SESSION 13

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Duality (Chapter ⇒ 5)

General Optimization problem

$$\left.\begin{array}{ll} \text{Minimise} & f_0(x) \\ \text{Subject to} & f_i(x) \le 0, \quad i = 1(1)m \\ & h_i(x) = 0, \quad i = 1(1)p \end{array}\right\} \text{Primal problem (1)}$$

Optimization variable $x \in \mathbb{R}^n$,

Domain $D = \bigcap_{i=0}^{m} \text{dom}(f_i) \bigcap_{i=0}^{p} \text{dom}(h_i)$

Lagrangian $L(x, \lambda, \nu) = f_0(x) + \sum_{i=0}^{m} \lambda_i f_i(x) + \sum_{i=0}^{p} \nu_i h_i(x)$

Dual function $g(\lambda, \nu) = \inf_x L(x, \lambda, \nu)$

Problem: L.P standard form
Linear programming

$$\begin{array}{ll} \text{minimise} & C^T x \\ \text{subject to} & Ax = b, \quad x \ge 0 \end{array}$$

$f_0(x) = c^T x$, $f_i(x) = -x_i$, $i = 1(1)m$, $h(x) = Ax - b$

$x = (x_1, x_2, \ldots, x_n)^T$

$$L(x, \lambda, \nu) = C^T x - \sum_{i=1}^{m} \lambda_i x_i + \nu^T (Ax - b)$$

$$= -b^T \nu + (c + A^T \nu - \lambda)^T x$$

Dual function $= g(\lambda, \nu) = \inf_x L(x, \lambda, \nu)$

Solved analytically, as the linear function bounded below only solved analytically, as the linear function bounded below only when it equals to zero.

Thus $g(\lambda,\nu) = \begin{cases} -b^T\nu & \text{when } (c + A^T\nu - \lambda = 0) \\ -\infty & \text{other-wise} \end{cases}$

$\therefore$ Lower bound of the optimization problem is $(-b^T\nu)$

Lagrange dual problem (Pg 223)

Each pair (λ,ν) with $\lambda \geq 0$, gives $g(\lambda,\nu)$ which is lower bound for the problem (1).

- Lower bound depends (λ,ν).

Question: what is the best lower bound of (1)

$$\left.\begin{aligned} &\text{maximize} && g(\lambda,\nu) \\ &\text{subject to} && \lambda \geq 0 \end{aligned}\right\} \quad (2)$$

Sum of affine funct^n is a concave function.

Let (λ^*,ν^*) are optimizer of ((2)) then they are called dual optimal.

Note:- Dual problem is convex optimization prob.^{n} even if primal problem is not convex.

Optimality Conditions (Section 5.5)

Dual feasible (λ,ν) of eq^n (2) $\boxed{g(\lambda,\nu) \leq p^*}$ where p^* is optimal value of (1).

$\boxed{\text{strong duality}}$ there exists arbitary good (λ, ν)

$$\text{Duality gap} = f_0(x) - g(\lambda, \nu)$$

for optimal λ^*, ν^*, $g(\lambda^*, \nu^*) = f_0(x^*)$, x^*

KKT conditions (5.5.3)

Since x^* minimizes $L(x, \lambda^*, \nu^*)$

that means

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{n} \nu_i^* \nabla h_i(x^*) = 0$$

$\therefore$ Therefore the orignal problem

$$f_i(x) \leq 0, \quad i = 1(1)m$$

$$h_i(x) = 0, \quad i = 1(1)p$$

$$\lambda_i \geq 0, \quad i = 1(1)m$$

$$\lambda_i^* f_i(x^*) = 0$$

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda^* \nabla f_i(x) + \sum_{i=1}^{n} \nu^* \nabla f_i(x) + \sum \nu_i \nabla h(x_i) = 0$$