defined on S.

Eg:- 1) $S = [0,1]$. Define for each $n \in \mathbb{N}$, $f_n(x) = x^n$, $x \in [0,1]$.

Eg:- 2) $S = \mathbb{R}$. Define for each $n \in \mathbb{N}$,

$$f_n(x) = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots \frac{x^n}{n!}, \quad x \in \mathbb{R}.$$

Eg:- 1)

$$\lim_{n \to \infty} f_n(x) = g(x) = \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 1, & \text{if } x = 1 \end{cases}$$

Pointwise limit → of $\{f_n\}_{n \in \mathbb{N}}$

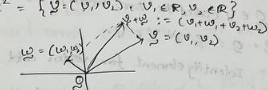
$f_n(x)$ is cont, but $g(x)$ not cont.

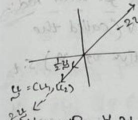
LINEAR ALGEBRA

Recall that $\mathbb{R}$ satisfies (A1) - (A6). Note that $\mathbb{C}$ also satisfies (A1)-(A6).

Motivating Example

$V = \mathbb{R}^2 = \{ \underset{\sim}{v} = (v_1, v_2) : v_1 \in \mathbb{R}, v_2 \in \mathbb{R} \}$





$\forall \alpha \in \mathbb{R}$, $\forall \underset{\sim}{u} \in V = \mathbb{R}^2$, $\alpha \underset{\sim}{u} = \alpha(u_1, u_2) = (\alpha u_1, \alpha u_2)$

Scalar mult — component wise multip

Note:- We have a binary operation

$V \times V \to V$ ← vector add

$(\underset{\sim}{v}, \underset{\sim}{w}) \mapsto (\underset{\sim}{v} + \underset{\sim}{w})$

called vector addition on $V = \mathbb{R}^2$ and another map, ~~R×V~~ $\mathbb{R} \times V \to V$, $(\alpha, \underset{\sim}{u}) \mapsto \alpha \underset{\sim}{u}$ called Scalar multiplication (of vectors) such that the following Properties hold: (that follow from (A1) to (A6))

(VS 1) Associativity of Vector Addition:

$\forall \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V$

$(\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w} = \underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w})$.

(VS2) Commutativity of vector addition:-

$\forall\ \underset{\sim}{u}, \underset{\sim}{v} \in V,\quad \underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$

(VS3) Existence of identity element for vector add:-

$\exists\ \underset{\sim}{0} \in V$ (called the zero vector) s.t

$\underset{\sim}{v} + \underset{\sim}{0} = \underset{\sim}{v}\ \forall\ \underset{\sim}{v} \in V.$

$\underset{\sim}{0}$ = Identity element for vector add.

(VS4) Existence of Inverse elements for Vector Add:-

For each $\underset{\sim}{v} \in V$, $\exists\ -\underset{\sim}{v} \in V$ (called the (vector) additive Inverse / Negative of $\underset{\sim}{v}$) s.t

$\underset{\sim}{v} + (-\underset{\sim}{v}) = \underset{\sim}{0}$

(VS5) Compatibility of Scalar Multiplication with the ~~Vector addition~~ Multiplication of Real Numbers

$\forall\ \alpha, \beta \in \mathbb{R}$ and $\forall\ \underset{\sim}{v} \in V$, ~~$\alpha(\beta \underset{\sim}{x}) = \alpha\beta(\underset{\sim}{x})$~~

$\alpha(\beta \underset{\sim}{v}) = (\alpha\beta)\underset{\sim}{v}$

(VS6) Existence of Identity element for scalar multiplication:-

the multiplicative Identity element 1 of $\mathbb{R}$ satisfies

$1\ \underset{\sim}{v} = \underset{\sim}{v}\quad \forall\ \underset{\sim}{v} \in V$

Identity element for scalar Multiplication

(VS7) Distributivity of Scalar Multiplication over / w.r.t Vector Addition:

$\forall\ \alpha \in \mathbb{R}$ and $\forall\ \underset{\sim}{u}, \underset{\sim}{v} \in V,$

$\alpha(\underset{\sim}{u} + \underset{\sim}{v}) = \alpha\underset{\sim}{u} + \alpha\underset{\sim}{v}.$

(VS8) Distributivity of Scalar Multiplication over / w.r.t Add of Real Numbers

$\forall\ \alpha, \beta \in \mathbb{R},\ \forall\ \underset{\sim}{v} \in V,$

$(\alpha + \beta)\underset{\sim}{v} = \alpha\underset{\sim}{v} + \beta\underset{\sim}{v}$

v tilde

Exc: Prove / verify (VS1) to (VS8) for $V = \mathbb{R}^2$.

(L-14)

Exc: S.T $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

is not a R.I on $[0,1]$.

A: For any partition P of $[0,1]$, $P = \{0, x_1, x_2, \dots, x_{n-1}\}$

$\forall\ x > y$, $\exists\ q \in (x,y)$ as well as $r \in (x,y)$

where $q \in \mathbb{Q}$, $r \notin \mathbb{Q}$

Hence $\inf\limits_{x \in [x_{i-1}, x_i]} f(x) = 0$, $\sup\limits_{x \in [x_{i-1}, x_i]} f(x) = 1$

$L(f,P) = 0$

lly $U(f,P) = 1(x_1 - x_0) + 1(x_2 - x_1) + \dots + 1(x_n - x_{n-1})$

$= 1$

Since this is true for any part P,

$\overline{\int_0^1} f(x)\,dx \neq \underline{\int_0^1} f(x)\,dx = 0.$

Exc:- F_1, F_2 are 2 primitives

$F_1(x) = F_2(x) + C\quad \forall\ x \in \mathbb{R}.$

P.

~~Pf: Define $G(x) = F_1(x) - F_2(x)$~~
~~Since F_1, F_2 are primitives they are diff~~
~~diff. Hence G is diff~~

Pf:- Define $G(x) = F_1(x) - F_2(x)$
Since F_1, F_2 are primitives they are diff.
Hence G is diff

$G'(x) = F_1'(x) - F_2'(x)$
$= f(x) - f(x) = 0$

i.e $G' = 0$ on $\mathbb{R}$
(already proved if $g' = 0$, then $g = \text{const}$)
$\Rightarrow G(x) = c$ for some $c \in \mathbb{R}$
$\Rightarrow F_1(x) = F_2(x) + c$

Q) Prove 2nd F.T.C

Statement:- Suppose I - open interval $f : I \to \mathbb{R}$ is a cont fn. Let $F : I \to \mathbb{R}$ be a primitive of f on I. Then $\forall\, c, x \in I$, we have

$$F(x) = F(c) + \int_c^x f(t)\,dt$$

Pf:- Let $A(x) = \int_c^x f(t)\,dt$
Since f is cont, it is R.I. So A is well-defined.
And by 1st FTC, $A'(x) = f(x)$ $\forall\, x \in I$
i.e, A is a primitive of f.
Also F is given to be the primitive of f by
Previous Q, $F(x) - A(x) = K$, a const,
Since $A(c) = \int_c^c f(t)\,dt = 0$, $K = F(c)$

i.e, $F(x) - A(x) = K = F(c)$
$\Rightarrow F(x) = A(x) + F(c)$
$\Rightarrow F(c) + \int_c^x f(t)\,dt$

14/11/24

LECTURE-17

* Fix $n \in \mathbb{N}$.
$V = \mathbb{R}^n = \{ \underset{\sim}{v} = (v_1, v_2, \dots v_n) \}$ and define vector addition component wise, ie,
$\underset{\sim}{v} + \underset{\sim}{w} := \{ v_1 + w_1, v_2 + w_2, \dots v_n + w_n \}$
where $\underset{\sim}{v} = (v_1, v_2, \dots v_n)$ and $\underset{\sim}{w} = (w_1, w_2, \dots w_n) \in V = \mathbb{R}^n$
Also define scalar multiplication
$\alpha \underset{\sim}{v} = \alpha (v_1, v_2, \dots v_n) = (\alpha v_1, \alpha v_2, \dots \alpha v_n)$
where $\alpha \in \mathbb{R} \Rightarrow$ Scalar, $\underset{\sim}{v} \in V = \mathbb{R}^n$.
It is easy to check that $V = \mathbb{R}^n$ also satisfies (VS1)-(VS8)

* Let $X = [0,1]$ and look at the space $V = \mathbb{R}^X$
$V = \mathbb{R}^X = \{ f \mid f : X \to \mathbb{R} \text{ is a map} \}$ of all real valued f's defined on $X = [0,1]$.

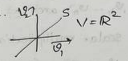
Eg: Suppose $0 < |A| < \infty$, $0 < |B| < \infty$, A, B are sets. then How many maps from $B \to A$? $\Rightarrow |A|^{|B|}$.
$A^B = \{ f : f : B \to A \text{ is a map} \}$

* Define vector addition and scalar multiplication pointwise, i.e, $\forall\, \alpha \in \mathbb{R}$ and $\forall\, f, g \in V$, define $f + g \in V$ and $\alpha f \in V$ as follows:
$(f+g)(x) = f(x) + g(x)$, $x \in X$
$(\alpha f)(x) = \alpha f(x)$, $x \in X$
It can be checked that V also satisfies (VS1) to (VS8)
With the zero function $\underset{\sim}{0}$ (i.e., $\underset{\sim}{0}(x) = 0$ $\forall\, x \in X$)

being the identity element for vector addition and the negative of a f^n $f \in V$ (i.e., $(-f)(x) := -f(x)$ $\forall x \in X$) being the vector additive inverse of f.

Ex:- Define $S = \{(v_1, v_2) \in V = \mathbb{R}^2 : v_1 = v_2\} \subseteq V = \mathbb{R}^2$

Let S borrow the vector addition and scalar multiplication from $V = \mathbb{R}^2$. One can check that S also satisfies (VS1)-(VS8) with these borrowed operation.



Vector Add	Scalar Mult
$S \times S \to S$	$\mathbb{R} \times S \to S$
$(\underset{\sim}{v}, \underset{\sim}{w}) \mapsto (\underset{\sim}{v} + \underset{\sim}{w})$	$(\alpha, \underset{\sim}{v}) \mapsto \alpha \underset{\sim}{v}$

Def^n: Let V be a non-empty set with a binary operation called vector addition on V, i.e, a map $V^2 = V \times V \to V$ defined by

$$(\underset{\sim}{u}, \underset{\sim}{v}) \mapsto \underset{\sim}{u} + \underset{\sim}{v} \quad \forall (\underset{\sim}{u}, \underset{\sim}{v}) \in V^2$$

and a scalar multiplication by real numbers, i.e, a map $\mathbb{R} \times V \to V$ defined by

$$(\alpha, \underset{\sim}{v}) \mapsto \alpha \underset{\sim}{v} \quad \forall \underset{\sim}{v} \in V$$

Such that the axioms (VS1) - (VS8) are satisfied. then we say that V is a vector space / linear space over $\mathbb{R}$.

Remarks: ① Elements of $V \to$ vectors

" of $\mathbb{R} \to$ scalars

② One can define a vector space over $\mathbb{C}$ or more generally over any "field" F (satisfies (A1) to (A6)) axioms.

For this course $F = \mathbb{R}$.

Eg: 1) $V = \mathbb{R}^n$, $n \in \mathbb{N}$ with component wise vector addition and component wise scalar multiplication, is a vector space over $\mathbb{R}$. We shall define $\mathbb{R}^0 := \{\underset{\sim}{0}\}$.

2) $X = [0,1]$, $V = \{f \mid f: X \to \mathbb{R}$ is a map.$\}$

Define both vector addition and scalar multiplication pointwise. then V is a vector space over $\mathbb{R}$.

Thm: Suppose V is a vector space over $\mathbb{R}$. Then $\forall$ $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V$ and $\forall \alpha, \beta, \gamma \in \mathbb{R}$, we have

(i) $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{w} \Rightarrow \underset{\sim}{v} = \underset{\sim}{w}$

(cancellation law for vector addition)

(ii) $\underset{\sim}{0}$ is unique $\Rightarrow$ Identity element for vector Add

(iii) $-\underset{\sim}{v}$ is unique (given a $\underset{\sim}{v}$).

(iv) $\underset{\sim}{v} + \underset{\sim}{v} = \underset{\sim}{v} \Rightarrow \underset{\sim}{v} = \underset{\sim}{0}$

(v) $0 \underset{\sim}{v} = \underset{\sim}{0}$

(vi) $\alpha \underset{\sim}{0} = \underset{\sim}{0}$

(vii) $\alpha \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow \alpha = 0$ or $\underset{\sim}{v} = \underset{\sim}{0}$

i.e $\alpha \underset{\sim}{v} = \underset{\sim}{0}$, $\alpha \in \mathbb{R} - \{0\} \Rightarrow \underset{\sim}{v} = \underset{\sim}{0}$

(cancellation law for scalar multiplic)

(viii) $(-\alpha) \underset{\sim}{v} = -(\alpha \underset{\sim}{v})$. In particular, $(-1)\underset{\sim}{v} = -\underset{\sim}{v}$

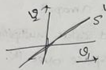
Exc:- (optional) Prove (i) - (viii) above in generality

Mandatory:- Prove them for $V = \mathbb{R}^n$.

Exc:- (Atleast for $V = \mathbb{R}^n$)

$(\alpha+\beta)\underset{\sim}{0} = (\alpha+\gamma)\underset{\sim}{v} \Rightarrow \underset{\sim}{v} = 0$ or $\beta = \gamma$ $\forall$ $\alpha, \beta, \gamma \in \mathbb{R}$ and $\forall$ $\underset{\sim}{v} \in V$.

Eg:- (Before one) $\underset{\text{vector space}}{\overset{\uparrow}{S}} = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 = v_2\} \subseteq \underset{\text{Vector space}}{\overset{\uparrow}{V = \mathbb{R}^2}}$



Def^n:- If V is a vector space over $\mathbb{R}$ and $\phi \neq S \subseteq V$, then S is called a (linear) Subspace/(vector) Subspace of V if S, with vector addition and scalar multiplication borrowed from V, forms a vector space. **Notation**: $\underset{(\text{subspace})}{S \subseteq V}$

Eg:- ① $V = \mathbb{R}^2$, $S = \{\underset{\sim}{v} = (v_1, v_2) \in \mathbb{R}^2 : v_1 = v_2\}$. Then S is a linear Subspace of V.

Exc:- ② $V = \mathbb{R}^3$, show that $S := \{\underset{\sim}{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} v_i = 0\}$ is a linear Subspace of V.

Eg:- ③ Let $X = [0,1]$. $V = \mathbb{R}^X = \{f \mid f : X \to \mathbb{R}$ is a map$\}$

V forms a vector space with operations defined point wise.

Exc:- Define $S = \{f \in V : f$ is cont$\}$

$= \{f \mid f : [0,1] \to \mathbb{R}$ is a cont map$\}$.

Show that S is an vector subspace of V.

Remark:- In Ex ③ above, if we define $S_1 = \mathcal{R}[0,1]$, then it can be shown that S_1 is also a linear Subspace of V. In fact,

$\underset{(\text{Subsp})}{S \subseteq} S_1 \underset{(\text{Subspace})}{\subseteq} V$

thm:- Let V be any vector space (over $\mathbb{R}$). Let S be A non-empty subset $S \subseteq V$ is a (linear/Vector) Subspace of V iff S is closed under taking linear combinations,

i.e; $\underbrace{\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}}_{\text{a linear combination}} \in S$ $\forall$ $\alpha, \beta \in \mathbb{R}$, $\forall$ $\underset{\sim}{u}, \underset{\sim}{v} \in S$.

of vectors $\underset{\sim}{u}, \underset{\sim}{v} \in V$ with scalar coeff $\alpha, \beta \in \mathbb{R}$.

Remark:- closed under taking linear combinations $\Leftrightarrow$ closed under both vector add, scalar multip

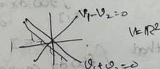
Cor:- (i) $\underset{(\text{Subsp})}{S \subseteq V} \Rightarrow \underset{\sim}{0} \in S$

(ii) $\underset{\substack{\uparrow \\ \text{Smallest possible subspace}}}{S_0} = \{\underset{\sim}{0}\} \underset{(S.S)}{\subseteq} V$ — trivial S.S of V.

(iii) $V \underset{(S.S)}{\subseteq} V$ (largest)

Any other S.S S, i.e, $\{\underset{\sim}{0}\} \subsetneq S \subsetneq V$, is called nontrivial S.S of V.

Exc:



$S = \{(v_1, v_2) \in \mathbb{R}^2 : (v_1+v_2)(v_1-v_2) = 0\}$

Show ~~that~~ $S \subseteq V$ but S is not a linear subspace of V.

Eg:- Let $X = [0,1]$, $V = \mathbb{R}^X = \{f \mid f : [0,1] \to \mathbb{R}\}$

Define $\mathcal{P} = \{f \in V : f \text{ is a polynomial } f^n\}$.

Exc: Show that $\underset{(S.S)}{\mathcal{P} \subseteq V}$.

Fix a $n \in \mathbb{N}$. Define $\mathcal{P}_n = \{f \in \mathcal{P} : \deg(f) \le n-1\}$

In part, $\mathcal{P}_1 = \{f \in \mathcal{P} : f \text{ is a const poly}\}$

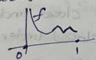
Exc: ~~show~~ that Fix $n \in \mathbb{N}$. show that
$$\mathcal{P}_n \underset{(S.S)}{\subseteq} \mathcal{P} \underset{(S.S)}{\subseteq} V$$

5/11/24

Improper Riemann Integral of Second Kind:

We Specialize the following Case:

Suppose $f : (0,1] \to \mathbb{R}$ is an unbdd func^n that "only blows up near 0".



i.e $\forall\, \varepsilon \in (0,1)$, $f : [\varepsilon, 1] \to \mathbb{R}$ is bdd. and $f \in \mathcal{R}([\varepsilon,1])$

Ex: $f : (0,1] \to \mathbb{R}$, defined by $f(x) = \frac{1}{\sqrt{x}}$; $x \in (0,1]$

In this situation, if the limit
$$I := \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 f(x)\,dx \in \mathbb{R} \text{ exists and is finite,}$$
then we say that f is Riemann Integrable on $(0,1]$ (and write $f \in \mathcal{R}(0,1]$) and define $\int_{(0,1]} f(x)\,dx = \int_0^1 f(x)\,dx := I$)

In this case we also say that $\int_{(0,1]} f(x)\,dx$ converges,

if $f \ge 0$, and write $\int_0^1 f(x)\,dx < \infty$. finite.

If for $f \ge 0$, $\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 f(x)\,dx = \infty$, we say that $\int_{(0,1]} f(x)\,dx$ diverges and we write $\int_0^1 f(x)\,dx = \infty$

Eg: $f(x) = \frac{1}{\sqrt{x}}$; $x \in (0,1]$. Note that $\forall\, \varepsilon \in (0,1)$, $f \in \mathcal{R}([\varepsilon,1])$.
$$\int_\varepsilon^1 f(x)\,dx = \int_\varepsilon^1 x^{-1/2}\,dx = \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_\varepsilon^1 = \left[2x^{1/2}\right]_\varepsilon^1$$
$$= 2 - 2\sqrt{\varepsilon}$$

Hence, $I := \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 f(x)\,dx = \lim_{\varepsilon \to 0^+} 2 - 2\sqrt{\varepsilon} = 2 < \infty$

$\Rightarrow f \in \mathcal{R}(0,1]$, and $\int_{(0,1]} f(x)\,dx = \int_{(0,1]} \frac{1}{\sqrt{x}}\,dx = 2$.

Exc: Define $\forall\, s \in (0,\infty)$, $f(x) = \frac{1}{x^s}$, $x \in (0,1]$.

Then $\int_{(0,1]} f(x)\,dx = \int_{(0,1]} \frac{1}{x^s}\,dx = \begin{cases} \text{converges} & \text{if } s < 1 \\ \text{diverges} & \text{is } s \ge 1 \end{cases}$

Q: What if the direct evaluation test fails?

Thm: (Comparision Test) : Suppose $f : (0,1] \to [0,\infty)$ and $g : (0,1] \to [0,\infty)$ are two unbdd func^n st $\forall\, \varepsilon \in (0,1)$, $f, g \in \mathcal{R}[\varepsilon, 1]$. If further $0 \le f(x) \le g(x)$ $\forall\, 0 < x \le 1$ then we have:

(i) $\int_{(0,1]} g(x) < \infty \Rightarrow \int_{(0,1]} f(x)\,dx < \infty$

(ii) $\int_{(0,1]} f(x)\,dx = \infty \Rightarrow \int_{(0,1]} g(x)\,dx = \infty$

Thm: (Ratio Test): Suppose $f: (0,1] \to [0,\infty)$ and $g: (0,1] \to (0,\infty)$ are two ~~cntbl~~ f^n's s.t. $\forall\, \epsilon \in (0,1)$ $f, g \in R[\epsilon, 1]$. If further $\lim_{x \to 0^+} \frac{f(x)}{g(x)} = c \in (0,\infty)$ — ①

then either both $\int_{(0,1]} f(x)dx$, $\int_{(0,1]} g(x)dx$ converges or both diverges.

Moreover, if $c = 0$ in ① then $\int_{(0,1]} g(x)dx < \infty$

$\Rightarrow \int_{(0,1]} f(x)dx < \infty$ but ~~not~~ vice versa may not be true.

6/11/24 **LECTURE - 18**

Goal: To understand $V = \mathbb{R}^n$ and its linear Subspaces

Def^n:- Let V be a vector space (over $\mathbb{R}$). We take $k \in \mathbb{N}$ and vector $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k \in V$.

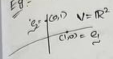
Any vector of the form

$\underset{\sim}{v} = \alpha_1 \underset{\sim}{v}_1 + \alpha_2 \underset{\sim}{v}_2 + \dots \alpha_k \underset{\sim}{v}_k =: \sum \alpha_i \underset{\sim}{v}_i$,

where $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$, is called a linear Combination of $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots \underset{\sim}{v}_k \in V$

Exc:- Let V be a vector space and $\phi \neq S \subseteq V$. Then show that $S \underset{(S.S)}{\underline{\subseteq}} V$ iff $\forall\, k \in \mathbb{N}$ and $\forall$ $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$ and $\forall$ $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots \underset{\sim}{v}_k \in S$, $\sum_{i=1}^{k} \alpha_i \underset{\sim}{v}_i \in S$. [Hint: If part $\to$ just use $k=2$ + Thm only if part $\to$ induction on k]

Eg:- $V = \mathbb{R}^2$ and $B = \{ \underset{\sim}{e}_1 := (1,0), \underset{\sim}{e}_2 := (0,1) \}$



Note that any vector $\underset{(v_1, v_2)}{\underset{\sim}{v}} \in V = \mathbb{R}$ can be written as

$\underset{\sim}{v} = (v_1, v_2) = (v_1, 0) + (0, v_2) = v_1(1,0) + v_2(0,1)$
$= v_1 \underset{\sim}{e}_1 + v_2 \underset{\sim}{e}_2 =: \sum_{i=1}^{2} v_i \underset{\sim}{e}_i$

We shall say that V is the "<u>linear Span</u>" $\bar{B} = Sp(B)$ of B.

Def^n:- Let V be any vector space. ~~If~~ For a nonempty finite set $B = \{ \underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots \underset{\sim}{v}_k \} \subseteq V$ of vectors, we define the linear span of B as the set

$\bar{B} = Sp(B) := \{ \sum \alpha_i v_i : \alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R} \}$ of all possible linear combinations of the elements of B

We also define $\bar{\phi} = Sp\{\phi\} = \{\underset{\sim}{0}\}$.

Exc:- Let V be a vector space. $B \subseteq V$ is any finite subset. Show that linear span of $\bar{B} \underset{(S.S)}{\subseteq} V$.

* In the above example, $V = \mathbb{R}^2$, $B = \{ \underset{\sim}{e}_1 := (1,0), \underset{\sim}{e}_2 = (0,1) \}$ and $\bar{B} = V = \mathbb{R}^2$ ($\bar{B} \subseteq V$ is obvious, $V \subseteq \bar{B}$ has been shown)

Def^n:- In the above setup, whenever $S = \underset{\text{Span}}{\bar{B}} = Sp\{ \underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots \underset{\sim}{v}_k \}$, we say that the vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots \underset{\sim}{v}_k$ generate S or simply say B generates S.

Eg: If $V = \mathbb{R}^2$, then $B = \{ \underset{\sim}{e}_1, \underset{\sim}{e}_2 \}$ generates V.

More generally, if $V = \mathbb{R}^n$, then $B = \{ \underset{\sim}{e}_1, \underset{\sim}{e}_2, \dots \underset{\sim}{e}_n \}$,

(where $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$
$e_2 = (0,1,0,\dots,0) \in \mathbb{R}^n$
$\vdots$
$e_n = (0,0,\dots,0,1) \in \mathbb{R}^n$)

generates V. Bcz $\overline{B}$ (linear span) of $B \subseteq V$. To show $V \subseteq \overline{B}$ $\rightarrow \overline{B} = V$ observe that any $v = (v_1, v_2, \dots v_n) \in V = \mathbb{R}^n$ is of the form

$$v = \sum_{i=1}^{n} v_i e_i = \left((v_1, 0, 0, \dots 0) + (0, v_2, 0, \dots 0) + \dots + (0, 0, \dots, v_n) \right)$$
$$= v_1 e_1 + v_2 e_2 + \dots v_n e_n \in \overline{B}$$

Claim:- If $V = \mathbb{R}^2$, then $B_1 = \{e_1, e_2, e_1 + e_2\}$
$= \{(1,0), (0,1), (1,1)\}$ also generates V.

Proof:- Clearly $\overline{B}_1 \subseteq V$. To show $V \subseteq \overline{B}_1$. Take any vector $v = (v_1, v_2) = V = \mathbb{R}^2$, then

$$v = v_1 e_1 + v_2 e_2 + 0(e_1 + e_2) \in \overline{B}_1$$

Therefore $\overline{B}_1 = V$.

However $(1,1) = e_1 + e_2$ is <u>redundant</u> (unnecessary) because e_1 and e_2 generate V anyway! This is happening bcz

$$1(e_1 + e_2) + (-1)e_1 + (-1)e_2 = 0, \quad \text{i.e,}$$

$e_1, e_2, e_1 + e_2$ are "linearly dependent".

Exc: Show that $e_1 = (1,0)$, $e_1 + e_2 = (1,1)$ also generate $V = \mathbb{R}^2$

Q?:- Can we remove one of these and yet generate $V = \mathbb{R}^2$?

Ans:- NO! (bcz e_1 and $e_1 + e_2$ are "linearly independent")

Def^n:- ① For any vector space V, a finite non empty collection $C = \{v_1, v_2, \dots v_k\} \neq \phi \subseteq V$ of vectors is called "linearly independent" (or $v_1, v_2, \dots v_k$ are called "linearly independent") if

$$\sum_{i=1}^{k} \alpha_i v_i = 0, \quad \alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$$
$$\rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_k = 0. \text{ Trivial linear comb.}$$

In other words, no nontrivial linear combination of $v_1, v_2, \dots v_k = 0$.

Eg.: (1) $V = \mathbb{R}^2$. $e_1, e_2, e_1 + e_2$ are not linearly independent (i.e linearly dependent) because

$$1(e_1 + e_2) + (-1)e_1 + (-1)e_2 = 0$$

Non trivial linear comb of $e_1 + e_2$, $e_1 + e_2$

Def^n: ② In the setup of ①, if $\exists\ \alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$ S.T $(\alpha_1, \alpha_2, \dots \alpha_k) \neq (0, 0, \dots 0)$ (i.e atleast one $\neq 0$).

and $\sum_{i=1}^{k} \alpha_i v_i = 0$ (non-trivial linear comb) then $v_1, v_2, \dots v_k$ are called linearly dependent or $C = \{v_1, v_2, \dots v_k\}$ is called linearly dependent.

** If $C = \phi$, then we define C to be <u>linearly independent</u>.

Eg: (2) $V = \mathbb{R}^n$. Fix $m \in \{1, 2, \dots n\}$. Let $C = \{e_1, e_2, \dots e_m\}$

Then C is linearly independent.

Ans:- Suppose $\underbrace{\sum_{i=1}^{m} \alpha_i \underset{\sim}{e_i}}_{\text{I}} = \underset{\sim}{0}$

$\Rightarrow (\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_m, 0, 0, \ldots 0) = \underset{\sim}{0}$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots \alpha_m = 0.$

$\Rightarrow \underset{\sim}{e_1}, \underset{\sim}{e_2}, \ldots \underset{\sim}{e_m}$ are lin independent

Exc:- $V = \mathbb{R}^n$. Show that $C = \{\underset{\sim}{e_1}, \underset{\sim}{e_1}+\underset{\sim}{e_2}, \underset{\sim}{e_1}+\underset{\sim}{e_2}+\underset{\sim}{e_3}, \ldots, \underset{\sim}{e_1}+\underset{\sim}{e_2}+\cdots \underset{\sim}{e_n}\}$ is linearly independent.

[Hint: $C = \{(1,0,0 \ldots 0), (1,1,0,0, \ldots 0), \ldots (1,1,1,1, \ldots 1)\}$]

Exc:- $V = \mathbb{R}^4$. Show that $\underset{\sim}{e_1}, \underset{\sim}{e_1}+2\underset{\sim}{e_2}, \underset{\sim}{e_1}+\underset{\sim}{e_2}, \underset{\sim}{e_3}, \underset{\sim}{e_4}$ are linearly dependent.

Thm:- Let V be any vector space. Then we have:

(i) If $B (\subseteq V)$ generates V ($\overline{B} = V$), then so does any finite superset of B which is also a subset of V.

(ii) If $C (\subseteq V)$ is linearly ind, then so is any subset of C.

(iii) If $\underset{\sim}{0} \in C$, then C is always lin dep.

(iv) $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots \underset{\sim}{v_k} \in V$ are linearly dep iff at least one of these vectors can be written as a linear comb of other vectors. In particular, for $k=2$, $\underset{\sim}{v_1}, \underset{\sim}{v_2}$ are lin dep iff one of them is scalar multiple of other.

Proof:- If part: Suppose WLOG (without loss of generality) that

$\underset{\sim}{v_1} = \sum_{i=2}^{k} \alpha_i \underset{\sim}{v_i}$, where $\alpha_1, \alpha_2, \ldots \alpha_k \in \mathbb{R}$.

Then $\underbrace{1}_{\neq 0}\underset{\sim}{v_1} + (-\alpha_2)\underset{\sim}{v_2} + (-\alpha_3)\underset{\sim}{v_3} + \cdots + (-\alpha_k)\underset{\sim}{v_k} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots \underset{\sim}{v_k}$ are lin dep.

Only if part:- Suppose $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots \underset{\sim}{v_k}$ are linearly dep $\Rightarrow \sum_{i=1}^{k} \alpha_i \underset{\sim}{v_i} = \underset{\sim}{0}$ for some $\alpha_1, \alpha_2, \ldots \alpha_k \in \mathbb{R}$ s.t. $(\alpha_1, \alpha_2, \ldots \alpha_k) \neq (0, 0, \ldots 0)$. Let $\alpha_j \neq 0$. Then

$$\alpha_j \underset{\sim}{v_j} = \sum_{\substack{i=1 \\ i \neq j}}^{n} (-\alpha_i) \underset{\sim}{v_i}$$

$$\Rightarrow \underset{\sim}{v_j} = \sum_{\substack{i=1 \\ i \neq j}}^{n} \underbrace{\left(-\frac{\alpha_i}{\alpha_j}\right)}_{\in \mathbb{R}} \underset{\sim}{v_i}$$

$\Rightarrow \underset{\sim}{v_j}$ is lin comb of rest.

The standard/canonical basis for $\mathbb{R}^n$.

* Note that $B = \{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \ldots \underset{\sim}{e_n}\} \subseteq V = \mathbb{R}^n$ is a lin ind set that generates $V = \mathbb{R}^n$. We have a name for such a set.

Def^n:- Let V be any vector space over $\mathbb{R}$. If

$B = \begin{cases} \{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots \underset{\sim}{v_k}\} & \text{if } k \in \mathbb{N} \\ \phi & \text{, if } k = 0 \end{cases}$ $\subseteq V$ (finite subset) $k \in \mathbb{N} \cup \{0\}$

is lin indep. Such that $\overline{B} = V$, then we say that V is a finite dimensional vector space over $\mathbb{R}$ and B is called a basis of / for V.

(If $B = \phi$, then $V = \{\underset{\sim}{0}\}$.)

$B \neq \phi$, then V = non trivial vector space

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Q^n: How to combine two types of improper Riemann Integrals?

Ans:- When $f:(0,\infty)\to\mathbb{R}$ is S.T it blows up near zero, and $f\in\mathcal{R}(0,1]$ and $f\in\mathcal{R}[1,\infty)$, then we say that f is Riemann Integrable on $(0,\infty)$ (and write $f\in\mathcal{R}(0,\infty)$), and define ~~$\int f(x)\,dx$~~

$$\int_{(0,\infty)} f(x)\,dx = \int_0^\infty f(x)\,dx = \int_{(0,1]} f(x)\,dx + \int_{[1,\infty)} f(x)\,dx.$$

Eg:- For each $\alpha\in(0,\infty)$, define a map $f:(0,\infty)\to(0,\infty)$ by $f(x):=x^{\alpha-1}e^{-x}$, $x\in(0,\infty)$.

Clearly (from class) that $f\in\mathcal{R}[1,\infty)$

Q^n:- Is $\int_0^1 f(x)\,dx<\infty$?

Case 1):- $\alpha\geq 1$

Then $f(x)=x^{\alpha-1}e^{-x}$; $x\in[0,1]$.

This is a cont f^n on closed, bdd interval: hence f is $\mathcal{R}[0,1] \Rightarrow \int_0^1 f(x)\,dx<\infty$.

Hence if $\alpha\geq 1$, then $f\in\mathcal{R}(0,\infty)$.

Case 2):- $\alpha<1$ means $\alpha\in(0,1)$.

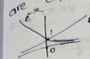
Define $g(x)=\dfrac{1}{x^{1-\alpha}}$, $x\in(0,1]$, and look at f restricted to $(0,1]$, i.e $f|_{(0,1]}$, i.e

$f(x)=\dfrac{1}{x^{1-\alpha}}e^{-x}$; $x\in(0,1]$

$f(x)=\dfrac{1}{x^{1-\alpha}}\cdot e^{-x}$

$g(x)=\dfrac{1}{x^{1-\alpha}}$

NOTE:- $f,g\in\mathcal{R}[\varepsilon,1]$ $\forall\ \varepsilon\in(0,1)$ and both are $(0,\infty)$-valued f^ns and $\lim\limits_{x\to0^+}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to0^+}e^{-x}=1\in(0,\infty)$.



Moreover, $g(x)=\dfrac{1}{x^s}$, $x\in(0,1]$, where $s=1-\alpha\in(0,1)$ is done as an exe.

Hence $g\in\mathcal{R}(0,1]$ and hence $\int_0^1 g(x)\,dx<\infty$.

So by ratio test $\int_0^1 f(x)\,dx$ is finite. Now, combining everything we get that $\forall\ \alpha\in(0,\infty)$, $\int_0^\infty x^{\alpha-1}e^{-x}\,dx$

$$=\underbrace{\int_0^1 x^{\alpha-1}e^{-x}\,dx}_{<\infty}+\int_1^\infty x^{\alpha-1}e^{-x}\,dx.$$

Def^n:- $\forall\ \alpha\in(0,\infty)$, define $\Gamma(\alpha):=\int_0^\infty x^{\alpha-1}e^{-x}\,dx$

(Gamma Function)

Note that $\Gamma:(0,\infty)\to(0,\infty)$.

*** Exc: Show that (i) $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ $\forall\ \alpha\in(0,\infty)$.

(ii) $\forall\ n\in\mathbb{N}$, S.t $\Gamma(n)=(n-1)!$

(Hint: Use (i) and induction on n).

Exc: S.T $\displaystyle\int_0^\infty \frac{1}{x^2+\sqrt{x}}\,dx<\infty$

(Hint:- Compare with $\frac{1}{\sqrt{x}}$ on $(0,1]$, with $\frac{1}{x^2}$ on $(1,\infty)$).

Breaking and camparision.

11/11/24

LECTURE-19

Thm:- Suppose V is a finite dimensional vector space over $\mathbb{R}$ (fdvs). Then TFAE (The following are equivalent). And $B \subseteq V$ (finite)

(i) $B = \{v_1, v_2, \dots v_k\}$ is a basis for V.

(ii) B is a minimum generating set for V, i.e no proper subset of B generates V.

(iii) B is a maximal linearly independent subset of V, i.e, $\forall\, v \in V \setminus B$, $B \cup \{v\}$ is linearly dependent.

(iv) For each vector $w \in V$ can be written uniquely as a linear combination of $v_1, v_2, \dots v_k$ i.e, for each $w \in V$, $\exists$ unique $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$ s.t

$$w = \sum_{i=1}^{k} \alpha_i v_i.$$

Eg:- (1) $V = \mathbb{R}^n$

Exc:- Show that the following sets are all bases of $V = \mathbb{R}^n$

(i) $B = \{e_1, e_2, \dots e_n\}$ = standard basis for $\mathbb{R}^n$.

(ii) $B_1 = \{e_1, e_1+e_2, e_1+e_2+e_3, \dots, e_1+e_2+\dots+e_n\}$

(iii) $B_2 = \{e_1, e_1+e_3, e_3, e_4, \dots, e_n\}$

Thm:- Let V be fdvs over $\mathbb{R}$. Then any two basis for V have same size

Def^n:- Let V be fdvs over $\mathbb{R}$ and B be any basis of V. Then we define the dimensional of V as

$\dim := |B|$

From the exc, it follows that $\dim(\mathbb{R}^n) = n$ $\forall\, n \in \mathbb{N}$. On the other hand $\dim(\{0\}) = 0$.

Therefore, it is customery to define

$\mathbb{R}^0 := \{0\}$.

Eg(2) $V = \mathbb{R}^3$. $S = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0\}$

It is possible to check that $S \underset{(S.S)}{\subseteq} V$.

Q^n:- $\dim(S) = ?$

$$S = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_3 = -v_1 - v_2\}$$
$$= \{(v_1, v_2, -v_1 - v_2) : v_1, v_2 \in \mathbb{R}\}$$

$\Rightarrow$ any $v = (v_1, v_2, v_3) \in S$ is of the form

$$v = (v_1, v_2, -v_1 - v_2) = (v_1, 0, -v_1) + (0, v_2, -v_2)$$
$$v = v_1(1,0,-1) + v_2(0,1,-1)$$

Claim:- $B = \{(1,0,-1), (0,1,-1)\}$ is a basis for S.

Proof:- We just showed that $S \subseteq \bar{B}$.

$B \subseteq S \Rightarrow \bar{B} \subseteq S \Rightarrow S = \bar{B}$.

To show B in lin ind since $S \underset{(S.S)}{\subseteq} V$

Take $\alpha_1, \alpha_2 \in \mathbb{R}$ s.T $\alpha_1(1,0,-1) + \alpha_2(0,1,-1) = (0,0,0)$

$(\alpha_1, \alpha_2, -\alpha_1 - \alpha_2) = (0,0,0)$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0$

$\Rightarrow B$ is lin ind and $\bar{B} = S$.

$\Rightarrow B$ is a basis for S.

In part, dim of $S = 2$. $= |B|$

$\dim(S) = |B| = 2$.

Exc:- $V = \mathbb{R}^3$. Define $S_1 := \{ \underset{\sim}{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 = v_2 = v_3 \}$

$S_2 := \{ \underset{\sim}{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 = v_2 \}$.

(i) Show that $\underset{(S.S)}{S_1 \subseteq} \underset{(S.S)}{S_2 \subseteq} V$.

(ii) Find a basis for S_1 and show that $\dim(S_1) = 1$.

(iii) Find a basis for S_2 and S.T $\dim(S_2) = 2$.

Thm: Suppose V is fdvs and $\underset{(S.S)}{S \subseteq} V$. Then any basis for S can be extended to a basis for V. In part, $\dim(S) \leq \dim(V)$

Def: Let V be a fdvs and S_1, S_2 are two subspaces of V. Define $S_1 + S_2$ to be

$$S_1 + S_2 := \{ \underset{\sim}{v_1} + \underset{\sim}{v_2} : \underset{\sim}{v_1} \in S_1, \underset{\sim}{v_2} \in S_2 \}$$

Exc:- Show that $S_1 \cap S_2$ and $S_1 + S_2$ are both subspaces of V.

* Suppose $V = \mathbb{R}^2$. Let $\underset{(S.S)}{S \subseteq} V$. $\Rightarrow \dim(S) \leq \dim(V) = 2$

$\Rightarrow \dim(S) \in \{0, 1, 2\}$

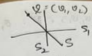
If $S = \{\underset{\sim}{0}\}$, then $\dim(S) = 0$

If $S = V = \mathbb{R}^2$, then $\dim(S) = 2$

For all other subspaces (i.e, nontrivial subspaces of V), $\dim(S) = 1$.

Take any $\underset{\sim}{v} \in S \setminus \{\underset{\sim}{0}\}$.

Then it can be shown that $S = \{ \overline{\underset{\sim}{v}} \}$. This means that $S = \{ \alpha \underset{\sim}{v} = (\alpha v_1, \alpha v_2) : \alpha \in \mathbb{R} \}$



$\dim(S) = 1 \Rightarrow S$ is a st. line passing through $\underset{\sim}{0}$.

Take $S_1 = \{ (v_1, v_2) \in \mathbb{R}^2 : v_2 = 0 \}$

$= \{ (v_1, 0) : v_1 \in \mathbb{R} \}$

$S_2 = \{ (v_1, v_2) \in \mathbb{R}^2 : v_1 = 0 \}$

$= \{ (0, v_2) : v_2 \in \mathbb{R} \}$

Clearly $S_1 \cap S_2 = \{(0,0)\} = \{\underset{\sim}{0}\}$.

$S_1 + S_2 = V = \mathbb{R}^2$ (bcz Any vector $\underset{\sim}{v} = (v_1, v_2)$ can be written as $\underset{\sim}{v} = \underset{S_1}{(v_1, 0)} + \underset{S_2}{(0, v_2)} \Rightarrow S_1 + S_2 = \mathbb{R}^2$)

Thm:- Let V be a fdvs with $\dim(V) = n \geq 2$. Let $\underset{(S.S)}{S \subseteq} V$. Then $\dim(S) = 0 \Rightarrow S = \{\underset{\sim}{0}\}$, ~~dim(S) = n~~

$\dim(S) = n \Rightarrow S = V$,

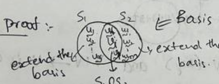
For all other subspaces S of V, i.e whenever $0 < \dim(S) < n$, we have $\{\underset{\sim}{0}\} \underset{\neq}{\subseteq} S \underset{\neq}{\subseteq} V$, i.e; S is a nontrivial subspace of V.

Thm:- let V be a fdvs and $\underset{(S.S)}{S_1 \subseteq V}$, $\underset{(S.S)}{S_2 \subseteq V}$. Then

$$\dim(S_1 + S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$$

(Inclusion-Exclusion Formulae for Dimension)

Sketch of Proof :-



$\Leftarrow$ Basis Venn diagram

$\dim(S_1 \cap S_2) = l$, $\dim(S_1) = k + l$, $\dim(S_2) = l + m$

$\dim(S_1 + S_2) = k + l + m$

$\Rightarrow \dim(S_1 + S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$.

Application :- Let $V = \mathbb{R}^2$, $S_1 = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 - v_2 = 0\}$

$S_2 = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 + v_2 = 0\}$

Easy to check ① $S_1 \underset{(SS)}{\subseteq} V$

② $S_2 \underset{(SS)}{\subseteq} V$

③ $S_1 \cap S_2 = \{\underset{\sim}{0}\}$

④ $\dim(S_1) = \dim(S_2) = 1$

Then what is $\dim(S_1 + S_2)$

$\Rightarrow \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2) = 1 + 1 - 0 = 2$.

$\Rightarrow \boxed{S_1 + S_2 = V = \mathbb{R}^2}$

(checking this directly is not very easy).

Exc:- $V = \mathbb{R}^3$, $S_1 = \{\underset{\sim}{v} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0\}$.

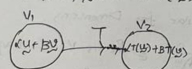
$S_2 = \{\underset{\sim}{v} \in \mathbb{R}^3 : v_1 = v_2 = v_3\}$

S.t $S_1 + S_2 = V$. (Show all steps)

Linear Map / Linear Transformation :-

Defⁿ :- ① Let V_1, V_2 be fdvs. A map $T : V_1 \to V_2$ is called a linear map / transformation if $\forall\, \alpha, \beta \in \mathbb{R}$, $\forall\, \underset{\sim}{u}, \underset{\sim}{v} \in V_1$

$T(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) = \alpha T(\underset{\sim}{u}) + \beta T(\underset{\sim}{v})$



② If $V_1 = V_2 = V$ (say) and $T : V \to V$ is a linear map, then T is called a linear operator.

③ If $V_2 = \mathbb{R}$ (also a fdvs over $\mathbb{R}$ with $\dim(V_2) = 1$) and $T : V_1 \to \mathbb{R}$ is a linear map, then T is called a linear fⁿal.

④ If $T : V_1 \to V_2$ is a bijective linear map, then T is called a vectorspace isomorphism. and V_1, V_2 are called isomorphic vector spaces.

Notation:- $V_1 \cong V_2$.

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LECTURE - 20

Thm :- ⓐ If two fdvs V_1, V_2 are isomorphic (i.e, $V_1 \cong V_2$) then they have same dimension $\dim(V_1) = \dim(V_2)$

ⓑ If further $\dim(V_1) = n \in \mathbb{N} \cup \{0\}$, then $V_1 \cong V_2 \cong \begin{cases} \{\underset{\sim}{0}\} & \text{if } n = 0 \\ \mathbb{R}^n & \text{if } n \in \mathbb{N} \end{cases}$

(Recall that $\mathbb{R}^0 := \{\underset{\sim}{0}\}$)

Eg :- (1) $V_1 = \mathbb{R}^2$, $V_2 = \mathbb{R}$

Define $T : V_1 \to V_2$ by $T(u_1, u_2) = 2u_1 - 3u_2 \;\; \forall\, (u_1, u_2) \in V_1 = \mathbb{R}^2$,

Then T is a linear functional because $\forall\, \alpha, \beta \in \mathbb{R}$ and $\underset{\sim}{u}, \underset{\sim}{v} \in V_1 = \mathbb{R}^2$,

$$
\begin{aligned}
T(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) &= T(\alpha(u_1, u_2) + \beta(v_1, v_2)) \\
&= T(\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2) \\
&= 2(\alpha u_1 + \beta v_1) - 3(\alpha u_2 + \beta v_2) \\
&= \alpha(2u_1 - 3u_2) + \beta(2v_1 - 3v_2) \\
&= \alpha T(u_1, u_2) + \beta T(v_1, v_2) \\
&= \alpha T(\underset{\sim}{u}) + \beta T(\underset{\sim}{v})
\end{aligned}
$$

Exc:

(2) $V_1 = \mathbb{R}^3$, $V_2 = \mathbb{R}^2$, ST $T : V_1 \to V_2$ defined by ~~Show that~~ $T(v_1, v_2, v_3) = (v_1 + v_2, v_2 + v_3)$; $(v_1, v_2, v_3) \in V_1$ is a linear transformation from V_1 to V_2.

(3) $V_1 = V_2 = \mathbb{R}^2$. S.T $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2)$, $(v_1, v_2) \in \mathbb{R}^2$ is a linear operator on $\mathbb{R}^2$.

(ii) Take $V_1 := \{ (\omega_1,\omega_2,\omega_3) \in \mathbb{R}^3 : \omega_1+\omega_2+\omega_3 = 0 \} \underset{(S.S)}{\subseteq} \mathbb{R}^3$.

$V_2 := \{ (u_1,u_2) \in \mathbb{R}^2 : u_1+u_2 = 0 \} \underset{(S.S)}{\subseteq} \mathbb{R}^2$

$V_3 := \mathbb{R}^2$

Exc:- Show the following

(a) $T_1 : V_1 \to V_2$ defined by $T_1(\omega_1,\omega_2,\omega_3) = (\omega_1+\omega_2, \omega_3)$ $\forall\ \omega_1,\omega_2,\omega_3 \in V_1$ is a linear map from $V_1 \to V_2$.

(b) $T_2 : V_1 \to V_3$ defined by $T_2(\omega_1,\omega_2,\omega_3) = (\omega_1, \omega_2)$ and $\omega_1,\omega_2,\omega_3 \in V_1$ is a vector space isomorphism : V_1 to V_3.

* W.K.T any non-trivial finite dimensional vector space over $\mathbb{R}$ looks like $\mathbb{R}^n$, $n \in \mathbb{N}$ (as non-trivial)

Fix $m, n \in \mathbb{N}$

Q^n: What are all possible linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$?

From now on whenever we write a vector ($\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ etc) it is always a column vector. For example, $\underset{\sim}{u} \in \mathbb{R}^n$ means $\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \leftarrow n\times 1$ matrix or a column vector in $\mathbb{R}^n$.

To write a row vector, I shall use a transpose notation:

$\underset{\sim}{u}^T = (u_1, u_2, \ldots, u_n) \leftarrow 1\times n$ matrix or row vector in $\mathbb{R}^n$
$(u_1\ u_2 \cdots u_n)$

Fix $m, n \in \mathbb{N}$

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map.

Let us denote by $\underbrace{\underset{\sim}{e}_1^{(n)}, \underset{\sim}{e}_2^{(n)}, \ldots, \underset{\sim}{e}_n^{(n)}}_{\text{column vectors} \in \mathbb{R}^n}$ the standard basis elements for $\mathbb{R}^n$ and by $\underbrace{\underset{\sim}{e}_1^{(m)}, \underset{\sim}{e}_2^{(m)}, \ldots, \underset{\sim}{e}_m^{(m)}}_{\text{Column Vectors} \in \mathbb{R}^m}$, the standard basis elements for $\mathbb{R}^m$.

For each $j \in \{1, 2, \ldots, n\}$,

$$\underset{\mathbb{R}^n}{T(\underset{\sim}{e}_j^{(n)})} \ (\in \mathbb{R}^m) = \underset{j^{\text{th}}\text{ column of } A}{\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}} \text{ (Say)} \quad (\text{here } a_{ij} \in \mathbb{R}\ \forall\ i = 1, 2, \ldots, m)$$

$$= a_{1j}\underset{\sim}{e}_1^{(m)} + a_{2j}\underset{\sim}{e}_2^{(m)} + \cdots + a_{mj}\underset{\sim}{e}_m^{(m)}$$

$$\left[\underset{\sim}{e}_1^{(m)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m,\ \underset{\sim}{e}_2^{(m)} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m, \ldots \right]$$

Now take any column vector $\underset{\sim}{x} \in \mathbb{R}^n$.

Clearly $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{j=1}^{n} x_j \underset{\sim}{e}_j^{(n)}$.

Since $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, we have

$$\underbrace{T(\underset{\sim}{x})}_{\in \mathbb{R}^m} = T\left(\sum_{j=1}^{n} x_j \underset{\sim}{e}_j^{(n)}\right)$$

$$\xrightarrow{\text{linearity}} \sum_{j=1}^{n} x_j\, T(\underset{\sim}{e}_j^{(n)}) \quad \text{※ vectors (not matrices)}$$

$$= \sum_{j=1}^{n} x_j \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \sum_{j=1}^{n} a_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{pmatrix} \leftarrow m\times 1 \text{ matrix}$$

$$= \underset{m\times n}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}} \underset{n\times 1}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} = A\underset{\sim}{x}, \text{ where}$$

$\therefore L = \frac{-1+\sqrt{13}}{2}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \leftarrow m \times n \text{ matrix with real entries}$$

$T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then

Summary: $\forall\, \underset{\sim}{x} \in \mathbb{R}^n$, $T(\underset{\sim}{x}) = A\underset{\sim}{x}$ for some $m \times n$ matrix with real entries.

Conversely, for any $m \times n$ matrix A with real entries the map $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $\underbrace{T(\underset{\sim}{x})}_{m\times 1} = \underbrace{\underset{m n}{A}\ \underset{n\times 1}{\underset{\sim}{x}}}_{m\times 1}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

This is bcz $\forall\, \underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$, $\forall\, \alpha, \beta \in \mathbb{R}$,

$A(\alpha \underset{\sim}{x} + \beta \underset{\sim}{y}) = \alpha A \underset{\sim}{x} + \beta A \underset{\sim}{y}$.

*Check!!

We have thus proved:

Thm: Fix $m, n \in \mathbb{N}$. $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if $\exists$ an ~~real~~ $m \times n$ matrix A with real entries s.t

$T(\underset{\sim}{x}) = A\underset{\sim}{x} \quad \forall\, \underset{\sim}{x} \in \mathbb{R}^n$

(Column Vector)

Remark: This A is also unique

Defn: A is called the matrix for the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ (wrt. the standard bases).

Eg: ① If $\underset{mn}{A} = 0$ (zero matrix), then $T: \mathbb{R}^n \to \mathbb{R}^m$ is the zero map, i.e $T(\underset{\sim}{x}) = \underset{\sim}{0} \in \mathbb{R}^m \quad \forall\, \underset{\sim}{x} \in \mathbb{R}^n$

② If $m = n$ and

$$\underset{n\times n}{A} = \underset{\substack{\text{Identity matrix} \\ \text{of order } n}}{I_n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

(i.e $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$), then $T: \mathbb{R}^n \to \mathbb{R}^n$ is the identity map, i.e., $T(\underset{\sim}{x}) = \underset{\sim}{x} \quad \forall\, \underset{\sim}{x} \in \mathbb{R}^n$.

③ $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(\underset{\sim}{x}) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix} \quad \forall\, \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$.

It is easy to check that T is a linear map.

Qn: What is the matrix for T?

Here $n = 3$ and $m = 2$ so the matrix for T would be a 2×3 matrix A.

$T(\underset{\sim}{e}^{(3)}_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1^{st}$ column of A.

$T(\underset{\sim}{e}^{(3)}_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2^{nd}$ column of A.

$T(\underset{\sim}{e}^{(3)}_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3^{rd}$ column of A.

$$\therefore A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}_{2\times 3}$$

[Sanity check: $A\underset{\sim}{x} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$
$= T(\underset{\sim}{x}) \quad \forall\, \underset{\sim}{x} \in \mathbb{R}^3$.]

Exe: ~~Find the~~ S.T $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 - x_2 \\ x_2 \end{pmatrix}$, $\forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ is a linear map

Then find the matrix for this linear combination?

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LECTURE - 21

Rank of a Matrix:- Suppose A is an $m\times n$ matrix (with real entries). $A = ((a_{ij}))_{1\le i\le m,\ 1\le j\le n}$ = entry in the i^{th} row, j^{th} column.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Denote by $\underbrace{A_{1*}, A_{2*}, \cdots A_{m*}}_{\text{row vectors}}$ the $1^{st}, 2^{nd}, \ldots m^{th}$ row of A.

eg:- $\underset{2\times3}{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$A_{1*} = (1,1,0) = (1\ 1\ 0)$ $\quad A_{*1} = \begin{pmatrix}1\\0\end{pmatrix}$

$A_{2*} = (0,1,1) = (0\ 1\ 1)$ $\quad A_{*2} = \begin{pmatrix}1\\1\end{pmatrix}$

$A_{*3} = \begin{pmatrix}0\\1\end{pmatrix}$

So, $A = \begin{bmatrix} A_{1*} \\ A_{2*} \\ \vdots \\ A_{m*} \end{bmatrix}$

Similarly, we shall use $\underbrace{A_{*1}, A_{*2}, \cdots A_{*n}}_{\text{column vectors}}$ to denote the $1^{st}, 2^{nd}, \ldots n^{th}$ column of A.

So, $A = [A_{*1}, A_{*2}, \cdots A_{*n}]$

clearly $A_{*1}, A_{*2}, \cdots A_{*n} \in \mathbb{R}^m$

Column Space:- $\mathcal{C}(A) := Sp\{A_{*1}, \cdots A_{*n}\} \underset{(S.S)}{\subseteq} \mathbb{R}^m$

Row Space of A: $\mathcal{R}(A) := Sp\{\underbrace{A_{1*}, A_{2*}, \cdots A_{m*}}_{\text{row vectors}}\} \underset{(S.S)}{\in} \mathbb{R}^n$

Thm:- $\dim(\mathcal{C}(A)) = \dim(\mathcal{R}(A))$

$\dim(\mathcal{C}(A))$ = Column Rank of A = Max no. of linearly indep columns of A

$\dim(\mathcal{R}(A))$ = Row rank of A = Max no. of linearly indep rows of A

Def^n: Max no. of linearly indep columns of A = Max no. of linearly indep rows of A

Def^n: The above dimensions are called the rank of the matrix A.

Notation: $\text{rank}(A) = \rho(A) := \dim(\mathcal{C}(A)) \overset{Thm}{=} \dim(\mathcal{R}(A))$

Remark: ① Rank plays a very imp role in prob/stat/data science.

② Clearly $\left.\begin{array}{l}\rho(A) \le m \\ \text{also } \rho(A) \le n\end{array}\right\} \Rightarrow \rho(A) \le \min\{m,n\}$.

Def^n: If $\rho(A) = \min\{m,n\}$, we call A a full rank matrix.

Thm:- Suppose A is an $n\times n$ matrix. Then $\rho(A) = n$ iff A is invertible / non-singular, $|A| \neq 0$

i.e., $\exists$ $n\times n$ matrix B s.t. $AB = BA = \underset{\substack{\text{Multiplicative}\\ \text{Identity for matrix multip}}}{I_n}$ $\quad (CI = IC = C)$ for any $n\times n$ matrix C.

Then $\boxed{B = A^{-1}}$ ← Inverse of A.

Remark: If $B = A^{-1}$, then $A = B^{-1}$ and hence the inverse of a matrix if exists is unique.

Remark:- $\rho(A) = n$ iff $\det(A) \neq 0$.

eg:- ① $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

1^{st} column is non-zero so linearly ind.

2^{nd} " " " " "

3^{rd} is not

$\rho(A) = 2$.

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ → non-zero & linea ind → linear ind.

$\mathcal{R}(A) = 2$

eg: ② $A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ $\quad \rho(A) = 3.$

(columns: non-zero ⇒ lin ind; lin ind; dep b/c dep on 2^{nd}; lin ind)

Exc: Find $\rho(A)$ for the following matrices.

(i) $A = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 1 & 10 \\ 3 & 0 & 15 \end{pmatrix}$ (ii) $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 0 & 5 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

* Suppose A is an $m\times n$ matrix, we define:

$\mathcal{N}(A) := \{ \underset{\text{Column vectors}}{\underset{\uparrow}{\underset{\sim}{x}}} \in \mathbb{R}^n : \underset{m\times n}{A}\, \underset{n\times 1}{\underset{\sim}{x}} = \underset{m\times 1}{\underset{\sim}{0}} \} \subseteq \mathbb{R}^n$

Thm: $\mathcal{N}(A)$ is a linear subspace of $\mathbb{R}^n$.

Pf: Take $\underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{N}(A)$, $\alpha, \beta \in \mathbb{R}$.

To show: $\alpha \underset{\sim}{x} + \beta \underset{\sim}{y} \in \mathcal{N}(A)$.

$\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n \Rightarrow \alpha \underset{\sim}{x} + \beta \underset{\sim}{y} \in \mathbb{R}^n$

Also $A\underset{\sim}{x} = A\underset{\sim}{y} = \underset{\sim}{0}$.

$\therefore \underset{m\times n}{A}(\underset{n\times 1}{\underbrace{\alpha \underset{\sim}{x} + \beta \underset{\sim}{y}}}) = \alpha(A\underset{\sim}{x}) + \beta A \underset{\sim}{y}$
$= \alpha(\underset{\sim}{0}) + \beta(\underset{\sim}{0}) = \underset{\sim}{0}$.

$\Rightarrow \alpha \underset{\sim}{x} + \beta \underset{\sim}{y} \in \mathcal{N}(A)$.

Def^n: ① $\mathcal{N}(A)$ is called the <u>null space</u> of A / <u>Kernel</u> of A.

② $\nu(A) := \dim(\mathcal{N}(A))$ is called the nullity of A.

Clearly $\nu(A) \le n$.

Q? Given an $m\times n$ matrix A, how to compute its nullity $\nu(A)$?

eg: ① Suppose $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, then null space $\mathcal{N}(A) = \{ \underset{\sim}{x} \in \mathbb{R}^3 : A\underset{\sim}{x} = \underset{\sim}{0} \}$

$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 + x_2 = 0,\ x_2 + x_3 = 0 \right\}$

It is easy to guess that $\nu(A) = \dim(\mathcal{N}(A)) = 3 - \underset{\text{linear constraints}}{2} = 1$

eg: ② Suppose $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$ then null space

$\underset{\text{null space}}{\mathcal{N}(A)} = \{ \underset{\sim}{x} \in \mathbb{R}^3 : A\underset{\sim}{x} = \underset{\sim}{0} \}$

$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{2\times 1}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \right\}$

$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \begin{matrix} x_1 + x_2 = 0 \\ \underset{\text{redundant}}{2x_1 + 2x_2 = 0} \end{matrix} \right\}$

$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 + x_2 = 0 \right\}$

$\underset{\text{Nullity}}{\nu(A)} = \dim(\mathcal{N}(A)) = 3 - 1 = 2.$

Ans: Ingeneral $\underset{m\times n}{A}\, \underset{n\times 1}{\underset{\sim}{x}} = \underset{m\times 1}{\underset{\sim}{0}} \in \mathbb{R}^m$.

$\underset{m\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{m\times 1}{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}$

$\Rightarrow a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = 0.$
$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = 0$
$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = 0$

Therefore $N(A) = \{ \underset{\sim}{x} \in \mathbb{R}^n : A\underset{\sim}{x} = \underset{\sim}{0} \}$.

$= \{ (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n : \forall\ i = 1, 2, \dots m,\ \sum_{j=1} a_{ij} x_j = 0 \}$

No. of ind line constraints = max no. of line ind rows of A.

$= \rho(A)$

$\Rightarrow \nu(A) = \dim(N(A)) = n - \rho(A)$

Thm: (Rank- Nullity thm)

For any $m \times n$ matrix A, $\rho(A) + \nu(A) = n$, i.e

$\nu(A) = n - \rho(A)$

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Exc:- $u, v, w \in V$, $u + w = v + w$. Then S.T. $u = v$.

Soln:- $w \in V \Rightarrow \exists (-w) \in V$, S.T. $w + (-w) = 0$

$u + w = v + w$.

$\Rightarrow (u+w) + (-w) = v + w + (-w)$

$= u + (w + (-w)) = v + (w + (-w))$

$\Rightarrow u + 0 = v + 0$ (cancellation law)

$\Rightarrow u = v$.

Exc:- Suppose $\alpha, \beta, \gamma \in \mathbb{R}$. $(\alpha + \beta) v = (\alpha + \gamma) v$

$v \in V$, Then S.T $v = 0$ or $\beta = \gamma$.

Soln:- $\alpha v + \beta v = \alpha v + \gamma v$

$\Rightarrow \beta v = \gamma v$.

$\Rightarrow (\beta - \gamma) v = 0$.

But if $\beta \neq \gamma$, $(\beta - \gamma) \neq 0 \in \mathbb{R}$

$\Rightarrow \frac{1}{\beta - \gamma} \in \mathbb{R}$

$\frac{1}{\beta-\gamma} (\beta - \gamma) v = 0$

$\Rightarrow v = 0$

if $\beta = \gamma$, $\Rightarrow \beta - \gamma = 0$

then $\underset{0}{(\beta - \gamma)} v = 0$

Hence we are done.

Exc:- $S = \{ v = (v_1, v_2, v_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} v_i = 0 \}$. Then S is a linear subspace of $\mathbb{R}^3$.

Soln. $v = (v_1, v_2, v_3)$

Let $w = (w_1, w_2, w_3) \in S$.

$\alpha, \beta \in \mathbb{R}$

To show:- $\alpha v + \beta w \in S$.

$\alpha v + \beta w = \alpha (v_1, v_2, v_3) + \beta (w_1, w_2, w_3)$

$= (\alpha v_1 + \beta w_1, \alpha v_2 + \beta w_2, \alpha v_3 + \beta w_3)$

$(\alpha v_1 + \beta w_1) + (\alpha v_2 + \beta w_2) + (\alpha v_3 + \beta w_3)$

$= \alpha (v_1 + v_2 + v_3) + \beta (w_1 + w_2 + w_3) = 0$.

$\Rightarrow \alpha v + \beta w \in S$, is 'S' is a linear subspace of $\mathbb{R}^3$.

Exc: $B \subseteq V$, is a finite subset of V. Then S.T. $\overline{B} \underset{(S.S)}{\subseteq} V$.

Soln: If $B = \{ v_1, v_2, \dots v_n \} \subseteq V$ $c_i, d_i \in \mathbb{R}$

Span B: $\overline{B} = \{ \sum_{i=1}^{n} c_i v_i : c_i \in \mathbb{R} \}$

$w_1 = \sum_{i=1}^{n} c_i v_i$ $w_2 = \sum_{i=1}^{n} d_i v_i$

To show:- $\alpha w_1 + \beta w_2 \in \overline{B}$, $\forall\ \alpha, \beta \in \mathbb{R}$.

$\alpha w_1 + \beta w_2 = \alpha \sum_{i=1}^{n} c_i v_i + \beta \sum_{i=1}^{n} d_i v_i$

$= \sum_{i=1}^{n} \alpha c_i v_i + \sum_{i=1}^{n} \beta d_i v_i$

$= \sum_{i=1}^{n} (\alpha c_i + \beta d_i) v_i \in \overline{B}$.

Exc:- $V = \mathbb{R}^n$, $C = \{ l_1, l_1 + l_2, l_1 + l_2 + l_3, \dots, l_1 + l_2 + \dots + l_n \} \subseteq V$

S.T C is Linearly ind.

Sol^n:- proof :- Let $c_1, c_2, \ldots c_n \in \mathbb{R}$,

S.T $c_1 l_1 + c_2(l_1+l_2) + \cdots c_n(l_1+l_2+\cdots l_n) = 0$.

To show $c_1 = c_2 \cdots c_n = 0$.

$c_1 l_1 + c_2(l_1+l_2) + \cdots c_n(l_1+\cdots l_n) = 0$;

$(c_1+\cdots c_n) l_1 + (c_2+\cdots c_n) l_2 + \cdots (c_{n-1}+c_n) l_{n-1} + c_n l_n = 0$.

let $d_1 = c_1+\cdots c_n = 0$
$d_2 = c_2+\cdots c_n = 0$
$\vdots$
$d_n = c_n = 0$. $\Rightarrow c_1 = c_2 = \cdots c_n = 0$.

$\Rightarrow d_1 l_1 + d_2 l_2 + \cdots d_n l_n = 0$.

$\Rightarrow \{l_1, l_2, \ldots l_n\}$ L.I $\Rightarrow d_1 = d_2 = \cdots d_n = 0$.

Exc:- $S_1 = \{ v = (v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 = v_2 = v_3 \}$

$S_2 = \{ v = (v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 = v_2 \}$

Find a basis & dim of S_1 & S_2.

Proof:- $v \in S_1 \Rightarrow v = (v_1, v_2, v_3)$, $v_1 = v_2 = v_3$

$= (v_1, v_1, v_1)$

$= v_1(1,1,1) \in \langle (1,1,1) \rangle$

$\text{Span}\{(1,1,1)\} = \{ c(1,1,1) : c \in \mathbb{R} \}$

$\Rightarrow S_1 \subseteq \langle (1,1,1) \rangle \subseteq S_1 \Rightarrow S_1 = \langle (1,1,1) \rangle$

$\Rightarrow \{(1,1,1)\}$ is a basis of S_1 & $\dim S_1 = 1$.

$v \in S_2 \Rightarrow v = (v_1, v_2, v_3)$, $v_1 = v_2$.

$= (v_1, v_1, v_3)$

$= (v_1, v_1, 0) + (0, 0, v_3)$

$= v_1(1,1,0) + v_3(0,0,1) \in \text{Span}\{(1,1,0),(0,0,1)\}$

$\Rightarrow S_2 \subseteq \text{span}\{(1,1,0),(0,0,1)\} \subseteq S_2$ (check)

$\Rightarrow S_2 = \text{span}\{(1,1,0),(0,0,1)\}$

$\{(1,1,0),(0,0,1)\}$ LI.

$\Downarrow$

basis.

$\dim S_2 = 2$.

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Lecture-22

Systems of linear equations:-

Suppose we have a system of m linear eq^n's in n variables (i.e $x_1, x_2, \ldots x_n$):

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}b_n &= b_m \end{aligned}\right\} (*)$$

(Here $a_{ij} \in \mathbb{R}$, $b_i \in \mathbb{R}$ $\forall$ $i = 1,2,\ldots m$ and $\forall$ $j = 1,2,\ldots n$)

(*) can be rewritten as $A\underset{\sim}{x} = \underset{\sim}{b}$ ---(*)

$\underset{m\times n}{A} = ((a_{ij}))_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ = coefficient matrix

$\underset{m\times 1}{\underset{\sim}{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\underset{m\times 1}{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

Eg:- $2x_1 + 3x_2 = 7$, $4x_1 + 6x_2 = 15$. has no sol^n. This is an inconsistent system.

Def^n:- We say that the system (*) is consistent if it has a sol^n, i.e, $\exists$ $\underset{\sim}{y} \in \mathbb{R}^n$ (Column Vector) s.t $A\underset{\sim}{y} = \underset{\sim}{b}$.

On the other hand, if no such $\underset{\sim}{y} \in \mathbb{R}^n$ exists, then we say that (*) is an inconsistent system.

Ex:- $2x_1 + 3x_2 = 7$, $4x_1 + 6x_2 = 14$ is consistent because $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a solⁿ.

Remark :- If $\underset{m\times 1}{\underset{\sim}{b}} = \underset{\sim}{0}$, then $(*)$ is consistent bcz $\underset{\sim}{0} \in \mathbb{R}^n$ is a solⁿ.

Note that for each $\underset{\sim}{y} \in \mathbb{R}^n$,

$$\underset{m\times 1}{A \underset{\sim}{y}} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} y_j \\ \sum_{j=1}^{n} a_{2j} y_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} y_j \end{pmatrix} = \sum_{j=1}^{n} y_j \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{pmatrix} = \sum_{j=1}^{n} y_j A_{*j}$$

coefficient Matrix column

A_{*j} = a linear comb of columns of A (i.e $A_{*1}, A_{*2}, \ldots A_{*n}$)

Therefore, $\exists\ \underset{\sim}{y} \in \mathbb{R}^n$ s.t $A\underset{\sim}{y} = \underset{\sim}{b} \Leftrightarrow \underset{\sim}{b}$ is a linear combination of $A_{*1}, A_{*2}, \ldots A_{*n} \Leftrightarrow \underset{\sim}{b} \in Sp\{A_{*1}, \ldots A_{*n}\} = \mathcal{C}(A)$ (column space of A)

Def^n: For the system $(*)$, define the argumented matrix of $(*)$ to be the $m\times(n+1)$ matrix.

$[A \mid \underset{\sim}{b}]$ by adding a column $\underset{\sim}{b}$ to A.

Thm:- The system $(*)$ of linear equation is consistent iff $\rho[A \mid b] = \rho(A)$

Proof:- Only if part

Suppose $(*)$ is consistent. Then $\exists\ \underset{\sim}{y} \in \mathbb{R}^n$ s.T

$A\underset{\sim}{y} = \underset{\sim}{b} \longrightarrow \underset{\sim}{b} \in \mathcal{C}(A)$

$\Rightarrow \rho[A \mid \underset{\sim}{b}] = \rho(A)$

If part

Suppose $\rho[A \mid \underset{\sim}{b}] = \rho(A)$. This implies if $\underset{\sim}{b} \notin \mathcal{C}(A)$, then $\rho[A \mid \underset{\sim}{b}] > \rho(A)$

Hence it follows that $\underset{\sim}{b} \in \mathcal{C}(A) \Rightarrow (*)$ is consistent.

Remark :- If $(*)$ is Inconsistent, then $\rho[A \mid \underset{\sim}{b}] = \rho(A) + 1$

Ex: ① $2x_1 + 3x_2 = 7$, $4x_1 + 6x_2 = 15$. Here $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 7 \\ 15 \end{pmatrix}$

$\rho(A) = 1$, $[A \mid \underset{\sim}{b}] = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 6 & 15 \end{bmatrix} \Rightarrow \rho[A \mid \underset{\sim}{b}] = 2$.

$\Rightarrow$ System is inconsistent.

② $2x_1 + 3x_2 = 7$, $4x_1 + 6x_2 = 14$. Here $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$

$\rho(A) = 1$, $[A \mid \underset{\sim}{b}] = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 6 & 14 \end{bmatrix} \Rightarrow \rho[A \mid \underset{\sim}{b}] = 1 = \rho(A)$

$\Rightarrow$ System is consistent.

Def^n: The set $\{\underset{\sim}{x} \in \mathbb{R}^n : A\underset{\sim}{x} = \underset{\sim}{b}\}$ is called the solution space of / for the system $(*)$.

- When $\underset{\sim}{b} = 0 \in \mathbb{R}^m$, then the solution space of $(*)$ is $\{\underset{\sim}{x} \in \mathbb{R}^n : A\underset{\sim}{x} = \underset{\sim}{0}\} = \mathcal{N}(A)$ = Null space of A, which is a linear subspace of $\mathbb{R}^n$ of dim $n - \rho(A)$.
- In particular, if $\rho(A) = n$ (i.e; if A has full column rank), then $\boxed{A\underset{\sim}{x} = \underset{m\times 1}{\underset{\sim}{0}}}$ has unique solution, namely $\underset{\sim}{x} = \underset{n\times 1}{\underset{\sim}{0}}$. $\rightarrow$ homogenous sysⁿ of eqⁿs.

If $\underset{\sim}{b} \neq \underset{\sim}{0}$, then $(*)$ is called an inhomogenous system.

Thm: Suppose the system $(*)$ is consistent and $\underset{\sim}{y}$ is a solⁿ of $(*)$, i.e, $A\underset{\sim}{y} = \underset{\sim}{b}$. Then the solution space of $(*)$ is equal to $\underset{\sim}{y} + \mathcal{N}(A) = \{\underset{\sim}{y} + \underset{\sim}{v} : \underset{\sim}{v} \in \mathcal{N}(A)\}$

Remark: The solution space is a shift of $\mathcal{N}(A)$, which is a lineal subspace of $\mathbb{R}^n$. The solution space is linear subspace of $\mathbb{R}^n$ iff $\underset{\sim}{b} = \underset{\sim}{0}$.

Eg: $2x_1 + 3x_2 = 7,\ 4x_1 + 6x_2 = 14$

Here $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \Rightarrow \rho(A) = 1 \Rightarrow \mathcal{N}(A)$ has dim 1.

$\mathcal{N}(A)$ is linear subspace of $\mathbb{R}^2$ of dim $2-1=1$.

$\Rightarrow$ The sol^n space is a line (not passing through origin)

$$\mathcal{N}(A) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : 2u_1 + 3u_2 = 0 \right\} = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : u_2 = -\tfrac{2}{3}u_1 \right\}$$

$$= \left\{ \begin{pmatrix} u_1 \\ -\frac{2}{3}u_1 \end{pmatrix} : u_1 \in \mathbb{R} \right\}$$

Note that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a sol^n of $(*)$.

Therefore the solution space $= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \mathcal{N}(A)$

$$= \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} u_1 \\ -\frac{2}{3}u_1 \end{pmatrix} : u_1 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 2+u_1 \\ 1-\frac{2}{3}u_1 \end{pmatrix} : u_1 \in \mathbb{R} \right\}.$$

Exc: Draw graph of above sol^n space.

Exc: S.T the following sys of lin eq^n is consistent;

$x_1 - 2x_2 = 2$, $5x_1 - 10x_2 = 10$.

Write down its sol^n space, explicity and draw its graph.

Proof of Thm: $(A\underset{\sim}{x} = \underset{\sim}{b}) \to (*)$

To show: Solution space for $(*)$ $= \underset{\sim}{y} + \mathcal{N}(A)$

First shall show: $\underset{\sim}{y} + \mathcal{N}(A) \subseteq$ sol^n space.

Take $\underset{\sim}{z} \in \underset{\sim}{y} + \mathcal{N}(A)$

$\Rightarrow \underset{\sim}{z} = \underset{\sim}{y} + \underset{\sim}{v}$ for some $\underset{\sim}{v} \in \mathbb{R}^n$ S.T $A\underset{\sim}{v} = \underset{\sim}{0}$.

$\Rightarrow A\underset{\sim}{z} = A(\underset{\sim}{y} + \underset{\sim}{v}) = A\underset{\sim}{y} + A\underset{\sim}{v} = \underset{\sim}{b} + \underset{\sim}{0} = \underset{\sim}{b}$

$\Rightarrow \underset{\sim}{z} \in$ sol^n space of $(*)$

Shall show: Sol^n space $\subseteq \underset{\sim}{y} + \mathcal{N}(A)$.

Take $\underset{\sim}{u} \in$ Sol^n space, $\Rightarrow A\underset{\sim}{u} = \underset{\sim}{b}$

We also know that : $A\underset{\sim}{y} = \underset{\sim}{b}$.

Therefore $A[\underset{\sim}{u} - \underset{\sim}{y}] = A\underset{\sim}{u} - A\underset{\sim}{y} = \underset{\sim}{b} - \underset{\sim}{b} = 0$

$\Rightarrow \underset{\sim}{u} - \underset{\sim}{y} \in \mathcal{N}(A)$

$\Rightarrow \underset{\sim}{u} \in \underset{\sim}{y} + \mathcal{N}(A)$.

Cor: If $\rho[A|\underset{\sim}{b}] = \rho(A) = n$, then $(*)$ has unique Sol^n.

Proof: $\rho[A|\underset{\sim}{b}] = \rho(A) \Rightarrow (*)$ is consistent. Let $\underset{\sim}{y}$ be a sol^n of $(*)$.

Also $\rho(A) = n \Rightarrow \dim(\mathcal{N}(A)) = n - n = 0$.

$\Rightarrow$ Sol^n space of $(*) = \underset{\sim}{y} + \{\underset{\sim}{0}\} = \{\underset{\sim}{y}\}$

Cor: Suppose $(*)$ is consistent. Then $(*)$ has unique Sol^n iff $\rho(A) = n$.

Remark: If $\underset{\text{No. of eq}^n\text{s}}{m} < \underset{\text{No. of Variables}}{n}$, then $\rho(A) \le m < n$ and hence either $(*)$ is inconsistent or $(*)$ has non-unique Sol^{n}s.

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Exc: $V_1 := \{(\omega_1,\omega_2,\omega_3) \in \mathbb{R}^3 : \omega_1+\omega_2+\omega_3 = 0\} \subseteq \mathbb{R}^3$

$V_2 := \{(u_1,u_2) \in \mathbb{R}^2 : u_1+u_2 = 0\} \subset \mathbb{R}^2$

$V_3 := \mathbb{R}^2$. Show

a) $T_1 : V_1 \to V_2$. $T(\omega_1,\omega_2,\omega_3) = (\omega_1+\omega_3, \omega_3), \forall (\omega_1,\omega_2,\omega_3)$ is a linear map from V_1 to V_2.

b) $T_2 : V_1 \to V_3$. $T_2(\omega_1,\omega_2,\omega_3) = (\omega_1,\omega_2), \forall (\omega_1,\omega_2,\omega_3) \in V_1$ is a V-space isomorphism from V_1 onto V_3.

Proof:- a) $\omega = (\omega_1,\omega_2,\omega_3)$. $\alpha,\beta \in \mathbb{R}$

$u = (u_1,u_2,u_3) \in V_1$

T.S: $T(\alpha\omega+\beta u) = \alpha T(\omega) + \beta T(u)$

$T(\alpha\omega+\beta u) = T((\alpha\omega_1,\alpha\omega_2,\alpha\omega_3) + (\beta u_1,\beta u_2,\beta u_3))$

$= T(\alpha\omega_1+\beta u_1, \alpha\omega_2+\beta u_2, \alpha\omega_3+\beta u_3)$

$= (\alpha\omega_1+\alpha\omega_2, \alpha\omega_3) + (\beta u_1+\beta u_2, \beta u_3)$

$= \alpha(\omega_1+\omega_2, \omega_3) + \beta(u_1+u_2, u_3)$

$= \alpha T(\omega) + \beta T(u)$

b) $T_2(\alpha\omega+\beta u) = T_2(\alpha(\omega_1,\omega_2,\omega_3) + \beta(u_1,u_2,u_3))$

$= T_2(\alpha\omega_1+\beta u_1, \alpha\omega_2+\beta u_2, \alpha\omega_3+\beta u_3)$

$= (\alpha\omega_1+\beta u_1, \alpha\omega_2+\beta u_2) = \alpha T_2(\omega) + \beta T_2(u)$

$\Rightarrow T_2 : V_1 \to V_3$ is linear.

T.S:- T_2 is a vector space isomorphism.

T.S:- T_2 is injective. S.T $T_2(\omega) = T_2(u)$

for $\omega = (\omega_1,\omega_2,\omega_3)$ $\Rightarrow (\omega_1,\omega_2) = (u_1,u_2)$

$u = (u_1,u_2,u_3) \in V_1$ $\Rightarrow \omega_1 = u_1, \omega_2 = u_2$.

$\therefore u = \omega$

$\omega \in V_1 \Rightarrow \omega_1+\omega_2+\omega_3 = 0 \to (*)$

$u \in V_1 \Rightarrow u_1+u_2+u_3 = 0$

$\Rightarrow u_3 = -u_1-u_2$

$= -\omega_1-\omega_2 = \omega_3$ (from $*$) $\Rightarrow u_3 = \omega_3$.

Hence T_2 is injective.

T.S:- T_2 is onto.

Let $\omega = (\omega_1,\omega_2) \in V_3 (= \mathbb{R}^2)$

Let $\vartheta = (\omega_1, \omega_2, -(\omega_1+\omega_2))$, Then $\vartheta \in V_1$.

Now, $T_2\vartheta = T_2(\omega_1,\omega_2,-(\omega_1+\omega_2)) = (\omega_1,\omega_2) = \omega$

So, for any $\omega \in V_3$, $\exists\, \vartheta \in V_1$ s.t. $T_2\vartheta = \omega$

$\Rightarrow T_2$ is onto.

Hence, T_2 is a bijection.

Thus, T_2 is a V-space isomorphism.

Exc:- $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1+2x_2\\2x_1-x_2\\x_2\end{pmatrix}, \forall \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2$

S.T T is linear. Find the matrix for T.

T.S:- T is linear,

Let $\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}y_1\\y_2\end{pmatrix} \in \mathbb{R}^2$, $\alpha,\beta \in \mathbb{R}$

$T\left(\alpha\begin{pmatrix}x_1\\x_2\end{pmatrix} + \beta\begin{pmatrix}y_1\\y_2\end{pmatrix}\right) = T\begin{pmatrix}\alpha x_1+\beta y_1\\\alpha x_2+\beta y_2\end{pmatrix} = \begin{pmatrix}\alpha x_1+\beta y_1+2\alpha x_2+2\beta y_2\\2\alpha x_1+2\beta y_1-\alpha x_2-\beta y_2\\\alpha x_2+\beta y_2\end{pmatrix}$

$= \alpha\begin{pmatrix}x_1+2x_2\\2x_1-x_2\\x_2\end{pmatrix} + \beta\begin{pmatrix}y_1+2y_2\\2y_1-y_2\\y_2\end{pmatrix} = \alpha T\begin{pmatrix}x_1\\x_2\end{pmatrix} + \beta T\begin{pmatrix}y_1\\y_2\end{pmatrix}$

Matrix for T:-

$\{e_1^{(2)}, e_2^{(2)}\} \to$ Standard basis for $\mathbb{R}^2$

$\{e_1^{(3)}, e_2^{(3)}, e_3^{(3)}\} \to$ standard basis for $\mathbb{R}^3$

$e_1^{(2)} = \begin{pmatrix}1\\0\end{pmatrix}$ $e_2^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$

$e_1^{(3)} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_2^{(3)} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $e_3^{(3)} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$T e_1^{(2)} = T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix} = 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\0\end{pmatrix} + 1\begin{pmatrix}1\\0\\0\end{pmatrix}$

$= 2e_2^{(3)} + 0e_3^{(3)} + e_1^{(3)}$

$T e_2^{(2)} = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + (-1)\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}$

$= 2e_1^{(3)} - e_2^{(3)} + e_3^{(3)}$

$A \to$ Matrix for T. (A is a 3×2 matrix)

$$A = \begin{pmatrix}1 & 2\\ 2 & -1\\ 0 & 1\end{pmatrix}$$

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Lecture-23

$(*)\ldots \quad \underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}$

Thm: If $m=n$ and A is non-singular, then

$$\underset{n\times 1}{y} = \underset{n\times n}{A^{-1}}\,\underset{n\times 1}{b}$$

is the unique sol^n of the system $(*)$.

Exc: (proof)

How to compute A^{-1}?

Thm: If $\underset{n\times n}{A}$ is non-singular, then

$$A^{-1} = \frac{1}{|A|}\Big(\underbrace{(-1)^{i+j} M_{ji}}_{(i,j)^{th}\text{ element of } A^{-1}}\Big)_{\substack{1\le i\le n\\ 1\le j\le n}}$$

where $M_{ji} = (j,i)^{th}$ minor of A.

M_{ji} := Determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the j^{th} row and i^{th} column of A.

$$A = \begin{array}{c|c|c} 1x & x & x^2 \\ \hline x & x & x \\ 3^x & 4 & x\,6 \end{array}$$

$$M_{22} = \begin{matrix} 1x & x^2 \\ 3x & x4 \end{matrix}$$

$|A| = a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{n+1}a_{1n}M_{1n}$

Determinants

Elementary Row operations (are used to compute $|A|$)

- Exchange two rows $\to$ det changes the sign
- Multiply any row by a const $c\in\mathbb{R}$ $\to$ det get multiplied by c.
- Add a scalar multiple of one row to another row $\to$ det remains unchanged.

Similarly Elementary column operations can also be used to compute $|A| = \det(A)$

Thm: For two $n\times n$ matrices, A,B we have

$$|AB| = |A|\,|B|$$

Thm: For any $n\times n$ matrix A, TFAE :-

(i) $\rho(A) = n$

(ii) A is non-singular (Inverse exists)

(iii) $|A| \neq 0$

Trace

Def^n: For an $n\times n$ matrix $\underset{[(a_{ij})]}{A}$, $tr(A) = trace(A)$

$$tr(A) := \sum_{i=1}^{n} a_{ii}$$

Thm: For two $n\times n$ matrices A,B,

$$tr(AB) = tr(BA)$$

Cor: For three $n\times n$ matrice A,B,C,

$$tr(ABC) = tr(CAB) = tr(BCA)$$

Def^n: An $n\times n$ matrix A is called an idempotent matrix if $A^2 = A$.

eg:- $A = I_n$. $= \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$A^2 = I_n = A \rightarrow$ Idempotent.

<u>Thm</u>:- For an idempotent matrix A, $\rho(A) = tr(A)$.

<u>Exc</u>*:- Suppose X is an $n \times p$ matrix ST $\underbrace{X^T X}_{p \times p}$ is non-singular

(i) S.T $\underset{n\times n}{I_n} - \underset{n\times p}{X} \underset{p\times p}{(X^T X)^{-1}} \underset{p\times n}{X^T}$ is an idempotent matrix.

(this matrix plays imp role in stat/DS)

(ii) Suppose J_n is the $n \times n$ matrix with all entries $= 1$.

Using (i), show that $I_n - \frac{1}{n} J_n$ is an idempotent matrix.

[Hint: use $p = 1$, $\underset{n\times 1}{X} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$]

(iii) Using (ii), sht the $\rho(I_n - \frac{1}{n} J_n) = n-1$

<u>PARTITION MATRICES</u>:-

Suppose $\underset{4\times 4}{A} = \left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array} \right)$

$A_{11}(2\times 2)$, $A_{12}(2\times 2)$, $A_{21}(2\times 2)$, $A_{22}(2\times 2)$

$\rightarrow A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

A partitioned matrix with 2x2 blocks.

In general, let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Take integers $1 \le k \le m-1$, $1 \le l \le n-1$, $1 \le r \le p-1$

We partition A as

$$\underset{m\times n}{A} = \begin{pmatrix} \underset{k\times l}{A_{11}} & \underset{k\times(n-l)}{A_{12}} \\ \underset{(m-k)\times l}{A_{21}} & \underset{(m-k)\times(n-l)}{A_{22}} \end{pmatrix}$$

$$B_{n\times p} = \begin{pmatrix} \underset{l\times r}{B_{11}} & \underset{l\times(p-r)}{B_{12}} \\ \underset{(n-l)\times r}{B_{21}} & \underset{(n-l)\times(p-r)}{B_{22}} \end{pmatrix}$$

<u>Thm</u>:- AB (an $m\times p$ matrix) $= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$

<u>Remark</u>:- One can state this thm for any partitioned matrix, not just with 2x2 blocks.

* $\underset{4\times 4}{P} = \begin{pmatrix} \underset{2\times 2}{P_{11}} & \underset{2\times 2}{P_{12}} \\ \underset{2\times 2}{P_{21}} & \underset{2\times 2}{P_{22}} \end{pmatrix}$ $\quad \underset{4\times 4}{Q} = \begin{pmatrix} \underset{2\times 2}{Q_{11}} & \underset{2\times 2}{Q_{12}} \\ \underset{2\times 2}{Q_{21}} & \underset{2\times 2}{Q_{22}} \end{pmatrix}$

<u>Thm</u>:- $P + Q = \begin{pmatrix} P_{11}+Q_{11} & P_{12}+Q_{12} \\ P_{21}+Q_{21} & P_{22}+Q_{22} \end{pmatrix}$

eg:- $A = I_n$.
$A^2 = I_n = A$ → Idempotent. $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

Thm:- For an idempotent matrix A, $\rho(A) = tr(A)$.

Exc*:- Suppose X is an $n \times p$ matrix ST $\underbrace{X^T X}_{p \times p}$ is non-singular

(i) S.T $\underset{n\times n}{I_n} - \underset{n\times p}{X} \underset{p\times p}{(X^T X)^{-1}} \underset{p\times n}{X^T}$ is an idempotent matrix.

(this matrix plays imp role in stat/DS)

(ii) Suppose J_n is the $n \times n$ matrix with all entries $= 1$. Using (i), show that $I_n - \frac{1}{n} J_n$ is an idempotent matrix.

[Hint: use $p = 1$, $\underset{n\times 1}{X} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$]

(iii) Using (ii), S.T the $\rho(I_n - \frac{1}{n} J_n) = n-1$

PARTITION MATRICES:-

Suppose $\underset{4\times 4}{A} = \left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}\right)$ with blocks $A_{11}(2\times 2)$, $A_{12}(2\times 2)$, $A_{21}(2\times 2)$, $A_{22}(2\times 2)$

$\rightarrow A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

A partitioned matrix with 2x2 blocks.

In general, let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Take integers $1 \le k \le m-1$, $1 \le l \le n-1$, $1 \le r \le p-1$

We partition A as

$$\underset{m\times n}{A} = \begin{pmatrix} \underset{k\times l}{A_{11}} & \underset{k\times (n-l)}{A_{12}} \\ \underset{(m-k)\times l}{A_{21}} & \underset{(m-k)\times(n-l)}{A_{22}} \end{pmatrix}$$

$$B_{n\times p} = \begin{pmatrix} \underset{l\times r}{B_{11}} & \underset{l\times(p-r)}{B_{12}} \\ \underset{(n-l)\times r}{B_{21}} & \underset{(n-l)\times(p-r)}{B_{22}} \end{pmatrix}$$

Thm:- AB (an $m \times p$ matrix) $= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$

Remark: One can state this thm for any partitioned matrix, not just with 2x2 blocks.

* $\underset{4\times 4}{P} = \begin{pmatrix} \underset{2\times 2}{P_{11}} & \underset{2\times 2}{P_{12}} \\ \underset{2\times 2}{P_{21}} & \underset{2\times 2}{P_{22}} \end{pmatrix}$ $\quad \underset{4\times 4}{Q} = \begin{pmatrix} \underset{2\times 2}{Q_{11}} & \underset{2\times 2}{Q_{12}} \\ \underset{2\times 2}{Q_{21}} & \underset{2\times 2}{Q_{22}} \end{pmatrix}$

Thm:- $P + Q = \begin{pmatrix} P_{11}+Q_{11} & P_{12}+Q_{12} \\ P_{21}+Q_{21} & P_{22}+Q_{22} \end{pmatrix}$

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1. $V = \mathbb{R}^3$, $S_1 = \{ \underset{\sim}{v} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0 \}$. $S_2 = \{ \underset{\sim}{v} \in \mathbb{R}^3 : v_1 = v_2 = v_3 \}$

Prove $S_1 + S_2 = V$.

A. We already know S_1, S_2 are subspaces of V. So prove that dim of $(S_1 + S_2) = 3$.

* Thm: If W is a subspace of V with $\dim W = \dim V < \infty$, then $W = V$.

Step 1: $S_1 \cap S_2 = \{ \underset{\sim}{0} \}$.

Proof: Let $\underset{\sim}{v} \in S_1 \cap S_2$ then $\underset{\sim}{v} \in S_2 \Rightarrow v_1 = v_2 = v_3 = \alpha$

Also $\underset{\sim}{v} \in S_1$ Hence $v_1 + v_2 + v_3 = 0$. $\Rightarrow 3\alpha = 0$ $\Rightarrow \alpha = 0$.

$\underset{\sim}{v} = \underset{\sim}{0} = (0,0,0)$.

Hence $S_1 \cap S_2 = \{0\}$ and $\dim(S_1 \cap S_2) = 0$.

Step-2: To compute $\dim(S_1)$

* Rank-Nullity thm: If A is $m\times n$ matrix, then
$\rho(A) + \nu(A) = n$.

Define $A = [1 \ 1 \ 1]_{1\times 3}$

If $A\,\underset{\sim}{v} = \underset{\sim}{0}$

$$[1\ 1\ 1]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$\Rightarrow v_1+v_2+v_3 = 0$

Thus, $S_1 = N(A)$ - Null Space of A.

$\rho(A) = 1$ clearly, because A has exactly one non-zero row,

Hence from Rank N. T. $\nu(A) = 3 - \rho(A) = 3-1 = 2$.

Hence $\dim S_1 = \dim N(A) = \nu(A) = 2$.

Step-3: To compute $\dim(S_2)$

$S_2 = \{ \underset{\sim}{v} \in \mathbb{R}^3 : v_1 = v_2 = v_3 \}$

$\Rightarrow v_1 - v_2 = 0 \qquad v_2 - v_3 = 0$;

$1\cdot v_1 + (-1)v_2 + 0\cdot v_3 = 0 \qquad 0\cdot v_1 + 1\, v_2 + (-1)\, v_3 = 0$.

Define $B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

$$B\underset{\sim}{v} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 - v_2 \\ v_2 - v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Claim: $N(B) = S_2$

Let $\underset{\sim}{\omega} = N(B)$. Them $B\cdot\underset{\sim}{\omega} = 0$

$$B\underset{\sim}{\omega} = \begin{bmatrix} \omega_1 - \omega_2 \\ \omega_2 - \omega_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \omega_1 = \omega_2 = \omega_3$, thus $\underset{\sim}{\omega} \in S_2$.

$\Rightarrow N(B) \subseteq S_2$

Let $\underset{\sim}{v} \in S_2$. Then $v_1 = v_2 = v_3$

Now, $B\underset{\sim}{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \underset{\sim}{v} \in N(B)$

$\Rightarrow S_2 \subseteq N(B)$. Hence $S_2 = N(B)$.

$\dim S_2 = \dim N(B) = \nu(B) = 3 - \rho(B)$.

To show $\rho(B) = 2$:

$B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

clearly B has 2 rows, so we will prove the rows of B are linearly indep.

Let $\alpha(1,-1,0) + \beta(0,1,-1) = (0,0,0)$.

$\Rightarrow (\alpha, -\alpha, 0) + (0, \beta, -\beta) = (0,0,0)$

$\Rightarrow (\alpha, \beta-\alpha, -\beta) = (0,0,0)$.

$\Rightarrow \alpha = \beta = 0 \qquad (\alpha,\beta \neq 0$ for dep$)$

Hence they are lin ind as $\alpha = \beta = 0$.

$\Rightarrow \rho(B) = 2$.

$\dim(S_2) = 3 - 2 = 1$.

Step-4: $\dim(S_1+S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$

$= 2 + 1 - 0 = 3 = \dim V$.

Hence $S_1 + S_2 = V$.

2. Find $\rho(A)$ where $A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

A: 4^{th} c is twice 3^{rd} column.

Then $\rho(A) < 4$.

Now we will prove C_1, C_2, C_3 are lin ind. and so that Rank would be atleast 3.

$\rho(A) < 4$

$\Rightarrow$ Rank $= 3$.

Let $\alpha(1,0,0,0) + \beta(1,1,0,0) + \delta(1,1,1,0) = (0,0,0,0)$.

$(\alpha+\beta+\delta), (\beta+\delta), (\delta), (0) = (0,0,0,0)$.

$\Rightarrow \delta = 0, \ \beta = 0, \ \alpha = 0$.

Hence c_1, c_2, c_3 are lin ind.

$\Rightarrow \rho(A) \geq 3$. and $\rho(A) < 4$.

$\Rightarrow \rho(A) = 3$.

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PRACTICE FINAL EXAM

1. (a) Using Taylor's Thm $(k=1)$, we get $\forall\, a, x \in \mathbb{R}$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(\eta)}{2!}(x-a)^2 \text{ for some } \eta \in (a, x)$$

For $a = 0$, we have $\forall\, x \in \mathbb{R}$,

$$f(x) = f(0) + f'(0)x + \frac{f''(\eta)}{2}x^2$$

$$f(x) = \frac{f''(\eta)}{2}x^2 \text{ for some } \eta \text{ btw } 0, x.$$

$\therefore \forall\, x \in [-1,1]$, we have

$$|f(x)| = \frac{|f''(\eta)|}{2}x^2 \leq \frac{1}{2} \quad [\because f''(\eta) \leq 1].$$

(b) f is cont on $[-1,1] \Rightarrow f \in \mathcal{R}[-1,1]$.

$$\left|\int_{-1}^{1} f(x)dx\right| \leq \int_{-1}^{1} |f(x)|dx \leq \int_{-1}^{1} \frac{1}{2}\cdot dx = 1 \quad [\text{By } 1(a)]$$

$$\Rightarrow \left|\int_{-1}^{1} f(x)dx\right| \leq 1.$$

2. (a)(i) Suppose $0 < s < 1$, clearly $\phi_s \in \mathcal{R}[\varepsilon, 1]$ (cont) $\forall\, \varepsilon \in (0,1]$

Also $\displaystyle\int_{\varepsilon}^{1} \phi_s(x)\,dx = \int_{\varepsilon}^{1} x^{-s}dx = \left[\frac{x^{1-s}}{1-s}\right]_{\varepsilon}^{1}$

$$= \frac{1 - \varepsilon^{1-s}}{1-s}$$

$s < 1 \Rightarrow 1-s > 0 \Rightarrow \varepsilon^{1-s} \to 0$ as $\varepsilon \to 0^+$

$$\Rightarrow \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \phi_s(x)\,dx = \frac{1}{1-s}$$

$\Rightarrow \phi_s \in \mathcal{R}(0,1]$

$\left(\text{In fact, } \displaystyle\int \phi_s(x)\,dx = \int_0^1 \phi_s(x)\,dx = \frac{1}{1-s}\right)$

(ii) Suppose $s > 1 \Rightarrow$ clearly $\phi_s \in \mathcal{R}[1, A]$ (cont) $\forall\, A > 1$.

Also $\displaystyle\int_1^A \phi_s(x)\,dx = \int_1^A x^{-s}dx = \left[\frac{x^{1-s}}{1-s}\right]_1^A = \frac{A^{1-s}-1}{1-s} = \frac{1-A^{1-s}}{s-1}$

$s > 1 \Rightarrow s-1 > 0,\ 1-s < 0$

$$A^{1-s} = \frac{1}{A^{s-1}} \to 0 \text{ as } A \to \infty$$

$$\Rightarrow \lim_{A\to\infty} \int_1^A \phi_s(x)\,dx = \frac{1}{s-1} \Rightarrow \phi_s \in \mathcal{R}[1,\infty)$$

$\left(\text{In fact, } \displaystyle\int_1^\infty \phi_s(x)\,dx = \frac{1}{s-1} = \int_{[1,\infty)} \phi_s(x)\,dx\right)$

(b) $$g(x) = \frac{1}{x^3 + \sqrt[3]{x}},\ x \in (0,\infty)$$

$\forall\, x \in (0,1]$, $0 \leq g(x) = \dfrac{1}{x^3 + x^{1/3}} \leq \dfrac{1}{\sqrt[3]{x}}$ $[\because x^3 + \sqrt[3]{x} \geq \sqrt[3]{x}]$

$$= x^{-1/3} = \phi_{\frac{1}{3}}(x)$$

Since $0 < \frac{1}{3} < 1$, using (i) we get $\phi_{\frac{1}{3}}(x) \in \mathcal{R}(0,1]$.

and $0 < g(x) \leq \phi_{\frac{1}{3}}(x)$ $\forall\, x \in (0,1]$.

Therefore, by comparision test, $g \in \mathcal{R}(0,1]$

$\forall\, x \in [1,\infty)$, $0 \le g(x) \le \frac{1}{x^3+3\sqrt{x}} \le \frac{1}{x^3}$ $[\because x^3+3\sqrt{x} \ge x^3]$

$= \phi_3(x)$

Since $3>1$, $\overset{\text{using}}{(ii)} \Rightarrow \phi_3 \in \mathcal{R}[1,\infty)$

Therefore by comparision test, $g \in \mathcal{R}[1,\infty)$

. We have show $g \in \mathcal{R}(0,1]$ and $g \in \mathcal{R}[1,\infty)$.

$\Rightarrow g \in \mathcal{R}(0,1] \cap \mathcal{R}[1,\infty) \Rightarrow g \in \mathcal{R}(0,\infty)$.

(Proved)

<u>8. Exc</u> : Solve prob 2(b) using ratio test.

3. $T((x_1,x_2,x_3,x_4)^T) = (x_1-2x_2,\ x_2+x_4,\ x_1+2x_4)^T$

$\forall\, (x_1,x_2,x_3,x_4)^T \in \mathbb{R}^4$.

Take $\underset{\sim}{u} = (u_1,u_2,u_3,u_4)^T \in \mathbb{R}^4$, $\underset{\sim}{v} = (v_1,v_2,v_3,v_4)^T \in \mathbb{R}^4$

$\alpha, \beta \in \mathbb{R}$.

(a) <u>To show</u> :- $T(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) = \alpha T(\underset{\sim}{u}) + \beta T(\underset{\sim}{v})$

$T(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) = T(\alpha(u_1,u_2,u_3,u_4)^T + \beta(v_1,v_2,v_3,v_4)^T)$

$= T((\alpha u_1+\beta v_1,\ \alpha u_2+\beta v_2,\ \alpha u_3+\beta v_3\ \ \alpha u_4+\beta v_4)^T)$

$$= \begin{pmatrix} (\alpha u_1+\beta v_1)-2(\alpha u_2+\beta v_2) \\ (\alpha u_2+\beta v_2)+(\alpha u_4+\beta u_4) \\ (\alpha u_1+\beta v_1)+2(\alpha u_4+\beta v_4) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(u_1-2u_2)+\beta(v_1-2v_2) \\ \alpha(u_2+u_4)+\beta(v_2+v_4) \\ \alpha(u_1+2u_4)+\beta(v_1+2v_4) \end{pmatrix} = \alpha \begin{pmatrix} u_1-2u_2 \\ u_2+u_4 \\ u_1+2u_4 \end{pmatrix} + \beta \begin{pmatrix} v_1-2v_2 \\ v_2+v_4 \\ v_1+2v_4 \end{pmatrix}$$

$$= \alpha\, T\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} + \beta\, T\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$= \alpha\, T(\underset{\sim}{u}) + \beta\, T(\underset{\sim}{v})$

This shows T is a linear map from $\mathbb{R}^4$ to $\mathbb{R}^3$.

(b) $T: \mathbb{R}^4 \to \mathbb{R}^3$ is a linear map. Here $n=4$, $m=3$ and hence A is a 3×4 matrix.

$A_{*1} = 1^{st}$ column of A $= T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$

$A_{*2} = T\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\1\\0\end{pmatrix}$

$A_{*3} = T\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$A_{*4} = T\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}$

$$A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

(c) $\rho(A) = ?$

<u>claim</u>: A_{*1} and A_{*2} are lin ind.

<u>Proof</u>:- Take $\alpha, \beta \in \mathbb{R}$. S.T $\alpha A_{*1} + \beta A_{*2} = \underset{\sim}{0}$.

i.e $\alpha\begin{pmatrix}1\\0\\1\end{pmatrix} + \beta\begin{pmatrix}-2\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\Rightarrow \alpha - 2\beta = 0$, $\beta = 0$, $\alpha = 0$.

$\Rightarrow \alpha = 0, \beta = 0$ -

This shows A_{*1}, A_{*2} are lin ind.

Also $A_{*3} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ is not lin ind.

$4 \ge \rho(A) \ge 2$

Also $\begin{pmatrix}0\\1\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}-2\\1\\0\end{pmatrix}$

Hence A_{*3} and A_{*4} are lin comb of A_{*1}, A_{*2}.

$\therefore \rho(A) = 2$.

4. $V := \mathbb{R}^4$ and

$S_1 := \{(v_1, v_2, v_3, v_4)^T \in \mathbb{R}^4 : \overset{\Rightarrow v_1 = v_2,\ v_3 = v_4}{v_1 - v_2 = v_3 - v_4 = 0}\} \subset \mathbb{R}^4$

$S_2 := \{(u_1, u_2, u_3, u_4)^T \in \mathbb{R}^4 : u_1 + u_2 = u_3 + u_4 = 0\}$.

(a) S.T. S_1 is subspace of V.

$S_1 \underset{(S.S)}{\subseteq} V = \mathbb{R}^4$. clearly $S_1 \neq \emptyset$. $(\underset{\sim}{0} \in S_1)$

Take $\underset{\sim}{x} = (x_1, x_2, x_3, x_4)^T$, $\underset{\sim}{y} = (y_1, y_2, y_3, y_4)^T \in S_1$.

Hence $\underset{\sim}{x} = (x_1, x_1, x_3, x_3)^T$, $\underset{\sim}{y} = (y_1, y_1, y_3, y_3)^T$.

Take $\alpha, \beta \in \mathbb{R}$. $\begin{pmatrix} x_1 = x_2, & x_3 = x_4 \\ y_1 = y_2, & y_3 = y_4 \end{pmatrix} \rightarrow (*)$

To show $\alpha \underset{\sim}{x} + \beta \underset{\sim}{y} \in S_1$

$$\alpha \underset{\sim}{x} + \beta \underset{\sim}{y} = \alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \\ \alpha x_4 + \beta y_4 \end{pmatrix} \in \mathbb{R}^4.$$

$$(*) \Rightarrow \left.\begin{array}{l} \alpha x_1 + \beta y_1 = \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 = \alpha x_4 + \beta y_4 \end{array}\right\} \Rightarrow \alpha \underset{\sim}{x} + \beta \underset{\sim}{y} \in S_1.$$

Exc: S.T $S_2 \underset{(S.S)}{\subseteq} V = \mathbb{R}^4$

Remark: Suppose V is a $\overset{\text{Vector}}{\text{~~Sub~~space}}$ and $\emptyset \neq S \subseteq V$.

Then S is a linear subspace of V iff .

(i) $\underset{\sim}{x}, \underset{\sim}{y} \in S \Rightarrow \underset{\sim}{x} + \underset{\sim}{y} \in S$.

(ii) $\alpha \in \mathbb{R}$, $\underset{\sim}{x} \in S \Rightarrow \alpha \underset{\sim}{x} \in S$.

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4.(b)

$S_1 := \{(v_1, v_2, v_3, v_4)^T \in \mathbb{R}^4 : v_1 - v_2 = v_3 - v_4 = 0\}$.

Let $\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in S_1$ Then, $v_1 - v_2 = v_3 - v_4 = 0$

$\Rightarrow v_1 = v_2$, $v_3 = v_4$ then $\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_1 \\ v_3 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

i.e, any element in S_1 is a linear comb of $\downarrow$ or

$\underset{\sim}{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\underset{\sim}{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ $\qquad -v_1 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow S_1 \subseteq \text{Span}\{\underset{\sim}{u}_1, \underset{\sim}{u}_2\}$

Let $\underset{\sim}{\omega} \in \text{Span}\{\underset{\sim}{u}, \underset{\sim}{v}\}$. Then, $\underset{\sim}{\omega} = \alpha \underset{\sim}{u} + \beta \underset{\sim}{v}$ for some $\alpha, \beta \in \mathbb{R}$.

$\Rightarrow \underset{\sim}{\omega} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \beta \\ \beta \end{bmatrix}$. clearly $\omega_1 = \omega_2$,

$\omega_3 = \omega_4$. Hence $\underset{\sim}{\omega} \in S_1 \Rightarrow \text{Span}\{\underset{\sim}{u}_1, \underset{\sim}{u}_2\} \subseteq S_1$

$\Rightarrow S_1 = \text{Span}\{\underset{\sim}{u}_1, \underset{\sim}{u}_2\}$

Claim: $\underset{\sim}{u}_1, \underset{\sim}{u}_2$ are lin. ind

Pf: Let $\gamma, \delta \in \mathbb{R}$ be S.T $\gamma \underset{\sim}{u}_1 + \delta \underset{\sim}{u}_2 = \underset{\sim}{0}$

$\Rightarrow \gamma \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \delta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma \\ \gamma \\ \delta \\ \delta \end{bmatrix}$

$\Rightarrow \gamma = 0$, $\delta = 0$. Hence the claim

Thus $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2\}$ forms a basis for S_1 .

(c) Dim of $S_1 = ?$

From (b) $\dim S_1 = |\{\underset{\sim}{u}_1, \underset{\sim}{u}_2\}| = 2$.

(d) $\psi((v_1,v_2,v_3,v_4)^T) = (v_1,-v_2,v_3,-v_4)^T$

$\forall (v_1,v_2,v_3,v_4)^T \in S_1$. Prove ψ is a linear or vector space isomorphism from S_1 onto S_2.

Pf: ① ψ is from S_1 to S_2.

Let $\underset{\sim}{v} \in S_1$. Then, $(v_1,v_2,v_3,v_4)^T$.

$v_1 = v_2,\ v_3 = v_4$

$\psi(\underset{\sim}{v}) = (v_1-v_2, v_3-v_4)^T = (v_2,-v_2,v_4,-v_4)^T$

and $v_2+(-v_2)=0 \quad = v_4+(-v_2)$

Hence $\psi(\underset{\sim}{v}) \in S_2$

② ψ is linear (H.W)

③ ψ is one-one.

let $\underset{\sim}{u} = (u_1,u_2,u_3,u_4)^T$, $\underset{\sim}{v} = (v_1,v_2,v_3,v_4)^T \in S_1$ be

S.T $\psi(\underset{\sim}{u}) = \psi(\underset{\sim}{v})$

$\Rightarrow (u_1,-u_2,u_3,-u_4)^T = (v_1,-v_2,v_3,-v_4)^T$

$\Rightarrow u_1 = v_1,\ u_2 = v_2,\ u_3 = v_3,\ u_4 = v_4$

$\Rightarrow \underset{\sim}{u} = \underset{\sim}{v}$

Hence ψ is 1-1.

④ ψ is onto.

let $\underset{\sim}{u} = (u_1,u_2,u_3,u_4)^T \in S_2 \Rightarrow \begin{matrix} u_1+u_2=0 \\ u_3+u_4=0 \end{matrix}$

$\Rightarrow u_1-(-u_2)=0,\ u_3-(-u_4)=0$.

$\Rightarrow \underset{\sim}{x} = (u_1,-u_2,u_3,-u_4)^T \in S_1$ by def^n.

$\psi(\underset{\sim}{x}) = \psi(u_1,-u_2,u_3,-u_4)^T$

$= (u_1,u_2,u_3,u_4)^T = \underset{\sim}{u}$.

i.e for every $\underset{\sim}{u} \in S_2\ \exists\ \underset{\sim}{x} \in S_1$ S.T $\psi(\underset{\sim}{x}) = \underset{\sim}{u}$.

Hence ψ is onto.

Hence ψ is isomorphism. (1-1, onto, linear).

(e) S.T $S_1+S_2 = V = \mathbb{R}^4$.

In (a) we saw $S_1 \underset{(ss)}{\subseteq} V$, $S_2 \underset{(ss)}{\subseteq} V$ Hence

$S_1+S_2 \underset{(ss)}{\subseteq} V$.

$\dim(S_1) = 2$ (from (c))

$\dim(S_2) = 2$ (from (d) we know $\psi: S_1 \to S_2$ is an onto isomorphism. Hence $S_1 \cong S_2$. As isomorphism preserves linearity dim)

$S_1 \cap S_2 =$

$\underset{\sim}{0} \in S_1 \cap S_2 \Rightarrow \{0\} \subseteq S_1 \cap S_2$.

Let $\underset{\sim}{v} \in S_1 \cap S_2$. Then $\underset{\sim}{v} \in S_1$ and $\underset{\sim}{v} \in S_2$

$(v_1,v_2,v_3,v_4)^T$

$\Rightarrow v_1 = v_2,\ v_3 = v_4$ and $\begin{matrix} v_1 = -v_2 \\ v_3 = -v_4 \end{matrix}$

$\Rightarrow v_1 = v_2 = v_3 = v_4 = 0$.

$\left.\begin{matrix} \Rightarrow \underset{\sim}{v} = \underset{\sim}{0}. \\ \Rightarrow \{\underset{\sim}{0}\} \subseteq S_1 \cap S_2 \end{matrix}\right\} \Rightarrow S_1 \cap S_2 = \{\underset{\sim}{0}\}$.

$\Rightarrow \dim(S_1 \cap S_2) = 0$.

$\dim(S_1+S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$.

$\dim(S_1+S_2) = 2+2-0$

$\Rightarrow \dim(S_1+S_2) = 4 = \dim V$.

Hence $S_1+S_2 = V = \mathbb{R}^4$. Since any subspace of dim equal to dim V is V itself.