Statistics II: Introduction to Inference

Problem set 2

1. Let $X_1, \ldots, X_n$ be an i.i.d. sample from the $N(\mu, \sigma^2)$ distribution. Find the Fisher information matrix, $\mathbf{I}_n(\boldsymbol{\theta})$, for the parameter $\boldsymbol{\theta} = (\mu, \sigma^2)$.

   [Note: The Fisher information matrix for a vector valued parameter $\boldsymbol{\theta}$ is defined as

$$\mathbf{I}_n(\boldsymbol{\theta}) = E\left[\left(\frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\mathbf{X}}(\mathbf{X}; \boldsymbol{\theta})\right)\left(\frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\mathbf{X}}(\mathbf{X}; \boldsymbol{\theta})\right)^\top\right] = -E\left[\left(\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \log f_{\mathbf{X}}(\mathbf{X}; \boldsymbol{\theta})\right)\right]]$$

   Further, from the above definition of $\mathbf{I}_n(\boldsymbol{\theta})$ show that, when $X_1, \ldots, X_n$ are i.i.d., then $\mathbf{I}_n(\boldsymbol{\theta}) = n\mathbf{I}_1(\boldsymbol{\theta})$, where $\mathbf{I}_1(\boldsymbol{\theta})$ is the Fisher information matrix for one sample.

2. Let $X_1, \ldots, X_n$ be an i.i.d. sample from the following distributions. In each case, find the method of moments estimator (MOME) for $g(\theta)$:

   (a) `Gamma`(α, β), and $g(\boldsymbol{\theta}) = (\alpha, \beta)^\top$.

   (b) `Beta`(α, β) and $g(\theta) = \alpha/\beta$.

   (c) `Poisson`(λ) and $g(\theta) = \exp\{-\lambda\}$.

   (d) `Location-scale Exponential`(μ, σ) with p.d.f.

$$f_{\mathbf{X}}(x; \mu, \sigma) = \begin{cases} \sigma^{-1}\exp\{-\sigma^{-1}(x-\mu)\} & x > \mu, \\ 0 & \text{otherwise}, \end{cases}$$

   and $g(\theta) = (\mu, \sigma)$.

3. Let $X_1, \ldots, X_n$ be an i.i.d. sample from the following distributions. In each case, find the MLE for $g(\theta)$:

   (a) `Binomial`(m, θ), and $g(\theta) = \theta$.

   (b) `Binomial`(θ, p), and $g(\theta) = \theta$ when $n = 1$.

   (c) `Binomial`(m, θ), and $g(\theta) = P(X_1 + X_2 = 0)$.

   (d) `Hypergeometric`(m, r, θ) with p.m.f.

$$f_X(x; m, r, \theta) = \frac{\binom{m}{x}\binom{\theta-m}{r-x}}{\binom{\theta}{r}}, \quad \theta = m+1, m+2, \ldots; \quad \max\{0, r+m-\theta\} \le x \le \min\{m, r\},$$

   $g(\theta) = \theta$ and $n = 1$.

   (e) Double exponential: pdf $f_X(x; \theta) = 2^{-1}\exp\{-|x-\theta|\}$; with $x \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

   (f) `Uniform`(α, β), and $g(\theta) = \alpha + \beta$.

   (g) `Normal`(θ, θ^2), and $g(\theta) = \theta$.

   (h) `Inverse Gaussian`(θ_1, θ_2) and $g(\boldsymbol{\theta}) = (\theta_1, \theta_2)$.

4. Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy $Y_i = \beta x_i + \epsilon_i,\ i = 1, \ldots, n$ where $x_1, \ldots, x_n$ are fixed constants, and $\epsilon_1, \ldots, \epsilon_n$ are iid $\mathrm{N}\left(0, \sigma^2\right), \sigma^2$ unknown.

   (a) Find a two-dimensional sufficient statistic for $\left(\beta, \sigma^2\right)$.

   (b) Find the MLE of β, and show that it is an unbiased estimator of β.

   (c) Find the distribution of the MLE of β.

   (d) Show that $\sum Y_i / \sum x_i$ is an unbiased estimator of β.

   (e) Calculate the exact variance of $\sum Y_i / \sum x_i$ and compare it to the variance of the MLE.

   (f) Show that $\left[\sum \left(Y_i/x_i\right)\right]/n$ is also an unbiased estimator of β.

   (g) Calculate the exact variance of $\left[\sum \left(Y_i/x_i\right)\right]/n$ and compare it to the variances of the estimators in the previous two estimates.

5. Let $W_1, \ldots, W_k$ be unbiased estimators of a parameter θ with known variances $\mathrm{var}(W_i) = \sigma_i^2$, $i = 1, \ldots, k$. Find the best unbiased estimator of θ of the form $\sum_{i=1}^k a_i W_i$.

6. Suppose that when the radius of a circle is measured, a random error is made, which is modeled as $N(0, \sigma^2)$. If n repeated independent measurements are made, then find an unbiased estimator of area of the circle. Is it the UMVUE?