Statistics II: Introduction to Inference

Problem set 3

1. (**Additive properties**) *Prove the following statements using moment generating functions.*

   (a) Let $X_i \overset{ind}{\sim} \texttt{binomial}(n_i, p)$, for $i = 1, \ldots, k$, then $T = \sum_{i=1}^{k} X_i$ follows $\texttt{binomial}(\sum_i n_i, p)$.

   (b) Let $X_i \overset{ind}{\sim} \texttt{Poisson}(\lambda_i)$, for $i = 1, \ldots, n$, then $T = \sum_{i=1}^{n} X_i$ follows $\texttt{Poisson}(\sum_i \lambda_i)$.

   (c) Let $X_i \overset{ind}{\sim} \texttt{normal}(\mu_i, \sigma_i^2)$, for $i = 1, \ldots, n$, then $T = \sum_{i=1}^{n} X_i$ follows $\texttt{normal}\left(\sum_i \mu_i, \sum_i \sigma_i^2\right)$.

   (d) Let $X_i \overset{ind}{\sim} \texttt{Gamma}(\alpha_i, \beta)$, for $i = 1, \ldots, n$, then $T = \sum_{i=1}^{n} X_i$ follows $\texttt{Gamma}\left(\sum_i \alpha_i, \beta\right)$.

   (e) Let $X_i \overset{ind}{\sim} \chi^2_{n_i}$, for $i = 1, \ldots, k$, then $T = \sum_{i=1}^{k} X_i$ follows χ^2_N where $N = \sum_i n_i$.

2. Let $X \sim \texttt{normal}(\mu, \sigma^2)$ distribution, then $T = aX + b \sim \texttt{normal}(a\mu + b, a^2\sigma^2)$.

3. Let $X \sim \texttt{Gamma}(\alpha, \beta)$ distribution, then $T = aX \sim \texttt{Gamma}(\alpha, \beta/a)$.

4. Let $X \sim \texttt{beta}(n/2, m/2)$ distribution, then $T = mX/\{n(1-X)\} \sim F_{n,m}$.

5. Let $X \sim \texttt{uniform}(0, 1)$ distribution, and $\alpha > 0$ then $T = X^{1/\alpha} \sim \texttt{beta}(\alpha, 1)$.

6. Let $X \sim \texttt{Cauchy}(0, 1)$ distribution, then $T = 1/(1 + X^2) \sim \texttt{beta}(0.5, 0.5)$.

7. Let $X \sim \texttt{uniform}(0, 1)$ distribution, then $T = -2 \log X \sim \chi^2_2$.

8. Let X be distributed as some absolutely continuous distribution with cdf G_X, then $T = G_X(X) \sim \texttt{uniform}(0, 1)$.

9. Let the random variable X have pdf

$$f(x) = \sqrt{\frac{2}{\pi}} \exp\{-x^2/2\}, \qquad x > 0.$$

   (a) Find $E(X)$ and $\mathrm{var}(X)$.

   (b) Find an appropriate transformation $Y = g(X)$ and α, β, so that $Y \sim \texttt{Gamma}(\alpha, \beta)$.

10. Let X is distributed as $\texttt{Gamma}(\alpha, \beta)$ distribution, $\alpha, \beta > 0$. Then show that the r-th order population moment

$$E\left(X^r\right) = \beta^{-r} \frac{\Gamma(r + \alpha)}{\Gamma(\alpha)}, \qquad r > -\alpha.$$

11. Let the bivariate random variable (X, Y) has a joint pdf

$$f_{X,Y}(x, y) = \begin{cases} C(x + 2y) & \text{if} \quad 0 < y < 1,\ 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the marginal distribution of Y.

   (b) Find the conditional distribution of Y given $X = 1$.

(c) Compare the expectations of the above two distributions of Y.

(d) Find the covariance between X and Y.

(e) Find the distribution of $Z = 9/(2Y+1)^2$.

(f) What is $P(X > Y)$?

12. Let $X \sim \texttt{normal}(0,1)$. Define $Y = -X\mathbb{I}(|X| \leq 1) + X\mathbb{I}(|X| > 1)$. Find the distribution of Y.

(Hint: Apply the CDF approach)

13. Let $X \sim \texttt{normal}(0,1)$. Define $Y = \text{sign}(X)$ and $Z = |X|$. Here $\text{sign}(\cdot)$ is a $\mathbb{R} \to \{0,1\}$ function such that $\text{sign}(a) = 1$ if $a \geq 0$, and $\text{sign}(a) = -1$ otherwise.

(a) Find the marginal distributions of Y and Z.

(b) Find the joint CDF of (Y, Z). Hence or otherwise prove that Y and Z are independently distributed.

14. Suppose $X_1, \cdots, X_n \overset{\text{IID}}{\sim} \texttt{normal}\left(\mu_x, \sigma^2\right)$, $Y_1, \cdots, Y_m \overset{\text{IID}}{\sim} \texttt{normal}\left(\mu_y, \sigma^2\right)$, and all the random variables $\{X_1, \cdots, X_n, Y_1, \cdots, Y_m\}$ are mutually independent. Then find the distribution of $T := S_X^{\star 2}/S_Y^{\star 2}$, where $S_X^{\star 2}$ and $S_Y^{\star 2}$ are the unbiased sample variances of X and Y, respectively.

15. Let $X_1, \cdots, X_n$ be iid random variables with continuous CDF F_X, and suppose $E(X_1) = \mu$. Define the random variables $Y_1, \cdots, Y_n$ as follows:

$$Y_i = \begin{cases} 1 & \text{if} X_i > \mu \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $E(Y_1)$.

(b) Find the distribution of $\sum_{i=1}^{n} Y_i$.

16. Let $X_1, \cdots, X_n \overset{\text{IID}}{\sim} \texttt{normal}\left(\mu, \sigma^2\right)$, and S_n^2 be the sample variance. Find a function of S_n^2, say $g\left(S_n^2\right)$, which satisfies $E\left[g\left(S_n^2\right)\right] = \sigma$. (Hint: You may use problem 2.)

17. [1] Let $X_1, \cdots, X_n$ be iid with pdf f_X and CDF F_X. Find the CDF of r-th order statistics $X_{(r)}$. Hence derive the pdf of $X_{(r)}$.

18. Let Y have a $\texttt{Cauchy}(0,1)$ distribution.

(a) Find the CDF of Y.

(b) Hence provide a method of simulating random samples from $\texttt{Cauchy}(0,1)$ distribution, starting from $\texttt{uniform}(0,1)$ random variables.

[1] You may skip this problem.