

## Homework 3

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- You are strongly encouraged to work in groups, but solutions must be written independently.

## 1 Review

To help you better answer the questions in HW 3, we review the example of Bernoulli distributions as follows:

- The experiment of interest is flipping a fair coin;
- the sample space corresponding to this experiment is  $\Omega = \{\mathbf{heads}, \mathbf{tails}\}$ ;
- the probability  $\mathbb{P}$  is defined by  $\mathbb{P}(A) = \frac{\#A}{\#\Omega}$ , i.e.,  $\mathbb{P}(\{\mathbf{heads}\}) = \mathbb{P}(\{\mathbf{tails}\}) = \frac{1}{2}$ ;
- the random variable  $X$  is defined by

$$X(\mathbf{heads}) = 1, \quad X(\mathbf{tails}) = 0.$$

The CDF of  $X$  is

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{2}, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x \geq 1. \end{cases} \quad (1)$$

**Proof:**

1. When  $x < 0$ , we have  $A_x = \{\omega \in \Omega : X(\omega) \leq x\} = \emptyset$ ; then,  $F_X(x) = \mathbb{P}(A_x) = \mathbb{P}(\emptyset) = 0$ .
2. When  $0 \leq x < 1$ , we have  $A_x = \{\omega \in \Omega : X(\omega) \leq x\} = \{\mathbf{tails}\}$ ; then,  $F_X(x) = \mathbb{P}(A_x) = \mathbb{P}(\{\mathbf{tails}\}) = \frac{1}{2}$ .
3. When  $x \geq 1$ , we have  $A_x = \{\omega \in \Omega : X(\omega) \leq x\} = \Omega$ ; then,  $F_X(x) = \mathbb{P}(A_x) = \mathbb{P}(\Omega) = 1$ .

The proof is completed. □

In addition, the Wikipedia page on random variables is nice material for learning the concept of random variables.

## 2 Problem Set

1. Let  $(\Omega, \mathbb{P})$  be a probability space. Suppose  $B$  is an event and  $0 < \mathbb{P}(B) < 1$ . **Please prove the following:**

(a) (1 point) If  $A$  and  $B$  are independent, then  $A$  and  $B^c$  are also independent.

*Proof.* Using (b), we have that  $\mathbb{P}(B^c | A) = 1 - \mathbb{P}(B | A) = 1 - \mathbb{P}(B) = \mathbb{P}(B^c)$ . □

(b) (1 point)  $\mathbb{P}(A|B) + \mathbb{P}(A^c|B) = 1$ .

*Proof.* We will first prove that  $\mathbb{P}(A \cap B) = \mathbb{P}(B) - \mathbb{P}(A^c \cap B)$ .

$$\begin{aligned}
 \mathbb{P}(A \cap B) &= \mathbb{P}(B \cap A) && \text{(commutativity)} \\
 &= 1 - \mathbb{P}((B \cap A)^c) && \text{(def of complement)} \\
 &= 1 - \mathbb{P}(B^c \cup A^c) && \text{(De Morgan's Law)} \\
 &= 1 - \mathbb{P}(B^c \cup A^c \cap \Omega) && (E \cap \Omega = E) \\
 &= 1 - \mathbb{P}(B^c \cup A^c \cap (B \cup B^c)) && \text{(def of complement)} \\
 &= 1 - \mathbb{P}(B^c \cup (A^c \cap B) \cup (A^c \cap B^c)) && \text{(distributive law)} \\
 &= 1 - \mathbb{P}(B^c \cup (A^c \cap B^c) \cup (A^c \cap B)) && \text{(commutativity)} \\
 &= 1 - \mathbb{P}(B^c \cup (A^c \cap B)) && \text{(def of } \cup) \\
 &= 1 - \mathbb{P}(B^c) - \mathbb{P}(A^c \cap B) && \text{(additivity)} \\
 &= \mathbb{P}(B) - \mathbb{P}(A^c \cap B)
 \end{aligned}$$

Now, we can use this relation to show that  $\mathbb{P}(A|B) + \mathbb{P}(A^c|B) = 1$ .

$$\begin{aligned}
 \mathbb{P}(A | B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} && \text{(conditional probability)} \\
 &= \frac{\mathbb{P}(B) - \mathbb{P}(A^c \cap B)}{\mathbb{P}(B)} && \text{(substitute from above)} \\
 &= \frac{\mathbb{P}(B)}{\mathbb{P}(B)} - \frac{\mathbb{P}(A^c \cap B)}{\mathbb{P}(B)} && \text{(distributive prop.)} \\
 &= 1 - \mathbb{P}(A^c | B) && \text{(conditional probability)}
 \end{aligned}$$

□

2. (2 points) Let  $n$  be a positive integer, and  $\Omega \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$ . Suppose  $\mathbb{P}$  is a function of subsets of  $\Omega$  defined as follows

$$\mathbb{P}(A) \stackrel{\text{def}}{=} \frac{\#A}{\#\Omega}, \quad \text{for all } A \subset \Omega.$$

You have proved in HW 1 that  $(\Omega, \mathbb{P})$  is a probability space.

We define a random variable  $X$  as follows

$$X(\omega) = \omega, \quad \text{for all } \omega \in \Omega = \{1, 2, \dots, n\}.$$

**Please derive the CDF of the random variable  $X$  defined above. Please present your answer using a formula like the one in Eq. (1).**

*Proof.* Consider the following cases for the CDF of the random variable  $X$ :

- When  $x < 1$ , we have that  $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\emptyset) = 0$ .
- When  $1 \leq x < n$ , we have that  $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\{1, \dots, \lfloor x \rfloor\}) = \frac{\lfloor x \rfloor}{n}$ .
- When  $x \geq n$ , we have that  $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\Omega) = 1$ .

$$\text{Therefore, } F_X(x) = \begin{cases} 0 & \text{if } x < 1, \\ \frac{\lfloor x \rfloor}{n} & \text{if } 1 \leq x < n, \\ 1 & \text{if } x \geq n. \end{cases} \quad \square$$

3. (2 points) Let  $X$  be a random variable defined on the probability space  $(\Omega, \mathbb{P})$ . Suppose  $X$  satisfies the following

$x$	1	2	3	4	5
$\mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$	1/2	1/4	1/8	1/16	1/16

**Please derive the CDF of the random variable  $X$ . Please present your answer using a formula like the one in Eq. (1).**

*Proof.* Consider the following cases for  $F_X(x)$ :

- When  $x < 1$ , we have that  $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\emptyset) = 0$ .
- When  $1 \leq x < 2$ , we have that  $F_X(x) = \mathbb{P}(X \leq x) = 0 + 1/2 = 1/2$ .
- When  $2 \leq x < 3$ , we have that  $F_X(x) = \mathbb{P}(X \leq x) = 1/2 + 1/4 = 3/4$ .
- When  $3 \leq x < 4$ , we have that  $F_X(x) = \mathbb{P}(X \leq x) = 3/4 + 1/8 = 7/8$ .
- When  $4 \leq x < 5$ , we have that  $F_X(x) = \mathbb{P}(X \leq x) = 7/8 + 1/16 = 15/16$ .
- When  $x \geq 5$ , we have that  $F_X(x) = \mathbb{P}(X \leq x) = 15/16 + 1/16 = 1$ .

$$\text{Therefore, } F_X(x) = \begin{cases} 0 & \text{if } x < 1, \\ 1/2 & \text{if } 1 \leq x < 2, \\ 3/4 & \text{if } 2 \leq x < 3, \\ 7/8 & \text{if } 3 \leq x < 4, \\ 15/16 & \text{if } 4 \leq x < 5, \\ 1 & \text{if } x \geq 5. \end{cases} \quad \square$$

4. (2 points) Let  $X$  be a random variable defined on the probability space  $(\Omega, \mathbb{P})$ . Suppose  $X$  satisfies the following

$$\mathbb{P}(\{\omega \in \Omega : X(\omega) = 0\}) = 1.$$

**Please derive the CDF of the random variable  $X$ . Please present your answer using a formula like the one in Eq. (1).**

5. (2 points) Let  $X$  be a random variable defined on the probability space  $(\Omega, \mathbb{P})$ . Suppose the CDF of  $X$  is the following

$$F_X(x) = \begin{cases} 0, & \text{if } x < 1; \\ \log x, & \text{if } 1 \leq x < e; \\ 1, & \text{if } e \leq x. \end{cases}$$

**Please compute the values of the following:**

- (a)  $\mathbb{P}(\{\omega \in \Omega : X(\omega) < 2\})$ ;
- (b)  $\mathbb{P}(\{\omega \in \Omega : 0 < X(\omega) \leq 3\})$ ;
- (c)  $\mathbb{P}(\{\omega \in \Omega : 2 < X(\omega) < 2.5\})$ .

**Remark:** For simplicity, many textbooks suppress the  $\omega$  and represent  $\mathbb{P}(\{\omega \in \Omega : X(\omega) < 2\})$ ,  $\mathbb{P}(\{\omega \in \Omega : 0 < X(\omega) \leq 3\})$ , and  $\mathbb{P}(\{\omega \in \Omega : 2 < X(\omega) < 2.5\})$  as  $\mathbb{P}(X < 2)$ ,  $\mathbb{P}(0 < X \leq 3)$ , and  $\mathbb{P}(2 < X < 2.5)$ , respectively. When you read those textbooks, this remark helps you understand what they mean.