Homework 4

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1 Problem 1

Four mixing techniques, each with $n_i = 4$ observations.

$$\begin{array}{ll} \text{Technique 1:} & 3129,\ 3000,\ 2865,\ 2890 \\ \text{Technique 2:} & 3200,\ 3300,\ 2975,\ 3150 \\ \text{Technique 3:} & 2800,\ 2900,\ 2985,\ 3050 \\ \text{Technique 4:} & 2600,\ 2700,\ 2600,\ 2765 \end{array}$$

Group sample means (computed directly from the data):

$$\bar{y}_{1\cdot} = 2971.00, \quad \bar{y}_{2\cdot} = 3156.25, \quad \bar{y}_{3\cdot} = 2933.75, \quad \bar{y}_{4\cdot} = 2666.25.$$

Grand mean:

$$\bar{y}_{\cdot\cdot} = 2931.812.$$

(a) ANOVA test

Model:

$$y_{ij} = \beta_i + \varepsilon_{ij}, \qquad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2).$$

$$\begin{aligned} \text{SST} &= \sum_{i=1}^{4} \sum_{j=1}^{4} (y_{ij} - \bar{y}_{\cdot\cdot})^2 = 643649.3125, \\ \text{SSB} &= \sum_{i=1}^{4} n_i (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 = 489737.6875, \\ \text{SSE} &= \text{SST} - \text{SSB} = 153911.625. \end{aligned}$$

Degrees of freedom:

$$\text{df}_{\text{between}} = k - 1 = 3, \qquad \text{df}_{\text{within}} = N - k = 16 - 4 = 12.$$

Mean squares and F-statistic:

$$\begin{aligned} \text{MSB} &= \frac{\text{SSB}}{3} = 163245.8958, \\ \text{MSE} &= \frac{\text{SSE}}{12} = 12825.96875, \\ F &= \frac{\text{MSB}}{\text{MSE}} \approx 12.72. \end{aligned}$$

Critical value for $F_{0.05;3,12} \approx 3.49$. Since $F \approx 12.72 > 3.49$, we reject the null hypothesis that all technique means are equal.

Conclusion: There is a statistically significant effect of mixing technique on tensile strength

$$\text{(ANOVA, } p < 0.001\text{).}$$

Source	SS	df	MS	F
Between (Treatment)	489737.6875	3	163245.8958	12.72
Within (Error)	153911.625	12	12825.96875	
Total	643649.3125	15		

(b) Normal probability plot of residuals

Residuals are computed as $r_{ij} = y_{ij} - \hat{\beta}_i = y_{ij} - \bar{y}_{i\cdot}$. The list of residuals (grouped by technique) is:

$$\begin{array}{ll} \text{T1:} & +158,\ +29,\ -106,\ -81 \\ \text{T2:} & +43.75,\ +143.75,\ -181.25,\ -6.25 \\ \text{T3:} & -133.75,\ -33.75,\ +51.25,\ +116.25 \\ \text{T4:} & -66.25,\ +33.75,\ -66.25,\ +98.75 \end{array}$$

Estimated error standard deviation: $\hat{\sigma} = \sqrt{\text{MSE}} = \sqrt{12825.96875} \approx 113.27$.

Standardized residuals (each residual divided by $\hat{\sigma}$) are all within about ± 1.6, with no extreme outliers (none exceed $|2|$). Thus a normal probability plot would show approximate linearity, there is no strong evidence against normality for the residuals.

$$\boxed{\text{Residuals appear approximately normal.}}$$

(c) Residuals vs fitted values

Fitted values are the group means $\{2971.00, 3156.25, 2933.75, 2666.25\}$. Plotting residuals vs fitted values (numerically checking spread) shows no clear trend or funnel shape and variances appear comparable across groups.

$$\boxed{\text{No clear evidence of heteroscedasticity or nonlinearity in residuals vs fitted.}}$$

(d) Pairwise two-sample t-tests (unadjusted)

All groups have equal sample sizes $n_i = 4$. The estimated standard error for the difference of two group means:

$$\text{SE}(\bar{y}_i - \bar{y}_j) = \sqrt{\text{MSE}\left(\frac{1}{4} + \frac{1}{4}\right)} = \sqrt{0.5 \cdot \text{MSE}} = \sqrt{0.5 \cdot 12825.96875} \approx 80.08.$$

Degrees of freedom: $N - k = 12$.

Compute differences and t-statistics (two-sided):

Pair	Difference ($\bar{y}_i - \bar{y}_j$)	t-stat	Approx. p-value
1 vs 2	-185.25	-2.31	≈ 0.039
1 vs 3	$+37.25$	0.47	≈ 0.65
1 vs 4	$+304.75$	3.81	≈ 0.003
2 vs 3	$+222.50$	2.78	≈ 0.017
2 vs 4	$+490.00$	6.12	< 0.0001
3 vs 4	$+267.50$	3.34	≈ 0.006

Unadjusted at $\alpha = 0.05$, the significant pairs are: (1 vs 2), (1 vs 4), (2 vs 3), (2 vs 4), (3 vs 4). Only (1 vs 3) is non-significant.

(e) Bonferroni adjustment

There are $m = \binom{4}{2} = 6$ pairwise comparisons. The Bonferroni adjusted per-comparison level is $\alpha^* = 0.05/6 \approx 0.00833$ (or multiply p-values by 6).

Applying Bonferroni:

- (1 vs 4): unadjusted $p \approx 0.003 \Rightarrow$ adjusted $p \approx 0.018 \Rightarrow$ still < 0.05: **significant**.
- (2 vs 4): unadjusted $p \ll 0.001 \Rightarrow$ adjusted $p \ll 0.01$: **significant**.
- (3 vs 4): unadjusted $p \approx 0.006 \Rightarrow$ adjusted $p \approx 0.036$: **significant**.
- (1 vs 2): unadjusted $p \approx 0.039 \Rightarrow$ adjusted $p \approx 0.234$: not significant after Bonferroni.
- (2 vs 3): unadjusted $p \approx 0.017 \Rightarrow$ adjusted $p \approx 0.102$: not significant after Bonferroni.
- (1 vs 3): not significant.

$$\boxed{\text{Conclusion (Bonferroni): Technique 4 is significantly lower than Techniques 1,2,3.}}$$

(f) Why different results (unadjusted vs Bonferroni)?

Unadjusted pairwise tests control Type I error per comparison but not the family-wise error rate. Bonferroni controls the Family Wise Error Rate by making each test dividing α by the number of comparisons, reducing the chance of any false positive across the family of comparisons at the cost of reduced power. Hence Bonferroni yields fewer significant comparisons.

2 Problem 2

Data summary: $k = 6$ treatment levels, $n_i = 5$ replicates per level, $N = 30$. Given:

$$\text{SSTotal} = 900.25, \qquad \text{SSTreat} = 750.5.$$

Compute SSE:

$$\text{SSE} = \text{SSTotal} - \text{SSTreat} = 900.25 - 750.5 = 149.75.$$

(a) Estimate of error variance

Degrees of freedom for error: $N - k = 30 - 6 = 24$.

$$\widehat{\sigma}^2 = \text{MSE} = \frac{\text{SSE}}{N-k} = \frac{149.75}{24} \approx 6.2395833333.$$

$$\boxed{\widehat{\sigma}^2 \approx 6.2396.}$$

(b) Proportion of variability explained by treatment

$$R^2 = \frac{\text{SSTreat}}{\text{SSTotal}} = \frac{750.5}{900.25} \approx 0.8339.$$

$$\boxed{\text{Treatments explain approximately 83.39\% of the total variability.}}$$

(c) ANOVA F-test

$$\text{MS}_{\text{treat}} = \frac{\text{SSTreat}}{k-1} = \frac{750.5}{5} = 150.1.$$

$$\text{MSE} = \frac{\text{SSE}}{N-k} = \frac{149.75}{24} \approx 6.2396.$$

$$F = \frac{\text{MS}_{\text{treat}}}{\text{MSE}} \approx \frac{150.1}{6.2396} \approx 24.06.$$

With df $(5, 24)$, this F is extremely large; the p-value is $\ll 0.001$. Thus the treatment effect is highly significant.

$$\boxed{\text{Reject } H_0\text{; there is a strong treatment effect } (F \approx 24.06,\ p \ll 0.001).}$$

3 Problem 3

Consider a factor with k levels, level means $\beta_1, \dots, \beta_k$, and observations $y_{ij} = \beta_i + \varepsilon_{ij}$ with ε_{ij} independent, mean zero, variance σ^2.

(a)

- *Effect model*: $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$ with the constraint $\sum_{i=1}^k \tau_i = 0$. Without a constraint on the τ_i's, the parameters $\mu, \tau_1, \dots, \tau_k$ are non-identifiable because adding a constant to μ and subtracting it from each τ_i yields the same fitted values.

(b) Removing non-identifiability

Use the $\mu + \tau_i$ form but impose a constraint (e.g., $\sum_i \tau_i = 0$ or set a reference level $\tau_1 = 0$). This yields unique estimates for μ and the τ_i.

(c) Variance of a contrast estimator

A linear contrast has the form $L = \sum_{i=1}^{k} c_i \beta_i$ with constants c_i (often with $\sum c_i = 0$). The unbiased estimator is

$$\widehat{L} = \sum_{i=1}^{k} c_i \hat{\beta}_i = \sum_{i=1}^{k} c_i \bar{y}_{i\cdot}.$$

Under independence and $\mathrm{Var}(\bar{y}_{i\cdot}) = \sigma^2/n_i$,

$$\mathrm{Var}(\widehat{L}) = \sigma^2 \sum_{i=1}^{k} \frac{c_i^2}{n_i}.$$

If $n_i = n$ for all i, this simplifies to

$$\mathrm{Var}(\widehat{L}) = \frac{\sigma^2}{n} \sum_{i=1}^{k} c_i^2.$$

In practice replace σ^2 by MSE to get an estimated variance:

$$\widehat{\mathrm{Var}}(\widehat{L}) = \mathrm{MSE} \sum_{i=1}^{k} \frac{c_i^2}{n_i}.$$

(d) Hypothesis test for a contrast

To test $H_0 : \sum_{i=1}^{k} c_i \beta_i = 0$, use the statistic

$$T = \frac{\widehat{L}}{\sqrt{\mathrm{MSE} \sum_{i=1}^{k} \frac{c_i^2}{n_i}}}.$$

Under the normal-error model and H_0, $T \sim t_{N-k}$. Reject for large $|T|$. A $(1-\alpha)$ confidence interval for the contrast is

$$\widehat{L} \pm t_{1-\alpha/2,\, N-k} \sqrt{\mathrm{MSE} \sum_{i=1}^{k} \frac{c_i^2}{n_i}}.$$