Session - 6

Jensen's inequality

Ref: Section 3.1.8 Boyd's

basic inequality: if f is convex, then for $0 \leq \theta \leq 1$,

convex

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

$x_1, x_2, \ldots, x_n$, $0 \leq \theta_1, \ldots, \theta_n \leq 1$

extension: if f is convex, then $f(\theta_1 x_1 + \cdots + \theta_n x_n) \leq \theta_1 f_1(x) + \cdots + \theta_n f_n(x)$

In general

$$f(\mathbf{E}\, z) \leq \mathbf{E}\, f(z)$$

for any random variable z

basic inequality is special case with discrete distribution

$$\mathbf{prob}(z = x) = \theta, \qquad \mathbf{prob}(z = y) = 1 - \theta$$

$0 \le p_1, \cdots, p_n \le 1, \quad \sum p_i = 1.$

Write Jensen's equality.

convex

$$f(p_1 x_1 + \cdots + p_n x_n) \le p_1 f(x_1) + \cdots + p_n f(x_n)$$

$$\Rightarrow f(E(X)) \le \sum p_i f(x_i) = E(f(X))$$

$P(X = x_1) = p_1$

$P(X = x_2) = p_2$

$P(X = x_n) = p_n$

1

Z : travel time : constant

$$E(Z) = Z$$

$$Var(Z) = 0$$

X : random noise

$$E(X) = 0$$

$$V(X) = 1$$

$$\underline{E(f(X+Z))} \geq f(\underline{E(Z+X)})$$

$$= f(E(Z) + E(X))$$

$$= \underline{f(Z)}$$

Larger the variance of V, the larger $E(f(x_0+V))$

Not true,

Counter example:

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$$

Random variable

$$W = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases} \qquad V = \begin{cases} 4 & \text{w.p. } 1/10 \\ -4/9 & \text{w.p. } 9/10 \end{cases}$$

Compute: $Var(V)$, $Var(W)$ and compare.

$E(f(V))$, $E(f(W))$ and compare.

Composition with scalar functions

Ref: 3.2.4

composition of $g : \mathbf{R}^n \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$:

convex

$$\boxed{f(x) = h(g(x))}$$

convexity

$f''(x) \geq 0$

f is convex if $\begin{array}{l} g \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing} \\ g \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing} \end{array}$

g: convex
h: convex
h. non decr

- proof (for $n = 1$, differentiable g, h)

$$f''(x) = \underbrace{h''(g(x))}_{\geq 0} \overset{\geq 0}{g'(x)^2} + h'(g(x))g''(x) \geq 0$$

- note: monotonicity must hold for extended-value extension $\tilde{h}$

examples

$h(x) = \exp(x)$, convex.

- $\exp g(x)$ is convex if g is convex
- $1/g(x)$ is convex if g is concave and positive (verify this)

Minime → Optimum.

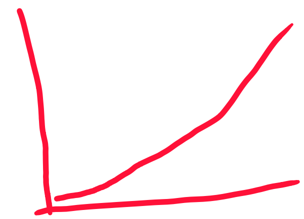
Maximum

Optimization problem in standard form

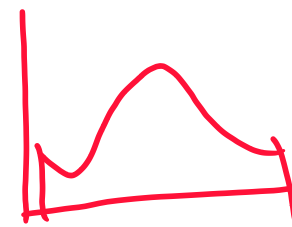
$$
\begin{array}{lll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 0, & i = 1, \dots, m \\
 & h_i(x) = 0, & i = 1, \dots, p
\end{array}
$$

$f(x) = e^{-x^2/2}$

$x \in R^+$



- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \dots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions



$x \in \mathbb{R}$

optimal value:

$$p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0, \ i = 1, \dots, m, \ h_i(x) = 0, \ i = 1, \dots, p\}$$

- $p^\star = \infty$ if problem is infeasible (no x satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

feasible solution: x which satisfies the constraints

Optimal and locally optimal points

- x is **feasible** if $x \in \textbf{dom}\, f_0$ and it satisfies the constraints
- a feasible x is **optimal** if $f_0(x) = p^\star$; $X_{\rm opt}$ is the set of optimal points
- x is **locally optimal** if there is an $R > 0$ such that x is optimal for

$$\begin{array}{ll} \text{minimize (over } z) & f_0(z) \\ \text{subject to} & f_i(z) \leq 0, \quad i = 1, \ldots, m, \quad h_i(z) = 0, \quad i = 1, \ldots, p \\ & \|z - x\|_2 \leq R \end{array}$$

Compute the min.

$f_0(x) = \frac{1}{x}$

$x \in R_{++}$

examples (with $n = 1$, $m = p = 0$)

- $f_0(x) = 1/x$, $\textbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = 0$, no optimal point
- $f_0(x) = -\log x$, $\textbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -\infty$
- $f_0(x) = x \log x$, $\textbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -1/e$, $x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x$, $p^\star = -\infty$, local optimum at $x = 1$

H.W. → check.

$f_0(x) = x \log x$ where $x \in R_+$

$$\frac{\partial}{\partial x} f_0(x) = 0$$

$$\frac{\partial^2}{\partial x^2} f_0(x) ,$$

Minimum of $f_0(x) = -\frac{1}{e}$

Minimiser $x = \frac{1}{e}$