Problem Set 2

1. Let $X_1, \ldots, X_n$ be an i.i.d. sample from the $N(\mu, \sigma^2)$ distribution. Find the Fisher information matrix, $\mathbf{I}_n(\boldsymbol{\theta})$, for the parameter $\boldsymbol{\theta} = (\mu, \sigma^2)$.

[Note: The Fisher information matrix for a vector valued parameter $\boldsymbol{\theta}$ is defined as

$$\mathbf{I}_n(\boldsymbol{\theta}) = E\left[\left(\frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\mathbf{X}}(\mathbf{X}; \boldsymbol{\theta})\right)\left(\frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\mathbf{X}}(\mathbf{X}; \boldsymbol{\theta})\right)^\top\right] = -E\left[\left(\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \log f_{\mathbf{X}}(\mathbf{X}; \boldsymbol{\theta})\right)\right]$$

Further, from the above definition of $\mathbf{I}_n(\boldsymbol{\theta})$ show that, when $X_1, \ldots, X_n$ are i.i.d., then $\mathbf{I}_n(\boldsymbol{\theta}) = n\mathbf{I}_1(\boldsymbol{\theta})$, where $\mathbf{I}_1(\boldsymbol{\theta})$ is the Fisher information matrix for one sample.

Solution:

The PDF of a normal distribution $N(\mu, \sigma^2)$ is given by:

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The log-likelihood function for a single observation is:

$$\log f(x; \mu, \sigma^2) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2}$$

The log-likelihood function for n i.i.d. observations is:

$$\log L(\mu, \sigma^2) = \sum_{i=1}^n \log f(x_i; \mu, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2$$

Now, we compute the first partial derivatives with respect to μ and σ^2:

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^n (x_i - \mu)$$

$$\frac{\partial \log L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^n (x_i - \mu)^2$$

Next, we compute the second partial derivatives:

$$\frac{\partial^2 \log L}{\partial \mu^2} = -\frac{n}{\sigma^2}$$
$$\frac{\partial^2 \log L}{\partial (\sigma^2)^2} = \frac{n}{2(\sigma^2)^2} - \frac{1}{(\sigma^2)^3} \sum_{i=1}^{n} (x_i - \mu)^2$$
$$\frac{\partial^2 \log L}{\partial \mu \partial \sigma^2} = \frac{\partial^2 \log L}{\partial \sigma^2 \partial \mu} = -\frac{1}{(\sigma^2)^2} \sum_{i=1}^{n} (x_i - \mu)$$

Now, we take the expected values of the negative second partial derivatives:

$$-E\left[\frac{\partial^2 \log L}{\partial \mu^2}\right] = \frac{n}{\sigma^2}$$
$$-E\left[\frac{\partial^2 \log L}{\partial (\sigma^2)^2}\right] = -E\left[\frac{n}{2(\sigma^2)^2} - \frac{1}{(\sigma^2)^3} \sum_{i=1}^{n} (x_i - \mu)^2\right] = -\frac{n}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^3} E\left[\sum_{i=1}^{n} (x_i - \mu)^2\right]$$

Since $E\left[\sum_{i=1}^{n} (x_i - \mu)^2\right] = n\sigma^2$:

$$-E\left[\frac{\partial^2 \log L}{\partial (\sigma^2)^2}\right] = -\frac{n}{2(\sigma^2)^2} + \frac{n\sigma^2}{(\sigma^2)^3} = \frac{n}{2(\sigma^2)^2}$$
$$-E\left[\frac{\partial^2 \log L}{\partial \mu \partial \sigma^2}\right] = -E\left[-\frac{1}{(\sigma^2)^2} \sum_{i=1}^{n} (x_i - \mu)\right] = \frac{1}{(\sigma^2)^2} E\left[\sum_{i=1}^{n} (x_i - \mu)\right] = 0$$

Thus, the Fisher information matrix is:

$$\mathbf{I}_n(\boldsymbol{\theta}) = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2(\sigma^2)^2} \end{bmatrix}$$

For one sample ($n = 1$), the Fisher information matrix is:

$$\mathbf{I}_1(\boldsymbol{\theta}) = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2(\sigma^2)^2} \end{bmatrix}$$

From the above, it is clear that:

$$\mathbf{I}_n(\boldsymbol{\theta}) = n\mathbf{I}_1(\boldsymbol{\theta})$$

(a) Gamma(α, β):

The first two moments of a Gamma distribution are:

$$E[X] = \alpha\beta$$
$$E[X^2] = \alpha\beta^2 + (\alpha\beta)^2$$

The sample moments are:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$
$$\frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Equating the population moments to the sample moments:

$$\alpha\beta = \bar{X}$$
$$\alpha\beta^2 + (\alpha\beta)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Substituting the first equation into the second:

$$\alpha\beta^2 + (\bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$
$$\alpha\beta^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X})^2 = S^2$$

where S^2 is the sample variance.

Dividing $\alpha\beta^2$ by $\alpha\beta$

$$\frac{\alpha\beta^2}{\alpha\beta} = \frac{S^2}{\bar{X}}$$
$$\beta = \frac{S^2}{\bar{X}}$$

Substituting β into first equation, Dividing $\alpha\beta^2$ by $\alpha\beta$

$$\frac{\alpha\beta^2}{\alpha\beta} = \frac{S^2}{\bar{X}}$$

$$\beta = \frac{S^2}{\bar{X}}$$

Substituting β into first equation,

$$\alpha = \frac{\bar{X}}{\beta} = \frac{\bar{X}}{\frac{S^2}{\bar{X}}} = \frac{\bar{X}^2}{S^2}$$

Therefore, the method of moments estimators for α and β are:

$$\hat{\alpha} = \frac{\bar{X}^2}{S^2}$$

$$\hat{\beta} = \frac{S^2}{\bar{X}}$$

Thus, the MOME for $g(\boldsymbol{\theta}) = (\alpha, \beta)^\top$ is $\left(\frac{\bar{X}^2}{S^2}, \frac{S^2}{\bar{X}}\right)^\top$

(b) Beta(α, β):

The first two moments of a Beta distribution are:

$$E[X] = \frac{\alpha}{\alpha+\beta}$$
$$E[X^2] = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

Equating the population moments to the sample moments:

$$\frac{\alpha}{\alpha+\beta} = \bar{X}$$
$$\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

From the first equation:

$$\alpha = \bar{X}(\alpha+\beta)$$
$$\alpha(1-\bar{X}) = \bar{X}\beta$$
$$\beta = \frac{\alpha(1-\bar{X})}{\bar{X}}$$

Now, let $m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. The sample variance is $S^2 = m_2 - \bar{X}^2$. We also know that

$$\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} = E[X^2] = S^2 + \bar{X}^2$$

Substituting β:

$$\frac{\alpha(\alpha+1)}{(\alpha+\frac{\alpha(1-\bar{X})}{\bar{X}})(\alpha+\frac{\alpha(1-\bar{X})}{\bar{X}}+1)} = \frac{\alpha(\alpha+1)}{(\frac{\alpha}{\bar{X}})(\frac{\alpha}{\bar{X}}+1)} = m_2$$
$$\frac{\alpha+1}{\frac{\alpha}{\bar{X}}(\frac{\alpha}{\bar{X}}+1)}\frac{\bar{X}^2}{\alpha} = m_2$$

The solution is:

$$\hat{\alpha} = \bar{X}\left[\frac{\bar{X}(1-\bar{X})}{S^2} - 1\right]$$
$$\hat{\beta} = (1-\bar{X})\left[\frac{\bar{X}(1-\bar{X})}{S^2} - 1\right]$$

Then, the MOME for $g(\theta) = \alpha/\beta$ is:

$$\frac{\hat{\alpha}}{\hat{\beta}} = \frac{\bar{X}}{1-\bar{X}}$$

© **Poisson(λ):**

The first moment of a Poisson distribution is:

$$E[X] = \lambda$$

Equating the population moment to the sample moment:

$$\lambda = \bar{X}$$

Therefore, the method of moments estimator for λ is:

$$\hat{\lambda} = \bar{X}$$

The MOME for $g(\theta) = \exp\{-\lambda\}$ is:

$$e^{-\bar{X}}$$

(d) Location-scale Exponential(μ, σ):

The first two moments of the given Exponential distribution are:

$$E[X] = \mu + \sigma$$
$$E[X^2] = \mu^2 + 2\mu\sigma + 2\sigma^2$$

Equating the population moments to the sample moments:

$$\mu + \sigma = \bar{X}$$
$$\mu^2 + 2\mu\sigma + 2\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

From the first equation:

$$\sigma = \bar{X} - \mu$$

Now, let $m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. We also know that

$$\sigma^2 = E[X^2] - E[X]^2 = m_2 - \bar{X}^2$$

Substituting σ:

$$(\bar{X} - \mu)^2 = m_2 - \bar{X}^2$$
$$\bar{X} - \mu = \sqrt{m_2 - \bar{X}^2}$$

Substituting μ:

$$\mu = \bar{X} - \sqrt{m_2 - \bar{X}^2}$$

The MOME:

$$\hat{\mu} = \bar{X} - \sqrt{m_2 - \bar{X}^2}$$
$$\hat{\sigma} = \sqrt{m_2 - \bar{X}^2}$$

The MOME for $g(\theta) = (\mu, \sigma)$ is:

$$(\bar{X} - \sqrt{m_2 - \bar{X}^2}, \sqrt{m_2 - \bar{X}^2})$$

(a) Binomial(m, θ):

The likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} \binom{m}{x_i} \theta^{x_i}(1-\theta)^{m-x_i} = \left(\prod_{i=1}^{n} \binom{m}{x_i}\right) \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{\sum_{i=1}^{n}(m-x_i)}$$

The log-likelihood function is:

$$\log L(\theta) = \sum_{i=1}^{n} \log \binom{m}{x_i} + \left(\sum_{i=1}^{n} x_i\right) \log\theta + \left(\sum_{i=1}^{n}(m-x_i)\right) \log(1-\theta)$$

Taking the derivative with respect to θ:

$$\frac{d\log L}{d\theta} = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{\sum_{i=1}^{n}(m-x_i)}{1-\theta}$$

Setting the derivative to zero:

$$\frac{\sum_{i=1}^{n} x_i}{\theta} = \frac{\sum_{i=1}^{n}(m-x_i)}{1-\theta}$$

$$\sum_{i=1}^{n} x_i - \theta\sum_{i=1}^{n} x_i = \theta\sum_{i=1}^{n} m - \theta\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i = \theta\sum_{i=1}^{n} m = \theta nm$$

$$\theta = \frac{\sum_{i=1}^{n} x_i}{nm} = \frac{\bar{X}}{m}$$

$$\sum_{i=1}^{n} x_i = \theta \sum_{i=1}^{n} m = \theta nm$$

$$\theta = \frac{\sum_{i=1}^{n} x_i}{nm} = \frac{\bar{X}}{m}$$

Therefore, the MLE for θ is:

$$\hat{\theta} = \frac{\bar{X}}{m}$$

The MLE for $g(\theta) = \theta$ is:

$$\hat{\theta} = \frac{\bar{X}}{m}$$

(b) Binomial(θ, p):

In this case, $n = 1$, and we have only one observation, X_1. The likelihood function is:

$$L(\theta) = \binom{\theta}{x_1} p^{x_1} (1-p)^{\theta - x_1}$$

For θ to be an integer, you have to use discrete optimization.

$theta$ must be greater than or equal to the observed value.

© Binomial(m, θ):

$g(\theta) = P(X_1 + X_2 = 0) = P(X_1 = 0, X_2 = 0) = P(X_1 = 0)P(X_2 = 0) = (1-\theta)^m(1-\theta)^m = (1-\theta)^{2n}$

Because the MLE of θ is $\bar{X}/m$. The MLE of $(1-\theta)^{2m}$ is $(1-\bar{X}/m)^{2m}$.

(d) Hypergeometric(m, r, θ):

The likelihood function is:

$$L(\theta) = \frac{\binom{m}{x}\binom{\theta-m}{r-x}}{\binom{\theta}{r}}$$

$$g(\theta) = \theta$$

Since $n = 1$, the MLE is the value of θ that maximizes $L(\theta)$. The solution is $\theta = 1$

(e) Double Exponential:

The likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{2} \exp\{-|x_i - \theta|\} = \frac{1}{2^n} \exp\left\{-\sum_{i=1}^{n} |x_i - \theta|\right\}$$

To maximize $L(\theta)$, we need to minimize $\sum_{i=1}^{n} |x_i - \theta|$. This is minimized when θ is the median of x_i.

(f) Uniform(α, β):

$$f(x;\alpha,\beta) = \frac{1}{\beta-\alpha}, \quad \alpha \le x \le \beta$$

$$L(\alpha,\beta) = \prod_{i=1}^{n} \frac{1}{\beta-\alpha} = \frac{1}{(\beta-\alpha)^n}$$

To maximize $L(\alpha,\beta)$, we must minimize $\beta - \alpha$.

$$\hat{\alpha} = \min(X_i)$$

$$\hat{\beta} = \max(X_i)$$

$g(\theta) = \alpha + \beta$.

$$\min(X_i) + \max(X_i)$$

(g) Normal(θ, θ^2):

The likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(x_i-\theta)^2}{2\theta^2}} = (2\pi\theta^2)^{-n/2} \exp\left\{ -\sum_{i=1}^{n} \frac{(x_i-\theta)^2}{2\theta^2} \right\}$$

The log-likelihood function is:

$$\log L(\theta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\theta^2) - \sum_{i=1}^{n} \frac{(x_i-\theta)^2}{2\theta^2}$$

Taking the derivative with respect to θ:

$$\frac{d\log L}{d\theta} = -\frac{n}{\theta} + \sum_{i=1}^{n} \frac{(x_i-\theta)}{\theta^2} + \sum_{i=1}^{n} \frac{(x_i-\theta)^2}{\theta^3}$$

Setting the derivative to zero:

$$\theta = \bar{X}$$

Problem 3(h): MLE for Inverse Gaussian

Given $X_1, \ldots, X_n \sim \text{Inverse Gaussian}(\theta_1, \theta_2)$ and $g(\boldsymbol{\theta}) = (\theta_1, \theta_2)$.

The likelihood function is:

$$L(\theta_1, \theta_2) \propto \prod_{i=1}^{n} \left(\frac{\theta_2}{2\pi x_i^3} \right)^{1/2} e^{-\frac{\theta_2 (x_i - \theta_1)^2}{2\theta_1^2 x_i}}$$

Taking the logarithm:

$$\log L \propto \sum_{i=1}^{n} \left[\frac{1}{2} \log(\theta_2) - \frac{1}{2} \log(2\pi x_i^3) - \frac{\theta_2 (x_i - \theta_1)^2}{2\theta_1^2 x_i} \right]$$

Maximizing with respect to θ_1 and θ_2 leads to:

$$\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$$

Solving for θ_2:

$$\frac{\partial}{\partial \theta_2} \log L = \frac{n}{2\theta_2} - \sum_{i=1}^{n} \frac{(x_i - \theta_1)^2}{2\theta_1^2 x_i} = 0$$

$$\hat{\theta}_2 = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\bar{x}^2 x_i} \right)^{-1}$$

Therefore, the MLE for $g(\boldsymbol{\theta})$ is:

$$(\bar{x}, \left(\frac{1}{n} \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\bar{x}^2 x_i} \right)^{-1})$$

Problem 4(a): Sufficient Statistic for Linear Model

Given $Y_i = \beta x_i + \epsilon_i$, with $\epsilon_i \sim N(0, \sigma^2)$.

The likelihood function is:

$$L(\beta, \sigma^2) \propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(Y_i - \beta x_i)^2}{2\sigma^2}}$$

$$f(y_1, \ldots, y_n | \beta, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (Y_i - \beta x_i)^2 \right\}$$

Expanding the square:

$$\sum_{i=1}^{n} (Y_i - \beta x_i)^2 = \sum_{i=1}^{n} (Y_i^2 - 2\beta x_i Y_i + \beta^2 x_i^2)$$

By the factorization theorem, a two-dimensional sufficient statistic for (β, σ^2) is:

$$(\sum_{i=1}^{n} x_i Y_i, \sum_{i=1}^{n} Y_i^2)$$

Problem 4(b): MLE of Beta

The log-likelihood is:

$$l(\beta, \sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \beta x_i)^2$$

Taking the derivative with respect to β:

$$\frac{\partial l}{\partial \beta} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \beta x_i)x_i = 0$$

Solving for β, we obtain the MLE:

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}$$

To show unbiasedness:

$$E[\hat{\beta}] = E\left[\frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}\right] = \frac{\sum_{i=1}^{n} x_i E[Y_i]}{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n} x_i(\beta x_i)}{\sum_{i=1}^{n} x_i^2} = \beta\frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^2} = \beta$$

Thus, the MLE $\hat{\beta}$ is an unbiased estimator of β.

Problem 4©: Distribution of MLE of Beta

Since $Y_i = \beta x_i + \epsilon_i$ and $\epsilon_i \sim N(0, \sigma^2)$, it follows that $Y_i \sim N(\beta x_i, \sigma^2)$.

The MLE is a linear combination of normal random variables:

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}$$

Therefore, $\hat{\beta}$ is also normally distributed.

The mean of $\hat{\beta}$ is:

$$E[\hat{\beta}] = \beta$$

The variance of $\hat{\beta}$ is:

$$Var(\hat{\beta}) = Var\left(\frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}\right) = \frac{\sum_{i=1}^{n} x_i^2 Var(Y_i)}{(\sum_{i=1}^{n} x_i^2)^2} = \frac{\sigma^2 \sum_{i=1}^{n} x_i^2}{(\sum_{i=1}^{n} x_i^2)^2} = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

Thus, the distribution of the MLE of β is:

$$\hat{\beta} \sim N\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}\right)$$

Problem 4(d): Unbiased Estimator of Beta

Let $\tilde{\beta} = \frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} x_i}$.

$$E[\tilde{\beta}] = E\left[\frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} x_i}\right] = \frac{\sum_{i=1}^{n} E[Y_i]}{\sum_{i=1}^{n} x_i} = \frac{\sum_{i=1}^{n} (\beta x_i)}{\sum_{i=1}^{n} x_i} = \beta \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i} = \beta$$

Therefore, $\tilde{\beta} = \frac{\sum Y_i}{\sum x_i}$ is an unbiased estimator of β.

Problem 4(e): Variance Comparison

The variance of $\tilde{\beta} = \frac{\sum Y_i}{\sum x_i}$ is:

$$Var(\tilde{\beta}) = Var\left(\frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n x_i}\right) = \frac{\sum_{i=1}^n Var(Y_i)}{(\sum_{i=1}^n x_i)^2} = \frac{n\sigma^2}{(\sum_{i=1}^n x_i)^2}$$

The variance of the MLE is:

$$Var(\hat{\beta}) = \frac{\sigma^2}{\sum_{i=1}^n x_i^2}$$

The ratio is:

$$\frac{Var(\tilde{\beta})}{Var(\hat{\beta})} = \frac{n\sigma^2}{(\sum x_i)^2} / \frac{\sigma^2}{\sum x_i^2} = \frac{n \sum x_i^2}{(\sum x_i)^2}$$

By Cauchy-Schwarz inequality:

$$(\sum x_i)^2 \leq n \sum x_i^2$$

So,

$$\frac{n \sum x_i^2}{(\sum x_i)^2} \geq 1$$

Thus, $Var(\tilde{\beta}) \geq Var(\hat{\beta})$. The MLE is more efficient.

Problem 4(f): Another Unbiased Estimator

Consider the estimator:

$$\beta^* = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{x_i}$$

To show unbiasedness:

$$E[\beta^*] = E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{x_i}\right] = \frac{1}{n}\sum_{i=1}^{n}\frac{E[Y_i]}{x_i} = \frac{1}{n}\sum_{i=1}^{n}\frac{\beta x_i}{x_i} = \frac{1}{n}\sum_{i=1}^{n}\beta = \beta$$

Thus, $\beta^* = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{x_i}$ is an unbiased estimator of β.

Problem 4(g): Variance Comparison

The variance of $\beta^* = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{x_i}$ is:

$$Var(\beta^*) = Var\left(\frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{x_i}\right) = \frac{1}{n^2}\sum_{i=1}^{n}Var\left(\frac{Y_i}{x_i}\right) = \frac{1}{n^2}\sum_{i=1}^{n}\frac{Var(Y_i)}{x_i^2} = \frac{\sigma^2}{n^2}\sum_{i=1}^{n}\frac{1}{x_i^2}$$

Comparing to:

$$Var(\hat{\beta}) = \frac{\sigma^2}{\sum_{i=1}^{n}x_i^2}$$

$$Var(\tilde{\beta}) = \frac{n\sigma^2}{(\sum_{i=1}^{n}x_i)^2}$$

The MLE, $\hat{\beta}$, is the best linear unbiased estimator (BLUE). It has the smallest variance.

Problem 5: Best Unbiased Estimator

Let $W_1, \dots, W_k$ be unbiased estimators of θ with known variances $Var(W_i) = \sigma_i^2$.

Consider an estimator of the form:

$$\hat{\theta} = \sum_{i=1}^{k} a_i W_i$$

For $\hat{\theta}$ to be unbiased, we require:

$$E[\hat{\theta}] = \sum_{i=1}^{k} a_i E[W_i] = \sum_{i=1}^{k} a_i \theta = \theta$$

This means:

$$\sum_{i=1}^{k} a_i = 1$$

We want to minimize the variance of $\hat{\theta}$ subject to this constraint:

$$Var(\hat{\theta}) = Var\left(\sum_{i=1}^{k} a_i W_i\right) = \sum_{i=1}^{k} a_i^2 Var(W_i) = \sum_{i=1}^{k} a_i^2 \sigma_i^2$$

Using Lagrange multipliers, we get:

$$a_i = \frac{1/\sigma_i^2}{\sum_{j=1}^{k} 1/\sigma_j^2}$$

Thus, the best unbiased estimator is:

$$\hat{\theta} = \sum_{i=1}^{k} \frac{W_i/\sigma_i^2}{\sum_{j=1}^{k} 1/\sigma_j^2}$$

Problem 6: Unbiased Estimator of Area

Let R_i be the measured radius with $R_i = r + \epsilon_i$, where $\epsilon_i \sim N(0, \sigma^2)$ and r is the true radius.
The area of the circle is:

$$A = \pi r^2$$

An obvious estimator is $\hat{A} = \pi \bar{R}^2$, where $\bar{R} = \frac{1}{n} \sum_{i=1}^{n} R_i$.

$$E[\pi \bar{R}^2] = \pi E\left[\left(\frac{1}{n} \sum R_i\right)^2\right] = \pi\left(r^2 + \frac{\sigma^2}{n}\right)$$

So $\pi \bar{R}^2$ is biased.
To find an unbiased estimator:

$$E[R_i^2] = Var(R_i) + E[R_i]^2 = \sigma^2 + r^2$$

Thus, $r^2 = E[R_i^2] - \sigma^2$.
We estimate σ^2 with:

$$S^2 = \frac{1}{n-1} \sum (R_i - \bar{R})^2$$

So, the unbiased estimator is:

$$\hat{A}_{unbiased} = \pi(\bar{R}^2 - \frac{S^2}{n})$$

Since the normal distribution is complete and sufficient, this estimator is the UMVUE.

Problem Set 3

1. (Additive Properties)

(a) Binomial:

Let $X_i \overset{ind}{\sim} \mathrm{Binomial}(n_i, p)$ for $i = 1, \ldots, k$. The moment generating function (MGF) of X_i is:

$$M_{X_i}(t) = (1 - p + pe^t)^{n_i}$$

Since the X_i's are independent, the MGF of $T = \sum_{i=1}^k X_i$ is:

$$M_T(t) = \prod_{i=1}^k M_{X_i}(t) = \prod_{i=1}^k (1 - p + pe^t)^{n_i} = (1 - p + pe^t)^{\sum_{i=1}^k n_i}$$

This is the MGF of a Binomial distribution with parameters $\sum_{i=1}^k n_i$ and p. Therefore:

$$T \sim \mathrm{Binomial}\left(\sum_{i=1}^k n_i, p\right)$$

(b) Poisson:

Let $X_i \overset{ind}{\sim} \text{Poisson}(\lambda_i)$ for $i = 1, \dots, n$. The MGF of X_i is:

$$M_{X_i}(t) = e^{\lambda_i(e^t-1)}$$

Since the X_i's are independent, the MGF of $T = \sum_{i=1}^{n} X_i$ is:

$$M_T(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} e^{\lambda_i(e^t-1)} = e^{\sum_{i=1}^{n} \lambda_i(e^t-1)} = e^{\left(\sum_{i=1}^{n} \lambda_i\right)(e^t-1)}$$

This is the MGF of a Poisson distribution with parameter $\sum_{i=1}^{n} \lambda_i$. Therefore:

$$T \sim \text{Poisson}\left(\sum_{i=1}^{n} \lambda_i\right)$$

© **Normal:**

Let $X_i \overset{ind}{\sim} \text{Normal}(\mu_i, \sigma_i^2)$ for $i = 1, \ldots, n$. The MGF of X_i is:

$$M_{X_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}$$

Since the X_i's are independent, the MGF of $T = \sum_{i=1}^n X_i$ is:

$$M_T(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2} = e^{\sum_{i=1}^n \mu_i t + \frac{1}{2}\sum_{i=1}^n \sigma_i^2 t^2} = e^{\left(\sum_{i=1}^n \mu_i\right)t + \frac{1}{2}\left(\sum_{i=1}^n \sigma_i^2\right)t^2}$$

This is the MGF of a Normal distribution with mean $\sum_{i=1}^n \mu_i$ and variance $\sum_{i=1}^n \sigma_i^2$. Therefore:

$$T \sim \text{Normal}\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$$

(d) Gamma:

Let $X_i \overset{ind}{\sim} \text{Gamma}(\alpha_i, \beta)$ for $i = 1, \ldots, n$. The MGF of X_i is:

$$M_{X_i}(t) = (1 - \beta t)^{-\alpha_i}, \quad t < \frac{1}{\beta}$$

Since the X_i's are independent, the MGF of $T = \sum_{i=1}^{n} X_i$ is:

$$M_T(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} (1 - \beta t)^{-\alpha_i} = (1 - \beta t)^{-\sum_{i=1}^{n} \alpha_i}$$

This is the MGF of a Gamma distribution with parameters $\sum_{i=1}^{n} \alpha_i$ and β. Therefore:

$$T \sim \text{Gamma}\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$$

(e) Chi-squared:

Let $X_i \overset{ind}{\sim} \chi^2_{n_i}$ for $i = 1, \ldots, k$. Recall that a Chi-squared distribution with n_i degrees of freedom is a special case of the Gamma distribution with $\alpha_i = n_i/2$ and $\beta = 2$. Therefore, $X_i \sim \text{Gamma}(n_i/2, 2)$. Using the result from part (d):

$$T = \sum_{i=1}^{k} X_i \sim \text{Gamma}\left(\sum_{i=1}^{k} \frac{n_i}{2}, 2\right)$$

Which is a Chi-squared distribution with $\sum_{i=1}^{k} n_i$ degrees of freedom. Therefore:

$$T \sim \chi^2_N, \quad \text{where } N = \sum_{i=1}^{k} n_i$$

2. Let $X \sim \text{Normal}(\mu, \sigma^2)$. Then $T = aX + b$. The MGF of X is:

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

The MGF of T is:

$$M_T(t) = E[e^{tT}] = E[e^{t(aX+b)}] = e^{bt} E[e^{(at)X}] = e^{bt} M_X(at)$$
$$M_T(t) = e^{bt} e^{\mu(at) + \frac{1}{2}\sigma^2 (at)^2} = e^{(a\mu + b)t + \frac{1}{2}(a^2\sigma^2)t^2}$$

This is the MGF of a Normal distribution with mean $a\mu + b$ and variance $a^2\sigma^2$. Therefore:

$$T \sim \text{Normal}(a\mu + b, a^2\sigma^2)$$

3. Let $X \sim \mathrm{Gamma}(\alpha, \beta)$. Then $T = aX$. The MGF of X is:

$$M_X(t) = (1 - \beta t)^{-\alpha}, \quad t < \frac{1}{\beta}$$

The MGF of T is:

$$M_T(t) = E[e^{tT}] = E[e^{t(aX)}] = E[e^{(at)X}] = M_X(at) = (1 - \beta(at))^{-\alpha}$$
$$M_T(t) = (1 - (a\beta)t)^{-\alpha}$$

This is the MGF of a Gamma distribution with parameters α and β/a. Therefore:

$$T \sim \mathrm{Gamma}\left(\alpha, \frac{\beta}{a}\right)$$

4. Let $X \sim \text{beta}(n/2, m/2)$. Then $T = \frac{mX}{n(1-X)}$. To find the distribution of T, we use the transformation method.

First, find the CDF of T:

$$F_T(t) = P(T \leq t) = P\left(\frac{mX}{n(1-X)} \leq t\right)$$

$$F_T(t) = P\left(X \leq \frac{nt}{m+nt}\right)$$

Now, differentiate $F_T(t)$ to find the PDF $f_T(t)$. The PDF of X is:

$$f_X(x) = \frac{x^{\frac{n}{2}-1}(1-x)^{\frac{m}{2}-1}}{B\left(\frac{n}{2}, \frac{m}{2}\right)}$$

where $B(\cdot, \cdot)$ is the beta function. Using the transformation formula, we find that:

$$T \sim F_{n,m}$$

5. Let $X \sim \mathrm{Uniform}(0, 1)$ and $\alpha > 0$. Then $T = X^{1/\alpha}$. The CDF of X is $F_X(x) = x$ for $0 < x < 1$. The CDF of T is:

$$F_T(t) = P(T \leq t) = P(X^{1/\alpha} \leq t) = P(X \leq t^\alpha) = F_X(t^\alpha) = t^\alpha$$

for $0 < t < 1$. Differentiating $F_T(t)$ with respect to t, we get the PDF of T:

$$f_T(t) = \alpha t^{\alpha-1}$$

for $0 < t < 1$. This is the PDF of a Beta distribution with parameters α and 1. Therefore:

$$T \sim \mathrm{Beta}(\alpha, 1)$$

6. Let $X \sim \mathrm{Cauchy}(0,1)$. Then $T = \frac{1}{1+X^2}$. The PDF of X is:

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

The CDF of T is:

$$F_T(t) = P(T \leq t) = P\left(\frac{1}{1+X^2} \leq t\right) = P\left(X^2 \geq \frac{1}{t} - 1\right)$$

$$F_T(t) = P\left(X \leq -\sqrt{\frac{1}{t} - 1}\right) + P\left(X \geq \sqrt{\frac{1}{t} - 1}\right)$$

Differentiating $F_T(t)$ to find the PDF $f_T(t)$ and simplifying, we find that:

$$T \sim \mathrm{Beta}(0.5, 0.5)$$

6. Let $X \sim \mathrm{Cauchy}(0,1)$. Then $T = \frac{1}{1+X^2}$. The PDF of X is:

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

The CDF of T is:

$$F_T(t) = P(T \le t) = P\left(\frac{1}{1+X^2} \le t\right) = P\left(X^2 \ge \frac{1}{t} - 1\right)$$
$$F_T(t) = P\left(X \le -\sqrt{\frac{1}{t} - 1}\right) + P\left(X \ge \sqrt{\frac{1}{t} - 1}\right)$$

Differentiating $F_T(t)$ to find the PDF $f_T(t)$ and simplifying, we find that:

$$T \sim \mathrm{Beta}(0.5, 0.5)$$

7. Let $X \sim \mathrm{Uniform}(0,1)$. Then $T = -2\log X$. The CDF of X is $F_X(x) = x$ for $0 < x < 1$. The CDF of T is:

$$F_T(t) = P(T \le t) = P(-2\log X \le t) = P\left(\log X \ge -\frac{t}{2}\right) = P\left(X \ge e^{-\frac{t}{2}}\right)$$

$$F_T(t) = 1 - e^{-\frac{t}{2}}, \quad t > 0$$

Differentiating $F_T(t)$ with respect to t, we get the PDF of T:

$$f_T(t) = \frac{1}{2}e^{-\frac{t}{2}}, \quad t > 0$$

This is the PDF of a Chi-squared distribution with 2 degrees of freedom. Therefore:

$$T \sim \chi_2^2$$

8. Let X be distributed as some absolutely continuous distribution with CDF G_X. Then $T = G_X(X)$. The CDF of T is:

$$F_T(t) = P(T \leq t) = P(G_X(X) \leq t)$$

Since G_X is a CDF, it is non-decreasing, so we can write:

$$F_T(t) = P(X \leq G_X^{-1}(t)) = G_X(G_X^{-1}(t)) = t, \quad 0 < t < 1$$

This is the CDF of a Uniform distribution on $(0, 1)$. Therefore:

$$T \sim \text{Uniform}(0, 1)$$

9. Let the random variable X have PDF $f(x) = \sqrt{\frac{2}{\pi}} \exp\{-x^2/2\}$, $x > 0$.

(a) Find $E(X)$ and $\mathrm{Var}(X)$.

The given PDF is that of a half-normal distribution.

The expected value of X is:

$$E(X) = \int_0^\infty x\sqrt{\frac{2}{\pi}} \exp\{-x^2/2\}dx = \sqrt{\frac{2}{\pi}} \int_0^\infty x \exp\{-x^2/2\}dx$$

Let $u = x^2/2$, then $du = xdx$. So:

$$E(X) = \sqrt{\frac{2}{\pi}} \int_0^\infty \exp\{-u\}du = \sqrt{\frac{2}{\pi}}[-e^{-u}]_0^\infty = \sqrt{\frac{2}{\pi}}$$

To find the variance, we first need $E(X^2)$:

$$E(X^2) = \int_0^\infty x^2\sqrt{\frac{2}{\pi}} \exp\{-x^2/2\}dx$$

Using integration by parts or recognizing that this is the second moment of a half-normal distribution, we have $E(X^2) = 1$.

Therefore:

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 1 - \left(\sqrt{\frac{2}{\pi}}\right)^2 = 1 - \frac{2}{\pi}$$

(b) Find an appropriate transformation $Y = g(X)$ and α, β, so that $Y \sim \mathrm{Gamma}(\alpha, \beta)$.
Consider $Y = X^2/2$. Then $X = \sqrt{2Y}$. The PDF of X is:

$$f_X(x) = \sqrt{\frac{2}{\pi}} \exp\{-x^2/2\}$$

The PDF of Y can be found using the transformation formula:

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| = \sqrt{\frac{2}{\pi}} \exp\{-y\} \left| \frac{1}{\sqrt{2y}} \right| = \frac{1}{\sqrt{\pi y}} \exp\{-y\}, \quad y > 0$$

This is the PDF of a Gamma distribution with parameters $\alpha = 1/2$ and $\beta = 1$. Therefore:

$$Y \sim \mathrm{Gamma}\left(\frac{1}{2}, 1\right)$$

10. Let X be distributed as $\text{Gamma}(\alpha, \beta)$. Then show that $E(X^r) = \beta^r \frac{\Gamma(r+\alpha)}{\Gamma(\alpha)}$, $r > -\alpha$.

The expected value of X^r is given by:

$$E(X^r) = \int_0^\infty x^r \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{r+\alpha-1}e^{-x/\beta} dx$$

Let $y = x/\beta$, then $x = \beta y$ and $dx = \beta dy$. Then:

$$E(X^r) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty (\beta y)^{r+\alpha-1} e^{-y} \beta dy = \frac{\beta^{r+\alpha}}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^{r+\alpha-1} e^{-y} dy$$

$$E(X^r) = \frac{\beta^r}{\Gamma(\alpha)} \Gamma(r+\alpha) = \beta^r \frac{\Gamma(r+\alpha)}{\Gamma(\alpha)}$$

11. Let the bivariate random variable (X, Y) have a joint PDF $f_{X,Y}(x, y) = C(x + 2y)$ if $0 < y < 1, 0 < x < 2$, and 0 otherwise.

(a) Find the marginal distribution of Y.

First, we need to find the value of C. Since the joint PDF must integrate to 1, we have:

$$\int_0^1 \int_0^2 C(x + 2y)dxdy = 1$$

Evaluating the integral:

$$C \int_0^1 \left[\frac{x^2}{2} + 2xy\right]_0^2 dy = C \int_0^1 (2 + 4y)dy = C\left[2y + 2y^2\right]_0^1 = C(2 + 2) = 4C = 1$$

So, $C = 1/4$.

Now, to find the marginal distribution of Y, we integrate the joint PDF over x:

$$f_Y(y) = \int_0^2 \frac{1}{4}(x + 2y)dx = \frac{1}{4}\left[\frac{x^2}{2} + 2xy\right]_0^2 = \frac{1}{4}(2 + 4y) = \frac{1}{2} + y$$

for $0 < y < 1$.

(b) Find the conditional distribution of Y given $X = 1$.

First, find the marginal distribution of X:

$$f_X(x) = \int_0^1 \frac{1}{4}(x + 2y)dy = \frac{1}{4}\left[xy + y^2\right]_0^1 = \frac{1}{4}(x + 1)$$

for $0 < x < 2$.

Then, the conditional distribution of Y given $X = 1$ is:

$$f_{Y|X}(y|x = 1) = \frac{f_{X,Y}(1, y)}{f_X(1)} = \frac{\frac{1}{4}(1 + 2y)}{\frac{1}{4}(1 + 1)} = \frac{1 + 2y}{2}$$

for $0 < y < 1$.

© Compare the expectations of the above two distributions of Y.

The expectation of the marginal distribution of Y is:

$$E[Y] = \int_0^1 y\left(\frac{1}{2} + y\right)dy = \int_0^1 \left(\frac{1}{2}y + y^2\right)dy = \left[\frac{1}{4}y^2 + \frac{1}{3}y^3\right]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

The expectation of the conditional distribution of Y given $X = 1$ is:

$$E[Y|X = 1] = \int_0^1 y\left(\frac{1 + 2y}{2}\right)dy = \int_0^1 \left(\frac{1}{2}y + y^2\right)dy = \left[\frac{1}{4}y^2 + \frac{1}{3}y^3\right]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

(d) Find the covariance between X and Y.

We need:

$$E[XY] = \int_0^1 \int_0^2 xy\frac{1}{4}(x+2y)dxdy = \frac{1}{4}\int_0^1 \int_0^2 (x^2y + 2xy^2)dxdy$$

$$E[XY] = \frac{1}{4}\int_0^1 \left[\frac{1}{3}x^3y + x^2y^2\right]_0^2 dy = \frac{1}{4}\int_0^1 \left(\frac{8}{3}y + 4y^2\right)dy = \frac{1}{4}\left[\frac{4}{3}y^2 + \frac{4}{3}y^3\right]_0^1 = \frac{1}{4}\left(\frac{4}{3} + \frac{4}{3}\right)$$

Now, we need:

$$E[X] = \int_0^2 xf_X(x)dx = \int_0^2 x\frac{1}{4}(x+1)dx = \frac{1}{4}\int_0^2 (x^2+x)dx = \frac{1}{4}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^2 = \frac{1}{4}\left(\frac{8}{3} + 2\right)$$

Then:

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{2}{3} - \left(\frac{7}{6}\right)\left(\frac{7}{12}\right) = \frac{2}{3} - \frac{49}{72} = \frac{48-49}{72} = -\frac{1}{72}$$

(e) Find the distribution of $Z = 9/(2Y+1)^2$.

$$P(Z \le z) = P\left(\frac{9}{(2Y+1)^2} \le z\right) = P\left((2Y+1)^2 \ge \frac{9}{z}\right) = P\left(2Y+1 \ge \frac{3}{\sqrt{z}}\right)$$

$$P\left(Y \ge \frac{\frac{3}{\sqrt{z}}-1}{2}\right) = \int_{\frac{\frac{3}{\sqrt{z}}-1}{2}}^{1} \left(\frac{1}{2}+y\right)dy$$

(f) What is $P(X > Y)$?

$$P(X > Y) = \int_0^1 \int_y^2 f_{X,Y}(x,y)dxdy = \int_0^1 \int_y^2 \frac{1}{4}(x+2y)dxdy$$

$$P(X > Y) = \frac{1}{4}\int_0^1 \left[\frac{x^2}{2} + 2xy\right]_y^2 dy = \frac{1}{4}\int_0^1 \left(2 + 4y - \frac{y^2}{2} - 2y^2\right)dy$$

$$P(X > Y) = \frac{1}{4}\int_0^1 \left(2 + 4y - \frac{5}{2}y^2\right)dy = \frac{1}{4}\left[2y + 2y^2 - \frac{5}{6}y^3\right]_0^1 = \frac{1}{4}\left(2+2-\frac{5}{6}\right) = \frac{1}{4}\left(\frac{19}{6}\right) = \frac{19}{24}$$

12. Let $X \sim \mathrm{Normal}(0,1)$. Define $Y = -X\mathbb{I}(|X| \leq 1) + X\mathbb{I}(|X| > 1)$. Find the distribution of Y. (Hint: Apply the CDF approach)

$$Y = \begin{cases} X, & X < -1 \\ -X, & -1 \leq X \leq 1 \\ X, & X > 1 \end{cases}$$

Let's find the CDF of Y, $F_Y(y) = P(Y \leq y)$.

- If $y < -1$:
  $F_Y(y) = P(X \leq y)$
- If $-1 \leq y \leq 1$:
  $F_Y(y) = P(X \leq -y)$
- If $y > 1$:
  $F_Y(y) = P(X \leq y) + P(X \geq -y)$

13. Let $X \sim \text{Normal}(0, 1)$. Define $Y = \text{sign}(X)$ and $Z = |X|$. Here $\text{sign}(\cdot)$ is a $\mathbb{R} \to \{-1, 1\}$ function such that $\text{sign}(a) = 1$ if $a \geq 0$, and $\text{sign}(a) = -1$ otherwise.

(a) Find the marginal distributions of Y and Z.

Y is a discrete random variable taking values -1 and 1.

$$P(Y = 1) = P(X \geq 0) = 0.5$$
$$P(Y = -1) = P(X < 0) = 0.5$$

$Z = |X|$ has a half-normal distribution.

$$f_Z(z) = 2f_X(z) = 2\frac{1}{\sqrt{2\pi}}e^{-z^2/2}, \quad z > 0$$

(b) Find the joint CDF of (Y, Z). Hence or otherwise prove that Y and Z are independently distributed.

$$P(Y = 1, Z = z) = P(X \geq 0, |X| = z) = P(X = z) = 0.5f_Z(z)$$
$$P(Y = -1, Z = z) = P(X < 0, |X| = z) = P(X = -z) = 0.5f_Z(z)$$
$$P(Y, Z) = P(Y)P(Z) \text{ thus independent}$$

14. Suppose $X_1, \cdots, X_n \overset{\text{IID}}{\sim} \text{Normal}(\mu_x, \sigma^2), Y_1, \cdots, Y_m \overset{\text{IID}}{\sim} \text{Normal}(\mu_y, \sigma^2)$, and all the random variables $\{X_1, \cdots, X_n, Y_1, \cdots, Y_m\}$ are mutually independent. Then find the distribution of $T := S_X^{\star 2}/S_Y^{\star 2}$, where $S_X^{\star 2}$ and $S_Y^{\star 2}$ are the unbiased sample variances of X and Y, respectively.

$S_X^{*2} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ and $S_Y^{*2} = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \bar{Y})^2$.

We know that $\frac{(n-1)S_X^{*2}}{\sigma^2} \sim \chi^2_{n-1}$ and $\frac{(m-1)S_Y^{*2}}{\sigma^2} \sim \chi^2_{m-1}$.

Thus $T = \frac{S_X^{*2}}{S_Y^{*2}}$ can be written as $T = \frac{\frac{\sigma^2}{n-1}\chi^2_{n-1}}{\frac{\sigma^2}{m-1}\chi^2_{m-1}}$.

Thus $\frac{S_X^{*2}}{S_Y^{*2}} = \frac{m-1}{n-1}\frac{\chi^2_{n-1}}{\chi^2_{m-1}}$.

Also we know that $F_{n-1,m-1} = \frac{\chi^2_{n-1}/(n-1)}{\chi^2_{m-1}/(m-1)}$.

Thus we have $\frac{S_X^{*2}}{S_Y^{*2}} \sim F_{n-1,m-1}$

Problem 15(a): Expectation of Y1

Given $X_1, \ldots, X_n$ i.i.d. with CDF F_X and $E(X_1) = \mu$. Define $Y_i = 1$ if $X_i > \mu$, and $Y_i = 0$ otherwise.
We want to find $E(Y_1)$. By definition:

$$E(Y_1) = 1 \cdot P(X_1 > \mu) + 0 \cdot P(X_1 \leq \mu) = P(X_1 > \mu)$$

Since $P(X_1 > \mu) = 1 - P(X_1 \leq \mu) = 1 - F_X(\mu)$, we have:

$$E(Y_1) = 1 - F_X(\mu)$$

If the distribution is continuous, then $P(X_1 = \mu) = 0$, so $P(X_1 \leq \mu) = F_X(\mu)$. If μ is the median, then $F_X(\mu) = 0.5$, and $E(Y_1) = 0.5$.

Problem 15(b): Distribution of Sum of Yi

Let $S = \sum_{i=1}^{n} Y_i$. Since Y_i are i.i.d. Bernoulli random variables with success probability $p = E(Y_1) = 1 - F_X(\mu)$, and S is the sum of n independent Bernoulli trials, S follows a binomial distribution:

$$S \sim \text{Binomial}(n, p)$$

Therefore, the probability mass function of S is:

$$P(S = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

where $k = 0, 1, \ldots, n$ and $p = 1 - F_X(\mu)$. Thus, the distribution of $\sum_{i=1}^{n} Y_i$ is Binomial$(n, 1 - F_X(\mu))$.

Problem 16: Function of Sample Variance

Given $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ and S_n^2 is the sample variance. We want to find $g(S_n^2)$ such that $E[g(S_n^2)] = \sigma$.

We know that $\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$.

Therefore, $E[\frac{(n-1)S_n^2}{\sigma^2}] = n - 1$, which implies $E[S_n^2] = \sigma^2$.

Let $g(S_n^2) = c \cdot S_n$. Then, we need to find c such that $E[cS_n] = \sigma$.

Since $\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$, $S_n = \sigma\sqrt{\frac{\chi^2_{n-1}}{n-1}}$. Then:

$E[S_n] = \sigma E[\sqrt{\frac{\chi^2_{n-1}}{n-1}}]$.

Let $m = n - 1$. Then, $E[S_n] = \sigma E[\sqrt{\frac{\chi^2_m}{m}}]$.

Let $A = E[\sqrt{\frac{\chi^2_m}{m}}]$. Then, $c = \frac{1}{A}$.

$g(S_n^2) = \frac{S_n}{E[\sqrt{\frac{\chi^2_{n-1}}{n-1}}]}$ so $E[g(S_n^2)] = \sigma$.

Problem 17(a): CDF of r-th Order Statistic

Let $X_1, \ldots, X_n$ be i.i.d. with pdf f_X and CDF F_X. Let $X_{(r)}$ be the r-th order statistic. The CDF of $X_{(r)}$ is given by:

$$F_{X_{(r)}}(x) = P(X_{(r)} \leq x) = \sum_{i=r}^{n} P(\text{at least } r \text{ of } X_1, \ldots, X_n \text{ are } \leq x)$$

$$F_{X_{(r)}}(x) = \sum_{i=r}^{n} \binom{n}{i} [F_X(x)]^i [1 - F_X(x)]^{n-i}$$

Problem 17(b): PDF of r-th Order Statistic

Differentiating the CDF of $X_{(r)}$ with respect to x to get the pdf:

$$f_{X_{(r)}}(x) = \frac{d}{dx} F_{X_{(r)}}(x)$$

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} [F_X(x)]^{r-1} [1 - F_X(x)]^{n-r} f_X(x)$$

Problem 18(a): CDF of Cauchy(0,1)

Let Y have a Cauchy(0,1) distribution. The pdf is:

$$f_Y(y) = \frac{1}{\pi(1+y^2)}, \quad -\infty < y < \infty$$

The CDF of Y is:

$$F_Y(y) = \int_{-\infty}^{y} \frac{1}{\pi(1+t^2)} dt = \frac{1}{\pi} \arctan(y) \Big|_{-\infty}^{y}$$

$$F_Y(y) = \frac{1}{\pi} \arctan(y) - \frac{1}{\pi}(-\frac{\pi}{2}) = \frac{1}{\pi} \arctan(y) + \frac{1}{2}$$

Problem 18(b): Simulating Cauchy Random Samples

Let $U \sim \text{Uniform}(0,1)$. We want to find a transformation $Y = g(U)$ such that $Y \sim \text{Cauchy}(0,1)$. Using the inverse transform method:

$$F_Y(y) = u$$

$$\frac{1}{\pi} \arctan(y) + \frac{1}{2} = u$$

$$\arctan(y) = \pi(u - \frac{1}{2})$$

$$y = \tan(\pi(u - \frac{1}{2}))$$

Therefore, to simulate a Cauchy(0,1) random variable, generate $U \sim \text{Uniform}(0,1)$ and set $Y = \tan(\pi(U - \frac{1}{2}))$.