Indian Statistical Institute
BSDS: 2024-26
First Year: Semester – II
Economics-II
Home Task 3

Qn. 3.1: Derive the jth raw moment of Lognormal distribution with parameters μ and σ^2 and hence find its variance and coefficient of variation.

Solution to Qn. 3.1:

$$\mathrm{E}\left(\mathrm{X}^{\mathrm{j}}\right)=\mathrm{E}\left(\mathrm{e}^{\mathrm{jLn(X)}}\right)=\mathrm{E}\left(\mathrm{e}^{\mathrm{jY}}\right);\text{where } \mathrm{Y}=\mathrm{Ln}(\mathrm{X})\sim\mathrm{N}(\mu,\sigma^2)$$

$$=\int_0^\infty u^j d\Lambda(u|\mu,\sigma^2)=\int_0^\infty e^{jLn(u)}d\Lambda(u|\mu,\sigma^2)$$

$$=\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma u}e^{-\frac{1}{2\sigma^2}\{(Ln(u))^2-2\mu Ln(u)+\mu^2-2\sigma^2 jLn(u)\}}du$$

$$=\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma u}e^{-\frac{1}{2\sigma^2}\left\{(Ln(u))^2-2\left(\mu+j\sigma^2\right)Ln(u)+\left(\mu+j\sigma^2\right)^2-j^2\sigma^4-2\mu j\sigma^2\right\}}du$$

$$=\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma u}e^{-\frac{1}{2\sigma^2}\left(Ln(u)-\left(\mu+j\sigma^2\right)\right)^2+j\mu+\frac{j^2\sigma^2}{2}}du$$

$$=e^{j\mu+\frac{j^2\sigma^2}{2}}\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma u}e^{-\frac{1}{2\sigma^2}\{Ln(u)-(\mu+j\sigma^2)\}^2}du=\mathrm{e}^{\mathrm{j}\mu+\frac{1}{2}\mathrm{j}^2\sigma^2}.$$

Since

$$\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma u}e^{-\frac{1}{2\sigma^2}\{Ln(u)-(\mu+j\sigma^2)\}^2}du=1.$$

Because it is the integral of a lognormal density function with parameters $(\mu+j\sigma^2,\sigma^2)$ over the entire range.

$$\text{Mean}=\mathrm{E}(\mathrm{X})=\mathrm{e}^{\mu+\frac{1}{2}\sigma^2}\text{ and }\mathrm{E}(\mathrm{X}^2)=\mathrm{e}^{2\mu+2\sigma^2}.$$

$$\mathrm{V}(\mathrm{X})=\mathrm{e}^{2\mu+2\sigma^2}-\left(\mathrm{e}^{\mu+\frac{1}{2}\sigma^2}\right)^2=\mathrm{e}^{2\mu+\sigma^2}\left(\mathrm{e}^{\sigma^2}-1\right).$$

$$\mathrm{CV}(\mathrm{X})=\frac{\sqrt{\mathrm{V}(\mathrm{X})}}{\mathrm{E}(\mathrm{X})}=\sqrt{\mathrm{e}^{\sigma^2}-1}=\eta,\text{ say}.$$

Qn. 3.2: Prove that Coefficient of Skewness and Coefficient of Kurtosis of Lognormal distribution can be written as

$$\text{Coefficient of Skewness} = \gamma_1 = \frac{\mu_3}{(\mu_2)^{3/2}} = \eta^3 + 3\eta > 0.$$

$$\text{Coefficient of Kurtosis} = \gamma_2 = \frac{\mu_4}{(\mu_2)^2} - 3 = \eta^8 + 6\eta^6 + 15\eta^4 + 16\eta^2 > 0,$$

where η is the coefficient of variation of the same Lognormal distribution.

Solution to Qn. 3.2:

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = \frac{E\left(X - E(X)\right)^3}{\left(E\left(X - E(X)\right)^2\right)^{\frac{3}{2}}} = \frac{E(X^3) - 3E(X^2)E(X) + 3E(X)\left(E(X)\right)^2 - \left(E(X)\right)^3}{\left[E(X^2) - 2E(X)E(X) + \left(E(X)\right)^2\right]^{3/2}}$$

$$= \frac{E(X^3) - 3E(X^2)E(X) + 2\left(E(X)\right)^3}{\left[E(X^2) - \left(E(X)\right)^2\right]^{3/2}}$$

$$= \frac{e^{3\mu + \frac{1}{2}3^2\sigma^2} - 3e^{2\mu + \frac{1}{2}2^2\sigma^2} e^{\mu + \frac{1}{2}\sigma^2} + 2\left(e^{\mu + \frac{1}{2}\sigma^2}\right)^3}{\left[e^{2\mu + \frac{1}{2}2^2\sigma^2} - \left(e^{\mu + \frac{1}{2}\sigma^2}\right)^2\right]^{\frac{3}{2}}}$$

$$= \frac{e^{\frac{1}{2}3^2\sigma^2} - 3e^{\frac{1}{2}2^2\sigma^2} e^{\frac{1}{2}\sigma^2} + 2\left(e^{\frac{1}{2}\sigma^2}\right)^3}{\left[e^{\frac{1}{2}2^2\sigma^2} - \left(e^{\frac{1}{2}\sigma^2}\right)^2\right]^{3/2}} = \frac{e^{\frac{1}{2}3^2\sigma^2} - 3e^{\frac{1}{2}2^2\sigma^2} e^{\frac{1}{2}\sigma^2} + 2\left(e^{\frac{1}{2}\sigma^2}\right)^3}{\left[e^{\frac{1}{2}2^2\sigma^2} - \left(e^{\frac{1}{2}\sigma^2}\right)^2\right]^{3/2}}$$

$$= \frac{e^{\frac{9}{2}\sigma^2} - 3e^{\frac{5}{2}\sigma^2} + 2e^{\frac{3}{2}\sigma^2}}{\left[e^{2\sigma^2} - e^{\sigma^2}\right]^{3/2}} = \frac{e^{\frac{3}{2}\sigma^2}\left(e^{3\sigma^2} - 3e^{\sigma^2} + 2\right)}{e^{\frac{3}{2}\sigma^2}\left[e^{\sigma^2} - 1\right]^{3/2}} = \frac{\left(e^{\sigma^2} - 1\right)^2\left(e^{\sigma^2} + 2\right)}{\left[e^{\sigma^2} - 1\right]^{3/2}}$$

$$= \frac{\eta^4\left(e^{\sigma^2} - 1 + 3\right)}{\left[e^{\sigma^2} - 1\right]^{3/2}} = \frac{\eta^4(\eta^2 + 3)}{\eta^3} = (\eta^2 + 3)\eta = \eta^3 + 3\eta > 0.$$

The other one is left to the students to solve.