Pareto Distribution

M Pal

Introduction

- Back in 1897 the Italian economist Vilfredo Federico Damaso Pareto, an Italian civil engineer, sociologist, economist, political scientist, and philosopher, found a certain regularity in the distribution of incomes in the capitalist countries and also in countries with feudal and early capitalist conditions. Pareto wanted to draw certain conclusion of an economic and sociological nature from this law.
- Pareto was a student of Leon Walras. He made several important contributions to economics, particularly in the study of income distribution and in the analysis of individuals' choices.
- In his pioneering paper in 1897 (The New Theories of Economics, *Journal of Political Economy*, Vol. 5, No. 4 (Sep., 1897), pp. 485-502), he argued that in all countries and times, the distribution of income (and wealth) is highly skewed, with a few holding most of the wealth.

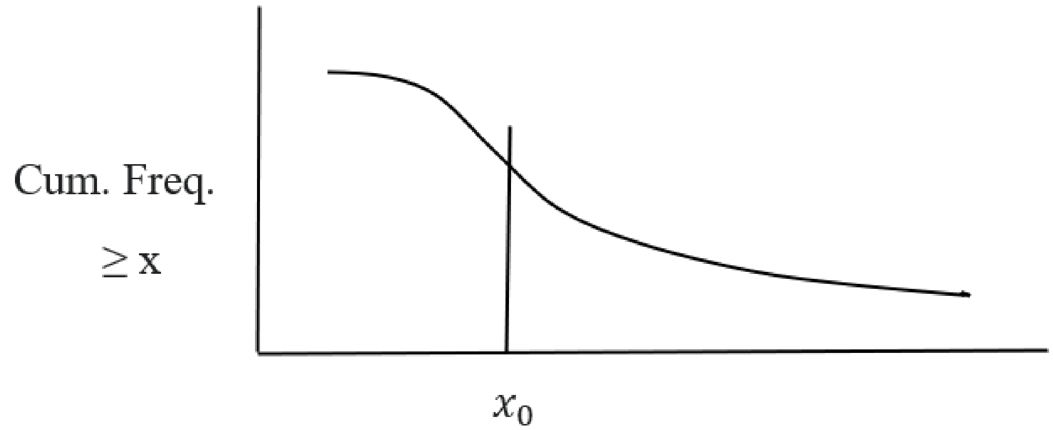
Pareto Law

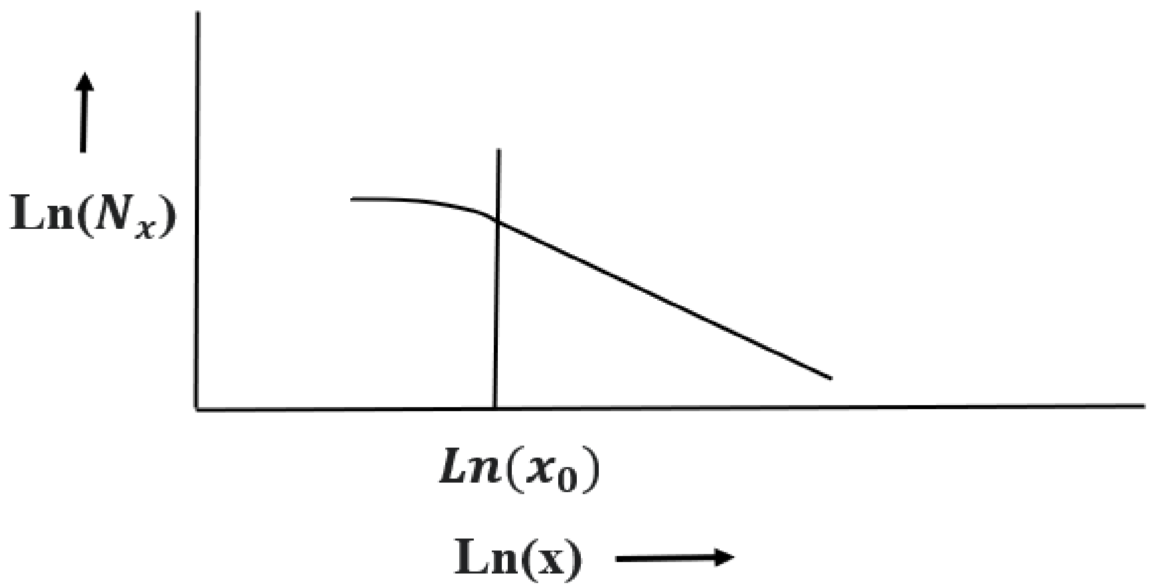
- On the basis of the statistics of various countries Pareto constructed cumulative frequency distribution which indicated how many persons had incomes not below a certain sums indicated in the series. Then he drew the diagrams of such frequency distributions, marking along the axis of abscissae the incomes, x, and on the axis of ordinates y, the number of persons with incomes x or greater than x. Pareto found that in most cases which he investigated, the curves are similar in shape to each other (hyperbola).
- **Pareto Law:** The law, as described by Pareto, is as follows: In all places and all times, the distribution of income in a stable economy is approximately given by the formula

$$N_x = \frac{A}{(x+b)^{\alpha}} \text{, for all } x \geq x_0, \text{ say,}$$

- where N_x represents the number of individuals having an income greater than x; b is a constant which for aggregate incomes is in general zero, or very near it; α is another constant whose value lies between 1 and 2. The form of the curve in the immediate neighbourhood of this minimum income is still undetermined, for statistics do not furnish us sufficient information for its determination.

The form of the Frequency Distribution





Weak Pareto Law Vs. Strong Pareto Law

- Since b is close to zero, $Ln(N_x)$ is approximately linearly related to Ln(x) for x ≥ x_0. Pareto's data covered many countries e.g., England, Germany, Ireland, Italian cities, Peru for the periods 13th to 19th century. The slope of the line was found to be approximately -1.5.
- Since b is very close to zero, we can write

$$N = \frac{A}{x^{\alpha}} \ or = Ax^{-\alpha}.$$

- This law is known as Strong Pareto law. This implies

$$\frac{x^{-\alpha}}{1 - F(x)} = 1 \qquad (x \geq x_0 > 1, \alpha > 0).$$

- The Weak Pareto law refers to

$$\lim_{x \to \infty} \frac{x^{-\alpha}}{1 - F(x)} = 1.$$

- Almost all popular income distributions obey the Weak Pareto law.

Universality of Pareto Law

- **Universality of Pareto Law:**
- Pareto law implies that the distribution of income is highly skewed, and it is universal, i.e., the law prevails in all countries and times. There are observations at $\mu+700\sigma$. Coefficient of skewness was found to be as high as 11. High degree of skewness implies extremely unequal distribution. If one believes that the Pareto law is universal, then one has to believe that the observed high degree of inequality holds everywhere and is, in a sense, natural and universal. Any attempt to modify the income distribution is foredoomed to be a failure and the society would eventually return to its Paretian pattern.
- After Pareto published his paper, there was a huge uproar among the Economists. They were divided into two groups – one group, who were pro-capitalists, supported it and the other group, who were pro-Marxist, opposed it vehemently. They had their own arguments.

Evidence for Universality

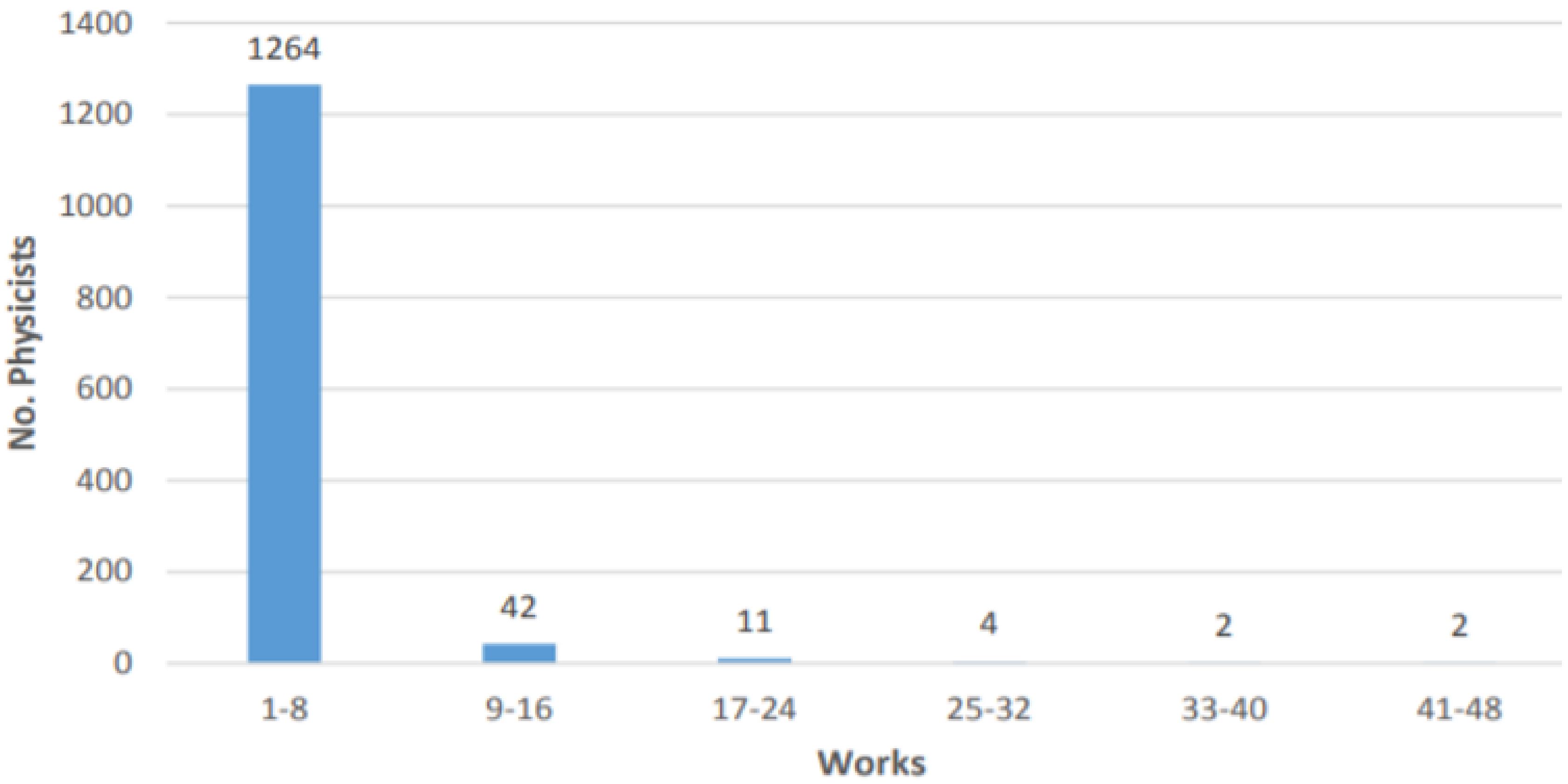
- The pro-Pareto group argued that the data on income of all the countries, except for a few, have been seen to obey the Pareto law. In fact, it was seen that the upper tail of the income distribution is approximately Paretian in every non-socialist country.

- The pro-Pareto group argued that there is a law called "**Lotka's inverse square law of scientific productivity**". On the basis of data on publication, Lotka derived what he termed an "inverse square law", according to which of any set of authors, about 60% produce one paper, whereas the percent producing 2 is $1/2^2$ or about 25%, the percent producing 3 equals $1/3^2$ or about 11.1%, the percent producing 4 is $1/4^2$ or about 6.3%, etc.

- Thus, of 1000 authors, 600 produce 1 paper, 250 produce 2 papers, 111 produce 3 papers, and 63 produce 4 papers. Lotka later defined the general formula for the relation he found between the frequency of y of persons making x contributions as: $x^n y$ = const. This is generally known as **Power law**.

- They argued that any effort which requires talent follows inverse square law of scientific productivity or more generally power law. Since the distribution of talent is very skewed, it is reflected in all scientific productivities.

Frequency Distribution of Works



Source: https://arxiv.org/ftp/arxiv/papers/1601/1601.04950.pdf

Evidence Against Universality of Pareto Law

- The anti-Pareto groups put forward the data of income of few countries like the then Soviet Union, Cuba and Vietnam and showed that there are a ceilings to income which are fairly moderate in those countries and hence Pareto law is not valid.
- They also argued that income, in particular, has two components – wage income and property income.
- Wage income is not extremely skewed. It is property income which gives rise to long tail. Property income is proportional to property owned. The extremely long tail of the income distribution is largely due to extremely unequal distribution of property among the population, and they may not have much to do with the distribution of abilities. Property income does not require any talent, because property income is inherited.
- So, the argument of “inverse square law of scientific productivity” does not stand here.

80/20 Rule

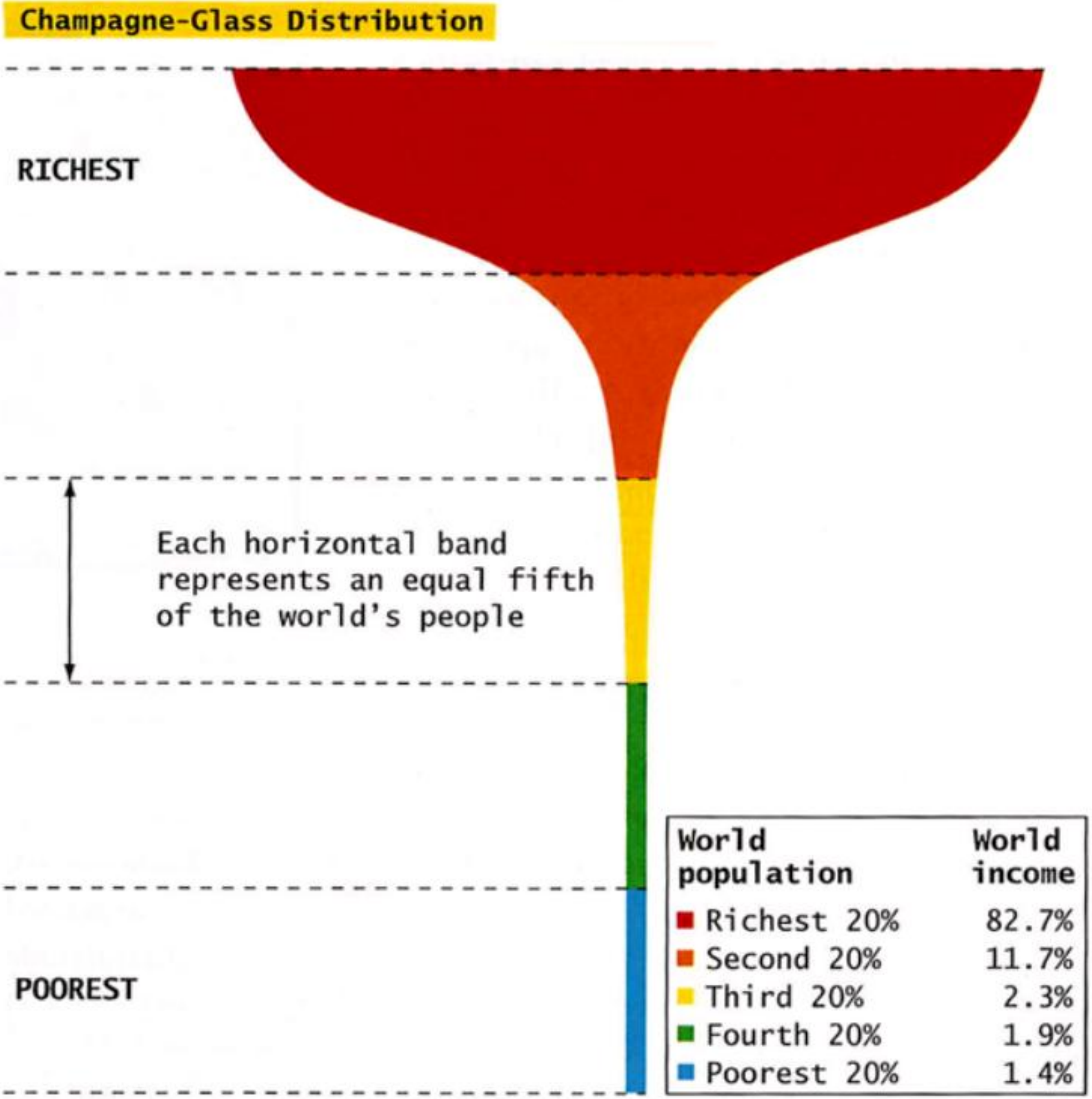
- For Pareto, the most illustrative example was that 20% of the landowners of the Italy of the 19th century owned 80% of land. Today, the ideas of Pareto have been developed and applied to different areas of economics and business such as management, HR, marketing and investment appraisal.
- The **Pareto** stated that for many outcomes, roughly 80% of consequences come from 20% of the causes. This principle is known as the **80/20 rule**, the **law of the vital few,** or the **principle of factor sparsity.** Pareto showed that approximately 80% of the land in Italy was owned by 20% of the population.
- Mathematically, the 80/20 rule is roughly described by Pareto distribution **for a particular set of parameters**, and many natural phenomena have been shown to exhibit such a distribution. It is an adage of business management that "80% of sales come from 20% of clients".
- Source: https://en.wikipedia.org/wiki/Pareto_principle

Champagne Glass Effect

- The Pareto principle also could be seen as applying to taxation. In the US, the top 20% of earners paid roughly 80–90% of Federal income taxes in 2000 and 2006, and again in 2018.
- A chart that gave the effect a very visible and comprehensible form, the so-called "champagne glass" effect, was contained in the 1992 United Nations Development Program Report, which showed that distribution of global income is very uneven, with the richest 20% of the world's population generating 82.7% of the world's income. Among nations, the Gini index shows that wealth distributions vary substantially around this norm.

Distribution of world GDP, 1989

Quintile Groups of population	Income
Richest 20%	82.70%
Second 20%	11.75%
Third 20%	2.30%
Fourth 20%	1.85%
Poorest 20%	1.40%



https://thesocietypages.org/graphicsociology/2009/05/27/champagne-glass-distribution-of-wealth/

80/20 Rule (Continued)

- An important property of Pareto distributions is that they have a fat tail. In the real world, this means that the wealthiest one percent of population possesses a substantially larger portion of the national income and wealth. Accordingly, greater understanding of the overall concentration of income and wealth requires increased attention be paid to why the distributions of top earners universally follow the Pareto distribution.

- To summarize, the empirical rules should not be summarily discarded. After all, many of the discoveries have come from investigation of empirical data. In fact, in “Data Mining” we precisely try to find out hidden patterns in the data. But one must apply these rules prudently, since these rules are not universal. Even the 80/20 rule is valid for a particular value of the parameter of pareto distribution, and that the 80/20 rule need not be valid for all the countries in the world.

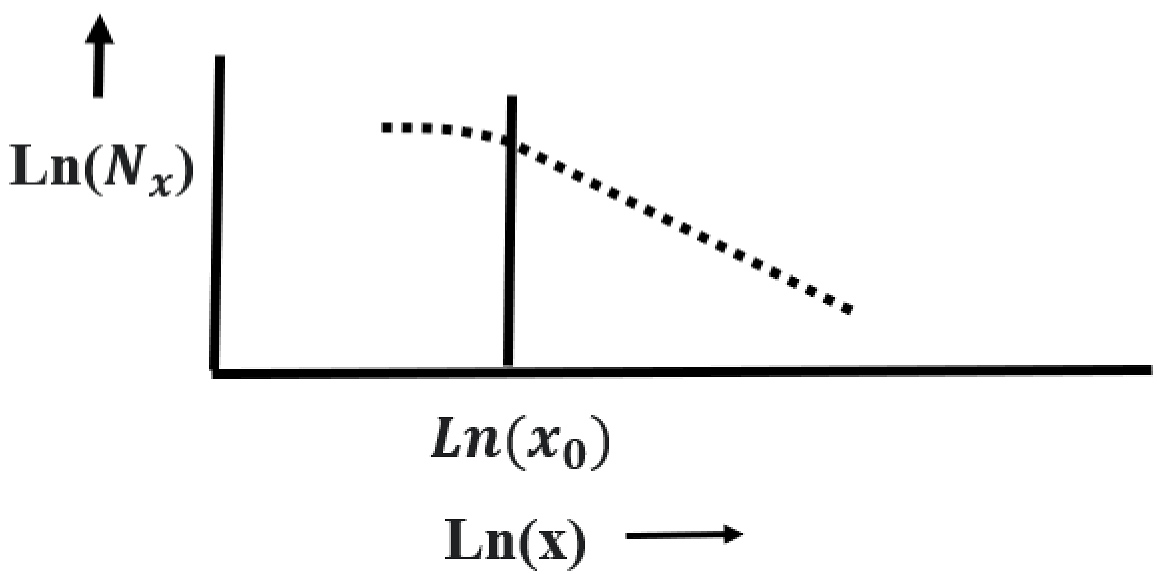
Graphical Test for Pareto Distribution

- **Graphical Test for Pareto Distribution**
- Given a grouped size distribution of income we first try to judge in a preliminary way whether the Pareto law would fit for this distribution, and if so, over what range of income.

Income Boundaries $(x_{i-1} - x_i)$	Number of Earners (n_i)	Income (Lower Boundary)	Cum. Freq. (≥ type) (N_x)	Ln(x)	$Ln(N_x)$
$x_1 - x_2$	n_1	x_1	$n_1 + n_2 + \cdots + n_K = N_1$	$\text{Ln}(x_1)$	$\text{Ln}(N_{x_1})$
$x_2 - x_3$	n_2	x_2	$n_2 + n_3 + \cdots + n_K = N_2$	$\text{Ln}(x_2)$	$\text{Ln}(N_{x_2})$
---	---	---	---	---	---
$x_i - x_{i+1}$	n_i	x_i	$n_i + n_{i+1} + \cdots + n_K = N_i$	$\text{Ln}(x_i)$	$\text{Ln}(N_{x_i})$
---	---	---	---	---	---
$x_K - x_{K+1}$	n_K	x_K	$n_K = N_K$	$\text{Ln}(x_K)$	$\text{Ln}(N_{x_K})$
Total	$\text{n} = \sum n_i$	---	---	---	---

Graphical Test for Pareto Distribution

- Pareto law does not hold for the entire region. Usually, the graph bends at some place after the median. From graph one finds whether the Pareto law holds, and at which point the graph bends.



Derivation of Pareto Distribution

- **Derivation of Pareto Distribution from Pareto Law**
- Pareto law says that

$$N = Ax^{-\alpha},$$

- where N is the number of persons having income ≥ x and x > x_0. Suppose the total number of persons in the community having income ≥ x_0 is M. So, N/M is proportion of persons having income ≥ x. Assuming M to be large enough, we can assume this proportion to be close to the corresponding probability. I.e.,

$$P(X \geq \mathrm{x}) = 1 - F(x) = \frac{N}{M} = \frac{Ax^{-\alpha}}{Ax_0{}^{-\alpha}} = \frac{x^{-\alpha}}{x_0{}^{-\alpha}}$$

$$= \left(\frac{x}{c}\right)^{-\alpha}, replacing\ x_0\ by\ c\ for\ convenience.$$

- Or,

$$F(x) = 1 - \left(\frac{x}{c}\right)^{-\alpha}, for\ x \geq c.$$

- We differentiate F(x) to get the pdf of X as

$$f(x) = \frac{dF(x)}{dx} = \alpha c^{\alpha} x^{-\alpha-1}, for\ x \geq c.$$

- According to Pareto, 1< α ≤ 2.

Home Tasks

- **Home Task 1:** Prove that

$$\text{(i)}E(X) = \frac{\alpha c}{\alpha - 1}, \text{(ii)}V(X) = \begin{cases} \infty & if\ \alpha \in (1.2] \\ \left(\frac{c}{\alpha - 1}\right)^2 \frac{\alpha}{\alpha - 2} & if\ \alpha > 2 \end{cases}, \text{and in general}$$

$$\text{(iii)}E(X^r) = \begin{cases} \infty & if\ \alpha \leq r, \\ \frac{\alpha c^r}{\alpha - r} & if\ \alpha > r \end{cases}$$

- **Home Task 2:** Prove that $(i)\ GM = c \times exp\left(\frac{1}{\alpha}\right)$, and $(ii)\ HM = c \times \left(1 + \frac{1}{\alpha}\right)$
- **Home Task 3:** If X ~ Pareto(c, α) then show that Y = ln(X/C) ~ Exponential(α), i.e., f(y) = $\alpha e^{-\alpha y}$.
- It then follows that X = e^{cY}. We shall later see that the lognormal distribution is also an exponential of a random variable, more precisely a normal random variable.

Lorenz Curve (LC) of Pareto distribution

- **Lorenz Curve of Pareto distribution**
- The Lorenz curve is often used to characterize income and wealth distribution. For any distribution, the Lorenz curve L(F) is written in terms of the CDF (F) as

$$L(F) = \frac{\int_{c}^{x(F)} xf(x)dx}{\int_{c}^{\infty} xf(x)dx}.$$

- L(F) is thus the first moment distribution of X and is often symbolized as $F_1(F)$ or simply F_1.
- **Home Task 4:** Prove that $F_1 = 1 - (1 - F)^{1-1/\alpha}$.

Properties of LC of Pareto distribution

- Since

$$F_1 = 1 - (1 - F)^{1-\frac{1}{\alpha}},$$

- Which is the equation of LC, the position of the curve depends on parameter α only. Given F, as α increases, F_1 also increases. A LC corresponding to higher α is uniformly interior to LC with a lower α. So α is the inequality parameter.
- Also,

$$\text{LR} = \frac{1}{2\alpha - 1}.$$

- So, as α increases LR decreases. Thus, Inequality varies inversely with α.

Home Tasks

- **Home Task 5:** Show that LR for Pareto(α) is $\frac{1}{2\alpha-1}$. (Note that inequality varies inversely with α).
- **Home Task 6:** Suppose X_1 and X_2 are two IID non-negative r.v.s with common mean μ. Then show that $E|X_1 - X_2| =$ 2μ(LR). $\Rightarrow LR = \frac{\Delta_1}{2\mu}$. This is the continuous version of LR.

Lorenz Ratio (LR) of Pareto distribution

- The Lorenz Ratio or Gini Coefficient is a measure of the deviation of the Lorenz curve from the equi-distribution line which is a line connecting [0, 0] and [1, 1], which is shown in black ($\alpha = \infty$) in the Lorenz plot below.
- Specifically, the Gini coefficient is twice the area between the Lorenz curve and the equi-distribution line. The Gini coefficient for the Pareto distribution is then calculated, for $\alpha \geq 1$, to be

$$LR = 1 - 2\left(\int_0^1 F_1 dF\right) = \frac{1}{2\alpha - 1}.$$

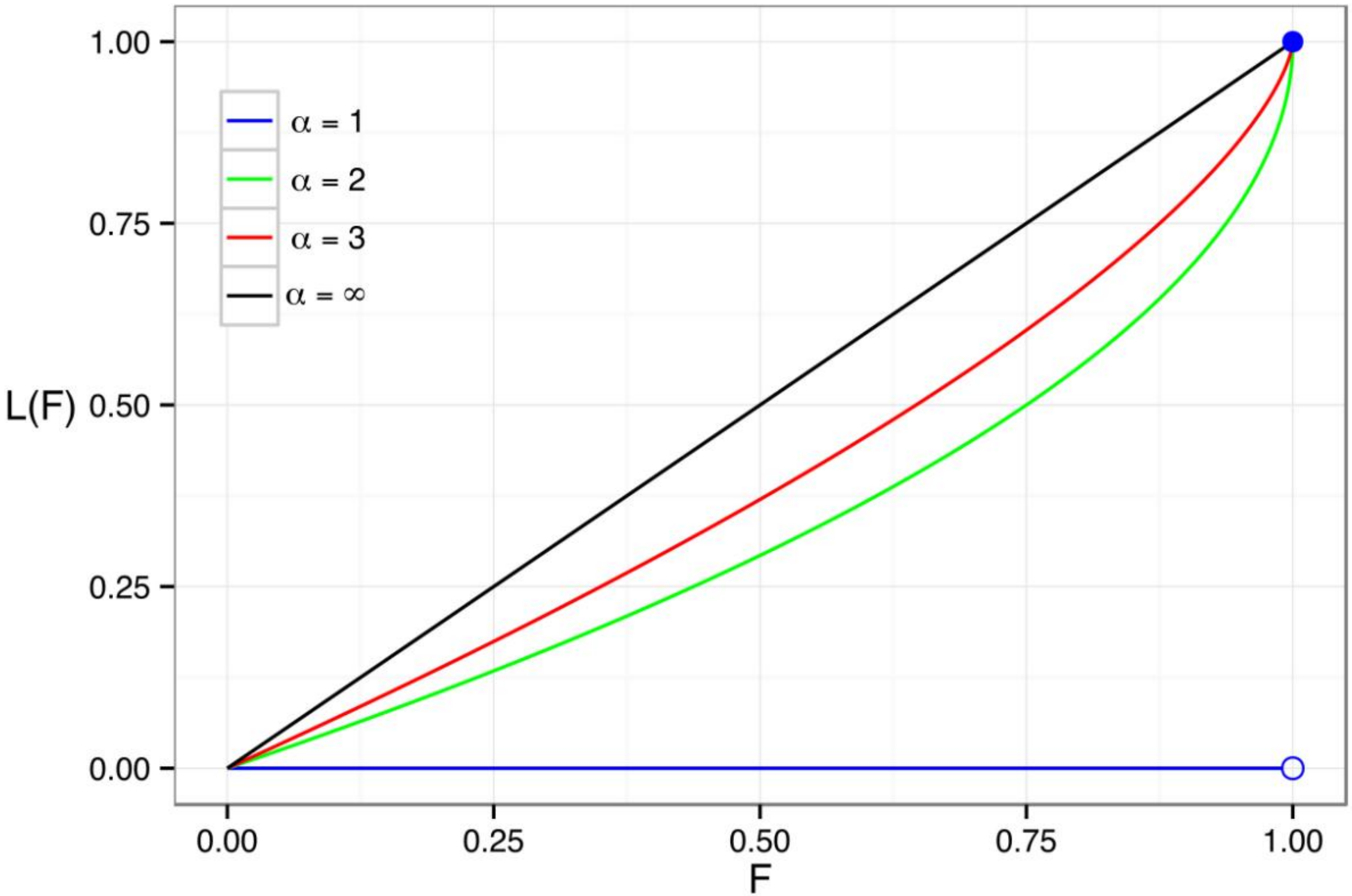


Diagram 1: Lorenz Curves of Pareto Distribution for Different Values of α.

Thank You