MATHEMATICS II

(Analysis + linear Algebra) $\boxed{\text{Apostol vol I} \to \text{chp-10}}$

sequences & series (of real numbers & complex numbers)

(Q) $\mathbb{Z} \xrightarrow{f} \mathbb{R}$ $a<b \to f(a)<f(b)$ $\to \mathbb{C} \to$ countable (or) not
prove

Defn: A function $f : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called an (infinite) sequence (real sequence/complex sequence)

Denote $\{f(n)\}_{n\in\mathbb{N}} = \{a_n\}_{n\in\mathbb{N}}$ $a_n = n^{th}$ term of sequence
$a_n \in \mathbb{R}$ (or) $\mathbb{C}$

Examples: $\{n\}_{n\in\mathbb{N}}$ $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$

$a_1 = 1$
$a_2 = 1$ $a_n = a_{n-1} + a_{n-2}$ $n \geq 3$ Fibonacci series

Defn: A (real (or) complex) sequence $\{a_n\}$ has a limit L ($L \in \mathbb{R}$ (or) $\mathbb{C}$) if for every positive number $\varepsilon > 0$ there exists $N \in \mathbb{N}$ (N may depend on ε) such that $|a_n - L| < \varepsilon$ whenever $n \geq N$

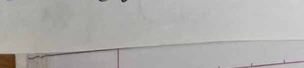
Notations: $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$

$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ $a_n \to \sum_{n=0}^{\infty} \frac{1}{n!} = e$

if $\{a_n\}$ has a limit L then say that $\{a_n\}_{n\in\mathbb{N}}$ is a convergent sequence else $\{a_n\}_{n\in\mathbb{N}}$ is divergent sequence.

$\{a_n\}_{n\in\mathbb{N}}$ is convergent if $\exists$ $L \in \mathbb{R}$ s.t $\forall$ $\varepsilon > 0$ $\exists$ $N = N(\varepsilon) \in \mathbb{N}$ with $n \geq N$ where $|a_n - L| < \varepsilon$

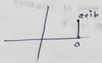
$\{a_n\}_{n\in\mathbb{N}}$ is divergent if $\forall$ $L \in \mathbb{R}$ $\exists$ $\varepsilon > 0$ such that $\forall$ $N \in \mathbb{N}$ $\exists$ $n \geq N$ with $|a_n - L| \geq \varepsilon$ } $\to$ divergent



Defn: $\lim_{n\to\infty} a_n = +\infty$ if given any $M \in \mathbb{Z}$ $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n \geq M$

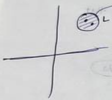
Ex: $a_1 = 1$ $a_3 = 2$ $a_{2n-1} = n$
$a_2 = -1$ $a_4 = -2$ $a_{2n} = -n$

$a_n \in \mathbb{R}$, $\lim_{n\to\infty} a_n = -\infty$ if given any $M \in \mathbb{Z}$ $\exists N \in \mathbb{N}$ s.t $\forall n \geq N$ hence $a_n \leq M$



$a_n = \{b_n + ic_n\}_{n\in\mathbb{N}}$
$Re(a_n) = b_n$
$Im(a_n) = c_n$

$L = l_1 + il_2$ if given $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t $|a_n - L| < \varepsilon$ whenever $n \geq N$



$\{z \in \mathbb{C} \mid |z - L| < \varepsilon\}$ → prove

$a_n \in \mathbb{C}$ $\lim_{n\to\infty} a_n = L$
$(\Rightarrow)$ $Re(a_n) \to Re(L)$ as $n \to \infty$
$Im(a_n) \to Im(L)$

PROOF

$a_n \in \mathbb{C}$ say that $\lim_{n\to\infty} a_n = \infty$ if $\lim_{n\to\infty} |a_n| = +\infty$

$f: \mathbb{N} \to \mathbb{R}^k$ $\{f(n)\}_{n\in\mathbb{N}} = \{a_n\}_{n\in\mathbb{N}}$

$a_n = (a_{n1}, a_{n2}, \ldots a_{nk})$, $L = (l_1, \ldots l_k)$

$\lim_{n\to\infty} a_n = L$ iff $\lim_{n\to\infty} a_{ni} = l_i$ $\underline{1 \leq i \leq k}$.

Defn: $\forall$ $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t $\|a_n - L\| < \varepsilon$ wherever $n \geq N$

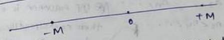
$\|v\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_k^2}$ $\left(\sum_{i=1}^{k} (a_{ni} - l_i)^2\right)^{1/2}$

Examples:
$\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$ $(\alpha > 0)$
$\lim_{n\to\infty} x^n = 0$ $(|x| < 1)$
$\lim_{n\to\infty} n^{1/n} = 1$
$\lim_{n\to\infty} \frac{(\log(n))^\alpha}{n^p} = 0$ $\forall$ $\underline{\alpha, p > 0}$.
$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$ $\forall$ $a \in \mathbb{R}$
$\lim_{n\to\infty} \sqrt[n]{p} = 1$ $(p > 0)$
$\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$ $p > 0, \alpha \in \mathbb{R}$.

proof

Defn: A real sequence $\{a_n\}$ is $\overbrace{\text{increasing (decreasing)}}^{\text{(monotonic)}}$ if $a_n \leq a_{n+1}$ $(a_n \geq a_{n+1})$ $\forall n$.

bounded if $\exists$ $M > 0$ s.t $|a_n| < M$ $\forall$ $n \in \mathbb{N}$



Ex: $(-1)^n$ is bdd but not converge.

Thm: (10.1) Monotonic seq. converges $(\Leftrightarrow)$ bdd
conv seq $\Rightarrow$ bdd seq (doesn't require monotonic)

Given a (real/complex) $\{a_n\}_{n\in\mathbb{N}}$ define $S_n = a_1 + a_2 + \ldots + a_n$ $\forall n$
S_n = n^{th} partial sum of $\{a_n\}_{n\in\mathbb{N}}$. $\{S_n\}_{n\in\mathbb{N}}$ is called an (infinite) series denoted by $\sum_{n=1}^{\infty} a_n$

if $\lim_{n\to\infty} S_n = S$ exists ($S \in \mathbb{R}$ or $\mathbb{C}$) then we say that the series $\underbrace{\sum_{n=1}^{\infty} a_n \text{ is convergent}}_{\text{converges to } S}$ & denote $\sum_{n=1}^{\infty} a_n = S$. else

PROOF

divergent series. $\left(\sum_{n=1}^{\infty} \frac{1}{2^n} = 1\right)\left(\sum_{n=1}^{\infty} \frac{1}{n} = \infty\right)\left(S_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \to \infty\right)$

$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6}$

~~$\{a_n\}$ diverges given~~

Thm 10.2) let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ & conv (real (or) complex) series $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$) Then $\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n)$ is a converging & is equal $\alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$ → proof

Thm 10.3) $\sum a_n$ converges, $\sum b_n$ diverges ⇒ ($\sum (a_n+b_n)$ diverges) → proof

Thm 10.4) $\{a_n\}$ $\{b_n\}$ 2 sequences (of reals (or) complex) → proof

s.t $a_n = b_n - b_{n+1}$ ∀ $n \in \mathbb{N}$

Then $\sum_{n=1}^{\infty} a_n$ converges ⟺ the seq $\{b_n\}_{n \in \mathbb{N}}$ converges.

& in such a case $\sum_{n=1}^{\infty} a_n = b_1 - \lim_{n\to\infty} b_n$?

Thm: $\{a_n\}$ is bdd seq (→ Real (or) complex) ∃ a subsequence of $\{a_n\}$ which converges. (B-W Thm) → proof

subsequences $a_1, a_2, \dots a_n, \dots$ If sequence is of ∞ then corresponding sub-seq is ∞ with order as given

$1 \le n_1 < n_2 < n_3 < n_4 < \dots$

$\{a_{n_1}, a_{n_2}, a_{n_3}, \dots\}$

Q) $\{a_n\}_{n \in \mathbb{N}}$ $a_n \in \mathbb{Q}$ seq of rational number & ∃ $M \in \mathbb{N}$

s.t $|a_n| \le M$ ∀ n (Not works for above Thm ?) ↳ (something extra that it converges to irrational)

irrational

($\sqrt{2} = 1.4142 \dots$

a_1	a_2	a_3	a_4	$\dots$	a_n
1.4	1.41	1.414	1.4142		$1.4142\dots n^{th}$ digit

$\lim_{n\to\infty} a_n = \sqrt{2}$

$\lim_{n\to\infty} a_n = L$ then every subseq $\{a_{n_k}\}$ of $\{a_n\}$ also converges to L.

$\{a_n\}, \{b_n\}$ are conv sequences then

$\lim_{n\to\infty} a_n = a$

$\lim_{n\to\infty} b_n = b$

convergent ← $\left\{ \begin{array}{l} \{a_n+b_n\} \\ \{a_n-b_n\} \\ \{ca_n\} \\ \{a_nb_n\} \\ \{\frac{1}{a_n}\} \end{array} \right.$ → proof, convergent, $a_n \neq 0$ & $a \neq 0$

proof of Thm 10.3) Taken $\sum (a_n+b_n)$ → converges

$\sum (a_n+b_n) + (-a_n)$ } → should converge from Thm 10.2.

that is $\sum b_n$ } → converges → $\boxed{\text{contradiction}}$

as given $\{b_n\}$ diverges

proof of Thm 10.4)

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} (S_n) = \lim_{n\to\infty} (b_1 - b_{n+1})$

$\lim_{n\to\infty} (b_1 - b_{n+1})$ exists ⇒ $\lim_{n\to\infty} (b_{n+1} - b_1)$ exists

Also $\lim_{n\to\infty} (b_1)$ exists ⇒ $\lim_{n\to\infty} (b_1 + b_{n+1} - b_1)$ exists

$\qquad\qquad = \lim_{n\to\infty} b_{n+1}$

$\sum_{n \in \mathbb{N}} a_n = \lim_{n\to\infty} (b_1 - b_{n+1}) = \lim_{n\to\infty} b_1 - \lim_{n\to\infty} b_{n+1}$

$= b_1 - \lim_{n\to\infty} b_n.$

Thm 10.5: If 'x' is a complex number with $|x|<1$ then the geometric series $\sum_{n=0}^{\infty} x^n$ converges & has sum $\frac{1}{1-x}$. if $|x| \ge 1$ this series $\sum_{n=0}^{\infty} x^n$ diverges

2

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}+\frac{1}{2}i\right)^n = \frac{1}{1-\left(-\frac{1}{2}+\frac{1}{2}i\right)}$$

PROOFS $S_n = 1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}$ $(x\neq 1)$

so, $(1-x\neq 0)$

$(1+x+x^2+\dots+x^n)(1-x) = 1+x+x^2+\dots+x^n - (x+x^2+\dots+x^{n+1})$

$= 1-x^{n+1}$

$S_n = \frac{1-x^{n+1}}{1-x}$ $(x\neq 0)$.

$\lim S_n$ exists $\Leftrightarrow \left\{\frac{x^{n+1}}{1-x}\right\}_{n\in\mathbb{N}}$ limit exists

$|x|<1$

$\{x^{n+1}\}_{n\in\mathbb{N}}$ $\{x^n\}_{n\in\mathbb{N}}$

we are looking at the sequence $\{a_n = x^{n+1}\}_{n\in\mathbb{N}}$

want to show $\lim_{n\to\infty} x^{n+1} = 0$

ETP: given $\epsilon>0$ $\exists$ $N\in\mathbb{N}$ s.t $\forall n\geq N$ we should have $|x^{n+1}-0| = |x^{n+1}| < \epsilon$

$|x|^{n+1}<\epsilon$

$\alpha\in\mathbb{C}$ $|\alpha^n| = |\alpha|^n$

$|\alpha\beta| = |\alpha||\beta|$

$0\leq\alpha=|x|<1$

$\underline{\alpha^n\to 0}$

$\alpha<1$ $\exists N$

$\alpha^n<\epsilon$ for any given ϵ

$N\log\alpha < \log\epsilon$

$\boxed{n\geq N > \frac{\log\epsilon}{\log\alpha}}$

$\alpha^n < \epsilon$ $\forall n\geq N$

$\sum_{n=0}^{\infty} x^n$ converges is equal to $\frac{1}{1-x}$ for all x s.t $|x|<1$

$|x|\geq 1 \Rightarrow \sum_{n=0}^{\infty} x^n$ diverges $S_n = 1+\dots+x^n = \frac{1-x^{n+1}}{1-x}$

$$\boxed{\sum_{n=0}^{\infty} x^n = \lim_{n\to\infty}\frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}}$$

$= \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$.

$\{x^{n+1}\}$ diverges.

$|x|\geq 1$ $|x^{n+1}| = |x|^{n+1}\to\infty$

Propn: $\sum_{n=1}^{\infty} a_n$ converges $(a_n\in\mathbb{R}$ or $\mathbb{C})$

$\Rightarrow a_n\to 0$ as $n\to\infty$.

Proof: $S_{n+1}-S_n = a_{n+1}$

$(a_1+\dots+a_{n+1}) - (a_1+a_2+\dots+a_n)$

$\lim_{n\to\infty} a_{n+1} = \lim(S_{n+1}-S_n) = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = 0$

Q) $\frac{3^n+(-2)^n}{3^{n+1}+(-2)^{n+1}} = a_n$ Does $\{a_n\}$ converges (or) diverges

$$\lim_{n\to\infty}\frac{3^n\left(1+\left(-\frac{2}{3}\right)^n\right)}{3^{n+1}\left(1+\left(-\frac{2}{3}\right)^{n+1}\right)} = \lim_{n\to\infty}\left(\frac{1}{3}\right)\left[\frac{1+\left(-\frac{2}{3}\right)^n}{1+\left(-\frac{2}{3}\right)^{n+1}}\right]$$

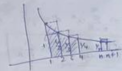
converges to $\frac{1}{3}$

$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ i.e. $\lim_{n\to\infty} \left(1+\frac{1}{2}+\dots+\frac{1}{n}\right) \to \infty$

$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} \geq \int_1^{n+1} f(x)\,dx$

$= \int_1^{n+1} \frac{1}{x}\,dx$

$= \log x \Big|_1^{n+1} = \log n+1 \to \infty$ as $n\to\infty$



Example: $a_n = \frac{1}{n^2+n} = \frac{1}{n} - \frac{1}{n+1} = b_n - b_{n+1}$

↳ by Thm 10.4

$\sum_{n=1}^{\infty} a_n = b_1 - \lim_{n\to\infty} b_n = 1$ $\quad \left[\therefore \sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1\right]$

Example: x is not a negative integer (x fixed)

$\frac{1}{(n+x)(n+x+1)(n+x+2)} = \frac{1}{2}\left[\frac{1}{(n+x)(n+x+1)} - \frac{1}{(n+x+1)(n+x+2)}\right]$ $\forall\, n \in \mathbb{N}$

↳ Telescoping series

$\sum_{n\in\mathbb{N}} \frac{1}{(n+x)(n+x+1)(n+x+2)} = \frac{1}{2(x+1)(x+2)}$

Thm: $x \in \mathbb{C}$ $\sum_{n=0}^{\infty} x^n$ (i) converge $|x|<1$

(ii) diverge $|x|>1$

Thm 10.6: if $\sum_{n\in\mathbb{N}} a_n$ converges (a_n real/complex) then

$\lim_{n\to\infty} a_n = 0$

Proof: $S_{n+1} - S_n = a_{n+1}$ $\quad \lim_{n\to\infty} S_n$ & $\lim_{n\to\infty} S_{n+1}$ exist & are equal

$\lim_{n\to\infty} a_{n+1} = \lim S_{n+1} - \lim S_n = 0$

$\lim_{n\to\infty} a_n$

$1+x+x^2+\dots+x^n+\dots = \frac{1}{1-x}$ if $|x|<1$ $x\in\mathbb{C}$

$1+x^2+x^4+\dots+x^{2n}+\dots = \frac{1}{1-x^2}$ if $|x|<1$

$x+x^3+x^5+\dots+x^{2n+1}+\dots = \frac{x}{1-x^2}$ if $|x|<1$

$1-x+x^2-x^3+\dots+(-1)^n(x)^n+\dots = \frac{1}{1+x}$ if $|x|<1$

$1-x^2+x^4-x^6+\dots+(-1)^n(x)^{2n}+\dots = \frac{1}{1+x^2}$ if $|x|<1$

$x-x^3+x^5-x^7+\dots+(-1)^n(x)^{2n+1}+\dots = \frac{x}{1+x^2}$ if $|x|<1$

$1+2x+4x^2+8x^3+\dots+2^nx^n+\dots = \frac{1}{1-2x}$ if $|x|<\frac{1}{2}$

$1+4x^2+16x^4+\dots+\dots = \frac{1}{1-4x^2}$ if $|x|<\frac{1}{2}$

power series

$\sum_{n=0}^{\infty} a_n x^n$ $a_n \in \mathbb{R}/\mathbb{C}$ real/complex series.

$\sum_{n=0}^{\infty} a_n x^n$ $a_n \in \mathbb{R}$ $\forall\, n=0,1,\dots$

$1+x+x^2+\dots+x^n+\dots = \frac{1}{1-x}$ $|x|<1$ $\quad -1<x<1$

$1+2x+3x^2+\dots+nx^{n-1}+\dots = \frac{1}{(1-x)^2}$ $|x|<1$

$1-x+x^2-x^3+x^4-\dots = \frac{1}{1+x}$ $|x|<1$

$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}\dots = \log(1+x)$ $|x|<1$ (also true for $x=1$)

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots = \log(2)$

$1-x^2+x^4-x^6+x^8-\dots = \frac{1}{(1+(x)^2)} = \frac{1}{1+x^2}$

$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots = \arctan(x)$ $|x|<1$ (also true for $x=\pm1$)

Thm 10.7] Assume $a_n \geq 0 \ \forall n \in \mathbb{N}$. Then $\sum_{n\in\mathbb{N}} a_n$ converges

$\Leftrightarrow \{a_1 + \dots + a_n\}_{n\in\mathbb{N}} = \{S_n\}_{n\in\mathbb{N}}$ is a bounded above sequence

Proof is obvious

Thm 10.8] Assume $a_n \geq 0$, $b_n \geq 0 \ \forall n \in \mathbb{N}$, $c > 0$ (real) then if $a_n \leq c b_n \ \forall n \in \mathbb{N}$ and $\sum_{n\in\mathbb{N}} b_n$ converges then $\sum_{n\in\mathbb{N}} a_n$ converges.

Proof: $t_n = b_1 + \dots + b_n$

$S_n = a_1 + \dots + a_n \qquad S_n \leq c t_n \ \forall n \in \mathbb{N}$

$c t_n = c b_1 + \dots + c b_n$

$c t_n$ is an increasing seq which converges to cb.

$t_n \to b$ i.e $\sum b_n = b$

then $c t_n \to cb$ i.e $\sum c b_n = cb$

$S_n \leq c t_n \leq cb \ \forall n \in \mathbb{N}$

S_n is increasing seq & which is bdd above

$\Rightarrow \lim S_n$ exists $= \sum a_n$

Ex: $\frac{1}{k!} \leq \frac{1}{2^{k-1}} \qquad \sum_{k=1}^{\infty} \frac{1}{k!} \leq \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 2$

$\frac{1}{1\cdot 2\cdot 3 \dots k} \leq \frac{1}{\underbrace{2 \dots 2}_{k-1}} \qquad e - 1 \qquad e \leq 3$

Ex: showed $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1$

we can show $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

$a_n = \frac{1}{n^2+n}$

$b_n = \frac{1}{n^2}$

$\frac{b_n}{a_n} = \frac{n^2+n}{n^2} \xrightarrow{n\to\infty} 1$

Thm 10.9] Assume $a_n > 0$, $b_n > 0 \ \forall n \in \mathbb{N}$ suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

then $\sum_{n\in\mathbb{N}} a_n$ conv $\Leftrightarrow \sum_{n\in\mathbb{N}} b_n$ conv

proof: $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \qquad$ choose $\epsilon = 1/2$ then $\exists N \in \mathbb{N}$

s.t $\forall n \geq N$ we have,

$\left|\frac{a_n}{b_n} - 1\right| < 1/2$. $\Leftrightarrow 1 - \frac{1}{2} < \frac{a_n}{b_n} < 1 + \frac{1}{2}$

$\Leftrightarrow -\frac{1}{2} < \frac{a_n}{b_n} - 1 < 1/2$.

$\frac{1}{2} < \frac{a_n}{b_n} < \frac{3}{2}$

$b_n < 2a_n$ and $a_n < \frac{3}{2} b_n$

$\sum a_n$ conv $\Rightarrow \sum b_n$ conv

$\sum b_n$ conv $\Rightarrow \sum a_n$ conv

Example: showed $\sum_{n\in\mathbb{N}} \frac{1}{n^2+n}$ conv $\Rightarrow \sum_{n\in\mathbb{N}} \frac{1}{n^2}$ conv

$\Rightarrow \sum_{n\in\mathbb{N}} \frac{1}{n^s}$ converges for all real $s \geq 2$

$s \geq 2$, $\frac{1}{n^s} \leq \frac{1}{n^2}$

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ conv $\forall s \geq 2$

$\boxed{\zeta = \text{Riemann zeta function}}$ [$\zeta(s)$ actually conv for all real $s > 1$]

$\zeta(1) = \infty \qquad s < 1 \Rightarrow \frac{1}{n^s} > \frac{1}{n}$

$\Rightarrow \zeta(s)$ div $s < 1$

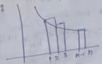
Thm 10.11] let f be a positive decreasing (continous) function $f: \mathbb{R}_{\geq 1} \longrightarrow \mathbb{R}_{>0}$

$\mathbb{R}_{\geq 1} = [1, +\infty) \qquad \mathbb{R}_{>0} = (0, \infty)$

Define $S_n = \sum_{k=1}^{n} f(k)$, $t_n = \int_1^n f(x)\,dx$

then both sequences $\{S_n\}$ & $\{t_n\}$ converge (or) both diverge.

Proof:



$$\underbrace{f(1)+\dots+f(n-1)}_{S_{n-1}} \geq \overset{\geq t_n}{\int_1^n f(x)\,dx} \geq \underbrace{f(2)+\dots+f(n)}_{S_n - f(1)}$$

$\therefore S_{n-1} \geq t_n \geq S_n - \underbrace{f(1)}_{\text{const}}$

(apply prev thm)

~~Example~~: $t_n = \int_1^n \frac{1}{x^s}\,dx \quad f(x) = \frac{1}{x^s}$

$$= \begin{cases} \frac{n^{1-s}-1}{1-s} & \text{if } s \neq 1 \\ \log(n) & \text{if } s = 1 \end{cases}$$

$s > 1 \Rightarrow \underline{1-s < 0}$

$n^{1-s} \to 0$ as $\underline{n \to \infty}$.

$\Rightarrow \{t_n\}$ converges

$\Rightarrow \{S_n\}$ converges

$\Rightarrow \sum_{n\in\mathbb{N}} \frac{1}{n^s} = \zeta(s)$ conv if $\underline{s > 1}$

if $s \leq 1$ $t_n \to \infty$ as $n \to \infty \Rightarrow \{t_n\}$ div

$\Downarrow$

$\sum_{n\in\mathbb{N}} \frac{1}{n^s} = \zeta(s) \Leftarrow \{S_n\}$ div

div for $\underline{s \leq 1}$.

3/2/25:

$\sum_{n\in\mathbb{N}} a_n$ is a series s.t $\exists$ $0<x<1$ with $0 \leq a_n \leq x^n$ $\forall$ $n \geq N$ ($\exists$ some N) $\iff 0 \leq a_n^{1/n} \leq x$ $\forall$ $n \geq N$

then $\sum_{n\in N}^{\infty} a_n$ converges $\left(\because \sum_{n=N}^{\infty} a_n \leq \sum_{n=N}^{\infty} x^n \leq \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \text{ (as } 0<x<1)\right)$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ also converges $\left(\sum_{n=1}^{\infty} a_n = (a_1+a_2+\dots+a_{N-1}) + \sum_{n=N}^{\infty} a_n\right)$

Thm 10-12: (<u>ROOT TEST</u>) let $\sum_{n\in\mathbb{N}} a_n$ be a series of non-ve reals s.t $a_n^{1/n} \to R$ as $n \to \infty$ $\left(\lim_{n\to\infty} a_n^{1/n} = R\right)$ then

a) if $R<1$ then $\sum_{n\in\mathbb{N}} a_n$ converges

b) if $R>1$ then $\sum_{n\in\mathbb{N}} a_n$ diverges

c) if $R=1$ then the series may (or) may not converge (the root test is inconclusive)

Proof:

a) Suppose $R<1$ then choose $x \in \mathbb{R}$ s.t $R<x<1$

$\lim_{n\to\infty} a_n^{1/n} = R$ given $\varepsilon = x - R$ $\exists$ $N \in \mathbb{N}$ s.t $\forall$ $n \geq N$

$|a_n^{1/n} - R| < x - R$

$a_n^{1/n} (n \geq N)$

R-ε R R+ε

$\Rightarrow 0 \leq a_n^{1/n} < x$

$\Rightarrow 0 \leq a_n < x^n$

$\Rightarrow \sum_{n=N}^{\infty} a_n \leq \sum_{n=N}^{\infty} x^n \leq \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$ $\underline{(\text{as } x<1)}$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ $\underline{\text{converges}}$

$\forall n \geq N$

1 R-ε R R+ε

ε

b) suppose $R>1 \Rightarrow$ for infinitely many n $a_n^{1/n} \geq 1 \Rightarrow a_n \geq 1$

$\Rightarrow \sum_{n\in N} a_n$ diverges $\leftarrow$

c) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\quad \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$\left[\begin{array}{l} a_n = 1/n, \quad a_n^{1/n} = \frac{1}{n^{1/n}} \\ \lim_{n\to\infty} n^{1/n} = 1 \text{ FACT to be proved.} \end{array}\right]$

$\lim_{n\to\infty} (1+a/n)^n = e^a$

$\lim_{n\to\infty} a_n = 1$

$\lim_{n\to\infty} a_n^2 \geq 1$

$b_n = \frac{1}{n^2}$. $\quad (b_n)^{1/n} = \frac{1}{(n^{1/n})^2}$

$\Rightarrow \boxed{\lim_{n\to\infty} \frac{1}{(n^{1/n})^2} = 1}$

Example: $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ $\quad a_n = \frac{1}{(\log n)^n}$ $\quad (a_n)^{1/n} = \frac{1}{\log n} \to 0$ as $n \to \infty$

converges.

Example: $\sum_{n\in\mathbb{N}} \left(\frac{n}{n+1}\right)^{n^2}$ $\quad a_n = \left(\frac{n}{n+1}\right)^{n^2}$

converges $\quad a_n^{1/n} = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} \to 1/e < 1$

Thm 10.13 (Ratio test): $\sum_{n=1}^{\infty} a_n$ is a series of +ve terms

s.t $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ then

a) $L<1 \Rightarrow \sum_{n\in\mathbb{N}} a_n$ conv

b) $L>1 \Rightarrow \sum_{n\in\mathbb{N}} a_n$ div

c) $L=1 \Rightarrow$ test is inconclusive

proof: $\sum \frac{1}{n}$ div $\quad \sum \frac{1}{n^2}$ conv

c) $a_n = 1/n$ $\quad \frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$ as $n\to\infty$

$b_n = \frac{1}{n^2}$ $\quad \frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \to 1$ as $n\to\infty$

$\frac{n}{n+1} = \frac{1}{1+1/n} \to 1$ $\quad \frac{n^2}{(n+1)^2} = \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} \to 1$ as $n\to\infty$

a) $L<1$ choose $L<x<1$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \Rightarrow \exists N\in\mathbb{N}$ s.t $\forall n \geq N \Rightarrow \frac{a_{n+1}}{a_n} < x$

$\frac{a_{n+1}}{a_n} < x \;\forall n\geq N \Rightarrow \frac{a_{n+1}}{x} < a_n \;\forall n\geq N$

$\Rightarrow \frac{a_{n+1}}{x^{n+1}} < \frac{a_n}{x^n} < \cdots < \frac{a_N}{x^N} = c$

$\forall n \geq N$

$a_n < cx^n$

$\sum_{n=N}^{\infty} a_n \leq \sum_{n=N}^{\infty} cx^n \leq \sum_{n=0}^{\infty} cx^n$

(since $x<1$) $= \frac{c}{1-x}$

$\sum_{n=1}^{\infty} a_n$ converges

b) $L>1 \Rightarrow$ $\quad \frac{a_{n+1}}{a_n} > 1 \;\forall n\geq N$

$a_{n+1} > a_n \;\forall n\geq N$ $\quad \Rightarrow \sum a_n$ div.

$> \cdots > a_N$

Ex: $\sum \frac{n!}{n^n}$ $\quad \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+1/n)^n} \to \frac{1}{e} < 1$

so converges

Thm (Leibnitz Rule): $\{a_n\}$ monotonically decreasing sequence with $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges. Further,

$0 < (-1)^n (s - s_n) < a_{n+1} \;\forall n\in\mathbb{N}$

$\left(\sum_{n=1}^{\infty} (-1)^{n-1} a_n = s,\right.$ $\quad s_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n$

proof: $s_{2n+1} = a_1 - a_2 + \cdots - a_{2n} + a_{2n+1}$

$s_{2n+1} - s_{2n-1} = a_{2n+1} - a_{2n} < 0$

$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} > 0$

$s_2 \quad s_4 \quad s_6 \quad \cdots \quad s_5 \quad s_3 \quad s_1$

$\{s_{2n-1}\}_{n\in\mathbb{N}}$ is a decreasing sequence.

$\{s_{2n}\}_{n\in\mathbb{N}}$ is an increasing sequence.

both sequences are bounded above s_1 and bounded below s_2

$$\left.\begin{aligned}\lim_{n\to\infty} s_{2n} &= s' \\ \lim_{n\to\infty} s_{2n-1} &= s''\end{aligned}\right\}$$ both these limits exist.

$$s_{2n+1} - s_{2n} = a_{2n+1}$$

$$\lim s_{2n+1} - \lim s_{2n} = \lim a_{2n+1} = \underline{0}$$

$$\lim s_{2n+1} = \lim s_{2n}$$

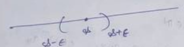
$$\underline{s' = s''}$$

$$\lim_{n\to\infty} S_{2n-1} = S = \lim_{n\to\infty} S_{2n}$$

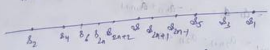
$$\Rightarrow \underline{\lim_{n\to\infty} S_n = S}$$

$\epsilon > 0$, $\quad n \geq N_1 \Rightarrow |s_{2n-1} - s| < \epsilon \qquad N = \max(N_1, N_2)$

$\qquad n \geq N_2 \Rightarrow |s_{2n} - s| < \epsilon$



$\Rightarrow \lim_{n\to\infty} S_n = s \qquad \lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = s.$



$$s_{2n} < s_{2n+2} < s < s_{2n+1} < s_{2n-1}$$

$$\left\{\begin{aligned} &\Rightarrow 0 < s - s_{2n} \leq s_{2n+1} - s_{2n} = a_{2n+1} \\ &\Rightarrow 0 < s_{2n-1} - s \leq s_{2n-1} - s_{2n} = a_{2n}\end{aligned}\right.$$

$$\Rightarrow 0 < (-1)^n (s - s_n) < a_{n+1}$$

<u>Example</u>: $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$ converges.

<u>liebniz rule</u>: $\frac{\log n}{n} \to 0$ as $n\to\infty$ & $\frac{\log(n)}{n}$ is a decreasing <u>fn</u>

FACT: $\lim_{n\to\infty} \frac{(\log n)^{\alpha}}{n^{\beta}} = 0$ for $\alpha > 0, \beta > 0$ (will proove on Friday)

$f(x) = \frac{\log x}{x}$ for $\underline{x > 0}$

$f'(x) = \frac{1 - \log x}{x^2} < 0$ if $\underline{x > e}$.

$\Rightarrow f(n+1) < f(n) \ \forall \ \underline{n \geq 3}$.

<u>Example</u>: Define a real sequence $\{a_n\}_{n\in\mathbb{N}}$ as <u>follows</u>

$a_{2n-1} = \frac{1}{n} \quad \forall n \in \mathbb{N} \qquad a_1 = 1, a_3 = \frac{1}{2}, a_5 = \frac{1}{3}, a_7 = \frac{1}{4}, \dots$

$a_{2n} = \int_n^{n+1} \frac{dx}{x} \qquad a_2 = \int_1^2 \frac{dx}{x},\ a_4 = \int_2^3 \frac{dx}{x}, \dots$

consider a series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$

$$s_{2n-1} = a_1 - a_2 + a_3 - a_4 + \dots + a_{2n-1}$$

$$= \underbrace{\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \tfrac{1}{n}\right)}_{(a_1 + a_3 + \dots + a_{2n-1})} - \underbrace{\left(\int_1^n \frac{dx}{x}\right)}_{(a_2 + a_4 + \dots + a_{2n-2})}$$

$$a_2 + a_4 + \dots + a_{2n-2} = \int_1^2 \frac{dx}{x} + \int_2^3 \frac{dx}{x} + \int_3^4 \frac{dx}{x} + \dots + \int_{n-1}^n \frac{dx}{x}$$

$$= \int_1^n \frac{dx}{x}$$

$$s_{2n-1} = 1 + \tfrac{1}{2} + \dots + \tfrac{1}{n} - \log(n)$$

$a_n \to 0$ as $n\to\infty$ and $\{a_n\}$ is a decreasing <u>sequence</u>

$$\begin{array}{ccccc} a_{2n-1} & > & a_{2n} & > & a_{2n+1} \\ \shortparallel & & \shortparallel & & \shortparallel \\ \frac{1}{n} & & \int_n^{n+1} \frac{dx}{x} & & \frac{1}{n+1}\end{array}$$

<u>leibniz rule applies</u>: $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ <u>converges</u>.

$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = c$ hence $s_{2n-1} \to c$ as $n\to\infty$

$\lim_{n\to\infty} \left(1 + \tfrac{1}{2} + \dots + \tfrac{1}{n} - \log n\right) = c$ = Euler's constant

$c \approx 0.5772156649$

$$\sum_{k=1}^{n} \frac{1}{k} - \log(n) = c + o(1) \qquad o(1) \to 0 \text{ as } n \to \infty$$

$\sum_{k=1}^{\infty} (-1)^{k-1}/k$ converges $\quad t_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1} - \frac{1}{2n}$

$$= \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \tfrac{1}{2n}\right) - 2\left[\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \dots + \tfrac{1}{2n}\right]$$

$$t_{2n} = \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \tfrac{1}{2n}\right) - \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \tfrac{1}{n}\right)$$

$$= \log 2n + c + o(1) - (\log(n) + c + o(1))$$

$$= \log 2 + o(1) \qquad o(1) \to 0 \text{ as } n \to \infty$$

$\lim_{n\to\infty} t_{2n} = \log(2)$

$\sum_{k=1}^{\infty} (-1)^{k-1}/k = \log(2)$

$\{a_n\}_{n\in\mathbb{N}}$ sequence of real numbers

10.9 (383)

extended real numbers $= \mathbb{R} \cup \{\pm\infty\} = \mathbb{R} \cup \{+\infty, -\infty\}$

14, 15, 18, 19, 20, 21, 22, 28, 34, 35

$E = \{x$ is an extended real number such that $\exists$ a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ of $\{a_n\}_{n\in\mathbb{N}}$ s.t. $\lim_{k\to\infty} a_{n_k} = x\}$

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Example: $\{a_n = (-1)^n\}_{n\in\mathbb{N}}$ $\quad E = \{\pm 1\}$, $-1 < a < 1$

Example: suppose $\{a_n\}$ is a conv. sequence $\Rightarrow E = \{L\}$

$\lim_{n\to\infty} a_n = L$ conv

Example: $\{a_n = (-1)^n n\}_{n\in\mathbb{N}}$ $\quad E = \{\pm\infty\}$

Example: let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence consisting of all the rationals. $\quad E =$ extended real no's

We can talk about/define the notion of supremum/infimum of any subset of $\mathbb{R}$ (the extended real number system)

Defn: $\lim_{n\to\infty} \sup a_n = \sup(E)$ $\quad E =$ all subseq. limits of $\{a_n\}$

$\lim_{n\to\infty} \inf a_n = \inf(E)$

Soln's — 171
pdf — 402

Suppose $\lim_{n\to\infty} a_n = L \iff \lim_{n\to\infty} \sup a_n = L = \lim_{n\to\infty} \inf a_n$

Thm ($\lim \sup$): $\lim_{n\to\infty} \sup a_n \in E$ $\qquad \sup_{n\in\mathbb{N}} (-1/n) = 0$

if $x > \lim_{n\to\infty} \sup a_n$ then $\exists N \in \mathbb{N}$ s.t $\forall n \geq N \Rightarrow a_n < x$.

Finally, $\lim_{n\to\infty} \sup a_n$ is the unique numbers with these 2 properties. $\exists$ a simillar defn of lim inf.

Root test: Given a series of real numbers $\sum_{n=1}^{\infty} a_n$

(Rudin Principles of Math-Analysis)

let $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$ then

if $\alpha < 1 \Rightarrow \sum a_n$ conv

$\alpha > 1 \Rightarrow \sum a_n$ div

$\alpha = 1 \Rightarrow$ inconclusive

Cauchy sequence: $\{a_n\}_{n\in\mathbb{N}}$ seq of real (or) complex no's is cauchy if given any $\epsilon > 0$ $\exists N \in \mathbb{N}$ s.t $\forall n, m \geq N$ have $|a_n - a_m| < \epsilon$.

conv $\Rightarrow$ cauchy & for real/complex cauchy $\Rightarrow$ conv

$\sum_{n=1}^{\infty} a_n$ series of real/complex converges $\iff \lim_{n\to\infty} S_n$ exist.

$S_n = n^{th}$ partial sum.

$\iff \{S_n\}$ is a conv. sequence. $\iff \{S_n\}$ is called a cauchy sequence

$\iff$ given any $\epsilon > 0$ $\exists N \in \mathbb{N}$ s.t $\forall m, n \geq N$ $(m \geq n \geq N)$

$|S_m - S_n| < \epsilon \qquad |S_m - S_n| = \left|\sum_{k=n+1}^{m} a_k\right| < \epsilon$

$S_m = a_1 + \dots + a_m \qquad S_n = a_1 + \dots + a_n$

$\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow$ $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ s.t $\forall m \geq n \geq N$ we have $\left|\sum_{k=n+1}^{m} a_k\right| < \epsilon$

Thm: Assume $\sum_{n=1}^{\infty} |a_n|$ converges, a_n are complex (or) real. then $\boxed{\sum_{n=1}^{\infty} a_n \text{ converges}}$ and $\boxed{\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|}$

Proof:

(1) $\sum_{n=1}^{\infty} |a_n|$ conv $\Leftrightarrow$ given any $\epsilon > 0$ $\exists N \in \mathbb{N}$ s.t $\forall m \geq n \geq N$ we have

$|a_{n+1} + \dots + a_m| \leq |a_{n+1}| + \dots + |a_m| < \epsilon$ (Δle ineq)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ conv

(2) case (i): Assume $a_n \in \mathbb{R}$ $\forall n$

define $b_n = a_n + |a_n| = \begin{cases} 2|a_n| & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases}$

$\Rightarrow 0 \leq b_n \leq 2|a_n| \Rightarrow \sum b_n$ conv since $\sum |a_n|$ conv

$a_n = b_n - |a_n|$

$\sum a_n$ conv & $\sum a_n = \sum b_n - \sum |a_n|$

case (ii): general case $a_n = u_n + iv_n$

$|v_n|, |u_n| \leq |a_n|$ $\quad \sum |a_n|$ conv $\Rightarrow \sum |u_n|$ conv, $\sum |v_n|$ conv

$\Rightarrow \sum u_n$ conv, $\sum v_n$ conv

$\Rightarrow \sum (u_n + iv_n) = \sum a_n$ conv

Thm: $\sum_{n=1}^{\infty} a_n$ (series of complex (or) real no's) is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ conv (Note: abs conv $\Rightarrow$ conv)

Note: conv $\not\Rightarrow$ abs conv $\quad \sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ conv

conv but not abs conv = conditionally convergence.

simple fact: $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are abs conv then so $\sum_{n=1}^{\infty} \alpha a_n + \beta b_n$ $(\alpha, \beta \in \mathbb{C})$

Q: conv, abs conv

Thm: $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$ Proof:

$\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|$ Δle inequality

$\Rightarrow \left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{\infty} |a_k|$ Now let $n \to \infty$ (n is there on LHS) ??

$\left|\sum_{k=1}^{\infty} a_k\right| \leq \sum_{k=1}^{\infty} |a_k| = c$

$b_n = \sum_{k=1}^{n} a_k$

$b_n \to b = \sum_{k=1}^{\infty} a_k$

$|b_n| \leq c = \sum_{k=1}^{\infty} |a_k|$

$-c \leq b_n \leq c \quad \forall n \in \mathbb{N}$

$\Rightarrow -c \leq \lim b_n = b \leq c$

$|b| \leq c$.

$\sum a_n$ Rearrangement of $\sum a_n$

$f: \mathbb{N} \to \mathbb{N}$ bijection $b_n = a_{f(n)}$

If $\sum a_n$ conv does $\sum b_n$ conv?

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \dots \; \} \sum a_n$

$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \dots \; \} \sum b_n$

$\sum a_n$ conv, $\sum_{n\in\mathbb{N}} a_n < 1-\frac{1}{2}+\frac{1}{3} = 5/6$

$\left(\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k}\right) > 0$

$\frac{1}{4k-3} > \frac{1}{4k}$, $\frac{1}{4k-1} > \frac{1}{4k}$

Thm: $\sum_{n\in\mathbb{N}} a_n$ converges abs $\Rightarrow$ any rearrangement of $\sum a_n$ conv $\rightarrow \sum b_n$

$\sum b_n = \sum a_n$.

Thm: Let $\sum_{n\in\mathbb{N}} a_n$ be a series of real numbers which converges conditionally.

Let $-\infty \le \alpha \le \beta \le +\infty$

Then $\exists$ a rearrangement of $\sum a_n$ (say $\sum b_n$) such that

$\limsup_{n\to\infty} b_n = \beta$ and $\liminf_{n\to\infty} b_n = \alpha$

Various limits: ① $\boxed{\lim_{n\to\infty} \frac{1}{n^p} = 0 \text{ for } p>0}$

Proof: Given $\epsilon>0$ to find $N\in\mathbb{N}$ s.t $\frac{1}{n^p} < \epsilon$ $\forall\, n \ge N$

$\frac{1}{n^p} < \epsilon \Rightarrow \frac{1}{\sqrt[p]{\epsilon}} < n$

choose N s.t $N > \frac{1}{\sqrt[p]{\epsilon}}$

Given $\epsilon > 0$ choose $N\in\mathbb{N}$ s.t $N > \frac{1}{\sqrt[p]{\epsilon}}$

then if $n \ge N$ have $n \ge N > \frac{1}{\sqrt[p]{\epsilon}} \Leftrightarrow \frac{1}{n^p} < \epsilon$

② $\boxed{\lim_{n\to\infty} \sqrt[n]{p} = 1 \text{ (for } p>0)}$

Proof: Given $\epsilon>0$ to find $N\in\mathbb{N}$ s.t $\forall\, n\ge N$, $|\sqrt[n]{p}-1| < \epsilon$

case (i): Assume $p>1 \Rightarrow \sqrt[n]{p} > 1$ $\forall\, n\in\mathbb{N}$, define $x_n = \sqrt[n]{p}-1 > 0$

$\Rightarrow 1+x_n = \sqrt[n]{p} \Rightarrow (1+x_n)^n = p$

Now, $1+nx_n \le (1+x_n)^n = p$

$\Rightarrow 0 < x_n \le \frac{p-1}{n} \Rightarrow \lim_{n\to\infty} x_n = 0 \quad \because 0 \le \lim_{n\to\infty} x_n \le \lim_{n\to\infty} \frac{p-1}{n} = 0$

$\Rightarrow 0 = \lim_{n\to\infty} x_n = \lim_{n\to\infty}(\sqrt[n]{p}-1) = \lim_{n\to\infty} \sqrt[n]{p} - 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{p} = 1.$

case (ii): Assume $0<p<1$ (if $p=1$ then OK), $a = 1/p > 1$.

$\sqrt[n]{p} = \frac{1}{\sqrt[n]{a}}$ $\quad \lim_{n\to\infty} \sqrt[n]{a} = 1$ (by above case, since $a>1$)

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{p} = 1$

③ $\lim_{n\to\infty} n^{1/n} = 1 = \lim_{n\to\infty} \sqrt[n]{n}$

Proof: $x_n = (n)^{1/n} - 1 > 0$ $((n)^{1/n} > 1)$ $\forall\, n \ge 2$

$\Rightarrow (n)^{1/n} = 1+x_n \Rightarrow n = (1+x_n)^n > 1 + nx_n + \binom{n}{2}x_n^2 > \binom{n}{2}x_n^2$

$\Rightarrow \frac{n(n-1)}{2}x_n^2 < n \Rightarrow \frac{n-1}{2}x_n^2 < 1$

$\Rightarrow 0 < x_n < \sqrt{\frac{2}{n-1}}$

$\Rightarrow \lim_{n\to\infty} x_n = 0$ $\left(\text{since } \lim_{n\to\infty}\sqrt{\frac{2}{n-1}} = 0\right)$

$\Rightarrow \lim_{n\to\infty} (n)^{1/n} = 1$

④ $p>0$, $\alpha\in\mathbb{R}$ then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

Proof: choose k an integer s.t, $(k>\alpha,\ k>0)$ choose $n>2k$

$(1+p)^n > \binom{n}{k}p^k = \frac{(n)(n-1)(n-2)\cdots(n-k+1)}{k!}p^k > \left(\frac{n}{2}\right)^k \frac{p^k}{k!} = \frac{n^k p^k}{2^k k!}$

$\Rightarrow (1+p)^n > \frac{n^k}{2^k}\cdot\frac{p^k}{k!}$

$\Rightarrow \frac{n^\alpha n^{k-\alpha}}{(1+p)^n} < \frac{2^k k!}{p^k} \Rightarrow 0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k n^{k-\alpha}} \to 0$ as $n\to\infty$

$\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

⑤ $|x|<1$ then $\lim_{n\to\infty} x^n = 0$

Proof: choose $\alpha=0$ in ④

$|x|<1 \Rightarrow \frac{1}{|x|}>1$

$1+p=\frac{1}{|x|}$ (or) $|x|=\frac{1}{1+p}$

$$0=\lim_{n\to\infty}\frac{n^\alpha}{(1+p)^n}=\lim_{n\to\infty}\frac{1}{(1+p)^n} \quad (p>0)$$

$$=\lim_{n\to\infty}|x|^n \Rightarrow x^n=0 \text{ as } n\to\infty$$

⑥ $e=\sum_{n=0}^{\infty}\frac{1}{n!}=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$

Ratio test: $\frac{a_{n+1}}{a_n}=\frac{n!}{(n+1)!}=\frac{1}{n+1}\to 0$ as $n\to\infty$

$\Rightarrow e$ converges.

so, $e=\sum_{n=0}^{\infty}\frac{1}{n!}$ makes sense.

$\sqrt{2}$ is irrational but algebraic no

rational $\Rightarrow$ algebraic

$x=p/q \Rightarrow qx-p=0$

non-algebraic no = transcendental no.

hence transcendental no $\Rightarrow$ irrational

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⑦ $\lim_{n\to\infty}(1+1/n)^n=e=\sum_{n=0}^{\infty}\frac{1}{n!}$

(a_n) is a real sequence

$\limsup_{n\to\infty} a_n$ = sup of all limits of conv subsequences of $\{a_n\}$

$\liminf_{n\to\infty} a_n$ = inf of all limits of conv subseq of $\{a_n\}$

$\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$

Proof:

$S_n=\sum_{k=0}^{n}\frac{1}{k!} \qquad t_n=(1+1/n)^n$

$t_n=(1+1/n)^n$

$=1+1+\binom{n}{2}\left(\frac{1}{n}\right)^2+\cdots+\binom{n}{n}\left(\frac{1}{n}\right)^n$

$=1+1+\frac{1}{2!}\underbrace{(1-1/n)}_{<1}+\frac{1}{3!}\underbrace{(1-1/n)(1-2/n)}_{<1}+\cdots+\frac{1}{n!}\underbrace{(1-1/n)\cdots\left(1-\frac{n-1}{n}\right)}_{<1}$

$<S_n \Rightarrow t_n<S_n \Rightarrow \limsup_{n\to\infty} t_n \le \limsup_{n\to\infty} S_n=\lim_{n\to\infty} S_n=e$

If $n\ge m$ then

$t_n=1+1+\frac{1}{2!}(1-1/n)+\cdots+\frac{1}{n!}(1-1/n)\cdots\left(1-\frac{n-1}{n}\right)\ge 1+1+\frac{1}{2!}(1-1/n)+\cdots+\frac{1}{m!}(1-1/n)\cdots\left(1-\frac{m-1}{n}\right)$

keep 'm' fixed, let $n\to\infty$

$\Rightarrow \liminf_{n\to\infty} t_n \ge 1+1+\frac{1}{2!}+\cdots+\frac{1}{m!}=S_m$

let $m\to\infty \Rightarrow \liminf_{n\to\infty} t_n \ge e \ge \limsup_{n\to\infty} t_n$

$\Rightarrow \liminf_{n\to\infty} t_n=\limsup_{n\to\infty} t_n=\lim_{n\to\infty} t_n=e$

so, $e=\lim_{n\to\infty}(1+1/n)^n$

$S_n=1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}$

$e=\sum_{k=0}^{\infty}\frac{1}{k!}$

$e-S_n=\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots$

$=\frac{1}{(n+1)!}\left[1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\cdots\right]$

$<\frac{1}{(n+1)!}\left[1+\frac{1}{n+1}+\frac{1}{(n+1)^2}+\cdots\right]$

$=\frac{1}{(n+1)!}\cdot\frac{1}{\frac{n}{(n+1)}}=\frac{1}{n!\,n}$

$e-S_n<\frac{1}{n!\,n}$

$n=10,\ S_{10}=1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{10!}$

$e-S_{10}<\frac{1}{10!\,10}<\frac{1}{10^7}$ check

Thm: 'e' is irrational

Proof: Suppose 'e' is rational, let $e = p/q$

$p, q \in \mathbb{Z} > 0$.

$0 < (e - s_q) < \frac{1}{q!\,q} \Rightarrow 0 < \underbrace{q!(e - s_q)}_{\substack{q!e - q!s_q \\ \text{integer}}} < 1/q$

$q!\,s_q = q!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!}\right)$ is an integer

$q!e$ is also an integer.

$q!e = q!\frac{p}{q} = (q-1)!\,p$ is integer.

what we got is $q!e - q!s_q$ is integer

↳ contradiction

↳ as there is no integer b/w 0 & $1/q$

Qn: Does $\exists$ a bijection from $\mathbb{N}$ to $\mathbb{Q}$ which is order preserving? i.e $f: \mathbb{N} \to \mathbb{Q}$ 1-1 onto & $m < n \Rightarrow f(m) < f(n)$

Sol: $a < c < b$, $c = \frac{a+b}{2}$, $\mathbb{Q}$ are dense than $\mathbb{N}$ so, doesn't exist a Bijection.

Apostol, Rudhin, Spivak (calculus), Mattuck (Analysis).

$\boxed{\text{lim inf \& lim sup are not in the course}}$ 4.

$E \subseteq \mathbb{R}$ (or $\mathbb{C}$) & consider a sequence of function $\{f_n\}_{n \in \mathbb{N}}$

$\forall n \in \mathbb{N}$, $f_n : E \to \mathbb{R}$ (or $\mathbb{C}$) E = domain of f_n $\forall n$.

$x \in E \Rightarrow$ sequence of real/complex no $\{f_n(x)\}_{n \in \mathbb{N}}$

Q: Does $\{f_n(x)\}$ converge.

let $S = \{x \in E \,/\, \lim_{n \to \infty} f_n(x) \text{ exists}\} \subseteq E$

Define $f : S \to \mathbb{R}$ (or $\mathbb{C}$) by $f(x) = \lim_{n \to \infty} f_n(x)$

'f' is called the pointwise limit function of the sequence $\{f_n\}$ on S.

f_n converges to f pointwise

Example: $f_n : [0,1] \to [0,1]$ $f_n(x) = x^n$

(1) $\lim_{n \to \infty} x^n = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

so, the pointwise limit f_n $f : [0,1] \to [0,1]$

as $f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$

f_n are cont $\forall n$ but f is not cont

f is discont at $x = 1$.

(2) $f_n(x) = nx(1-x^2)^n$ $\forall x \in [0,1]$, $f_n : [0,1] \to \mathbb{R}$

$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} nx(1-x^2)^n = 0$ $\forall x \in [0,1]$

so, the ptwise limit f of $\{f_n\}$ exists on $[0,1]$ &

$f \equiv 0$ on $[0,1]$

$\int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x^2)^n\,dx = n\int_0^1 (x)(1-x^2)^n\,dx$

$= \frac{-n}{2(n+1)} (1-x^2)^{n+1}\Big|_0^1$

$= \frac{n}{2(n+1)}$

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}$$

$$\neq$$

but $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 f(x)\,dx = 0$

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ptwise conv:

$\underset{n\in\mathbb{N}}{f_n} : E \to \mathbb{R}$ (or $\mathbb{C}$) $\quad f : E \to \mathbb{R}$ (or $\mathbb{C}$)

suppose $f_n \to f$ ptwise on E

let $x \in E$ then $\lim_{n\to\infty} f_n(x) = f(x)$ means for every $\epsilon > 0$

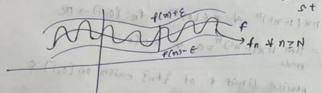
$\exists \; N \in \mathbb{N}$ s.t $\forall \; n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$

$N = N(\epsilon, x)$

Defn: let $f_n : E \to \mathbb{R}$ (or $\mathbb{C}$) be a sequence of functions such that f_n converges to f pointwise i.e $\lim_{n\to\infty} f_n(x) = f(x)$ exist $\forall \; x \in E$ f_n is said to f uniformly on E if given $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ (N depends only on ϵ but its indep of $x \in E$)

s.t $\forall \; n \geq N$ $\quad |f_n(x) - f(x)| < \epsilon \quad \forall \; x \in E$

Notation: $f_n \to f$ uniformly on E $\quad$ $\forall \epsilon > 0 \; \exists \; N \in \mathbb{N}$ s.t $\forall \; n \geq N$ $\forall \; x \in E$



We have, $\quad f(x) - \epsilon < f_n(x) < f(x) + \epsilon$

$\iff |f_n(x) - f(x)| < \epsilon$

Thm: Suppose $f_n \to f$ uniformly on E, suppose f_n is cont $\forall \; n \in \mathbb{N}$

$\Rightarrow f$ is cont on E.

Proof: "Unif limit of cont fns is cont"

ETP: $\forall \; p \in E$ $\quad f$ is cont at p

Given $\epsilon > 0$ we have to find a $\delta > 0$ s.t $\underset{x\in E}{|x-p| < \delta} \Rightarrow \boxed{|f(x) - f(p)| < \epsilon}$

Given this $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t $\forall \; n \geq N$ $\forall \; y \in E$ have

$\boxed{|f_N(y) - f(y)| < \epsilon/3}$

f_N is cont hence $\exists \; \delta > 0$ s.t $\underset{x\in E}{|x-p| < \delta} \Rightarrow |f_N(x) - f_N(p)| < \frac{\epsilon}{3}$

$|f(x) - f(p)| \leq \underbrace{|f(x) - f_N(x)|}_{< \epsilon/3} + \underbrace{|f_N(x) - f_N(p)|}_{< \epsilon/3} + \underbrace{|f_N(p) - f(p)|}_{< \epsilon/3}$ $\quad \epsilon$

$\Rightarrow |f(x) - f(p)| \leq \epsilon$

$\Rightarrow$ f is cont at p $\forall \; p \in E$ $\Rightarrow$ f is cont on E.