Indian Statistical Institute
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First Year: Semester – II
Economics-II
Home Task 2

Home Task 2.1: If $X \sim$ Pareto(c, α) then show that

$$\text{(i) } E(X) = \frac{\alpha c}{\alpha - 1}, \qquad \text{(ii) } V(X) = \begin{Bmatrix} \infty & \text{if } \alpha \in (1.2] \\ \left(\frac{c}{\alpha - 1}\right)^2 \frac{\alpha}{\alpha - 2} & \text{if } \alpha > 2 \end{Bmatrix},$$

$$\text{(iii) } E(X^r) = \begin{Bmatrix} \infty & \text{if } \alpha \le r, \\ \frac{\alpha c^r}{\alpha - r} & \text{if } \alpha > r \end{Bmatrix}, \text{(iv) } GM = c \times \exp\left(\frac{1}{\alpha}\right), \text{(v) } HM = c \times \left(1 + \frac{1}{\alpha}\right)$$

$$\text{(vi) } Y = \ln\left(\frac{X}{C}\right) \sim \text{Exponential}(\alpha), \text{i.e.}, f(y) = \alpha e^{-\alpha y},$$

$$\text{(vii) Prove that } F_1 = 1 - (1 - F)^{1-\frac{1}{\alpha}}, \text{(viii) } LR = \frac{1}{2\alpha - 1},$$

$$\text{(ix) The Pth quantile } x_P \text{ is } x_P = c(1 - P)^{-1/\alpha}.$$

Solution to Qn. 2.1:
(i), (ii) and (iii): The pdf and CDF of pareto distribution are

$$f(x) = \frac{\alpha c^\alpha}{x^{\alpha+1}}, \text{for } x \ge c \text{ and } F(x) = 1 - \frac{c^\alpha}{x^\alpha}.$$

$$E(X^r) = \int_c^\infty \frac{x^r \alpha c^\alpha}{x^{\alpha+1}} dx = \int_c^\infty \frac{\alpha c^\alpha}{x^{(\alpha-r)+1}} dx$$

Assume $\alpha > r$.

$$E(X^r) = \frac{\alpha c^\alpha}{(\alpha - r)c^{\alpha-r}} \int_c^\infty \frac{(\alpha - r)c^{(\alpha-r)}}{x^{(\alpha-r)+1}} dx$$

$$= \frac{\alpha c^r}{(\alpha - r)} \int_c^\infty \frac{\alpha' c^{\alpha'}}{x^{\alpha'+1}} dx = \frac{\alpha c^r}{(\alpha - r)}.$$

If $\alpha < r$, then $E(X^r) = \infty$, i.e., does not exist. Thus, if $\alpha = 1.5$, say, then E(X) exists, but V(X) does not exist.

Put r = 1 and 2 to get

$$\mathrm{E(X)} = \frac{\alpha c}{\alpha - 1}, \text{and } \mathrm{E(X^2)} = \frac{\alpha c^2}{(\alpha - 2)}$$

Thus, for $\alpha > 2$, we have

$$\mathrm{V(X)} = \frac{\alpha c^2}{(\alpha - 2)} - \left(\frac{\alpha c}{(\alpha - 1)}\right)^2 = \cdots = \frac{\alpha c^2}{(\alpha - 2)(\alpha - 1)^2}.$$

(iv) To show $\mathrm{GM} = c \times \exp\left(\frac{1}{\alpha}\right)$ we use the result (vi), i.e., $\mathrm{Y} = \mathrm{Ln}\left(\frac{X}{c}\right) \sim \mathrm{Exponential}(\alpha)$.

Logarithm of GM = AM of logarithms.

$E(Y) = E\big(Ln(X/c)\big) = \int_0^{\infty} y\, \alpha e^{-\alpha y} dy = \frac{1}{\alpha}\int_0^{\infty} z\, e^{-z} dz$ *choosing* $z = \alpha y$

$$= \frac{1}{\alpha}[-e^{-z} - ze^{-z}]_0^{\infty} = \frac{1}{\alpha}.$$

Hence, $E\big(Ln(X)\big) = Ln(c) + \frac{1}{\alpha} \Rightarrow \mathrm{GM}(X) = e^{Ln(c)+\frac{1}{\alpha}} = ce^{\frac{1}{\alpha}}$.

(v) To show $\mathrm{HM} = c \times \left(1 + \frac{1}{\alpha}\right)$

HM = Reciprocal of mean of reciprocals = $1/\mathrm{E}(1/X)$.

$$\mathrm{E}\left(\frac{1}{X}\right) = \int_c^{\infty} \frac{1}{t} . \frac{\alpha c^{\alpha}}{t^{\alpha+1}} dt = \int_c^{\infty} \frac{\alpha c^{\alpha}}{(\alpha + 1)c^{\alpha+1}} \frac{(\alpha + 1)c^{\alpha+1}}{t^{\alpha+2}} dt$$

$$= \frac{\alpha c^{\alpha}}{(\alpha + 1)c^{\alpha+1}}, \textit{since } \frac{(\alpha + 1)c^{\alpha+1}}{t^{\alpha+2}} \textit{is also a Pareto density.}$$

$$= \frac{\alpha}{(\alpha + 1)c} .$$

$$\textit{Hence}, \mathrm{HM} = \frac{(\alpha + 1)c}{\alpha} = c \times \left(1 + \frac{1}{\alpha}\right).$$

(vi) To show $\mathrm{Y} = \ln\left(\frac{X}{c}\right) \sim \mathrm{Exponential}\,(\alpha)$, i. e. , $f(y) = \alpha e^{-\alpha y}$.

$$y = \mathrm{Ln}\left(\frac{X}{c}\right) \Rightarrow x = ce^{y} \Rightarrow \frac{dx}{dy} = ce^{y}.$$

Suppose $g(x)$ is the pdf of X. Then

$$f(y) = g(x)\frac{dx}{dy} = \frac{\alpha c^{\alpha}}{(ce^{y})^{\alpha+1}} ce^{y} = \frac{\alpha}{(e^{y})^{\alpha}} = \alpha e^{-\alpha y}.$$

The range of Y is obviously from 0 to ∞. Hence, Y ~Exponential(α).

(vii) To prove that $\mathrm{F}_1 = 1 - (1 - \mathrm{F})^{1-\frac{1}{\alpha}}$.

$$E(X)F_1(x) = \int_c^x \frac{t\alpha c^\alpha}{t^{\alpha+1}} dt = \frac{\alpha c^\alpha}{(\alpha-1)c^{\alpha-1}} \int_c^x \frac{(\alpha-1)c^{(\alpha-1)}}{t^{(\alpha-1)+1}} dt$$

$$= \frac{\alpha c^\alpha}{(\alpha-1)c^{\alpha-1}} \int_c^x \frac{\alpha' c^{\alpha'}}{t^{\alpha'+1}} dt = \frac{\alpha c}{(\alpha-1)} \left[1 - \frac{c^{\alpha-1}}{x^{\alpha-1}}\right]$$

$$= \frac{\alpha c}{(\alpha-1)} \left[1 - \left(1 - F(x)\right)^{\frac{(\alpha-1)}{\alpha}}\right].$$

Since,

$$E(X) = \frac{\alpha c}{(\alpha-1)},$$

we have,

$$F_1 = 1 - (1-F)^{1-\frac{1}{\alpha}}.$$

(viii) To prove $LR = \frac{1}{2\alpha-1}$

$$LR = 1 - 2\int_0^1 F_1 dF$$

$$= 1 - 2\int_0^1 \left[1 - (1-F)^{1-\frac{1}{\alpha}}\right] dF$$

$$= 1 - 2\left[1 - \int_0^1 (1-F)^{\alpha^*} dF\right]$$

$$= 1 - 2\left[1 - \left[\frac{-(1-F)^{\alpha^*+1}}{\alpha^*+1}\right]_0^1\right]$$

$$= 1 - 2\left[1 - \frac{1}{\alpha^*+1}\right]$$

$$= 1 - 2\left[1 - \frac{1}{\frac{(\alpha-1)}{\alpha}+1}\right]$$

$$= 1 - 2\left[1 - \frac{\alpha}{2\alpha-1}\right] = \cdots = \frac{1}{2\alpha-1}.$$

(ix) To prove that the Pth quantile x_P is $x_P = c(1-P)^{-1/\alpha}$.

$$P = F(x_P) = 1 - \frac{c^\alpha}{(x_P)^\alpha}$$

$$\text{or } \frac{c^\alpha}{(x_P)^\alpha} = 1 - P$$

$$\text{or } (x_P)^\alpha = \frac{c^\alpha}{1-P}$$

$$\text{or } x_P = c(1-P)^{-1/\alpha}.$$

Home Task 2.2: Suppose $x_1, x_2, \ldots x_n$ are iid rs from $X \sim$ Pareto(c, α). Show that

$$E(X_{(1)}) = E(\text{Min}(x_1, x_2, \ldots x_n)) = \frac{n\hat{\alpha}\hat{c}}{n\hat{\alpha} - 1}.$$

Solution to Qn. 2.2:

Suppose $f_{X_{(1)}}(x)$ and $F_{X_{(1)}}(x)$ are respectively the pdf and DF of first order statistics of Pareto distribution of random samples $x_1, x_2, \ldots x_n$ and the f(x) and F(x) are respectively the pdf and DF of Pareto distribution.

$$1 - F_{X_{(1)}}(x) = P(X_{(1)} > x)$$

$$= P(X_{(1)} > x, X_{(2)} > x, \ldots, X_{(n)} > x)$$

$$= P(X_1 > x, X_2 > x, \ldots, X_n > x)$$

$$= P(X_1 > x).P(X_2 > x) \ldots P(X_n > x) = (1 - F(x))^n$$

$$F_{X_{(1)}}(x) = 1 - (1 - F(x))^n.$$

$$f_{X_{(1)}}(x) = \frac{dF_{X_{(1)}}(x)}{dx} = n(1 - F(x))^{n-1} f(x)$$

Since we have

$$f(x) = \frac{\alpha c^{\alpha}}{x^{\alpha+1}}, \text{for } x \geq c \text{ and } F(x) = 1 - \frac{c^{\alpha}}{x^{\alpha}}.$$

$$f_{X_{(1)}}(x) = \frac{dF_{X_{(1)}}(x)}{dx} = n\left(\frac{c^{\alpha}}{x^{\alpha}}\right)^{n-1} \frac{\alpha c^{\alpha}}{x^{\alpha+1}}$$

$$= \frac{n\alpha c^{n\alpha}}{x^{n\alpha+1}}.$$

Thus, $X_{(1)} \sim \text{Pareto}(c, n\alpha)$. Hence

$$E(X_{(1)}) = \frac{n\alpha c}{n\alpha - 1}.$$

Home Task 2.3: Suppose X_1 and X_2 are two IID non-negative r.v.s with common mean μ. Then show that

$$E|X_1 - X_2| = 2\mu(LR). \Rightarrow LR = \frac{\Delta_1}{2\mu}.$$

Solution to Qn. 2.3:

$$\Delta_1 = E|X_1 - X_2|$$

$$= \int_0^\infty \int_0^\infty |x_1 - x_2| dF(x_2) dF(x_1)$$

$$= \int_0^\infty \left[\int_0^{x_1} (x_1 - x_2) f(x_2) dx_2 + \int_{x_1}^\infty (x_2 - x_1) f(x_2) dx_2\right] f(x_1) dx_1$$

$$= I_1 + I_2 \text{, say.}$$

Since,

$$E(X_1 - X_2) = 0 = I_1 - I_2 \text{,}$$

we have

$$\Delta_1 = 2\int_0^\infty \int_0^{x_1} (x_1 - x_2) f(x_2)\, f(x_1) dx_2\, dx_1$$

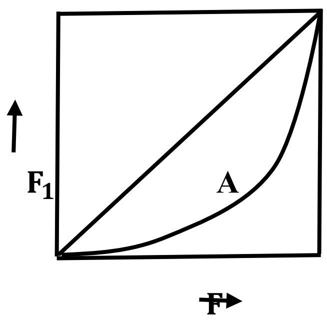
$$= 2\left[\int_0^\infty x_1 f(x_1) F(x_1) dx_1 - \int_0^\infty \mu f(x_1) F_1(x_1) dx_1\right],$$

where $\mu = E(X)$. Thus, we have

$$\Delta_1 = 2\left(\int_0^1 \mu F dF_1 - \int_0^1 \mu F_1 dF\right)\text{, because } dF_1 = \frac{x dF}{\mu}.$$

$$= 2\mu\left(\int_0^1 F dF_1 - \int_0^1 F_1 dF\right)$$

Now, consider the following Lorenz diagram



$$\Delta_1 = 2\mu\left[\left(\frac{1}{2} + A\right) - \left(\frac{1}{2} - A\right)\right] = 2\mu(2A) = 2\mu LR.$$

Hence $\qquad LR = \dfrac{\Delta_1}{2\mu}.$