

Continuous and Absolutely Continuous Random Variables

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Definition: A random variable X is *absolutely continuous* if there exists a function $f(x)$ such that

$$\Pr(X \in A) = \int_A f(x) dx$$

for all Borel sets A . (A *Borel set* is any member of the Borel σ -algebra on $(-\infty, \infty)$. The latter is the smallest σ -algebra for $\Omega = (-\infty, \infty)$ containing all subintervals of all types.)

The function $f(x)$ is called the *probability density function* (pdf) of the random variable X . It is ordinarily denoted $f_X(x)$. We often omit the word *function* and simply call f the *probability density*.

Facts:

1. Absolute continuity implies continuity, i.e. if X is absolutely continuous, then it is also continuous.

Proof: If X is absolutely continuous, then for any x , the definition of absolute continuity implies

$$\Pr(X=x) = \Pr(X \in \{x\}) = \int_{\{x\}} f(x') dx' = 0$$

where the last equality follows from the fact that integral of a function over a singleton set is 0. Since $\Pr(X=x) = 0$ for all x , X is continuous.

2. The converse is false, i.e. if X is continuous it is not necessarily absolutely continuous. An example of a continuous random variable that is not absolutely continuous will be given later.
3. Continuous random variables that are not absolutely continuous are rare. Hence from now on, unless otherwise specified, we assume that continuous random variables are absolutely continuous. That is, unless otherwise specified, we assume that every continuous random variable has a probability density function. Our textbook (Gubner) does not distinguish continuity and absolute continuity.
4. A random variable is continuous iff every countable set (finite or countably infinite) has probability zero.
5. A random variable is absolutely continuous iff every *set of measure zero* has zero probability. (See the definition below.)
6. Definition: A set F has *measure zero* if and only if it can be covered by a countable collection of intervals with arbitrarily small total length, i.e., for any number $\varepsilon > 0$, there is a sequence of intervals I_1, I_2, \dots such that $\bigcup_{i=1}^{\infty} I_i \supset F$ and $\sum_{i=1}^{\infty} |I_i| \leq \varepsilon$.
7. Equivalently, a set has measure zero if it has Borel measure zero.

Note that here we say Borel measure, not Borel probability measure.

We have previously discussed the Borel *probability* measure, which for a finite interval sample space $\Omega = [u, v]$ is the unique probability measure on the Borel σ -algebra for Ω such that $P((a, b)) = (b-a)/(v-u)$ for all a, b such that $u \leq a < b \leq v$.

Here on the other hand, the *Borel measure* is defined for sample space $\Omega = (-\infty, \infty)$, the σ -algebra is the Borel σ -algebra for Ω , and the Borel measure is the unique *measure* on the Borel σ -algebra such that $P((a, b)) = b-a$ for all $u \leq a < b \leq v$. It satisfies all the axioms of probability except that $P(\Omega) = \infty$, rather than 1. For this reason, it is called a *measure*, but not a *probability measure*.

8. Any finite or countable infinite set has measure zero. There are also some uncountable sets that have measure zero.