and Surjective Maps

1. A function $T : V \to W$ is injective if $T(u) = T(v)$ implies $u = v$.

2. Injectivity is equivalent to null space equals $\{0\}$.

3. A linear map is surjective if its range equals its codomain

4. A map to a smaller dimensional space cannot be injective

5. A map to a larger dimensional space cannot be surjective

Matrices

1. $T(v_k) = A_{1,k}w_1 + \cdots + A_{m,k}w_m$ (column is the result of the map on a basis vector)

2. $M(S+T) = M(S) + M(T)$

3. $M(\lambda T) = \lambda M(T)$

4. $\mathbb{F}^{m,n}$ is a vector space with dimension mn.

5. $M(ST) = M(S)M(T)$

6. If A is an mxn matrix and C is an nxp matrix, AC is an mxp matrix

7. Suppose A is an mxn matrix and C is an $nx1$ matrix such that $C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

$$AC = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$$

8. $M(v) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ where $v = c_1 v_1 + \cdots + c_n v_n$.

9. $M(T(v)) = M(T)M(v)$

10. Let $T \in \mathcal{L}(V)$. If $\mathcal{M}(T)$ is upper triangular, then span $(v_1, \ldots, v_n)$ is invariant under T.

10. An upper triangular matrix is invertible if and only if all the diagonal entries are non-zero.

11. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then $M(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$ and $M(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n))$ are inverses of each other.

12. Change of basis: $M(T, (u_1, \ldots, u_n)) = M(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n)) \cdot M(T, (v_1, \ldots, v_n)) \cdot M(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$

13. Let $T \in \mathcal{L}(V)$. $M(T)$ is diagonalizable if and only if V has a basis of eigenvectors of T.

Isomorphisms

1. For a linear map, T, another linear map S is the inverse of T if $ST = I$ and $TS = I$.

2. An invertible linear map has a unique inverse.

3. Invertibility is equivalent to injectivity and surjectivitiy.

4. Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

5. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$.

6. Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

7. Invertible is equivalent to columns being a basis for codomain

8. $A^{-1} = \frac{1}{\det A} C^T$

9. For an nxn matrix A, the following are equivalent
   a) A is invertible
   b) $\det A \neq 0$
   c) the rows and columns of A are linearly independent

Operators

Suppose $T \in \mathcal{L}(V)$

1. Injectivity, surjectivity, and invertibility are equivalent

2. Every operator has an upper triangular matrix

Polynomials

1. Fundamental Theorem of Algebra: every non-constant polynomial with complex coefficients has a zero.

2. Every non-constant polynomial has a unique factorization

$$p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$$

3. If $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is defined as $(pq)(z) = p(z)q(z)$.

4. A monic polynomial is a polynomial in which the highest degree term has a coefficient of one.

5. Suppose $T \in \mathcal{L}(V)$. The mimimal polynomial of T is the unique monic polynomial p of smallest degree such that $p(T) = 0$.

Invariance and Eigenvectors

1. U is invariant under T if $T(u) \in U$ for all $u \in U$.

2. Zero can be an eigenvalue, but zero cannot be an eigenvector

3. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The following are equivalent:
   a) λ is an eigenvalue of T
   b) $T - \lambda I$ is not injective
   c) $T - \lambda I$ is not surjective
   d) $T - \lambda I$ is not invertible

4. If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T, then the corresponding eigenvectors $v_1, \ldots, v_m$ are linearly independent.

5. If V is finite-dimensional, each operator on V has at most $\dim V$ distinct eigenvalues.

6. The eigenvalues of an upper triangular matrix are the diagonal values

Characteristic Polynomial

1. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then the characteristic polynomial of T has degree $\dim V$.

2. Cayley-Hamilton Theorem: Let $T \in \mathcal{L}(V)$ and let q be the characteristic polynomial of T. Then $q(t) = 0$.

Determinant

1. Three properties
   a) Multi-linear

$$D(v_1, \ldots, av_k + bv_k', \ldots, v_n) = aD(v_1, \ldots, v_k, \ldots, v_n) + bD(v_1, \ldots, v_k', \ldots, v_n)$$

   b) Alternating

$$D(v_1, \ldots, v_j, \ldots, v_k, \ldots, v_n) = 0 \text{ if } v_j = v_k$$

   c) Normalized

$$\det(e_1, \ldots, e_n) = 1$$

2. Invertible is equivalent to non-zero determinant

3. Leibniz (Permutation) Formula

$$\det A = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} A_{\sigma_i, i}$$

4. Interchanging two columns in a matrix flips the sign of the determinant.

5. If a matrix has two columns that are equal, its determinant is zero.

6. $\det(AB) = \det(BA) = (\det A)(\det B)$

7. Co-factor Expansion: $\det A = A_{j,1}C_{j,1} + \cdots + A_{j,n}C_{j,n}$ where $C_{j,k} = (-1)^{j+k} \det A_{\hat{j},\hat{k}}$

8. $\det A = \det A^T$

9. Let $A \in \mathbb{C}^{nxn}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $\det A = \lambda_1 \cdots \lambda_n$.

Elementary Matrices

1. Row exchange: I with two rows switched
2. Scaling: I except with $I_{k,k}$ equal to a scalar λ
3. Row Replacement: Add λ in kth entry of jth row in I

Reduced Row Echelon Form

1. Reduced row echelon form:
 a) All 0 rows come last
 b) For any nonzero row, its first nonzero entry (pivot) is strictly to the right of pivot in the previous row
 c) All pivot entries are 1
 d) All entries above the pivots are 0

2. Suppose $v_1, \ldots, v_n \in \mathbb{F}^n$, and let $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$. Then the following is true:
 a) $v_1, \ldots, v_n$ are linearly independent if and only if the echelon form of A has a pivot in every column
 b) $v_1, \ldots, v_n$ spans $\mathbb{F}^n$ if and only if the echelon form of A had a pivot in every row
 c) $v_1, \ldots, v_n$ is a basis for $\mathbb{F}^n$ if and only if the echelon form of A is I.

3. A matrix is invertible if and only if its echelon form is I.

Homework Results

1. $\text{span}\,(U_1 \cap U_2) \subset \text{span}\,(U_1) \cap \text{span}\,(U_2)$

2. $\dim(U_1 + \cdots + U_m) \leq \dim(U_1) + \cdots + \dim(U_m)$

3. Product of upper triangular matrices is upper triangular

4. $M(T(v)) = M(T)M(v)$

5. $(ST)^{-1} = T^{-1}S^{-1}$

6. Determinant of upper triangular matrix is product of diagonals

7. T and T^{-1} have the same eigenvectors (and eigenvalues are reciprocals of each other)