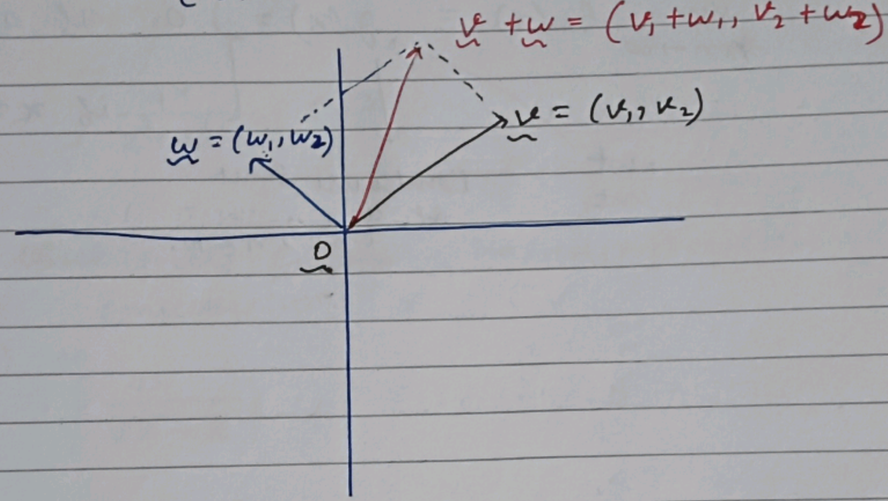
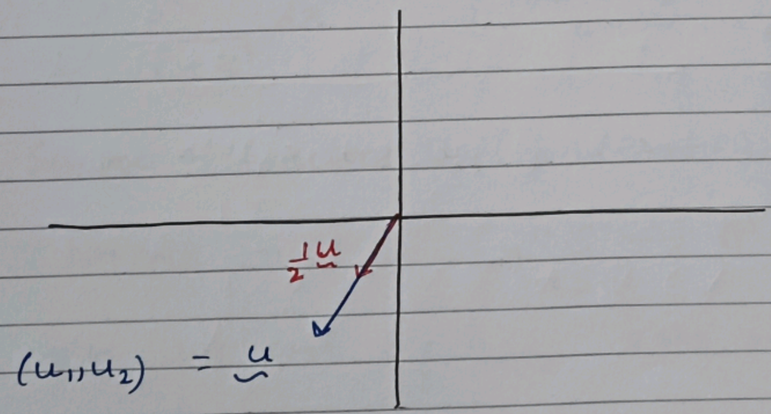
Linear Algebra

Recall that $\mathbb{R}$ satisfies $(A_1)-(A_6)$. Note that $\mathbb{C}$ also satisfies $(A_1)-(A_6)$.

Motivating Example

$V = \mathbb{R}^2 = \{ \underset{\sim}{v} = (v_1, v_2) : v_1 \in \mathbb{R}, v_2 \in \mathbb{R} \}$.





$\forall \alpha \in \mathbb{R}$ and $\forall \underset{\sim}{u} \in V = \mathbb{R}^2$,

$$\alpha \underset{\sim}{x} = \alpha (x_1, x_2) = (\alpha x_1, \alpha x_2)$$

$$\alpha \underset{\sim}{u} = \alpha (u_1, u_2) = (\alpha u_1, \alpha u_2)$$

Note: we have a binary operation

$$V \times V \to V$$

$$(\underset{\sim}{v}, \underset{\sim}{w}) \mapsto (\underset{\sim}{v} + \underset{\sim}{w})$$ vector addition

called vector addition on $V = \mathbb{R}^2$ and another map

$$\mathbb{R} \times V \to V$$

$$(\alpha, \underset{\sim}{u}) \longmapsto \alpha \underset{\sim}{u}$$

called the scaler multiplication (of vectors) s.t the following properties hold (that follow from (A1) - (A6)).

(VS1) Associativity of vector addition.

$\forall\ \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V,$

$$(\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w} = \underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w})$$

(VS2) Commutativity of Vector Addition

$\forall\ \underset{\sim}{u}, \underset{\sim}{v} \in V,\quad \underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$

(VS3) Existence of Identity element for vector addition.

$\exists\ \underset{\sim}{0} \in V$ (called the zero vector)

s.t. $\underset{\sim}{v} + \underset{\sim}{0} = \underset{\sim}{v}\ \forall\ \underset{\sim}{v} \in V$.

($\underset{\sim}{0}$ = Identity element for vector addition)

~~(VS4)~~

(VS4) Existence of Inverse Elements for vector addition.

For each $\underset{\sim}{v} \in V$, $\exists\ -\underset{\sim}{v} \in V$

(called the (vector) additive inverse / negative of $\underset{\sim}{v}$).

s.t. $\underset{\sim}{v} + (-\underset{\sim}{v}) = \underset{\sim}{0}$.

(VS5) Compatibility of scaler multiplication with ~~vector addition~~ the multiplication of Real numbers.

$\forall\ \alpha, \beta \in \mathbb{R}$ and $\forall\ \underset{\sim}{v} \in V$,

$$\alpha(\beta \underset{\sim}{x}) = (\alpha\beta)\underset{\sim}{x}$$

$$\alpha(\beta \underset{\sim}{v}) = (\alpha\beta)\underset{\sim}{v}$$

(VS6) Existence of Identity Element for scaler multiplication

The multiplicative identity element 1 of $\mathbb{R}$ satisfies

$$1 \times \underset{\sim}{x} = \underset{\sim}{x}$$

$$1 \times \underset{\sim}{v} = \underset{\sim}{v} \quad \forall\ \underset{\sim}{v} \in V.$$

Identity element of scalar multiplication

(VS 7) Distributivity of scalar multiplication over / w.r.t vector addⁿ.

$\forall\ \alpha \in \mathbb{R}$ and $\forall\ \underset{\sim}{u}, \underset{\sim}{v} \in V$,

$$\alpha(\underset{\sim}{u} + \underset{\sim}{v}) = \alpha\underset{\sim}{u} + \alpha\underset{\sim}{v}.$$

(VS 8) Distributivity of scalar multiplication over / w.r.t addition of Real numbers.

$\forall\ \alpha, \beta \in \mathbb{R}$ and $\forall\ \underset{\sim}{v} \in V$,

$$(\alpha + \beta)\underset{\sim}{v} = \alpha\underset{\sim}{v} + \beta\underset{\sim}{v}$$

v tilde

Ex: Prove / Verify (VS1) – (VS8) for $V = \mathbb{R}^2$.

TUT-C

Q) Show that $f : [0,1] \to \mathbb{R}$ $\quad f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$

is not R.I on $[0, 1]$.

Pf: For any partition P of $[0,1]$, $P = \{0, x_1, x_2, \ldots, x_{n-1}\}$

$\forall \ x < y$, $\exists \ q \in (x, y)$ as well as $r \in (x,y)$
where $q \in \mathbb{Q}$, $r \notin \mathbb{Q}$

Hence $\inf\limits_{x \in [x_{i-1}, x_i]} f(x) = 0$, $\sup\limits_{x \in [x_{i-1}, x_i]} f(x) = 1$

$L(f, P) = 0$

Similarly $U(f, P) = 1(x_1 - x_0) + 1(x_2 - x_1) + \ldots + (x_n - x_{n-1})$

Since this is true for any part P,

$$\underline{\int_0^1} f(x)\,dx = 0 \neq \overline{\int_0^1} f(x)\,dx = 1$$

Lec - 17 4/11/2024

Ex $n \in \mathbb{N}$

$V = \mathbb{R}^n = \{ \underset{\sim}{v} = (v_1, v_2, \ldots, v_n) \}$ and define

vector addition componentwise i.e. $\underset{\sim}{v} + \underset{\sim}{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$,

where $\underset{\sim}{v} = (v_1, v_2, \ldots, v_n)$,

$\underset{\sim}{w} = (w_1, w_2, \ldots, w_n) \in V = \mathbb{R}^n$

Also define scaler multiplication.

$\alpha \underset{\sim}{v} = \alpha (v_1, v_2, \ldots, v_n) = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n)$

where $\alpha \in \mathbb{R}$, $\underset{\sim}{v} \in V = \mathbb{R}^n$, It is easy to check that $V = \mathbb{R}^n$ also satisfies (VS1) - (VS8)

Ex : $X = [0,1]$ and look at the space

$V = \mathbb{R}^X = \{ f \mid f : X \to \mathbb{R} \text{ is a map} \}$

of all real-valued f^ns defined on

$X = [0,1]$.

Example $A, B \to$ sets

$0 < |A| < \infty$, $0 < |B| < \infty$ then

How many maps ?

$B \to A$

Ans) $\to |A|^{|B|}$

$A^B = \{ f \mid f : B \to A \text{ is a map} \}$

Define Vector Addition and scalar multiplication pointwise, i.e, $\forall\, \alpha \in \mathbb{R}$ and $\forall\, f, g \in V$, define $f+g \in V$ and $\alpha f \in V$ as follows:

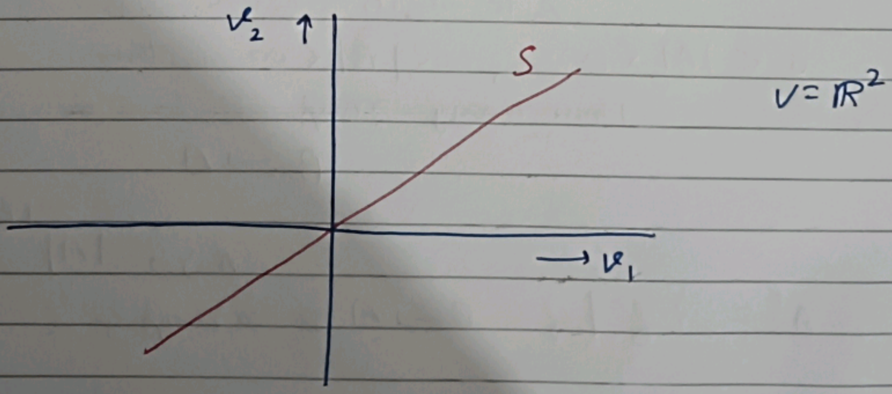
$$(f+g)(x) = f(x) + g(x) \quad , x \in X,$$

$$(\alpha f)(x) = \alpha f(x) \quad , x \in X.$$

It can be checked that V also satisfies (VS1)–(VS8) with the zero function $\underset{\sim}{0}$ (i.e., $\underset{\sim}{0}(x) = 0$ $\forall\, x \in X$) being the identity element for vector addition and the negative $(-f)$ of a f^n $f \in V$ (i.e., $(-f)(x) = -f(x)$ $\forall\, x \in X$) being the vector additive inverse of f.

<u>Ex</u>: Define $S = \{(v_1, v_2) \in V = \mathbb{R}^2 : v_1 = v_2\}$ $\subseteq V = \mathbb{R}^2$.

Let S borrow the vector addition and scalar multiplication from $V = \mathbb{R}^2$. One can check that S also satisfies (VS1) – (VS8) with these borrowed oper^n.



Vector Addⁿ

$S \times S \longrightarrow S$

$(\underset{\sim}{v}, \underset{\sim}{w}) \longmapsto (\underset{\sim}{v} + \underset{\sim}{w})$

Scalar Addⁿ

$\mathbb{R} \times S \longrightarrow S$

$(\alpha, \underset{\sim}{v}) \longmapsto \alpha \underset{\sim}{v}$

Defⁿ: Let V be a non empty set with a binary operation called vector addition on V, i.e., a map $V \times V \longrightarrow V$ defined by

$$(\underset{\sim}{u}, \underset{\sim}{v}) \longmapsto \underset{\sim}{u} + \underset{\sim}{v} \quad \forall (\underset{\sim}{u}, \underset{\sim}{v}) \in V^2$$

and a scalar multiplication by real numbers, i.e, a map $\mathbb{R} \times V \longrightarrow V$ defined by

$$(\alpha, \underset{\sim}{v}) \longmapsto \alpha \underset{\sim}{v} \quad \forall \underset{\sim}{v} \in V$$

such that the axioms (VS1) – (VS8) are satisfied.

Then we say that V is a vector space / linear space over $\mathbb{R}$.

Remarks: ① Elements of $V \rightarrow$ vectors

Elements of $\mathbb{R} \rightarrow$ scalers

② One can define a vector space over $\mathbb{C}$ or more generally over any 'field' F.

Satisfies (A_1)–(A_6)

for this course $F = \mathbb{R}$.

Ex: Let $n \in \mathbb{N}$,

1) $V = \mathbb{R}^n$ with componentwise vector add^n and componentwise scalar multiplication, is a vector space over $\mathbb{R}$.

We shall define $\mathbb{R}^0 = \{\underset{\sim}{0}\}$

2) $X = [0, 1]$.

$V = \{ f \mid f : X \to \mathbb{R} \text{ is a map} \}$.

Define both vector addition and scalar multip^n pointwise.

Then V is a vector space over $\mathbb{R}$.

Thm: Suppose V is a vector space over $\mathbb{R}$.

Then $\forall\ \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V$ and $\forall\ \alpha, \beta, \gamma \in \mathbb{R}$, we have

(i) $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{w}$

$\Rightarrow \underset{\sim}{v} = \underset{\sim}{w}$ (cancellation law for vector add^n)

(ii) $\underset{\sim}{0}$ is unique

(iii) $-\underset{\sim}{v}$ is unique (given a $\underset{\sim}{v}$).

(iv) $\underset{\sim}{v} + \underset{\sim}{v} = \underset{\sim}{v} \Rightarrow \underset{\sim}{v} = 0$

(v) $0 \underset{\sim}{v} = \underset{\sim}{0}$

(vi) $\alpha \underset{\sim}{0} = \underset{\sim}{0}$

(vii) $\alpha \underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \alpha = 0$ or $\underset{\sim}{x} = \underset{\sim}{0}$

$\alpha \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow \alpha = 0$ or $\underset{\sim}{v} = \underset{\sim}{0}$

i.e. $\alpha \underset{\sim}{x} = \underset{\sim}{0}$, $\alpha \in \mathbb{R} - \{0\} \Rightarrow \underset{\sim}{x} = \underset{\sim}{0}$
$\alpha \underset{\sim}{v} = \underset{\sim}{0}$, $\alpha \in \mathbb{R} - \{0\} \Rightarrow \underset{\sim}{v} = \underset{\sim}{0}$
(Cancellation Law for scalar Multipⁿ)

(viii) $(-\alpha) \underset{\sim}{v} = -(\alpha \underset{\sim}{v})$.
In particular, $(-1) \underset{\sim}{v} = -\underset{\sim}{v}$

Exe (optional): Prove (i) – (viii) above
in generality.)

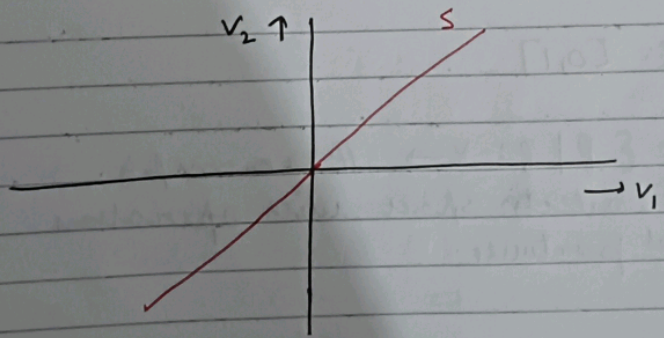
Mandatory: Prove them for $V = \mathbb{R}^n$.

Exe : (Atleast for $V = \mathbb{R}^n$)

$(\alpha + \beta) \underset{\sim}{v} = (\alpha + \gamma) \underset{\sim}{v} \Rightarrow \underset{\sim}{v} = 0$ or $\beta = \gamma$

$\forall\ \alpha, \beta, \gamma \in \mathbb{R}$ and $\forall\ \underset{\sim}{v} \in V$.

3)



$V = \mathbb{R}^2$

$S = \{ (v_1, v_2) \in \mathbb{R}^2 : v_1 = v_2 \} \subseteq V = \mathbb{R}^2$

vector space

Vector space to start with

Def^n: If V is a vector space over $\mathbb{R}$ and $\emptyset \neq S \subseteq V$, then S is called a (linear) subspace / (vector) subspace of V if S, with vector add^n and scalar mult^n borrowed from V, forms a vector space.

Notat^n: $S \subseteq V$ (subspace)

Ex: ① $V = \mathbb{R}^2$.

$$S = \{ \underset{\sim}{v} = (v_1, v_2) \in \mathbb{R}^2 : v_1 = v_2 \}.$$

Then S is a linear subspace of V.

② $V = \mathbb{R}^3$

Ex Show that

$$S = \{ \underset{\sim}{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} v_i = 0 \}$$

is a linear subspace of V.

③ let $X = [0,1]$

$V = \mathbb{R}^X = \{ f \mid f : X \to \mathbb{R} \text{ is a map} \}$

V forms a vector space with operations defined pointwise.

Ex: Define $S = \{ f \in V : f \text{ is continuous} \}$

$= \{ f \mid f : [0,1] \to \mathbb{R} \text{ is a cont. map} \}$.

Show that S is a vector subspace of V.

Remark: In Ex-3 above, if we define $S_1 = \mathbb{R}[0,1]$, then it can be shown that S_1 is also a linear subspace of V. Infact, $S \underset{(\text{Subsp})}{\subseteq} S_1 \underset{(\text{Subsp})}{\subseteq} V$.

Thm: Let V be any vector space (over $\mathbb{R}$). Let S be a non empty subset $S \subseteq V$ is a (linear/vector) subspace of V iff S is closed under taking linear combⁿs., i.e., $\underbrace{\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}}_{\substack{\text{a linear comb}^n \text{ of vector} \\ \underset{\sim}{u}, \underset{\sim}{v} \in V \\ \text{with scalar} \\ \text{coeff } \alpha, \beta \in \mathbb{R} \\ \text{respectively.}}} \in S$ $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \underset{\sim}{u}, \underset{\sim}{v} \in S$.

Remark: Closed under taking linear combⁿs $\Leftrightarrow$ Closed under both vector addⁿ and scalar multⁿ.

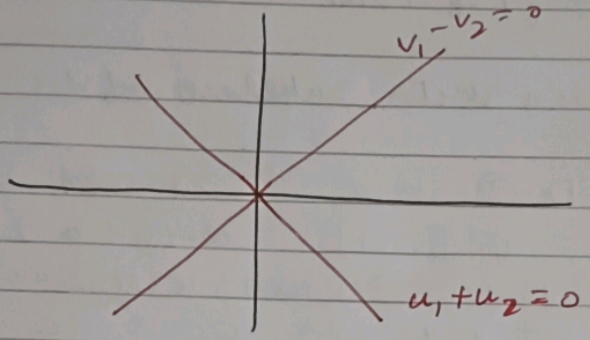
Cor: (1) $S \underset{(\text{Subsp})}{\subseteq} V \Rightarrow \underset{\sim}{0} \in S$

(2) $\underbrace{S_0}_{\text{smallest possible subspace}} = \{ \underset{\sim}{0} \} \underset{(\text{Subsp})}{\subseteq} V$

Trivial subsp. of V.

(3) $V \underset{(\text{Subsp})}{\subseteq} V$

Any other subspace S, i.e. $\{0_V\} \subsetneq S \subsetneq V$, is called a non trivial subspace of V.



$V = \mathbb{R}^2$

$$S = \{(v_1, v_2) \in \mathbb{R}^2 : (v_1 + v_2)(v_1 - v_2) = 0\}$$

Ex: Show $S \subseteq V$ but S is not a linear subspace of V.

Ex (4) Let $X = [0,1]$

$$V = \mathbb{R}^X = \{f \mid f : [0,1] \to \mathbb{R} \text{ is a map}\}$$

Define $\mathcal{P} = \{f \in V : f \text{ is a polynomial } f^n\}$

Ex: Show that $\underset{(\text{subsp})}{\mathcal{P} \subseteq V}$

Ex $n \in \mathbb{N}$

Define $\mathcal{P}_n = \{f \in \mathcal{P} : \text{degree}(f) \leq n-1\}$

In part, $\mathcal{P}_1 = \{f \in \mathcal{P} : f \text{ is a const } f^n\}$.

→ Just understand everything for $V = \mathbb{R}^n$; $n \in \mathbb{N}$

<u>Exe</u> : Fix $n \in \mathbb{N}$. show that

$$\underset{(\text{subsp})}{\mathcal{P}_n \subseteq} \ \underset{(\text{subsp})}{\mathcal{P} \subseteq} \ V$$

L-18 6/11/24

To understand $V = \mathbb{R}^n$ and its linear subspaces.

Def^n: Let V be a vector space (over $\mathbb{R}$) we take $k \in \mathbb{N}$ and vectors $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_k} \in V$. Any vector of the form

$$\underset{\sim}{v} = \alpha_1 \underset{\sim}{v_1} + \alpha_2 \underset{\sim}{v_2} + \dots + \alpha_k \underset{\sim}{v_k} = \sum_{i=1}^{k} \alpha_i v_i$$

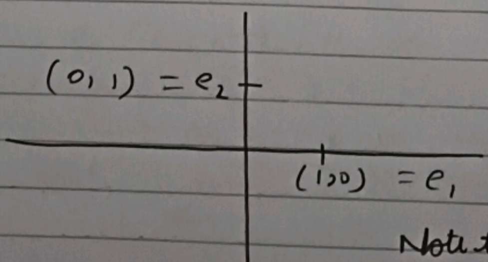
where $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ is called a linear comb^n of $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_k} \in V$.

Exe Let V be a vector space and $\phi \neq S \subseteq V$, Then show that $\underset{(\text{subset})}{S \subseteq V}$ iff $\forall k \in \mathbb{N}$ and $\forall \alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$ and $\forall \underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_k} \in S$, $\sum_{i=1}^{k} \alpha_i v_i \in S$

[Hint If part $\rightarrow$ just use $k=2$ & the only if part $\rightarrow$ use induction on k]

Ex $V = \mathbb{R}^2$ and

$$B = \{ e_1 = (1,0), e_2 = (0,1) \}$$



$V = \mathbb{R}^2$

Note that any $\underset{\sim}{v} \in V = \mathbb{R}^2$ $\;\overset{\shortparallel}{(v_1, v_2)}$ can be written as

$$\begin{aligned} \underset{\sim}{v} = (v_1, v_2) &= (v_1, 0) + (0, v_2) \\ &= v_1 (1,0) + v_2 (0,1) \\ &= v_1 \underset{\sim}{e_1} + v_2 \underset{\sim}{e_2} \\ &= \sum_{i=1}^{2} v_i e_i . \end{aligned}$$

$$= \sum_{i=1}^{2} v_i e_i .$$

we shall say that V is the linear span..

$$\bar{B} = sp(B) = \text{span of } B.$$

Def^n: Let V be any vector space
For a non empty set $B = \{v_1, v_2, \dots, v_k\} \subseteq V$
of vectors we define the linear span of B as the set
$\bar{B}$ = span of B = $sp(B)$.

$$\bar{B} = sp(B) = \left\{ \sum_{i=1}^{k} \alpha_i v_i : \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R} \right\}$$

we also define $sp(\phi) = \bar{\phi} = \{\underset{\sim}{0}\}$

Ex: Let V be a vector space. $B \subseteq V$ is any finite subset
show that $\bar{B} \subseteq V$
(subspace)

In this example, $V = \mathbb{R}^2$

$$B = \{\underset{\sim}{e_1} = (1,0), e_2 = (0,1)\} \text{ and}$$

$\bar{B} = V = \mathbb{R}^2$. ($\bar{B} \subseteq V$ is obvious
$V \subseteq \bar{B}$ has been shown)

Def^n: In the above setup whenever $S = \bar{B} = sp\{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_k}\}$
we say that the vectors $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_k}$ generate
S or simply say B generates S.

Ex: If $V = \mathbb{R}^2$, then $B = \{e_1, e_2\}$ generates V.

More generally if $V = \mathbb{R}^n$, then $B = \{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_n}\}$

(where $\underset{\sim}{e_1} = (1, 0, 0, \dots, 0) \in \mathbb{R}^n$

$$\underset{\sim}{e_2} = (0, 1, 0, \dots, 0) \in \mathbb{R}^n$$

$$\vdots$$

$$\underset{\sim}{e_n} = (0, 0, 0, \dots, 1) \in \mathbb{R}^n$$

generates V, why?

Clearly $\overline{B} \subseteq V$. To show $V \subseteq \overline{B}$.

observe that any $\underset{\sim}{v} = (v_1, v_2, \dots, v_n) \in V = \mathbb{R}^n$

is of the form $\underset{\sim}{v} = (v_1, v_2, \dots, v_n)$

$= (v_1, 0, \dots, 0) + (0, v_2, 0, \dots, 0) + \dots$

$\dots + (0, 0, \dots, 0, v_n)$

$= v_1 \underset{\sim}{e_1} + v_2 \underset{\sim}{e_2} + \dots + v_n \underset{\sim}{e_n} \in \overline{B}$

Claim: If $V = \mathbb{R}^2$, then

$B_1 = \{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underbrace{\underset{\sim}{e_1} + \underset{\sim}{e_2}}\}$ $\longrightarrow$ Redundant.

$= \{(1,0), (0,1), (1,1)\}$ also generates V.

Proof: clearly $\overline{B_1} \subseteq V$

To show $V \subseteq \overline{B_1}$

Take any vector $\underset{\sim}{v} = (v_1, v_2) = V = \mathbb{R}^2$

Then $\underset{\sim}{v} = v_1 \underset{\sim}{e_1} + v_2 \underset{\sim}{e_2} + 0(e_1 + e_2) \in \overline{B_1}$

Therefore $\overline{B_1} = V$

However $(1,1) = \underset{\sim}{e_1} + \underset{\sim}{e_2}$ is redundant.

b/c $\underset{\sim}{e_1}$ and $\underset{\sim}{e_2}$ generate V anyway.

This is happening b/c ....

$1(\underset{\sim}{e_1}+\underset{\sim}{e_2}) \neq (-1)\underset{\sim}{e_1}+(-1)\underset{\sim}{e_2}=\underset{\sim}{0}$

Exe Show that $\underset{\sim}{e_1}=(1,0)$ and $\underset{\sim}{e_1}+\underset{\sim}{e_2}=(1,1)$ also generate $V=\mathbb{R}^2$

Q^n: can we remove one of them and yet generate $V=\mathbb{R}^2$?

Ans) NO! (B/c $\underset{\sim}{e_1}$ and $\underset{\sim}{e_1}+\underset{\sim}{e_2}$ are linearly independent)

Def^{n}s: for any vector space V, a finite non empty collection $C=\{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots, \underset{\sim}{v_k}\} (\neq \emptyset) \subseteq V$ of vector is c/d linearly independent (or $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots, v_k$ are c/d linearly independent) if

$$\sum_{i=1}^{k} \alpha_i \underset{\sim}{v_i}=\underset{\sim}{0}, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$$

Trivial linear comb^n

$$\Rightarrow \alpha_1=\alpha_2=\cdots=\alpha_k=0$$

In other words no non trivial linear comb^n of $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots, \underset{\sim}{v_k}$ is equal to $\underset{\sim}{0}$.

Ex (i) $V=\mathbb{R}^2$

$\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_1}+\underset{\sim}{e_2}$ are not linearly independent

b/c $\underbrace{1(\underset{\sim}{e_1}+\underset{\sim}{e_2})+(-1)e_1+(-1)e_2=0}_{\text{Non Trivial lin comb}^n \text{ of } e_1, e_2, \& e_1+e_2}$

(2) In the setup of (1) if $\exists\ \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ s.t $(\alpha_1, \alpha_2, \ldots, \alpha_k) \neq (0,0,\ldots,0)$

and $\sum_{i=1}^{k} \alpha_i \underset{\sim}{v_i}=\underset{\sim}{0}$ ↳ Non Trivial linear comb^n

$\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots, \underset{\sim}{v_k}$ are c/d linearly dependent or

$C=\{v_1, v_2, \ldots, v_k\}$ is c/d linearly dependent.

(3) If $C \neq \phi$, then we define C to be linearly independent.

Ex

(2) $V = \mathbb{R}^n$, fix $m \in \{1, 2, \dots, n\}$ $m \leq n$

Let $C = \{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_m}\}$

Then C is linearly independent.

Suppose $\sum_{i=1}^{m} \alpha_i \underset{\sim}{e_i} = \underset{\sim}{0}$

$\Rightarrow (\alpha_1, \alpha_2, \dots, \alpha_m, 0, \dots, 0) = \underset{\sim}{0}$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_m = 0$

$\Rightarrow \underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_m}$ are linearly independent.

(3) Ex $V = \mathbb{R}^n$ show that $C = \{\underset{\sim}{e_1}, \underset{\sim}{e_1} + \underset{\sim}{e_2}, \underset{\sim}{e_1} + \underset{\sim}{e_2} + \underset{\sim}{e_3}, \dots, \dots, \underset{\sim}{e_1} + \underset{\sim}{e_2} + \dots + \underset{\sim}{e_n}\}$ is linearly independent.

[$C = \{(1, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, 1, \dots, 1)\}$]

(4) Ex $V = \mathbb{R}^4$ show that

$\underset{\sim}{e_1}, \underset{\sim}{e_1} + 2\underset{\sim}{e_2}, \underset{\sim}{e_1} + \underset{\sim}{e_2} + , \underset{\sim}{e_3}$ and $\underset{\sim}{e_4}$

are linearly dependent.

Thm: Let V be any vector space, Then we have :-

i) If $B (\subseteq V)$ generates V.
then so does any finite superset of B which is also a subset of V.

ii) If $C (\subseteq V)$ is linearly independent
then so is any subset of C.

iii) If $\underset{\sim}{0} \in C$, then C is always linearly dependent.

iv) $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_K} \in V$ are linearly dependent iff atleast one of these vectors can be written as a linear combn of the other vectors.
In particular for $K=2$, $\underset{\sim}{v_1}$ & $\underset{\sim}{v_2}$ are linearly dependent iff one of them is a scalar multiple of the other

without loss of generality.

Proof: If part suppose (WLOG)
that $\underset{\sim}{v_1} = \sum_{i=2}^{K} \alpha_i \underset{\sim}{v_i}$ where $\alpha_1, \alpha_2, \dots, \alpha_K \in \mathbb{R}$
Then $\underset{\neq 0}{1} \cdot \underset{\sim}{v_1} + (-\alpha_2)\underset{\sim}{v_2} + (-\alpha_3)\underset{\sim}{v_3} + \dots + (-\alpha_K)(\underset{\sim}{v_K}) = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_K}$ are linearly dependent.

Only if part.
→ suppose $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_K}$ are linearly dependent.

$\Rightarrow \sum_{i=1}^{K} \alpha_i v_i = \underset{\sim}{0}$ for $\alpha_1, \alpha_2 \dots, \alpha_K \in \mathbb{R}$

s.t $(\alpha_1, \alpha_2, \dots, \alpha_K) \neq (0, 0, 0, \dots, 0)$
Let $\alpha_j \neq 0$ Then $\alpha_j v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} (-\alpha_i)\underset{\sim}{v_i}$

$$\Rightarrow v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} \underbrace{\left(\frac{-\alpha_i}{\alpha_j} \right)}_{\in \mathbb{R}} v_i$$

$\rightarrow$ v_j is a linear comb^n of the rest.

$\square$

, The standard/canonical values for $\mathbb{R}^n$.

Note that $\beta = \{ \underset{\sim}{e_1}, \underset{\sim}{e_2}, \ldots, \underset{\sim}{e_n} \} \subseteq V = \mathbb{R}^n$

is a linearly independent set that generates $V = \mathbb{R}^n$ we have a name for such a set.

Def^{n}s: Let V be any vector space over $\mathbb{R}$, if $k \in \mathbb{N} \cup \{0\}$

$$B = \begin{cases} \{ \underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots, \underset{\sim}{v_k} \} & \text{if } k \in \mathbb{N} \\ \phi & \text{if } k = 0 \end{cases}$$

$\subseteq V$ (finite subset) is linearly independent

such that $\bar{B} = V$, then we say

that V is a finite dimensional vector space over $\mathbb{R}$ and B is c/d a basis of/for V.

(If B is empty, then $V = \{\underset{\sim}{0}\}$)

(If $B = \emptyset$, then $V = \{\underset{\sim}{0}\}$)

L-19 11-Nov

Thm: Suppose V is a finite dimensional vector space over $\mathbb{R}$ (f d v s) $B \subseteq V$ (finite).
Then TFAE (The following Are equivalent):-

(i) $B = \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k\}$ is a basis of for V.

ii) B is a minimal generating set for V, i.e. no proper subset of B generates V.

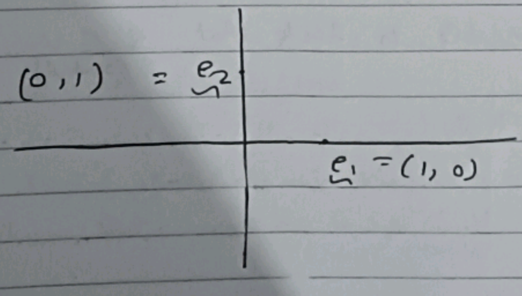
iii) B is a maximal linearly independent subset of V, i.e. $\forall\ \underset{\sim}{u} \in V \sim B$, $B \cup \{\underset{\sim}{u}\}$ is linearly dependent.

iv) Each vector $\underset{\sim}{w} \in V$ can be written uniquely as a linear comb^n of $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k$, i.e. for each $\underset{\sim}{w} \in V$, $\exists$ unique $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ s.t.

$$\underset{\sim}{w} = \sum_{i=1}^{k} \alpha_i \underset{\sim}{v}_i .$$

Ex (1) $V = \mathbb{R}^n$

Ex: show that the following sets are all bases of V:
i) $B = \{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \dots, \underset{\sim}{e}_n\}$ = standard Basis for $\mathbb{R}^n$.



ii) $B_1 = \{\underset{\sim}{e}_1, \underset{\sim}{e}_1 + \underset{\sim}{e}_2, \underset{\sim}{e}_1 + \underset{\sim}{e}_2 + \underset{\sim}{e}_3, \ldots,$

$\underset{\sim}{e}_1 + \underset{\sim}{e}_2 + \cdots + \underset{\sim}{e}_n\}$

iii) $B_2 = \{\underset{\sim}{e}_1, \underset{\sim}{e}_1 + \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4, \ldots, \underset{\sim}{e}_n\}$.

__Theorem:__ Let V be a finite dimensional vector space over $\mathbb{R}$. Then any two bases for V have the same size.

Def^n: Let V be a fdVS and B be any basis for V. Then we define the dimension of V as

$$\dim(V) = |B|.$$

From the exc., it follows that $\dim(\mathbb{R}^n) = n$ $\forall\ n \in \mathbb{N}$.

On the other hand $\dim(\{\underset{\sim}{0}\}) = 0$.

Therefore it is customary to define

$$\mathbb{R}^0 = \{\underset{\sim}{0}\}$$

__Ex:__ (2) $V = \mathbb{R}^3$.

$S = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0\}$.

It is ~~easy~~ possible to check that $\underset{(\text{subsp})}{S \subseteq} V$.

Qn: $\dim(S) = ?$

$S = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_3 = -v_1 - v_2\}$

$= \{(v_1, v_2, -v_1 - v_2) : v_1, v_2 \in \mathbb{R}\}$

$\Rightarrow$ any $\underset{\sim}{v} = (v_1, v_2, v_3) \in S$ is of the form

$\underset{\sim}{v} = (v_1, v_2, -v_1 - v_2)$

$\Rightarrow (v_1, 0, -v_1) + (0, v_2, -v_2)$

$= v_1(1, 0, -1) + v_2(0, 1, -1)$
(Here, $v_1, v_2 \in \mathbb{R}$)

Claim: $B = \{(1,0,-1), (0,1,-1)\}$ is a basis for S.

Proof: We just showed that ~~any vector $v \in \bar{B}$~~. $S \subseteq \bar{B}$.

$B \subseteq S \Rightarrow \bar{B} \subseteq S \Rightarrow S = \bar{B}$. since $S \subseteq V$ (subsp)

To show: B is linearly ind.
Take $\alpha_1, \alpha_2 \in \mathbb{R}$ such $\alpha_1(1,0,-1) + \alpha_2(0,1,-1) = (0,0,0)$
$\Rightarrow (\alpha_1, \alpha_2, -\alpha_1 - \alpha_2) = (0,0,0)$

$\Rightarrow \alpha_1 = 0$ and $\alpha_2 = 0$.
$\Rightarrow B$ is linearly independent and $\bar{B} = S$.
$\Rightarrow B$ is a basis for S.
In part, $\dim(S) = |B| = 2$

Ex: $V = \mathbb{R}^3$. Define
$S_1 = \{\underset{\sim}{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 = v_2 = v_3\}$,
$S_2 = \{\underset{\sim}{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 = v_2\}$
i) Show that $\underset{(\text{subsp})}{S_1} \subseteq \underset{(\text{subsp})}{S_2} \subseteq V$

ii) Find a basis for S_1 and show that $\dim(S_1) = 1$.

iii) Find a basis for S_2 and show that $\dim(S_2) = 2$.

Thm: Suppose V is fdvs and $S \underset{(subsp)}{\subseteq} V$. Then any basis for S can be extended to a basis for V.
In part., dimension$(S) \leq \dim(V)$.

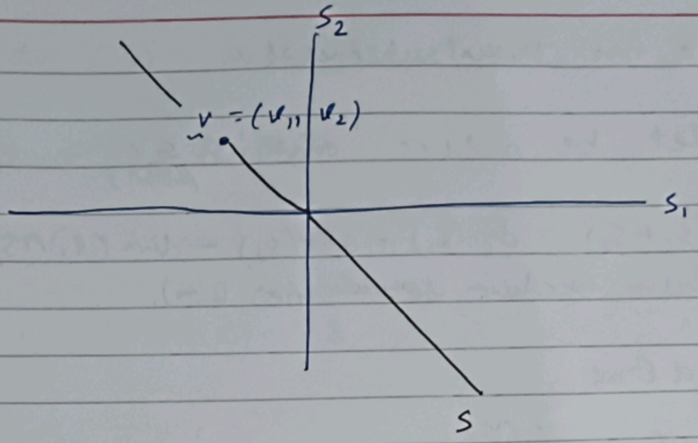
Def^n: ~~Let~~ Let V be a a fdvs and S_1, S_2 are two subspaces of V.
Define $S_1 + S_2 : \{ \underset{\sim}{v_1} + \underset{\sim}{v_2} : \underset{\sim}{v_1} \in S_1, \underset{\sim}{v_2} \in S_2 \}$

Ex: Show that $S_1 \cap S_2$ and $S_1 + S_2$ are both subspaces of V.

Suppose $V = \mathbb{R}^2$. Let $S \underset{(subsp)}{\subseteq} V$.

$\Rightarrow \dim(S) \leq \dim(V) = 2$.
$\Rightarrow \dim(S) \in \{0, 1, 2\}$.
If $S = \{\underset{\sim}{0}\}$, then $\dim(S) = 0$.
If $S = V = \mathbb{R}^2$, then $\dim(S) = 2$.
For all other subspaces (i.e., non trivial subspaces of V), $\dim(S) = 1$.

Suppose $\dim(S) = 1$.
Take any vector $\underset{\sim}{v} \in S - \{\underset{\sim}{0}\}$
Then it can be shown that $S = \overline{\{\underset{\sim}{v}\}}$.
This means that ~~$S = \{\alpha_1 v_1 + \alpha_2 v_2$~~
$$S = \{\alpha \underset{\sim}{v} = (\alpha v_1, \alpha v_2) : \alpha \in \mathbb{R}\}$$



$\dim(S) = 1 \Rightarrow$ S is a straight line passing through $\underset{\sim}{0}$.

Take $S_1 = \{(v_1, v_2) \in \mathbb{R}^2 : v_2 = 0\}$

$= \{(v_1, 0) : v_1 \in \mathbb{R}\}$

$S_2 = \{(v_1, v_2 \in \mathbb{R}^2 ; v_1 = 0\}$

Clearly $S_1 \cap S_2 = \{\underset{\sim}{0}\}. = \{(0,0)\}.$

$S_1 + S_2 = V = \mathbb{R}^2.$

(Because any vector $\underset{\sim}{v} = (v_1, v_2)$ ~~is a~~ can be written as $\underset{S_1}{\underset{\Uparrow}{(v_1, 0)}} + \underset{S_2}{\underset{\Uparrow}{(0, v_2)}} = \underset{\sim}{v}$)

Thm: Let V be f.d.v.s with $\dim(V) = n \geq 2$.
Let $\underset{(\text{subsp})}{S \subseteq V}$. Then $\dim(S) = 0 \Rightarrow S = \{\underset{\sim}{0}\}$,

$\dim(S) = n \Rightarrow S = V$,

For all other subspaces S of V, i.e. whenever ~~the dimension~~ $0 < \dim(S) < n$,
we have $\{\underset{\sim}{0}\} \subsetneq S \subsetneq V$,

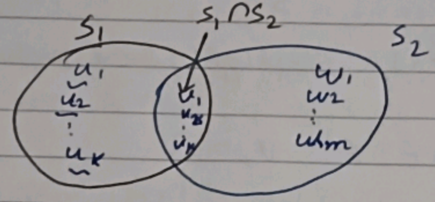
i.e S is a non trivial subspace of V.

Thm: Let be a fdvs and $\underset{(\text{subsp})}{S_1 \subseteq V}$, $\underset{(\text{subsp})}{S_2 \subseteq V}$.

$$\dim(S_1+S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$$

(Inclusion exclusion formula for Dim)

Sketch of Proof



Basis Venn Diagram

$\dim(S_1 \cap S_2) = l$ $\quad\quad$ $\dim(S_1) = k + l$

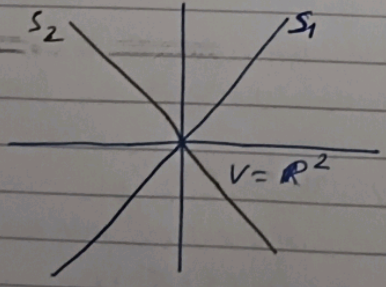
$\dim(S_2) = l + m$ $\quad\quad$ $\dim(S_1+S_2) = k + l + m$

$\Rightarrow \dim(S_1+S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$

Application: Let $V = \mathbb{R}^2$

$S_1 = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 - u_2 = 0\}$

$S_2 = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 + u_2 = 0\}$



Easy to check:

$$\left.\begin{array}{l} ① \ \underset{(\text{subsp})}{S_1 \subseteq V} \\ ② \ \underset{(\text{subsp})}{S_2 \subseteq V} \end{array}\right\} \underset{(\text{subsp})}{S_1 + S_2 \subseteq V}$$

③ $S_1 \cap S_2 = \{\underset{\sim}{0}\}$

(4) $\dim(S_1) = \dim(S_2) = 1$

Then $\dim(S_1 + S_2) = $

by Thm $\dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$

$= 1 + 1 - 0. = 2$

$\Rightarrow S_1 + S_2 = V = \mathbb{R}^2$

(checking this directly is not very easy).

Exc: $V = \mathbb{R}^3$

$S_1 = \{ \underset{\sim}{v} \in \mathbb{R}^3 ; v_1 + v_2 + v_3 = 0 \}$

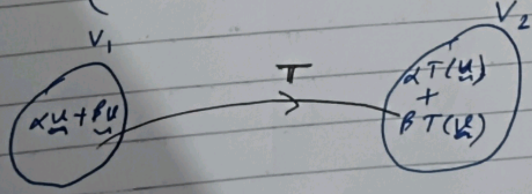
$S_2 = \{ \underset{\sim}{v} \in \mathbb{R}^3 ; v_1 = v_2 = v_3 \}$

Using same logic show that $S_1 + S_2 = V$.
(show all steps).
including previous exc.

Linear Map / Linear Transformation

Defn: (1) Let V_1, V_2 be fd VS A map $T: V_1 \to V_2$ is called a linear map / linear transformation if $\forall \alpha, \beta \in \mathbb{R}$, $\forall \underset{\sim}{u}, \underset{\sim}{v} \in V_1$

$$T(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) = \alpha T(\underset{\sim}{u}) + \beta T(\underset{\sim}{v})$$



(2) If $V_1 = V_2 = V$ (say) and

$T: V \to V$ is a linear ~~transform~~ map, then T is c/d a linear operator.

(3) If $V_2 = \mathbb{R}$ (also a f d v s over $\mathbb{R}$ with $\dim(V_2) = 1$) and $T: V_1 \to \mathbb{R}$ is a linear map, then T is c/d a linear functional.

(4) If $T: V_1 \to V_2$ is a bijective linear map, then T is c/d a vector space isomorphism, and V_1, V_2 are c/d isomorphic vector spaces. Notation $V_1 \cong V_2$.

L-20

13-NOV-2024

Thm : If two finite d.v.s V_1 & V_2 are isomorphic (i.e, $V_1 \cong V_2$), then they have $\dim(V_1) = \dim(V_2)$

b) If ~~dim(V₁) = n~~ further dimension of $V_1 = n \in \mathbb{N} \cup \{0\}$, then $V_1 \cong V_2 \cong \begin{cases} \{\underset{\sim}{0}\} & \text{if } n = 0 \\ \mathbb{R}^n & \text{if } n \in \mathbb{N}. \end{cases}$

(Recall : $\mathbb{R}^0 = \{\underset{\sim}{0}\}$).

Ex i) $V_1 = \mathbb{R}^2$, $V_2 = \mathbb{R}$.,

Define $T : V_1 \longrightarrow V_2$ by
$T(u_1, u_2) = 2u_1 - 3u_2 \quad \forall (u_1, u_2) \in V_1 = \mathbb{R}^2$
Then T is a linear functional b/c $\forall \alpha, \beta \in \mathbb{R}$ and $\underset{\sim}{u}, \underset{\sim}{v} \in V_1 = \mathbb{R}^2$,

$$\begin{aligned} T(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) &= T(\alpha(u_1, u_2) + \beta(v_1, v_2)) \\ &= T(\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2) \\ &= 2(\alpha u_1 + \beta v_1) - 3(\alpha u_2 + \beta v_2) \\ &= \alpha(2u_1 - 3u_2) + \beta(2v_1 - 3v_2) \\ &= \alpha T(u_1, u_2) + \beta T(v_1, v_2) \\ &= \alpha T(\underset{\sim}{u}) + \beta T(\underset{\sim}{v}) \end{aligned}$$

2) $V_1 = \mathbb{R}^3$, $V_2 = \mathbb{R}^2$

Exe : Show that $T : V_1 \longrightarrow V_2$ defined by

$$T(v_1, v_2, v_3) = (v_1 + v_2 , v_2 + v_3),$$
$$(v_1, v_2, v_3) \in V_1$$

is a linear transformation from V_1 to V_2.

3) $V_1 = V_2 = \mathbb{R}^2$

Exe : show that $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(v_1, v_2) = (v_1 + v_2 , v_1 - v_2)$, $(v_1, v_2) \in \mathbb{R}^2$

is a linear operator on $\mathbb{R}^2$.

4) Take

$V_1 = \{ \underbrace{(w_1, w_2, w_3)}_{\in \mathbb{R}^3} : w_1 + w_2 + w_3 = 0 \} \underset{(\text{subsp})}{\subseteq} \mathbb{R}^3$

$V_2 = \{ (u_1, u_2) \in \mathbb{R}^2 : u_1 + u_2 = 0 \} \underset{(\text{subsp})}{\subseteq} \mathbb{R}^2$

$V_3 = \mathbb{R}^2$

Exe : Show the following :

a) $T_1 : V_1 \longrightarrow V_2$ defined by $T_1 (w_1, w_2, w_3) = (w_1 + w_2, w_3)$

, $(w_1, w_2, w_3) \in V_1$. is a linear map from V_1 to V_2.

b) $T_2 : V_1 \longrightarrow V_3$ defined by

$T_2 (w_1, w_2, w_3) = (w_1, w_2)$, $(w_1, w_2, w_3) \in V_1$.

is a vector space isomorphism from V_1 to V_3.

We know that any non trivial finite dimensional vector space over $\mathbb{R}$ looks like $\mathbb{R}^n$, where $n \in \mathbb{N}$.

Fix $m, n \in \mathbb{N}$.

Qn): What are all possible linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$?

From now on whenever we write a vector ($\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ etc) it is always a column vector. For example $\underset{\sim}{u} \in \mathbb{R}^n$ means

$$\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \leftarrow n \times 1 \text{ matrix or a column vector in } \mathbb{R}^n.$$

To write a row vector, I shall use a transpose notation:

$$\underset{\sim}{u}^T = (u_1, u_2, \ldots, u_n) \leftarrow 1 \times n \text{ matrix or a row vector in } \mathbb{R}^n.$$

fix $m, n \in \mathbb{N}$

Suppose $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map.

Let us denote by $\overbrace{\underset{\sim}{e}_1^{(n)}, \underset{\sim}{e}_2^{(n)}, \ldots, \underset{\sim}{e}_n^{(n)}}^{\text{column vectors}}$ the standard basis elements for $\mathbb{R}^n$.

and by $\underbrace{\underset{\sim}{e}_1^{(m)}, \underset{\sim}{e}_2^{(m)}, \ldots, \underset{\sim}{e}_m^{(m)}}_{\text{column vectors} \in \mathbb{R}^m}$ these standard basis elements for $\mathbb{R}^m$.

For each $j = \{1, 2, \ldots, n\}$,

$$\underset{\in \mathbb{R}^n}{T(\underset{\sim}{e}_j^{(n)})} \; (\in \mathbb{R}^m) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \quad (\text{say})$$

(here $a_{ij} \in \mathbb{R} \; \forall i = 1, 2, \ldots, m$)

$$= a_{1j} \underset{\sim}{e}_1^{(m)} + a_{2j} \underset{\sim}{e}_2^{(m)} + \cdots + a_{mj} \underset{\sim}{e}_m^{(m)}$$

$$\left[\underset{\sim}{e}_1^{(m)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m \right.$$

$$\left. \underset{\sim}{e}_2^{(m)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m \text{ and so on} \right]$$

Now take any column vector $\underset{\sim}{x} \in \mathbb{R}^n$.

Clearly $$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{j=1}^{n} x_j \underset{\sim}{e}_j^{(n)}$$

Since $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, we have

$$\underbrace{T(\underset{\sim}{x})}_{\in \mathbb{R}^m} = T\left(\sum_{j=1}^{n} x_j \underset{\sim}{e}_j^{(n)} \right)$$

using linearity of the map $$= \sum_{j=1}^{n} x_j \, T(\underset{\sim}{e}_j^{(n)})$$

$$= \sum_{j=1}^{n} x_j \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{pmatrix} \leftarrow m \times 1 \text{ matrix}$$

$$= \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{m \times n} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{n \times 1}$$

$$= A \underset{\sim}{x} \quad \text{, where}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \leftarrow m \times n \text{ matrix with real entries}$$

$T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then

<u>Summary</u>: $\forall \ \underset{\sim}{x} \in \mathbb{R}^n$,

$T(\underset{\sim}{x}) = A\underset{\sim}{x}$ for some $m \times n$ matrix with real entries.

Conversely, for any $m \times n$ matrix A with real entries the map $T: \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$\underbrace{T(\underset{\sim}{x})}_{m \times 1} = \underbrace{\underset{m \times n}{A} \underset{n \times 1}{\underset{\sim}{x}}}_{m \times 1}$ is linear transform^n from $\mathbb{R}^n$ to $\mathbb{R}^m$.

column vector.

This is b/c $\forall \ \underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$, and $\forall \ \alpha, \beta \in \mathbb{R}$,

$$\underbrace{A(\alpha \underset{\sim}{x} + \beta \underset{\sim}{y}) = \alpha A \underset{\sim}{x} + \beta A \underset{\sim}{y}}$$

check!

We have this result:

<u>Thm</u>: Fix $m, n \in \mathbb{N}$, $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transform^n if & only if $\exists$ ~~a real~~ an $m \times n$ matrix A with real entries s.t.

$$T(\underset{\sim}{x}) = A\underset{\sim}{x} \quad \forall\ \underset{\sim}{x} \in \mathbb{R}^n.$$

(column vector)

<u>Remark</u>: This A is also unique.

<u>Def^n</u>: A is called the matrix for the linear transform^n $T : \mathbb{R}^n \to \mathbb{R}^m$

(w.r.t the standard basis)

eg. ① If $\underset{m \times n}{A} = 0$ (zero matrix),

then $T : \mathbb{R}^n \to \mathbb{R}^m$ is the zero map,

i.e. $T(\underset{\sim}{x}) = \underset{\sim}{0} \in \mathbb{R}^m \ \forall\ \underset{\sim}{x} \in \mathbb{R}^n$.

② If $m = n$ and

$$\underset{n \times n}{A} = \underset{\substack{\downarrow \\ \text{identity matrix} \\ \text{of order } n}}{I_n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 \cdots & 0 & 1 \end{pmatrix}$$

$\left(\text{i.e. } a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \right)$

then $T : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map, i.e.,

$$T(\underset{\sim}{x}) = \underset{\sim}{x} \quad \forall\ \underset{\sim}{x} \in \mathbb{R}^n.$$

(3) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is defined by $T(\underset{\sim}{x}) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$

$\forall\ \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$

It is easy to check that T is a linear map.

Qn: What is the matrix for T?

Here $n = 3$ and $m = 2$ So the matrix for T would be a 2×3 matrix A.

$T(\underset{\sim}{e}_1^{(3)}) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1^{st}$ column of A.

$T(\underset{\sim}{e}_2^{(3)}) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2^{nd}$ column of A.

$T(\underset{\sim}{e}_3^{(3)}) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3^{rd}$ column of A.

$\therefore \underset{(2 \times 3 \text{ matrix})}{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

[sanity check: $A\underset{\sim}{x}$

$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$

$= T(\underset{\sim}{x})\ \forall\ \underset{\sim}{x} \in \mathbb{R}^3$.]

Ex: Show that $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 - x_2 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

is a linear map. Find the matrix for this linear map.

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