Genesis of LN Distribution

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A Stochastic Model Leading to LN Distribution

- **A Stochastic Model Leading to LN Distribution:**
- The most popular stochastic model leading to LN distribution is the Law of Proportionate Effect Model due to **Kapteyn** and its modification by **Kalecki**.
- There are some stochastic models which lead to Pareto distribution, e.g., Champernowne's Markov Chain model. However, we shall not discuss these models here. The readers, if interested can go through "Barry C. Arnold: Pareto Distributions 2nd Edition, CRC Press, Taylor and Francis Group, Chapman and Hall, 2015."
- The essence of all Central Limit Theorems (CLTs) is the following:

Stochastic Model Leading to LN Distribution (Continued)

- **CLT:** Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of random variables. Then under fairly mild conditions, $\sum X_j$ is asymptotically normally distributed as $n \to \infty$. The simplest of these CLTs is the condition of $X_1, X_2, \dots, X_n, \dots$ being i.i.d. r.v.'s. Thus, if $X_1, X_2, \dots, X_n, \dots$ are i.i.d. r.v.'s, then $\sum X_j$ is asymptotically normal, if the common variance exists (Lindeberg-Levy theorem).

- **Corollary:** Let $Y_1, Y_2, \dots, Y_n, \dots$ be a sequence of independent r.v.'s taking only positive values. Consider $X_1 = \mathrm{Ln}(Y_1), X_2 = \mathrm{Ln}(Y_2), \dots, X_n = \mathrm{Ln}(Y_n), \dots$, then under mild conditions $\sum X_i$ is asymptotically normally distributed and hence $\prod(Y_i)$ is asymptotically LN.

- The proof follows, because $\sum \mathrm{Ln}(Y_i)$ is asymptotically normal $\Rightarrow$ $\mathrm{Ln}(\prod(Y_i))$ is asymptotically normal $\Rightarrow e^{\mathrm{Ln}(\prod(Y_i))}$ is asymptotically LN or $\prod(Y_i)$ is asymptotically LN.

Law of Proportionate Effect (LPE)

- **Law of Proportionate Effect (LPE):**
- Consider a r.v. say income of an individual or size of a biological organism. Suppose the initial value is X_0, and the value after n periods is X_n, n = 1, 2, 3, … .
- Assumption:

$$X_t - X_{t-1} \propto X_{t-1} \text{ or } X_t - X_{t-1} = \epsilon_t X_{t-1},$$

- where ϵ_t is a random variable.
- The variate X is said to be subject to the Law of Proportionate Effect (LPE) if the increment during any time interval is a random proportion of the size obtained at the beginning of the period.

$$X_t = X_{t-1}(1 + \epsilon_t); \quad 1 + \epsilon_t > 0 \ \forall t.$$

$$X_n = X_0(1 + \epsilon_1)(1 + \epsilon_2) \dots (1 + \epsilon_n).$$

- ϵ_t is very small, hence $(1 + \epsilon_t) > 0$ for all t = 1, 2, 3, …

Kalecki's Criticism to LPE

- By CLT, as $n \to \infty$, under fairly general conditions X_n asymptotically tends to LN distribution.

$$\mathrm{Ln}(X_n) = \mathrm{Ln}(X_0) + \mathrm{Ln}(1 + \epsilon_1) + \mathrm{Ln}(1 + \epsilon_2) + \cdots + \mathrm{Ln}(1 + \epsilon_n).$$

$$\text{or}, X'_n = X'_0 + \epsilon'_1 + \epsilon'_2 + \cdots + \epsilon'_n, \text{say}.$$

- **Kalecki's Criticism to LPE**

$$X'_n = X'_0 + \epsilon'_1 + \epsilon'_2 + \cdots + \epsilon'_n.$$

- If $X'_n, X'_0, \epsilon'_1, \epsilon'_2, \ldots \epsilon'_n$ ... are mutually independent, as is usually assumed, then

$$V(X'_n) = V(X'_0) + V(\epsilon'_1) + V(\epsilon'_2) + \cdots + V(\epsilon'_n).$$

- Now, for LN variate, V(Ln) is the natural measure of inequality. It appears that with the passage of time, $V(X'_n)$ will grow indefinitely. This is contrary to observations.

Kalecki's Modification to LPE

- **Kalecki's Modification to LPE**
- Assume that

$$V(X'_{n+1}) = V(X'_n)$$

$$V(X'_{n+1}) = V(X'_n) + V(\epsilon'_{n+1}) + 2\mathrm{Cov}(X'_n, \epsilon'_{n+1})$$

$$\therefore \ \mathrm{Cov}(X'_n, \epsilon'_{n+1}) = -\frac{1}{2}V(\epsilon'_{n+1}) < 0.$$

- There must be a negative correlation between X_n and ϵ_{n+1}.

$$X_{n+1} = X_n(1 + \epsilon_{n+1}).$$

- Government policy retarding further growth of high incomes and favouring the poor will stabilize V(Ln(X)).

Kalecki's Modification to LPE (Continued)

- Suppose the regression of ϵ'_t on X'_{t-1} is linear.

$$\epsilon'_t = A' - \alpha X'_{t-1} + \eta_t \text{ say, and } \eta_t \text{ is independent of } X'_{t-1}.$$

- Since

$$X'_t = X'_{t-1} + \epsilon'_t, \text{ for all t,}$$

- we have,

$$X'_t = A' + (1-\alpha)X'_{t-1} + \eta_t.$$

- (Note that $0 < 1 - \alpha < 1$.)

$$\therefore\ X_t = AX_{t-1}^{1-\alpha}e^{\eta_t}. \qquad \text{(Kalecki's model)}$$

- The questions are: Shall we still have X_t asymptotically LN? Shall we have $V(X'_t) = V(X'_{t+1})$ for large t?

Kalecki's Modification to LPE (Continued)

$$X'_t = A' + (1-\alpha)X'_{t-1} + \eta_t$$

$$= A' + (1-\alpha)[A' + (1-\alpha)X'_{t-2} + \eta_{t-1}]X'_{t-1} + \eta_t$$

$$= \cdots$$

$$= (A' + (1-\alpha)A' + (1-\alpha)^2 A' + \cdots) + \eta_t + (1-\alpha)\eta_{t-1} + (1-\alpha)^2\eta_{t-2} + \cdots$$

$$= \frac{A'}{\alpha} + \text{negligible term} + \sum_{r=0}^{\infty}(1-\alpha)^r \eta_{t-r}$$

- Thus, X'_t is constant plus weighted sum of uncorrelated η's. Hence, X'_t is asymptotically normal and X_t is asymptotically LN.

Kalecki's Modification to LPE (Continued)

$$V(X_t') = \sum_r (1-\alpha)^{2r} V(\eta_{t-r}) = \sigma_\eta^2 \sum_r (1-\alpha)^{2r},$$

- since η_t's are i.i.d. with finite variance σ_η^2, say. Moreover,

$$V(X_t') = \sigma_\eta^2 \sum_r (1-\alpha)^{2r},$$

is finite and independent of t.

- Hence,

$$V(X_t') = V(X_{t+1}') \text{ for large t.} \qquad \text{Q. E. D.}$$

Thank You