Testing of Hypothesis

1) Hypotheses
- Null hyp & Alternative hyp.
- Simple & composite hyp
- One - sided & two - sided hypo

2) Test of a hypothesis
- test function
- Acceptance, Rejection, critical region

3) Types of error
- level of significane
- size of a test
- power $f^{\underline{n}}$
- power

4) UMP / MP test.

Estimation $\to$ Point $\left\{\begin{array}{l} \text{UMVUE} \\ \text{MLE} \\ \text{MOM} \end{array}\right.$

Estimation $\to$ Interval

Statistical Hypothesis means a statment about a parameter

Ex: $X_1, \ldots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$H: \mu = 0$ vs $\left.\begin{array}{l} H_{11}: \mu > 0 \\ H_{12}: \mu < 0 \end{array}\right\}$ one sided

$H_{13}: \mu \neq 0$ – two sided

a) Null Hypothesis (H_0)

Alternative hypotheses (H_1 / H_A)

Ex: Q: If a coin is fair or not,

$X_1, \ldots, X_n \overset{iid}{\sim} Bern(\theta)$

$H_0: \theta = \frac{1}{2}$ vs $H_1: \theta \neq \frac{1}{2}$

Q: If a new medicine is better than the traditional one.

$$X_i = \begin{cases} 1 & \text{if a person taking the medicine is cured} \\ 0 & \text{o.w.} \end{cases}$$

$X_i \overset{iid}{\sim} \text{Bern}(\theta) \qquad i = 1, \dots n$

$H_0 : \theta = \theta_0 \qquad \text{vs} \qquad H_1 : \theta > \theta_0$

→ Test of Hypothesis is about accepting or rejecting the null hypothesis (H_0) only in light of the alternative (H_1)

Simple and composite hypothesis →

- When a hypothesis entirely specifies the population then it is called a 'simple hypothesis'.
- Composite hypothesis : A hypothesis which is not 'simple'.

$X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(\theta)$

$H_0 : \theta = \frac{1}{2}$ [simple]

$H_1 : \theta \neq 1/2$ [composite]

$H_1 : \theta = \frac{3}{4}$ [simple]

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2) \quad H_0 : \mu = 0$, [Composite]

Test of Hypothesis →

Q: How to test?

$X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(\theta) \qquad n = 10$

$H_0 : \theta = \frac{1}{2}$

$H_1 : \theta > \frac{1}{2}$

$\{0, \ldots\ldots, 0\}$

Reject if $\{x_1, \ldots, x_n\}$ is such that $\sum_{i=1}^{n} x_i > 5$

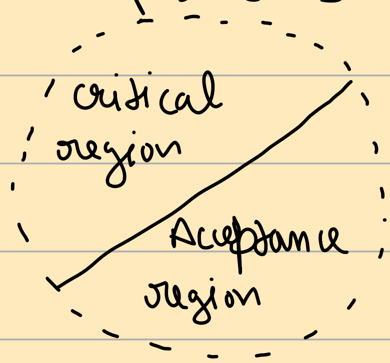
<u>Testing rule</u>: For any realisation $\{x_1, \ldots, x_n\}$ we should have an unique answer, whether we should accept or reject H_0.

<u>Test f^n</u> ϕ:

$$\mathcal{X} = \{\text{set of all realizations}\} \xrightarrow{\phi} \{\text{Accept, Reject}\}$$

[sample space]

$\phi : \mathcal{X} \rightarrow \{0, 1\}$ — $1 \rightarrow$ Acceptance, $0 \rightarrow$ Reject

$\mathcal{X} =$ 

$\phi(\{x_1 \ldots x_n\}) = 1 \iff$ for the realization $\{x_1 \ldots x_n\}$ we reject H_0.

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{10} x_i > 5 \iff \bar{x}_n > \frac{1}{5} \\ 0 & \text{otherwise} \end{cases}$$

Critical region : The points in $\mathcal{X}$ for which the test
(accept)
ϕ rejects the null hypothesis is called the critical region 'C'.

(acceptance region , A).

Errors →

Decision \ True state	H_0 is true	H_0 is false
Accept H_0	✓	Type II error
Reject H_0	Type I error	✓

$$P(\text{Type I error}) = P(\underbrace{\text{Reject } H_0}_{\sum_{i=1}^{10} X_i > 5} \mid \underbrace{H_0 \text{ is true}}_{\theta = \frac{1}{2}})$$

$X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(\theta)$

$\phi(\underset{\sim}{x}) = \mathbb{I}\left(\sum_{i=1}^{10} x_i > 5\right)$

$H_0: \theta = \frac{1}{2}$ vs $H_1: \theta > \frac{1}{2}$

$$P(\text{Type I error}) = \mathbb{P}\left(\sum X_i > 5 \mid \theta = \tfrac{1}{2}\right)$$
$$= \sum_{j=6}^{10} \binom{10}{j} \left(\frac{1}{2}\right)^{10} = 0.376$$

[Reject if $\sum X_i = 10$] $\quad P(\text{Type I error}) = \frac{1}{2^{10}}$

$H_0: \theta = \frac{1}{2}$ vs $H_1: \theta = \frac{3}{4}$

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$P(\text{Type II error}) = P(\text{Accept } H_0 \mid H_0 \text{ is false})$

$= P(\sum X_i = 10 \mid \theta = \frac{3}{4})$

$= \left(\frac{3}{4}\right)^{10}$

exp) $X_1, \ldots\ldots, X_n \overset{iid}{\sim} f_\theta$ $\quad \underset{\text{I}}{[\text{Bern}(\theta)]}, \underset{\text{II}}{N(\theta, 10)}$

$H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ $\quad (\theta_1 > \theta_0)$

$[\underset{\text{I}}{\theta_0 = 1/2}, \underset{\text{II}}{\theta_0 = 0}]$ $\quad [\underset{\text{I}}{\theta_1 = 3/4}, \underset{\text{II}}{\theta_1 = 1}]$

$\phi : \mathcal{X} \longrightarrow \{0, 1\}$

I $\quad \mathcal{X} = \left\{ \underset{\sim}{0}, \begin{pmatrix} 1 \\ \underset{\sim}{0} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \underset{\sim}{0} \end{pmatrix}, \ldots\ldots \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\}$

II $\quad \mathcal{X} = \mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_{10} \end{pmatrix} ; x_i \in \mathbb{R} \right\}$

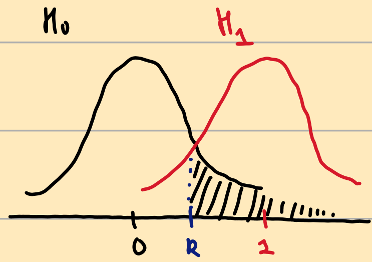
$\phi^* : \mathbb{R} \longrightarrow \{0, 1\} \; ; \; \phi^*(\bar{x}_n) = \begin{cases} 1 & \text{if } \bar{x}_n \geq k \\ 0 & \text{ow} \end{cases}$

Type I error : Rejection of a true H_0

Type II error : Acceptance of a false H_0

$$P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0 \text{ is true})$$

→ Distribution of $\bar{X}_n$ under



$$P[\bar{X}_n \geq k \mid X_i \sim N(0,10)]$$
$$= P[\bar{X}_n \geq k \mid \bar{X}_n \sim N(0,1)] \quad ; n = 10$$
$$= 1 - \Phi(k) = \Phi(-k)$$

$$P(\text{Type II error}) = P(\text{Accept } H_0 \mid \underbrace{H_0 \text{ false}}^{H_1 \text{ True}})$$
$$= P(\bar{X}_n < k \mid \bar{X}_n \sim N(1,1)) \Rightarrow P(\bar{X}_n - 1 < k - 1 \mid \bar{X}_n - 1 \sim N(0,1))$$
$$= \Phi(k-1)$$

	$P(\text{Type I error})$	$P(\text{Type II error})$
$k=1$	$\Phi(-k) = c$	$\Phi(k-1) = K$
$k=2$	$c' < c$	$K' > K$

Remark: If one increases (decreases) k then the $P(\text{Type I error})$ is reduced (increased) and $P(\text{Type II error})$ is increased (reduced).

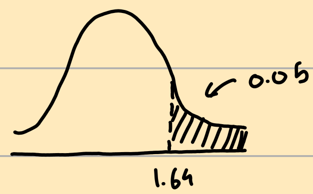
Usually one puts a threshold to the maximum Prob of type I error, which is called the 'level-of-significance' denoted by α

exp: $\alpha = 0.05$

Q: What will be the 'best value' of k in the above test if $\alpha = 0.05$?

→ Any test with k s.t $\Phi(-k) \leq 0.05$ is permitted

→ However, for maximum reduction of P(Type II error), one should choose k s.t $\Phi(-k) = 0.05 \Leftrightarrow k = 1.64$



Power $= 1 - P(\text{Type II error})$

$= P(\text{Rejecting } H_0 \text{ when } H_0 \text{ is false})$

- All test with $k \geq 1.64$ obeys the level-α condition
- But we choose the test with $k = 1.64$ because it is the most powerful test.

Size of test $= P(\text{Type I error})$

exp) $X_1, \ldots, X_n \sim \text{Bern}(\theta)$ $\quad n = 10$

$H_0 : \theta = \frac{1}{2}$ vs $H_1 : \theta = \frac{3}{4}$

$$\phi^*(S_n) = \begin{cases} 1 & \text{if } S_n \geq k \\ 0 & \text{ow} \end{cases} \qquad S_n = X_1 + \cdots + X_n$$

$\alpha = 0.05$

size $= \mathbb{P}(\text{Type I error}) = \mathbb{P}(S_n \geq k \mid S_n \sim \text{Bin}(10, 1/2))$

k	size
10	0.001
9	0.0107
8	0.0547
$\vdots$	$\vdots$

$k = 9$

If we take $k = 9$ then size $= 0.0107$

power $= 0.244$

To increase the power, we can Randomize the test.

$$\phi^{**}(S_n) = \begin{cases} 1 & \text{if } S_n \geq 9 \\ \gamma = 0.89 & \text{if } S_n = 8 \\ 0 & \text{if } S_n < 8 \end{cases}$$

Size $= \mathbb{P}(\text{Type I error of } \phi^{**})$

$= \mathbb{P}(\text{Reject } H_0 \mid S_n \sim \text{Bin}(10, 1/2))$

$= \mathbb{P}(S_n \geq 9 \mid S_n \sim \text{Bin}(10, 1/2)) + 8\,\mathbb{P}(S_n = 8 \mid S_n \sim \text{Bin}(10, 1/2))$

$= 0.0107 + \gamma \cdot (0.0547 - 0.0107) = 0.05$

$\gamma = 0.89$

$\phi^{**} : \mathbb{R} \to [0, 1]$

$X_1, \ldots, X_n \sim f_\theta$

$H_0 : \theta \in \underbrace{\Theta_0}$ vs $H_1 : \theta \in \underbrace{\Theta_1}$

$\theta = \theta_0$ | $\theta = \theta_1$

$\theta \leq \theta_0$ | $\theta > \theta_0$

$\theta = \theta_0$ | $\theta \neq \theta_0$

$$\begin{cases} \Theta_0 = \{\theta_0\} \\ \Theta_1 = \{\theta_1\} \end{cases}$$

$$\begin{cases} \Theta_0 = (-\infty, \theta_0] \\ \Theta_1 = (\theta_0, \infty) \end{cases}$$

$$\begin{cases} \Theta_0 = \{\theta_0\} \\ \Theta_1 = \mathbb{R} - \{\theta_0\} \end{cases}$$

$\phi : \underset{\sim}{x} \longrightarrow [0, 1]$

$\underset{\sim}{x}$ ↓ sample space

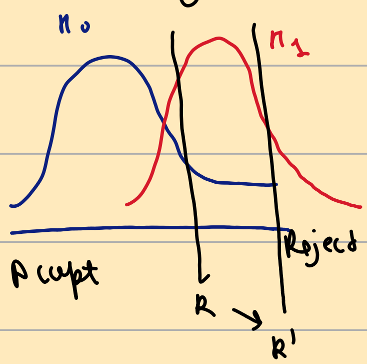
eg $\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{x}_n \geq k \\ 0 & \text{ow} \end{cases}$

ϕ : probability of rejecting H_0 given $\underset{\sim}{x}$.

Recall, size $= \mathbb{P}(\text{Type I error})$

$=$ prob of rejection when H_0 is true

power $=$ prob of rejection when H_0 is false



Fix a lvl-of-significance α

If size of a test $\phi \leq \alpha$,

then ϕ is called a level-α test

- Among the level-α test, one must look at the most powerful test.

Power function : $\beta_\phi(\theta) = E_\theta[\phi(\underset{\sim}{x})]$

$= P(\text{Rejecting } H_0 | \theta)$

<u>Ex1</u> $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, 1)$

$H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ $(\theta_1 > \theta_0)$

$\theta \neq \theta_0$

$\theta > \theta_0$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{x}_n \geq k \\ 0 & \text{ow} \end{cases}$$

$$\beta_\phi(\theta) = E_\theta[\phi(\underset{\sim}{x})] = P(\phi(\underset{\sim}{x}) = 1) = P_\theta(\bar{X}_n \geq k)$$

<u>Ex2</u> $X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(\theta)$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \Sigma x_i > k \\ \gamma & \text{if } \Sigma x_i = k \\ 0 & \text{if } \Sigma x_i < k \end{cases}$$

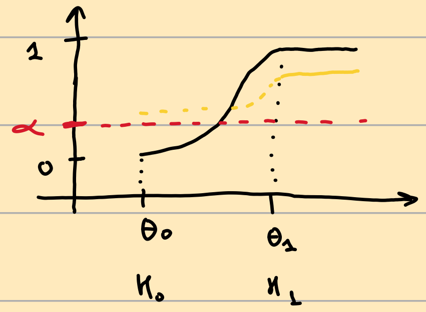
$\beta_\phi(\theta) = E_\theta[\phi(\underset{\sim}{x})]$

$= P_\theta(\Sigma x_i > k) + \gamma P_\theta(\Sigma x_i = k)$

$=$ prob of rejection

For $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$

$\beta_\phi(\theta_0) =$ size of ϕ $\quad$ $\beta_\phi(\theta_1) =$ Power



Most powerful level - α - test →

Consider the testing problem

$H_0 : \theta \in H_0$ vs $H_1 : \theta \in H_1$ $\qquad$ $H_0 : \theta_{01} \leq \theta \leq \theta_{02}$

A test ϕ is called a level-α test if

$$\sup_{\theta \in H_0} \beta_\phi(\theta) \leq \alpha$$

$$\text{size} = \sup_{\theta \in [\theta_{01}, \theta_{02}]} \beta_\phi(\theta)$$

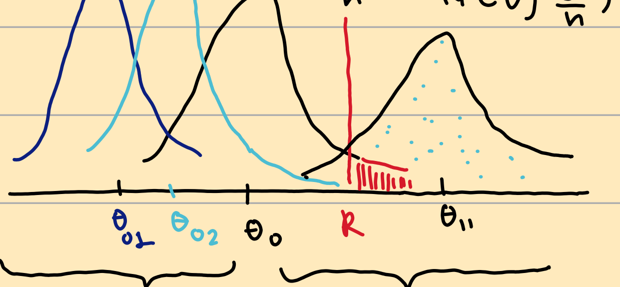
Eg $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, 1)$

$H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{x}_n \geq k \\ 0 & \text{ow} \end{cases}$$

$$\beta_\phi(\theta) = P_\theta(\bar{X}_n \geq k)$$

$\bar{X}_n \sim N(\theta, \frac{1}{n})$

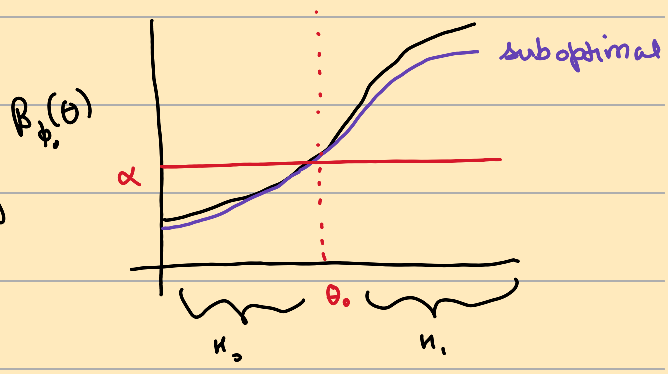


$= \alpha$

H_0 H_1

$$\sup_{\theta \le \theta_0} \beta_\phi(\theta) = P(\bar{X}_n \ge k \mid \theta_0) = \text{size}$$

A test is called a uniformly most powerful test (UMP test) if for any ϕ satisfying (*) [for any level-α test ϕ]



$$\beta_{\phi_0}(\theta) \ge \beta_\phi(\theta) \quad \forall \quad \theta \in H_1$$

when $\theta = \theta_0$ vs $H_1 : \theta = \theta_1$ } simple vs simple

then UMP $\equiv$ MP test

$$(*) \equiv \beta_\phi(\theta_0) \le \alpha$$

$$(**) \equiv \beta_{\phi_0}(\theta_1) \ge \beta_\phi(\theta_1)$$

Neyman Pearson's lemma: [Provides the MP test for simple vs simple hypothesis]

$$X_1, \dots, X_n \overset{iid}{\sim} f_\theta$$

Suppose we want to test

$H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$

(equiv, $X_i \overset{iid}{\sim} f_0$) (equiv, $X_i \overset{iid}{\sim} f_1$)

NP lemma says that a most powerful test ϕ_0 will be of the form

(*) $\phi_0(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \dfrac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})} > k \\ 0 & \text{if } \dfrac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})} < k \end{cases}$

and satisfies $\beta_{\phi_0}(\theta_0) = \alpha$ —— (**)

exp) $X_1, \; X_n \overset{iid}{\sim} N(\mu, \sigma_0^2)$

$H_0: \mu = \mu_0$ vs $H_1: \mu = \mu_1$ $(\mu_1 > \mu_0)$

when, $\mu_0 \Rightarrow N(\mu_0, \sigma_0^2)$

, $\mu_1 \Rightarrow N(\mu_1, \sigma_0^2)$

$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \underbrace{\dfrac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})}}_{\to \lambda(\underset{\sim}{x})} > k \\ 0 & \text{if } \lambda(\underset{\sim}{x}) < k \end{cases}$

$$\lambda(x) = \frac{(1/\sqrt{2\pi}\sigma_0)^n \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i-\mu_1)^2\right)}{(1/\sqrt{2\pi}\sigma_0)^n \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i-\bar{x}_n)^2 + n((\bar{x}_n)-\mu_0)^2\right)}$$

$$\Rightarrow \lambda(\underset{\sim}{x}) = \exp\left(\frac{-n}{2\sigma_0^2}(2\bar{x}_n-\mu_0-\mu_1)(\mu_0-\mu_1)\right)$$

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we need $\lambda(\underset{\sim}{x}) > k$

$\Rightarrow \log \lambda(\underset{\sim}{x}) > \log k$

$$\Rightarrow \frac{-n}{2\sigma_0^2}(2\bar{x}_n - \mu_0 - \mu_1)(\mu_0 - \mu_1) > \log k$$

$$\Rightarrow \bar{x}_n > \left(\frac{\sigma_0^2 \log k}{n(\mu_1 - \mu_0)} + \frac{\mu_0 + \mu_1}{2}\right) = k_0$$

except k everything is known

$\Rightarrow \bar{x}_n > k_0$

(**)

$\beta_{\phi_0}(\mu_0) = \alpha = 0.05$

from $\phi_0(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{x}_n > k_0 \\ 0 & \text{if } \bar{x}_n < k_0 \end{cases}$

$$\mathbb{P}\left(\bar{x}_n > k \mid \bar{x}_n \sim N\left(\mu_0, \frac{\sigma_0^2}{n}\right)\right) = \alpha$$

$$\mathbb{P}\left(\frac{\bar{X}_n - \mu_0}{\frac{\sigma_0}{\sqrt{n}}} > \frac{k_0 - \mu_0}{\frac{\sigma_0}{\sqrt{n}}}\right) = 0.05$$

$$1 - \Phi\left(\frac{k_0 - \mu_0}{\frac{\sigma_0}{\sqrt{n}}}\right) = 0.05$$

$$\frac{k_0 - \mu_0}{\sigma_0/\sqrt{n}} = 1.64$$

$$\phi_0(\bar{x}_n) = \begin{cases} 1 & \text{if } \bar{x}_n \geq \mu_0 + \frac{\sigma_0}{\sqrt{n}}(1.64) \\ 0 & \text{if } \bar{x}_n < \mu_0 + \frac{\sigma_0}{\sqrt{n}}(1.64) \end{cases}$$

Exp) $X_1, \dots, X_n \overset{iid}{\sim} Ber(\theta)$

$H_0: \theta = \theta_0 \qquad H_1: \theta = \theta_1$

$$\lambda(\underset{\sim}{x}) = \frac{\theta_1^{\sum_{i=1}^n x_i} (1-\theta_1)^{n - \sum_{i=1}^n x_i}}{\theta_0^{\sum_{i=1}^n x_i} (1-\theta_0)^{n - \sum_{i=1}^n x_i}}$$

$$= \frac{\theta_1^{n\bar{x}_n} (1-\theta_1)^{n(1-\bar{x}_n)}}{\theta_0^{n\bar{x}_n} (1-\theta_0)^{n(1-\bar{x}_n)}}$$

$\log \lambda(\underset{\sim}{x}) > \log k$

$$\Rightarrow \quad n\bar{x}_n > \frac{-n\log(1-\theta_1) + n\log(1-\theta_0) + \log k}{\log(\theta_1(1-\theta_0)) - \log(\theta_0(1-\theta_1))} = k_0$$

$$(*) \quad \phi_0(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } n\bar{x}_n > k_0 \\ \gamma & \text{if } n\bar{x}_n = k_0 \\ 0 & \text{if } n\bar{x}_n < k_0 \end{cases}$$

$(**) \quad \beta_{\phi_0}(\theta_0) = \alpha$

$\Rightarrow P(n\bar{x}_n > k_0) + \gamma P(n\bar{x}_n = k_0) = \alpha$

if $n = 10$

k_0	$P(\Sigma X_i \geq k_0)$
10	0.001
9	0.0107
8	0.0547

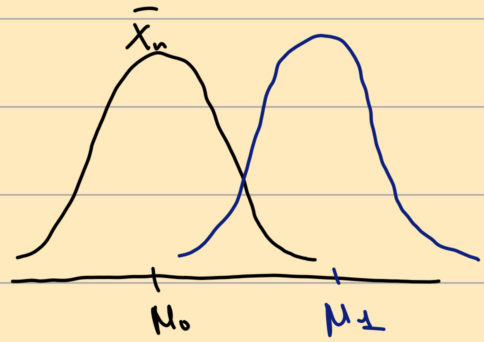
Generalization →

Ex. $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma_0^2)$

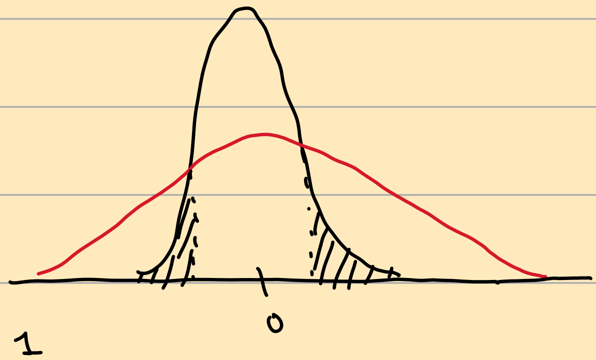
$H_0: \mu = \mu_0$ vs $H_1: \mu = \mu_1$ $(\mu_1 > \mu_0)$

$$\phi_0(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{x}_n \geq \mu_0 + \frac{\sigma_0}{\sqrt{n}} 1.64 \\ 0 & \text{ow} \end{cases}$$

Q1) $H_{12}: \mu = \mu_1 + \delta$ $\quad \delta > 0$



$\frac{f_1}{f_0} > k$



$H_0: \sigma^2 = 1$

$H_1: \sigma^2 = 4$

$\Rightarrow k_0 \nexists$ s.t $\mathbb{P}(|X| > k_0) = \alpha$

Q: $H_0: \mu \leq \mu_0$ vs $H_1: \mu = \mu_1$ $(\mu_1 > \mu_0)$

size $= \beta_\phi(\theta_0) = \alpha$

$$\text{size} = \sup_{\mu \leq \mu_0} \beta_\phi(\mu)$$

$$\beta_\phi(\mu) = 1 - \Phi\left(\frac{k_0 - \mu}{\sigma_0/\sqrt{n}}\right) \quad \phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{x}_n \geq k_0 \\ 0 & \text{ow} \end{cases}$$

observe, $\beta_\phi(\mu)$ is an increasing fn. of μ

$\frac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})} = \lambda(\underset{\sim}{x})$, for $\theta_1 > \theta_0$, is a monotone fn. of $T(\underset{\sim}{X}) \left(= \bar{X}_n\right)$

$\Downarrow$

some statistic

<u>Testing of hypothesis</u>

- Unbiased test
- p-value

From NP lemma, the MP test for testing $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$, given a random sample $X_1, \ldots, X_n$ is of the form

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \lambda(\underset{\sim}{x}) = \frac{f_{\theta_1}(\underset{\sim}{x})}{f_{\theta_0}(\underset{\sim}{x})} > k \\ \gamma & \text{if } \lambda(\underset{\sim}{x}) = k \\ 0 & \text{if } \lambda(\underset{\sim}{x}) < k \end{cases}$$

and satisfying $E_{\theta_0}(\phi(\underset{\sim}{x})) = \beta_\phi(\theta_0) = \alpha$ — $(**)$

Ex) $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$

$H_0: \lambda = \lambda_0$ vs $H_1: \lambda = \lambda_1$ $(\lambda_1 > \lambda_0)$

$$\lambda(\underset{\sim}{x}) = e^{-n(\lambda - \lambda_0)} \left(\frac{\lambda_1}{\lambda_0}\right)^{\Sigma X_i} > k$$

$$\Rightarrow \Sigma X_i > \frac{\log k + n(\lambda_1 - \lambda_0)}{\log \lambda_1 - \log \lambda_0} = k_0$$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if} \quad \Sigma x_i > k_0 \\ \gamma & \Sigma x_i = k_0 \\ 0 & \Sigma x_i < k_0 \end{cases}$$

and (k_0, γ) is such that

$$E_{\lambda_0}[\phi(\underset{\sim}{x})] = \alpha$$

$$\begin{aligned} \beta_\phi(\lambda) = \mathbb{P}_\lambda(\text{Rejecting } H_0) = {} & \mathbb{P}_\lambda(\text{Reject } H_0 \cap \Sigma X_i > k_0) \\ & + \mathbb{P}_\lambda(\text{Reject } H_0 \cap \Sigma X_i = k_0) \\ & + \mathbb{P}_\lambda(\text{Reject } H_0 \cap \sum_{i=1} X_i < k_0) \end{aligned}$$

[{A, B, C are mutually exclusive and exhaustive}

$$\mathbb{P}(D) = \mathbb{P}(D \cap A) + \mathbb{P}(D \cap B) + \mathbb{P}(D \cap C)$$
$$= \mathbb{P}(D \mid A)\mathbb{P}(A) + \mathbb{P}(D \mid B)\mathbb{P}(B) + \mathbb{P}(D \mid C)\mathbb{P}(C) \ ]$$

$$= \mathbb{P}(\text{reject } H_0 \mid \Sigma X_i > k_0)\,\mathbb{P}(\Sigma X_i > k_0)$$
$$+ \mathbb{P}(\text{reject } H_0 \mid \Sigma X_i = k_0)\,\mathbb{P}(\Sigma X_i = k_0)$$
$$+ \mathbb{P}(\text{reject } H_0 \mid \Sigma X_i < k_0)\,\mathbb{P}(\Sigma X_i < k_0)$$

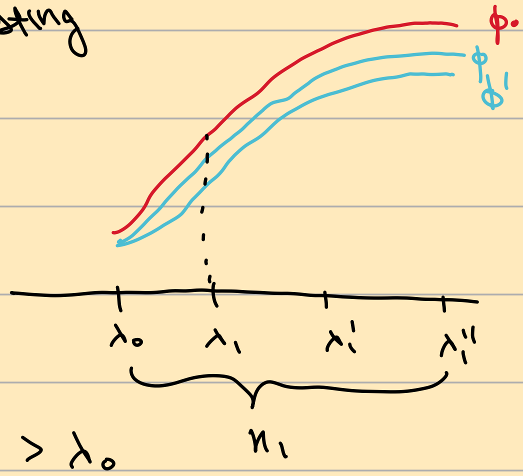
$$= 1 \cdot \mathbb{P}_\lambda(\Sigma X_i > k_0) + \gamma\, \mathbb{P}_\lambda(\Sigma X_i = k_0)$$

Q. Can we generalize to $H_0: \lambda = \lambda_0$ vs $H_1': \lambda > \lambda_0$

Same test follows

So, ϕ is UMP for testing

H_0 vs H_1'

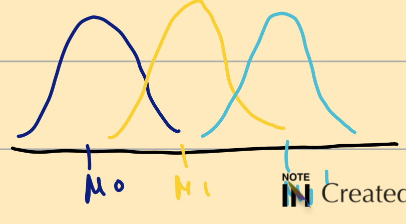


Q: Can we generalize to

$H_0: \lambda \le \lambda_0$ vs $H_1'': \lambda > \lambda_0$

size $= \alpha$

$$\sup_{\lambda \le \lambda_0} \beta_\phi(\lambda) = \sup_{\lambda \le \lambda_0} \left[\mathbb{P}_\lambda(\Sigma X_i > k_0) + \gamma \mathbb{P}_\lambda(\Sigma X_i = k_0) \right]$$



Verify that, $\beta_\phi(\lambda)$ is an $\uparrow$ fxn of λ

size $= \sup_{\lambda \le \lambda_0} \beta_\phi(\lambda) = \beta_\phi(\lambda_0) = \alpha$

Again, in order to solve the

size $= \alpha$ condition

we need to solve $\beta_\phi(\lambda_0) = \alpha$ for (k_0, γ)

So, some soln will appear

The MP test for testing $H_0: \lambda = \lambda_0$ vs $H_1: \lambda = \lambda_1$ $(\lambda_1 > \lambda_0)$

turns out to be the UMP test

$H_0: \lambda \le \lambda_0$ vs $H_1: \lambda > \lambda_0$

<u>Two-side testing</u>

Let $X_1, \dots, X_n$ be iid from $N(\mu, 1)$

$H_0: \mu = \mu_0$ vs $H_1: \mu \ne \mu_0$

$\mu \in \Theta_1 = \mathbb{R} - \{\mu_0\}$

$\Theta_1 = \Theta_{11} \cup \Theta_{12}$

$\Theta_{11} = (\mu_0, \infty)$

$\Theta_{12} = (-\infty, \mu_0)$

$\alpha = 0.05$

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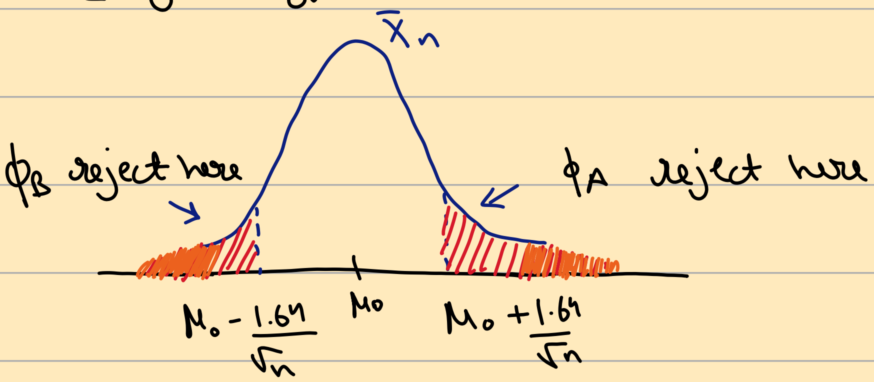
$$\phi_A(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{X}_n \geq \mu_0 + \frac{1.64}{\sqrt{n}} \\ 0 & \text{ow} \end{cases}$$

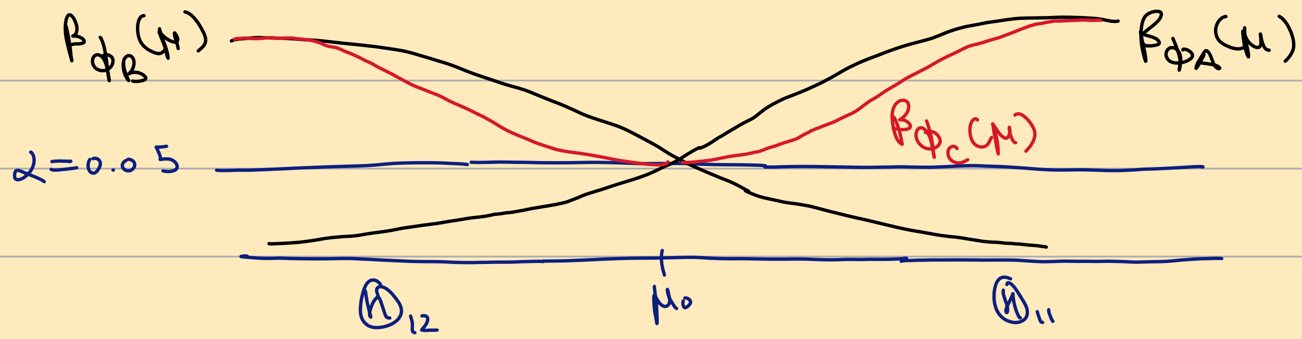
$$\phi_B(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{x}_n \leq \mu_0 - \frac{1.64}{\sqrt{n}} \\ 0 & \text{ow} \end{cases}$$



$$\phi_c(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{X}_n \geq \mu_0 + \frac{\overset{1.97}{\cancel{1.64}}}{\sqrt{n}} \text{ or } \bar{X}_n \leq \mu_0 - \frac{\overset{1.97}{\cancel{1.64}}}{\sqrt{n}} \\ 0 & \text{ow} \end{cases}$$

$$\phi_c(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{X}_n \geq k_0 \text{ or } \bar{X}_n \leq c_0 \\ 0 & \text{ow} \end{cases}$$

and solve (k_0, c_0) by size $= \alpha$ condition



ϕ_A is the UMP test for $H_0: \mu \le \mu_0$ vs $H_1: \mu > \mu_0$
ϕ_B " " " " " $H_0: \mu \ge \mu_0$ vs $H_1: \mu < \mu_0$

So, $\nexists$ a UMP level-α test for testing
$H_0: \mu = \mu_0$ vs $H_1: \mu \ne \mu_0$

$\exists$ some $\mu' \in \Theta_1$ for which

$$\underbrace{P(\text{Rejecting } H_0 \mid \mu')}_{\beta_{\phi_A}(\mu') \;\Downarrow\; \text{power}} < P(\text{Rejecting } H_0 \mid \mu_0) = \underbrace{\beta_{\phi_A}(\mu_0)}_{\Downarrow \; \text{size}}$$

<u>Unbaiasness</u>: A test ϕ is called unbaised for testing
$H_0: \theta \in \Theta_0$ vs $H_1: \theta \in \Theta_1$ if
for any $\theta' \in \Theta_0$ and $\theta'' \in \Theta_1$

$$\beta_\phi(\theta') = P(\text{Rejecting } H_0 \mid \theta')$$
(Prob of Type I error under θ')
$$< P(\text{Rejecting } H_0 \mid \theta'')$$
$$= \beta_\phi(\theta'') = \text{power at } \theta''$$

$$\sup_{\theta \in \Theta_0} \beta_\phi(\theta) \le \inf_{\theta \in \Theta_1} \beta_\phi(\theta)$$

UMPU test: Uniformly most powerful unbaised test

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, 1)$

P-value: $H_0: \mu = 0$ vs $H_1: \mu > 0$

$$\phi_A = \begin{cases} 1 & \text{if } \bar{X}_n \geq 1.64/\sqrt{n} \\ 0 & \text{ow} \end{cases}$$

Realization 1 : $\sqrt{n}\,\bar{X}_n = 1.8$ (Reject H_0)

Realization 2 : $\sqrt{n}\,\bar{y}_n = 3$ (Reject H_0)

P- value : Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, n)$

$H_0 : \mu = 0$ vs $H_1 : \mu > 0$

The MP test at level $\alpha = 0.025$ is

$$\phi_0(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{x}_n \geq 1.96 \\ 0 & \text{ow} \end{cases}$$

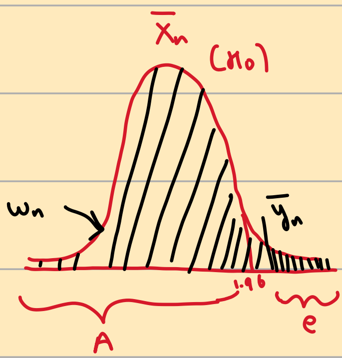
Consider two realization $\underset{\sim}{y_n} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ and $\underset{\sim}{z_n} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

such that $\bar{y}_n = \frac{1}{n}(y_1 + \cdots + y_n) = 2$ and $z_n = 3.5$

We reject H_0 based on $\underset{\sim}{y_n}$ as well as $\underset{\sim}{z_n}$

P- value offers a quantitive measurement of the strength of the decision -

$$p(\underset{\sim}{y_n}) = \mathbb{P}_{H_0}(\underbrace{\bar{X}_n \geq \bar{y}_n}_{H_1 : \mu > 0})$$



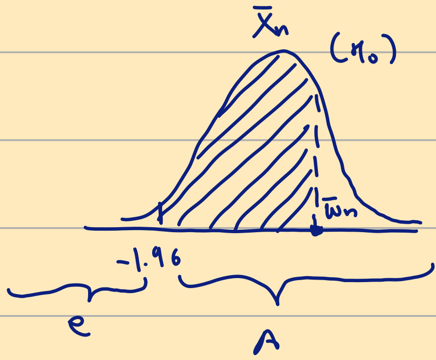
$$p(\underset{\sim}{z_n}) = \mathbb{P}_{H_0}(\bar{X}_n \geq \bar{z}_n)$$

$$p(\underset{\sim}{y_n}) > p(\underset{\sim}{z_n})$$

$\Rightarrow \underset{\sim}{z_n}$ indicates more extreme situation

$H_0 : \mu = 0 \quad$ vs $\quad H_1 : \mu < 0 \quad$ at level $\alpha = 0.025$

$$\phi_0(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \quad \bar{x}_n \le -1.96 \\ 0 & \text{ow} \end{cases}$$



$\underset{\sim}{w_n}$ s.t $\bar{w}_n = 0.5$

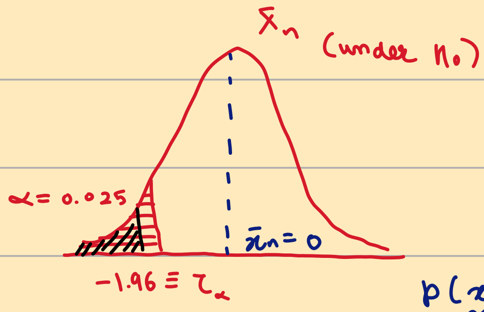
$$p(\underset{\sim}{w_n}) = P_{H_0}(\bar{X}_n \le \bar{w}_n)$$

For another realization $\underset{\sim}{v_n} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$,

suppose $p(\underset{\sim}{v_n}) = 0.05 = P_{H_0}(\bar{X}_n \le \bar{v}_n)$

Q: Should we accept H_0 at level $\alpha = 0.025$ against H_1, based on $\underset{\sim}{v_n}$?

→ If p-value for some realization, say $\underset{\sim}{x_n}$, is $\overset{(\le\alpha)}{>\alpha}$ then, we $\overset{\text{(reject)}}{\text{accept}}$ H_0 at level α.



$p(\underset{\sim}{x_n}) = 0.5$

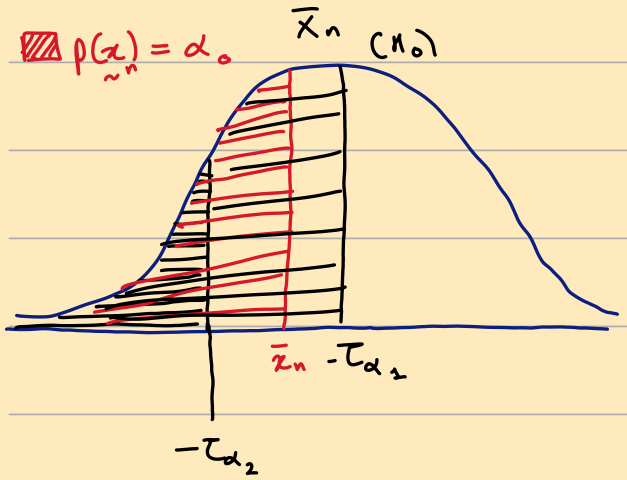
$$P_{H_0}(\bar{X}_n \le -\tau_\alpha) = 0.025$$

• If I take any $\bar{x}_n > -\tau_\alpha$, then $p(\underset{\sim}{x_n}) > 0.025$ and H_0 will be accepted.

• If I take any $\bar{x}_n < -\tau_\alpha$ then $p(\underset{\sim}{x_n}) < 0.025$. Also, based on $\underset{\sim}{x_n}$, H_0 is rejected.

Defn (p-value)

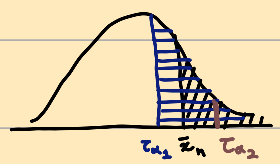
$$p(\underset{\sim}{x}) = \inf\{\alpha : H_0 \text{ is rejected based on } \underset{\sim}{x}_n \text{ against } H_1 \text{ at level-}\alpha\}$$



$\alpha_1 \geq \alpha_0$: Reject H_0 ag. H_1 at level α_1 based on $\underset{\sim}{x}_n$

$\alpha_2 < \alpha_0$: Accept H_0 ag. H_1 at level α_2 based on $\underset{\sim}{x}_n$

$H_0 : \mu = 0$ vs $H_1 : \mu > 0$



$p(\underset{\sim}{x}_n) = \alpha_0$

$\alpha_1 > \alpha_0 \Rightarrow$ reject H_0 based on $\bar{x}_n$

$\alpha_2 < \alpha_0 \Rightarrow$ Accepting H_0 based on $\underset{\sim}{x}_n$

Informal defn of p-value

$p(\underset{\sim}{x}_n) = \mathbb{P}_{H_0}$ (observing at least as extreme obsn as $\underset{\sim}{x}_n$)

$H_0 : \mu = 0$ vs $H_1 : \mu \neq 0$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{x}_n \geq c_1 \text{ or } \bar{x}_n \leq -c_2 \\ 0 & \text{ow} \end{cases}$$

Realization $\underset{\sim}{x}_n = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$p(\underset{\sim}{x}_n) = \min\{\mathbb{P}_{H_0}(\bar{X}_n \geq \bar{x}_n), \mathbb{P}_{H_0}(\bar{X}_n \leq \bar{x}_n)\}$$

$H_1 : \mu > 0$ $\quad \mathbb{P}_{H_0}(\bar{X}_n \geq \underset{\sim}{\bar{x}}_n)$

$H_2 : \mu < 0$ $\quad \mathbb{P}_{H_0}(\bar{X}_n \leq \bar{x}_n)$

Some important testing problems

(A) Testing of binomial proportion

$X_1, \ldots, X_n \overset{iid}{\sim} Bern(p)$

$H_0: p = p_0$ against $H_{11}: p > p_0$

$H_{12}: p < p_0$

$H_{13}: p \neq p_0$

The UMP test for H_0 against H_{11}

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i > K_\alpha \\ \gamma & \text{if } \sum_{i=1}^{n} x_i = K_\alpha \\ 0 & \text{if } \sum_{i=1}^{n} x_i < K_\alpha \end{cases}$$

with $E_{H_0}[\phi(\underset{\sim}{x})] = \alpha \Rightarrow P_{H_0}\left(\sum_{i=1}^{n} x_i > K_\alpha\right) + \gamma P\left(\sum_{i=1}^{n} X_i = K_\alpha\right) = \alpha$

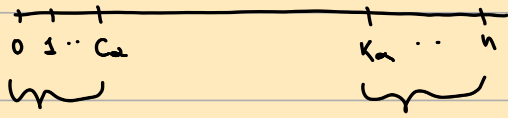
Under H_0, $\sum_{i=1}^{n} X_i \sim Bin(n, p_0)$

The test f^n for testing H_0 against $H_{13}: p \neq p_0$

$$\phi_3(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum x_i > K_\alpha \text{ or } \sum x_i < C_\alpha \\ \gamma_1 & \text{if } \sum x_i = K_\alpha \\ \gamma_2 & \text{if } \sum x_i = C_\alpha \\ 0 & \text{o}\omega \end{cases}$$

example constraint: $\gamma_1 = \gamma_2$

$E_{H_0}[\phi(\underset{\sim}{x})] = \alpha$ $\qquad$ $c_\alpha, k_\alpha \in \{0, \dots, n\}$



Large sample version:

Let $\{T_n\}$ be a seq. of r.v.s, $T_1, T_2, \dots$

The asymptotic distribution of T_n is defined when

$$\frac{T_n - a_n}{b_n} \xrightarrow{d} G, \quad \text{where } G \text{ is non-degenerate}$$

What is the asymptotic distn of $T_n = \sum_{i=1}^{n} x_i$ under $H_0: p = p_0$?

$$\frac{T_n - a_n}{b_n}$$

$a_n = np_0$

$$\frac{T_n - np_0}{\sqrt{np_0(1-p_0)}} \xrightarrow{d} N(0,1)$$

$b_n = \sqrt{np_0(1-p_0)}$

$G \equiv N(0,1)$

The asymptotic distn of T_n is normal with asymptotic mean np_0 and variance $np_0(1-p_0)$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } T_n > k_\alpha \\ \gamma & \text{if } T_n = k_\alpha \\ 0 & \text{ow} \end{cases}$$

$P(T_n > K_\alpha)$

$$= P\left(\frac{T_n - np_0}{\sqrt{np_0(1-p_0)}} > \frac{K_\alpha - np_0}{\sqrt{np_0(1-p_0)}} \right)$$

$$\approx 1 - \Phi\left(\frac{K_\alpha - np_0}{\sqrt{np_0(1-p_0)}} \right) \quad \text{when } n \text{ is large}$$

To obtain K_α we need to solve

$$1 - \Phi\left(\frac{K_\alpha - np_0}{\sqrt{np_0(1-p_0)}} \right) = \alpha$$

(B) Testing mean of normal population

Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$; σ^2 is unknown

$H_0: \mu = \mu_0$ against $H_0: \mu > \mu_0$

$H_{12}: \mu < \mu_0$

$H_{13}: \mu \neq \mu_0$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{x}_n \geq K_\alpha \\ 0 & \text{ow} \end{cases}$$

$$E_{H_0}(\phi(\underset{\sim}{X})) = P_{H_0}(\bar{X}_n \geq K_\alpha) = \alpha$$

$$\Rightarrow 1 - \Phi\left(\frac{K_\alpha - \mu_0}{\sigma/\sqrt{n}} \right) = \alpha$$

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We need a test statistic which depends on $\bar{X}_n$, but its distribution is completely known under H_0.

$$T_n = \frac{\bar{X}_n - \mu_0}{\hat{\sigma}/\sqrt{n}} \quad \text{where} \quad \hat{\sigma}^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (X_i - X_n)^2$$

What is the distribution of T_n under H_0?

$$\bar{X}_n \overset{H_0}{\sim} N(\mu_0, \sigma^2/n) \quad \text{indep} \quad \frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

$$\frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} \overset{H_0}{\sim} N(0, 1)$$

$$T_n = \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}\sqrt{\frac{\sum (X_i - \bar{X}_n)^2}{\sigma^2 (n-1)}}} \overset{H_0}{\sim} t_{(n-1)}$$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if} \quad T_n \geq K_\alpha \\ 0 & \text{ow} \end{cases}$$

To find K_α we need to solve size $= \alpha$

$$\underset{H_0}{P}(T_n \geq K_\alpha \mid T_n \sim t_{(n-1)}) = \alpha$$

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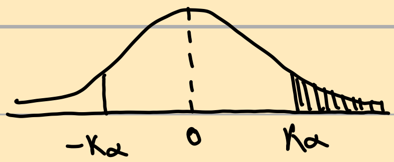
$t_{(n-1)}$



$\Rightarrow K_\alpha = t_{\alpha;(n-1)}$

[upper α-point of $t_{(n-1)}$ distn.]

Test fn. for H_0 ag. $H_{13}: \mu \neq \mu_0$

$$\phi_3(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } T_n \geq K_\alpha \text{ or } T_n \leq -K_\alpha ; |T_n| \geq K_\alpha \\ 0 & \text{ow} \end{cases}$$

Paired-t test :

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \cdots\cdots \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}\right)$$

$$X_1 \ldots X_n \overset{iid}{\sim} N(\mu_x, \sigma_x^2)$$

$$Y_1 \ldots Y_n \overset{iid}{\sim} N(\mu_y, \sigma_y^2)$$

X_i, Y_i are correlated

$Cov(X_i, Y_i) = \sigma_{xy}$

$H_0: \mu_x = \mu_y$ against $H_{11}: \mu_x > \mu_y$

$H_{12}: \mu_x < \mu_y$

$H_{13}: \mu_x \neq \mu_y$

or $\mu_x - \mu_y = 0$

$W_i = (X_i - Y_i)$, $i = 1, \ldots, n$

$E(W_i) = \mu_x - \mu_y$

$$var(W_i) = \sigma_x^2 + \sigma_y^2 - 2\sigma_{xy} = \sigma_w^2$$

$W_i \overset{iid}{\sim} N(\mu_w, \sigma_w^2)$ X

$\underset{\sim}{Z_1} \perp\!\!\!\perp \underset{\sim}{Z_2}$

$g(\underset{\sim}{Z_1}) \perp\!\!\!\perp g(\underset{\sim}{Z_2})$

$W_1 \perp\!\!\!\perp W_2$