The density histogram co-incides with the estimator $\hat{f}(z)$ we derived earlier - only at the midpoints of every bin - if all bins have the same width given by $\underline{2h}$.

$$\hat{f}(z) = \frac{1}{2nh} \sum_{i=1}^{n} 1\{x_i \in \underline{(z-h, z+h)}\}$$ is known as the

"average shifted histogram" (ASH)

$N(\mu, \sigma^2)$

$N(\hat{\mu}, \hat{\sigma}^2)$

location: mean
scale : 6

SRSWR

STATISTICS - II

problems:

i) calculate avg-height of all 18-years old boys (Jan 2025) in India. → avg of P_1 is property of population

→ proportion of P_2 is property we're interested in

ii) calculate the proportion of 2 children families in India (Jan 2025)

iii) calculate the expectation of $N_{[-2,3]}^{(0,1)}$ ← average of P_3

$$E(x) = \int x f_x(x)\, dx$$

$$f_x(x) = \frac{\phi(x)\, \mathbb{I}_{[-2,3]}(x)}{\Phi(3) - \Phi(-2)}$$

$$\int_{-2}^{3} \frac{\cancel{x}\, \phi(x)\, dx}{\Phi(3) - \Phi(-2)} = \frac{1}{\Phi(3) - \Phi(-2)} \int_{-2}^{3} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Population: a collection of objects / measurements we are interested in.

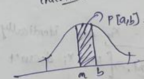
i) P_1 = { collection of heights of all 18-years old boys in India (Jan 2025) } : continous popln

ii) P_2 = { collection of no. of children in India (Jan 2025) }
= { 0, 1, 0, 0, 1, 2, ... } : a discrete variable

Fix $a < b$

let $P_{(a,b)}$ be the proportion of boys in P_1 whose heights are between $(a, b]$ ↗ proportion is fixed in every particular interval.

iii) $P_3 > N_{[-2,3]}^{(0,1)}$



$P[a,b]$ = proportion of P_3 taking values in $(a, b]$

$$= \int_{a}^{b} \frac{\phi(x)\, dx}{\Phi(3) - \Phi(-2)}$$

$$= F(b) - F(a)$$

Sample: subset of the populatn.

⇓ (independence)

representative of the populatn.

$x_1, x_2, \dots x_n$

indep. indentically distributed.

$$\boxed{\mathbb{P}[x_1 \in (a,b]] = P_{(a,b]} = \int_a^b \underbrace{f_x(x)}_{} dx}$$

↳ characterizes the population

i) $\overline{x_n}$ ii) $f_n = \dfrac{\#\{2 \text{ children in } x_1, x_2, \dots x_n\}}{n}$ iii) $\overline{x_n}$

$\mathbb{P}_2 = \{0, 0, 1, 5, 2, \dots\}$

ii) $f_n = \dfrac{\#\{2s \text{ in } x_1, x_2, \dots x_n\}}{n} \xrightarrow[(n\to\infty)]{} \mathbb{P}(X=2) =$ proportion of 2's in $\mathbb{P}_2$

0 as prop p_0

1 as prop p_1

$\vdots$

$$X_1 = \begin{cases} 0 & \text{with prob } p_0 \\ 1 & \text{with prob } p_1 \\ \vdots & \end{cases}$$

Statistical problem: You have to take sample,

$x_1, x_2, \dots x_n$

$\mathbb{P} = \mathcal{N}(0,1)$ $x_1 \in \mathbb{R}$

$\mathbb{P}(x_1 \in (a,b]) = P(a,b]$

$\underbrace{x_1, x_2, \dots x_n}$ are identically distributed.

independent → [one x_i doesn't effect the other one]

$\mathbb{P}[\mu \in (\bar{x}-a, \bar{x}+a)]$

{ After taking the samples - we will infer about the population from the sample.

↳ Statistical Interference:

statistical model → statistical interference ← The data

x_i; $i = 1, 2, \dots 15$ } → from germination of seeds example

$$\left.\begin{aligned} \mathbb{P}(x_i = 1) &= \theta \\ P(x_i = 0) &= 1-\theta \end{aligned}\right\} i = 1, 2, \dots 15$$

$\boxed{x_i \overset{iid}{\sim} Ber(\theta), \ i = 1, \dots 15}$ ⊛ → statistical model.

↑ parameter

The structure which was given by us by using the Example

[Ex-1]

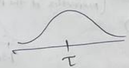
$T = \sum_{i=1}^{15} X_i \sim Bin(15, \theta)$

$\mathbb{P}(T = 10) = \binom{15}{10} \theta^{10} (1-\theta)^{15-10}$

$\mathbb{P}\left(\left(\underset{\sim}{\tfrac{2}{3} - 0.1}, \underset{\sim}{\tfrac{2}{3} + 0.1}\right) \ni \theta \right)$?

[Ex-2]

$g = \dfrac{4\pi^2 l}{T^2}$ $\boxed{T \sim N(\tau, \sigma^2)}$ ↳ parameters



$T_1, \dots T_{10}$ $T_i \overset{iid}{\sim} \mathcal{N}(\tau, \sigma^2)$

$\frac{1}{n}\sum_{i=1}^{10} x_i = \overline{T_{10}} \sim \mathcal{N}(\tau, \sigma^2/10)$

1] Model

2] Parameters [unknown]

3] By using the sample we will estimate the parameters

$\Big\}$ Parametric inference

$\sum_{i=1}^{n} T_i^2 \longrightarrow$ statistic (complete function of the given data)

Summary:

1] We are interested in some property of class (population)

2] For any $b \in \mathbb{R}$ if we know the proportion of measurements, (for any $b \in \mathbb{R}$) in the population falling below (or equal to) b, then $P(-\infty, b]$,

we can measure any property of the population. $(*)$

3] It is not possible to measure the entire population. so, we infer about the population of the sample.

- $\rightarrow x_1, x_2, \dots x_n$, these samples are random.
- $\rightarrow P(x_i \le b) = P(-\infty, b]$ (define in $(*)$)

x_i's are identically distributed.

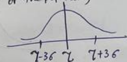
$x_1, x_2, \dots x_n$ are independent.

- $x_i \overset{iid}{\sim} F$, $i = 1, 2, \dots n$; $F(b) = P(-\infty, b]$
- $x_1, \dots x_n$ is a random sample from F.

4] The goal is to infer about the population from the sample.

$\rightarrow$ Towards this, one may assume a structure of F (model), except some parameters (unknown constants, properties of the populatn)

eg: $x_i \sim N(\mu, \sigma^2)$



$\pi N(\mu_1, \sigma_1^2) + (1-\pi) N(\mu_2, \sigma_2^2)$

$\rightarrow$ we use the samples to estimate the parameters, thereby estimate the populatn. [This procedure is called parametric inference]

(*) What is Non-parametric inference?

$\hookrightarrow$ Not getting any structure of F main focus on the problem in hand which we want like mean, variance and we'll estimate the required.

Parametric inference:
- point estimation (1)
- Interval estimation (3)
- Hypothesis estimation (2)

point Estimation: $x_1, x_2, \dots x_n$ is a random sample from $F_{\underset{\sim}{\theta}}$.

$P(-\infty, b]$ = populatn proportion in $(-\infty, b]$

populatn proportion in $(-\infty, b_1] \underbrace{\times}_{\text{And}} (-\infty, b_2] = P_{(-\infty,b_1]\times(-\infty,b_2]}$

$\Rightarrow F_{X,Y}(b_1, b_2) = P(\underbrace{X_i \le b_1}_{\text{Height}}, \underbrace{Y_i \le b_2}_{\text{weight}})$

And.

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \dots \begin{bmatrix} x_n \\ y_n \end{bmatrix}$

$F_{X,Y}(b_1, b_2) = \int_{-\infty}^{b_1}\int_{-\infty}^{b_2} \underbrace{f_{X,Y}(x,y)}_{\text{joint pdf}}\, dy\, dx$

$x_1, x_2, \dots x_n$ is a random sample from $F_{\underset{\sim}{\theta}}$

we are interested in function of $\underset{\sim}{\theta}$, say $\psi = \psi(\underset{\sim}{\theta})$

Eg: $x_i \overset{iid}{\sim} N(\mu, \sigma^2)$ $\quad \underset{\sim}{\theta} = \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}$ $\quad \psi: \mathbb{R}^2 \to \mathbb{R}$, $\psi(\underset{\sim}{\theta}) = \mu$.

[A function of the samples; $T(\underset{\sim}{X})$] (statistic) $\rightarrow T: \mathbb{R}^n \to \mathbb{R}$ is used to estimate $\psi = \psi(\underset{\sim}{\theta})$

$T(\underset{\sim}{X})$ is called an point estimation of $\psi(\underset{\sim}{\theta})$.

Notation: 1] For random variables use capital letters.

2] For realization use small letters.

3] For parameters use greek letters.

For a realization of the samples $x_1, x_2, \dots x_n$ the value of $T(\underset{\sim}{x})$ is called an estimate of $\psi(\underset{\sim}{\theta})$

which estimation is good?

Ex: $X_1, X_2, \dots X_n$ is a random sample from $\text{Bin}(\overset{\text{known}}{m}, \theta)$

$T_1(\underset{\sim}{X}) = \frac{\bar{X}_n}{m}$ $\qquad \frac{E(X_i)}{m} = \theta$

$\text{Var}(X_i) = m\theta(1-\theta)$

$T_2(\underset{\sim}{X}) = \left(S_n^2 + \frac{\bar{X}_n^2}{m}\right)/m$ $\qquad = m\theta - \frac{[E[X]]^2}{m}$

$x_{11}, \dots x_{n1}$	$T_1(x_1)$	$T_2(x_2)$
⋮	⋮	
$x_{110k} \dots x_{n10k}$	$T_1(x_{10k})$	$T_2(x_{10k})$

① $\bar{T_1}(\underset{\sim}{x})$ $\quad \bar{T_2}(\underset{\sim}{x})$

② $\text{Var}(T_1(x))$ $\quad \text{Var}(T_2(x))$

$E_\theta[T_1[\underset{\sim}{X}]] = E_\theta[\bar{X}_n/m]$

$= E_\theta\left[\frac{1}{mn}\sum_{i=1}^{n} X_i\right]$

$= \frac{1}{mn}\sum_{i=1}^{n} E_\theta(X_i)$

$= \frac{1}{mn}\sum_{i=1}^{n} m\theta = \underline{\underline{\theta}}$

$E_\theta(T_2[\underset{\sim}{X}]) = E_\theta\left[\frac{S_n^2}{m} + \frac{\bar{X}_n^2}{m^2}\right]$ $\qquad S_n^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}{n}$

$= E_\theta\left[\frac{1}{mn}\sum_{i=1}^{n} X_i^2 - \frac{\bar{X}_n^2}{m} + \frac{\bar{X}_n^2}{m^2}\right]$ $\qquad = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2$

$= \frac{1}{mn}\sum_{i=1}^{n} E_\theta(X_i^2) - \frac{1}{m}\left[1 - \frac{1}{m}\right] E_\theta(\bar{X}_n^2)$

$E_\theta(X_i^2) = \text{Var}_\theta(X_i) + E_\theta^2(X_i)$

$= \underline{m\theta(1-\theta) + m^2\theta^2}$

$E_\theta(\bar{X}_n^2) = E_\theta\left(\frac{1}{n^2}\left\{\sum_{i=1}^{n} X_i\right\}^2\right)$

$= E_\theta\left[\frac{1}{n^2}\left\{\sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_i X_j\right\}\right]$

$= \frac{1}{n^2}\sum_{i=1}^{n} E[X_i^2] + \frac{1}{n^2}\sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} E[X_i X_j]$

verify: $E_\theta(T_2(X)) = \theta - \underbrace{\left[\frac{m-1}{mn}\right]\theta(1-\theta)}_{\text{Bias}}$

So, $T_1(\underset{\sim}{X})$ is better estimate than $T_2(\underset{\sim}{X})$ as its expected value is underestimating the θ.

Summary:

$X_1, X_2, \dots X_n \overset{\text{iid}}{\sim} F_{\underset{\sim}{\theta}}\ (f_{\underset{\sim}{\theta}})$ both can be used.

Discrete distribution : $f_{\underset{\sim}{\theta}}$: pmf

continuous distribution : $f_{\underset{\sim}{\theta}}$: pdf

Problem: Estimate $\psi(\underset{\sim}{\theta}) = \psi$ $\qquad \psi : \mathbb{R}^k \to \mathbb{R}$

Defn: (Estimator) A fn of $X_1, X_2, \dots X_n$, $T(\underset{\sim}{X})$, which is used to estimate ψ is called an estimator.

Q: How to choose a good estimator?

Example: $X_1, X_2, \dots X_n$ is a random sample from $\text{Bin}(m, \theta)$

$T_1(\underset{\sim}{X}) = \frac{\bar{X}_n}{m}$, $\quad T_2(\underset{\sim}{X}) = \frac{S_n^2}{m} + \frac{\bar{X}_n^2}{m^2}$

$E_\theta(T_1(\underset{\sim}{X})) = \theta$, $\quad E_2(T_2(\underset{\sim}{X})) = \theta - \frac{(m-1)}{mn}\theta(1-\theta)$ verify

Defn (Bias): Let $T(\underset{\sim}{X})$ be a estimator of $\psi(\underset{\sim}{\theta}) = \psi$ then the bias in estimation is

$B_T(\psi(\underset{\sim}{\theta})) = E_\theta[T(\underset{\sim}{X}) - \psi(\underset{\sim}{\theta})]$

$= E_\theta[T(\underset{\sim}{X})] - \psi(\underset{\sim}{\theta})$

$B_T(\psi(\underset{\sim}{\theta}))$ for T_1 is $\underline{0}$ $\qquad B_T(\psi(\underset{\sim}{\theta}))$ for T_2 is $-\frac{(m-1)}{mn}\theta(1-\theta)$

Remarks:

[1] If $B_T(\psi(\underset{\sim}{\theta}))$ is positive then $T(\underset{\sim}{x})$ over estimates $\psi(\underset{\sim}{\theta})$

[2] If $B_T(\psi(\underset{\sim}{\theta}))$ is negative then $T(\underset{\sim}{x})$ under estimates $\psi(\underset{\sim}{\theta})$

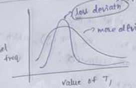
Defn: [unbiased estimation]

$T(\underset{\sim}{x})$ is called an unbiased estimator of $\psi(\theta)$ if $B_T(\psi(\underset{\sim}{\theta}))=0$, for all $\underset{\sim}{\theta}$.

(3) $T_3(\underset{\sim}{x}) = \frac{x_1}{m}$ $\quad E_\theta(T_3(\underset{\sim}{x}))=\theta$

$var(T_1(\underset{\sim}{x})) < var(T_3(\underset{\sim}{x}))$] T_1 is the better estimator than T_3.

(4) as T_1 is calculated by 'n' different samples & T_3 can be calculated by only one sample & that one sample can be bad sample



$$\boxed{var(T(x)) = E_\theta\left[[T(x)-\psi(\theta)]^2\right]}$$
$\quad\hookrightarrow E_\theta(T(x))$

$$\cancel{var(T_3(x)) = E_\theta[(T_3(x)-\psi(\theta))^2]}$$

$$\cancel{= E_\theta[(T_3(x))^2 + \psi^2(\theta) - 2T_3(x)\psi(\theta)]}$$

$$= var\left(\frac{x_1+x_2+\dots+x_n}{n}\right), \frac{1}{m^2}\theta(1-\theta)$$

$$var(T_3(\underset{\sim}{x})) = var\left(\frac{x_1}{m}\right)$$
$$= \frac{1}{m^2} var[x_1] = \frac{1}{m^2} m\theta(1-\theta) = \frac{1}{m}\theta(1-\theta) \rightarrow (1)$$

$$var[T_1(\underset{\sim}{x})] = var\left[\frac{\bar{x}_n}{m}\right] = \frac{1}{m^2} var(\bar{x}_n)$$
$$= \frac{1}{m^2}\frac{m\theta(1-\theta)}{n} = \frac{1}{mn}\theta(1-\theta) \rightarrow (2)$$

verify

Defn: (Relative efficiency)

Let $T_1(\underset{\sim}{x})$ & $T_2(\underset{\sim}{x})$ be two unbiased estimators of $\psi(\underset{\sim}{\theta})$ then the relative efficiency of T_1 and T_2 is.

$$\boxed{RE_{T_1,T_2}(\psi(\underset{\sim}{\theta})) = \frac{var(T_1(\underset{\sim}{x}))}{var(T_2(\underset{\sim}{x}))}}$$

Remark: $RE_{T_1,T_2}(\psi(\underset{\sim}{\theta}))>1$ then $T_2(\underset{\sim}{x})$ is more efficient (better) than $T_1(\underset{\sim}{x})$ in estimating $\psi(\underset{\sim}{\theta})$.

Example: $RE_{T_1,T_3}(\theta) = \frac{1}{n} < 1$

↳ As it's <1, T_1 is better than T_3.

Cramer-Rao lower bound: (CRLB):

(1) CRLB provides a lower bound of variance of unbiased estimators.

~~[$x_1, \dots x_n$ iid f_θ]~~

(2) CRLB holds for certain distributions which satisfy some regularity conditions

one of the regularity condition is: the sample space ($\mathcal{X}$) is free of the parameter (doesn't depend on parameter)

[set of values that can be realized by the sample.

Ex: $X \sim N(0,1)$

$\mathcal{X} = \mathbb{R}$ || $\mathcal{X}$ doesn't depend on $\underset{\sim}{\theta}$.

If $X \sim U(0,\theta)$ || $\mathcal{X}$ depends on $\underset{\sim}{\theta}$.

$\mathcal{X} = [0,\theta]$

Defn (Fisher Information): let $x_1, x_2, \dots x_n$ be a random sample from f_θ. then the fisher information of θ is

$$I_n(\theta) = nI(\theta) \quad \text{where,} \quad I(\theta) = E_\theta\left[\left\{\frac{\partial}{\partial\theta}\log f_\theta(x)\right\}^2\right]$$

Defn (CRLB): let $X_1, X_2, \dots X_n$ be a random sample from a distribution with pmf/pdf f_θ and the f_θ satisfies certain regularity conditions. Then for any estimator $T(\underset{\sim}{X})$,

$$\mathrm{var}_\theta(T(\underset{\sim}{X})) \geq \frac{\left[\frac{\partial}{\partial\theta} E_\theta(T(\underset{\sim}{X}))\right]^2}{n I(\theta)}$$

Note: If $T(\underset{\sim}{X})$ is unbiased for θ, then $\mathrm{var}_\theta(T(\underset{\sim}{X})) \geq [n I(\theta)]^{-1}$

Ex: (continue)

$$f_\theta(x_1) = \binom{m}{x_1}\theta^{x_1}(1-\theta)^{m-x_1}$$

$$\log f_\theta(x_1) = \log\binom{m}{x_1} + x_1\log\theta + (m-x_1)\log(1-\theta)$$

$$\frac{\partial}{\partial\theta}\log f_\theta(x_1) = \frac{x_1}{\theta} + \frac{m-x_1}{1-\theta}(-1)$$

$$\frac{\partial}{\partial\theta}\log f_\theta(x_1) = \frac{x_1}{\theta} + \frac{m-x_1}{\theta-1}$$

$$I(\theta) = E_\theta\left[\underbrace{\left(\frac{x_1}{\theta} - \frac{m-x_1}{1-\theta}\right)^2}_{\left(\frac{\partial}{\partial\theta}\log f_\theta(x_1)\right)^2}\right]$$

$$I(\theta) = E\left[\frac{x_1^2}{\theta^2} + \left(\frac{m-x_1}{1-\theta}\right)^2 - \frac{2x_1(m-x_1)}{\theta(1-\theta)}\right]$$

$$\boxed{I(\theta) = \frac{m}{\theta(1-\theta)}}$$ verify

CRLB of $T_1(\underset{\sim}{X}) = \frac{\bar{X}_n}{m}$ is $\frac{1}{nI(\theta)} = \frac{1}{nm/\theta(1-\theta)}$

$$= \frac{\theta(1-\theta)}{mn}$$

Recall: $\mathrm{var}_\theta\left[\frac{\bar{X}_n}{m}\right] = \frac{\theta(1-\theta)}{nm}$.

Defn: (Efficient estimator): If the variance of an unbiased estimator $T(\underset{\sim}{X})$ of $\psi(\theta)$ achieves the CRLB, then it is called an efficient estimator.

Q) can a biased estimator be better than an unbiased estimator?

Sol: Yes! Goal: $\underbrace{E_\theta\{[T(\underset{\sim}{X}) - \psi(\theta)]^2\}}_{\text{~~MSE_T(θ)~~ } MSE_T(\psi(\theta))}$ → minimize wrt T.

→ [Mean squared error MSE]

(less MSE → good estimator)

$$MSE_T(\psi(\theta))$$

$$= E_\theta\left[\{\underbrace{T(\underset{\sim}{x}) - E_\theta(T(\underset{\sim}{x}))}_{a} + \underbrace{E_\theta(T(\underset{\sim}{x})) - \psi(\theta)}_{b}\}^2\right]$$

$$= E_\theta\left[\{T(x) - E_\theta(T(\underset{\sim}{x}))\}^2 + \{E_\theta(T(\underset{\sim}{x})) - \psi(\theta)\}^2 + 2\underbrace{E_\theta\{\overbrace{E_\theta(T(\underset{\sim}{x})) - \psi(\theta)}^{B_T(\psi(\theta))}\}\{T(\underset{\sim}{x}) - E_\theta(T(\underset{\sim}{x}))\}}_{=0}\right]$$

$$= \underbrace{E_\theta\left[\{T(x) - E_\theta(T(x))\}^2\right]}_{\mathrm{var}(T(\underset{\sim}{x}))} + B_T^2(\psi(\theta)) + 0$$

$$= \mathrm{var}_\theta(T(\underset{\sim}{x})) + B_T^2(\psi(\theta)).$$

Problems:

Rao-Bl

3) $T_3 = \bar{X}_n$

$$E[\bar{X}_n] = E\left[\frac{X_1 + X_2 + \dots + X_n}{n}\right]$$

$$= \frac{1}{n}E[X_1 + X_2 + \dots + X_n] = \frac{1}{n}nE[X_1]$$

$$\boxed{E[\bar{X}_n] = E[X_1]}$$

2) $T_2 = \frac{1}{8}X_1 + \frac{3}{4(n-2)}(X_2 + \dots + X_{n-1}) + \frac{1}{8}X_n$

$E[T_2] = \frac{1}{8}E[X_1] + \frac{3}{4(n-2)}(n-2)E[X_i] + \frac{1}{8}E[X_n]$

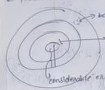
$= \left(\frac{1}{8} + \frac{3}{4} + \frac{1}{8}\right)E[X_i]$

$= E[X_i]$

---

efficiency is reciprocal of variability (variance)

precision → error should be less



Problems:

3) $var(T_3) = \frac{\sigma^2}{n}$ $T_3 = \bar{X}_n$

$var(\bar{X}_n) = var\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) = \frac{1}{n^2} var\left[\sum_{i=1}^{n} X_i\right]$

$= \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$

2) $var(T_2) = \frac{(n+16)}{32(n-2)}\sigma^2$

$var\left[\frac{1}{8}X_1 + \frac{3}{4(n-2)}(X_2 + \dots + X_{n-1}) + \frac{1}{8}X_n\right]$

sample size (n) → accuracy & precision

replication / repetition → precision

Asymptotics → ↑ing the sample size

Problem:

$\sigma^2 = E[X^2] - (E[X])^2$

$T_1 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$

$\frac{1}{n-1}\left(E[X^2] - (E[X])^2\right)$

$E[T_1] = \frac{1}{n-1} E\left[\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right]$

$\frac{1}{n-1}$

$= \frac{1}{n-1} E\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}_n^2\right]$

$E[X^2] =$

$= \frac{1}{n-1}\left[E[X_1^2 + X_2^2 + \dots + X_n^2] - n\,E[\bar{X}_n^2]\right]$

$= \frac{1}{n-1}\left[n\,E[X_i^2] - n\,E[\bar{X}_n^2]\right]$

$\cancel{\sigma^2 = (E[X])^2}$ -

$= \frac{n}{(n-1)}\left[E[X_i^2] - E[\bar{X}_n^2]\right]$

$\sigma^2 = E[X_i^2] - (E[X_i])^2 \Rightarrow \boxed{E[X_i^2] = \sigma^2 + \mu^2}$

$\frac{\sigma^2}{n^2} = E[\bar{X}_n^2] - [E[\bar{X}_n]]^2$

$E\left[\frac{X_1 + X_2 + \dots + X_n}{n}\right]$

$\frac{1}{n} n$

$\frac{\sigma^2}{n^2} + \mu^2$

$= \frac{n}{n-1}\left[\sigma^2 + \cancel{\mu^2} - \frac{\sigma^2}{n} - \cancel{\mu^2}\right] = \frac{n}{n-1}\left[\sigma^2\left[\frac{n-1}{n}\right]\right]$

$\therefore \boxed{E[T_1] = \sigma^2}$

$var(T_1) = E[T_1^2] - [E[T_1]]^2$