INDEX

Name : ____________ Subject : ____________

Std : ______ Div : ______ Roll No. ______

School / College : ____________

Sl. No.	Date	Name of the Experiment	Page No.	Marks Awarded	Remarks

14/8/24

Sets

$A, B, C, \ldots \rightarrow$ denotes sets.

$x, y, z \ldots \rightarrow$ elements

$x \in A \Rightarrow$ "x is a element of A".

$\Rightarrow$ Order of elements does not matter.

$\Rightarrow$ Elements do not repeat in a set.

Eg: The set of all letters of the word "Mississippi" $\Rightarrow \{M, s, p, i\}$

$\Rightarrow \{M, p, i, s\}$

Def: A set A is said to be a subset of a set B if $x \in A \Rightarrow x \in B$. $A \subseteq B$.

B is a superset of A. ie $B \supseteq A$.

Ex: B = Set of students in this course from Bang.

S = " " " " in this course.

then $B \subseteq S$ but $S \not\subseteq B$.

and $B \neq S$.

Def^n: Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$.

Exercise: 1) Suppose $A = \{x; \frac{x}{2}$ is an even integer$\}$.

$B = \{y, y$ is a integer divisible by $4\}$.

S.T $A = B$.

* $\mathbb{R}$ = the set of all real numbers.

$\mathbb{C}$ = " " " " complex ".

$\mathbb{Q}$ = " " " " rational ".

$= \{\frac{p}{q} : p, q$ integers, $q > 0$, $\gcd\{p, q\} = 1\}$

$\mathbb{Z}$ = the set of all integers = $\{0, \pm1, \pm2, \ldots\}$

$\mathbb{N}$ = " " " natural no.'s

$= \{1, 2, 3, \ldots\}$

* Take $a, b \in \mathbb{R}$ and $a < b$.

$(a,b) := \{x \in \mathbb{R} : a < x < b\}$ open interval

defined

bounded element.

$[a,b] := \{x \in \mathbb{R} : a \le x \le b\}$ closed interval.

$(a,b] := \{x \in \mathbb{R} : a < x \le b\}$ semi open or semi closed intervals.

$[a,b) := \{x \in \mathbb{R} : a \le x < b\}$

* $(a,\infty) := \{x \in \mathbb{R} : x > a\}$

$[a,\infty) := \{x \in \mathbb{R} : x \ge a\}$

$(-\infty,b) := \{x \in \mathbb{R} : x < b\}$ unbounded intervals.

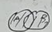
$(-\infty,b] := \{x \in \mathbb{R} : x \le b\}$.

$(-\infty,\infty) = \mathbb{R}$

* Def^n: $A \cap B := \{x : x \in A$ and $x \in B\}$

$A \cup B := \{x : x \in A$ or $x \in B\}$

inclusive = and/or. (always).

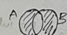


$A - B := \{x : x \in A$ and $x \notin B\}$

Set subtraction

$A \Delta B := \{x : (A-B) \cup (B-A)$

Symmetric difference.



Exc: Show that $A \Delta B = (A \cup B) - (A \cap B)$

2) (Not just using a venn Diagram).

$\Rightarrow$ $\emptyset$ = Null set or empty set or void set

U = Universal set (universe depend on context/situation)

Def^n: $A^c = \{x : x \in U$ and $x \notin A\}$

$= U - A$.

also context dependent.

* $(a,b) =]a,b[$

$A \setminus B = A - B \neq \{a - b : a \in A, b \in B\}$ sometimes

Ans: $A \Delta B = (A-B) \cup (B-A)$

$(A \cup B) - (A \cap B) = (A-B) \cup (B-A)$

$A - B = A - (A \cap B)$

$B - A = B - (A \cap B)$.

$A \cup B = A + B - (A \cap B)$.

$A \Delta B = (A - (A \cap B)) \cup (B - (A \cap B))$.

$= A + B - 2(A \cap B) - ((A - (A \cap B)) \cap (B - (A \cap B)))$

19/8/24

* Union and intersection can be defined for any family of sets - both for finitely many sets and even for infinitely many sets.

Suppose $A_1, A_2, \ldots$ are sets. Then $\forall n \ge 2$, we can define:

* $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cdots A_n$:=

$\{x : x \in A_1, x \in A_2, \ldots x \in A_n\}$

$= \{x : x \in A_i \ \forall \ i = 1, 2, \ldots n\}$

* $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$

$= \{x : x \in A_1$ or $x \in A_2$ or $\ldots$ $x \in A_n\}$

$= \{x : x \in A_i$ for some (or atleast) $i = 1, 2, \ldots n\}$.

* $\bigcap_{i=1}^{\infty} A_i = \bigcap_{i \in \mathbb{N}} A_i = A_1 \cap A_2 \cap \ldots$

$= \{x : x \in A_i \ \forall \ i \in \mathbb{N}\}$

<u>Exc</u>:3) Define $A_i = [i, \infty) \ \forall \ i \in \mathbb{N}$. Show that

$\bigcap_{i=1}^{\infty} A_i = \phi$. (Use Archimedean property)

* $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i \in \mathbb{N}} A_i = A_1 \cup A_2 \cup \cdots$

$= \{x : x \in A_i$ for some $i \in \mathbb{N}\}$.

<u>Exc</u>:4) Show that $\forall a, b \in \mathbb{R}$ with ~~$a<b$~~, $a+1<b$,

(A) $\bigcap_{n \in \mathbb{N}} (a, b + \frac{1}{n}) = (a, b]$ (Use Archimedean property)

(B) $\bigcap_{n \in \mathbb{N}} (a - \frac{1}{n}, b) = [a, b)$

(C) $\bigcup_{n \in \mathbb{N}} [a, b - \frac{1}{n}] = (a, b)$

<u>Archimedean property of positive Real Numbers</u>

For any $x, y \in \mathbb{R}$ with $x > 0, y > 0$, $\exists n \in \mathbb{N}$ (depend on x, y) such that $nx > y$.

<u>Properties of sets</u>: For sets $A, B, C, A_1, A_2, \ldots$ and $n \in \mathbb{N}$, we have

(1) <u>Commutative laws</u>: $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(2) <u>Associative Laws</u>: $(A \cup B) \cup C = A \cup (B \cup C)$.

$(A \cap B) \cap C = A \cap (B \cap C)$.

(3) <u>Distributive laws</u>: ~~$(A \cap B) \cup C =$~~

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(4) <u>De Morgan's Laws</u>: (Finitely many sets.)

$(\bigcap_{i=1}^{n} A_i)^c = \bigcup_{i=1}^{n} A_i^c$

$(\bigcup_{i=1}^{n} A_i)^c = \bigcap_{i=1}^{n} A_i^c$

(5) <u>De Morgan's Laws</u>: (Infinitely many sets)

$(\bigcap_{i \in \mathbb{N}} A_i)^c = \bigcup_{i \in \mathbb{N}} A_i^c$

$(\bigcup_{i \in \mathbb{N}} A_i)^c = \bigcap_{i \in \mathbb{N}} A_i^c$

<u>Exc</u>:5) If A, B, C are finite sets then show the following: (inclusion-exclusion principle)

(i) ~~$A \cup B = A$~~ $|A \cup B| = |A| + |B| - |A \cap B|$

$|A|$: No. of elements in A.

(ii) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C|$

$- |C \cap A| + |A \cap B \cap C|$

* (iii) For n finite sets $A_1, A_2, \ldots, A_n$, what can you say about the size of ~~$|A_1 \cup A_2 \cup A_3 \cup \ldots|$~~ $|A_1 \cup A_2 \cup \ldots \cup A_n|$?

Can you write an inclusion - exclusion formula that generalizes (i) and (ii)?

Exc 6) For sets A, B, C show the following: (use one of distributive laws.)

(i) $A = (A \cap B) \cup (A \cap B^c)$

(ii) $A \subseteq C$, $B \subseteq C \Rightarrow A \cup B \subseteq C$.

(iii) $A \subseteq B \Rightarrow B^c \subseteq A^c$

Properties of Real Numbers :-

* The set $\mathbb{R}$ has two binary operations called addition $(+)$ and multiplication $(\cdot)$ s.t for every pair $x, y \in \mathbb{R}$, we can form their sum $x+y$ and their product $x \cdot y = xy$, both of which are real numbers uniquely determined by (x,y).

In other words, $(x,y) \xmapsto{\text{maps to}} x+y$ and $(x,y) \longmapsto xy$ are both functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

These binary operations satisfy the following properties :

(A1) Commutative laws :- $x+y = y+x$, $x \cdot y = y \cdot x$ $\forall x, y \in \mathbb{R}$.

(A2) Associative laws :- $\left.\begin{array}{l}(x+y)+z = x+(y+z) \\ (xy)z = x(yz)\end{array}\right\} \forall x, y, z \in \mathbb{R}$

(A3) Distributive law :- $x(y+z) = xy + xz$. (of multiplication over add)

(A4) Existence of Identity Elements: $\exists$ two distinct real numbers $0 \in \mathbb{R}$ and $1 \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, we have $x + \underset{\text{additive identity}}{0} = x$ and $x \cdot \underset{\text{multiplicative identity}}{1} = x$.

(A5) Existence of Additive Inverses (Negatives):

$\forall x \in \mathbb{R}$ $\exists$ $\underset{\text{Additive Inverse}}{(-x)} \in \mathbb{R}$ s.t $x + (-x) = 0$.

(A6) Existence of Multiple Inverses :-

$\forall x \in \mathbb{R} - \{0\}$ $\underset{\text{Multiplicative Inverse of } x}{x^{-1}} \in \mathbb{R}$ s.t $x \cdot x^{-1} = 1$.

$\Rightarrow$ For all $a, b, c, d \in \mathbb{R}$, the following properties hold:

(T1) Cancellation Law for Addition :- If $a+b = a+c$, then $b = c$. In particular, the no. 0 in (A_4) is unique.

Also given $x \in \mathbb{R}$, its negative $-x$ is unique, and $-0 = 0$.

Exc-1) To Prove $A = B$ show that $A \subseteq B$, $B \subseteq A$.

To show $A \subseteq B$: let $a \in A$.

then $\frac{a}{2}$ is even int.

$\frac{a}{2} = 2m$ (for some $m \in \mathbb{Z}$)

$a = 4m$ = int div by 4.

$a \in B$ Thus $A \subseteq B$.

To show $B \subseteq A$:

let $b \in B$ then b is an int div by 4.

i.e, $b = 4n$ (for some $n \in \mathbb{Z}$)

$\frac{b}{2} = 2n$

Hence $\frac{b}{2}$ is an even int.

$\frac{b}{2} \in A$ Thus $B \subseteq A$

As $A \subseteq B$, $B \subseteq A$ gives $A = B$.

Exc-2) WKT $A \Delta B = (A \setminus B) \cup (B \setminus A)$

let $C = (A \setminus B) \cup (B \setminus A)$, $D = (A \cup B) \setminus (A \cap B)$

1) $C \subseteq D$:

let $x \in C \Rightarrow x \in (A/B) \cup (B \setminus A)$

$\Rightarrow (x \in A/B)$ or $(x \in B/A)$

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

(assume it seperately)

case ①: $x \in A$ and $x \notin B \Rightarrow x \in (A \cup B)$ and $x \notin (A \cap B)$

case ②: $x \in B$ and $x \notin A \Rightarrow x \in (B \cup A)$ and $x \notin (A \cap B)$

2) $D \subseteq C$:

let $y \in D$ then $y \in A \cup B$ and $y \notin A \cap B$

$\Rightarrow (y \in A$ or $y \in B)$ and $(y \notin A)$ or $(y \notin B)$

(if $y \in A \cap B \Rightarrow y \in A$ and $y \in B$. if $y \notin A \cap B \Rightarrow y \notin A$ or $y \notin B$)

① $y \in A$ and $y \notin B \Rightarrow y \in A/B$.

② $y \in B$ and $y \notin A \Rightarrow y \in B/A$ (or).

$y \in (A/B) \cup (B/A)$

21|8|24

(T2) Possibility of Subtraction: Given $a, b \in R$, $\exists$ unique $x \in \mathbb{R}$ such that $a + x = b$. This x is denoted by $b - a$.

(T3) $b - a = b + (-a)$. In part, $0 - a = -a$,
$b - 0 = b$

(T4) $-(-a) = a$

(T5) $a(b-c) = ab - ac$

(T6) $0 \cdot a = a \cdot 0 = 0$.

(T7) Cancellation law for multiplication:

If $ab = ac$ and $a \neq 0$, then $b = c$. In part, the number 1 in (A4) is unique. Also, give $x \in \mathbb{R}$, $x \neq 0$, x^{-1} in (A6) is unique. and $1^{-1} = 1$.

(T8) Possibility of Division: Given $a, b \in \mathbb{R}$ with $a \neq 0$, $\exists$ unique $y \in \mathbb{R}$ such that $ay = b$. This y is denoted by $\frac{b}{a}$.

(T9) If $a \neq 0$, then $\frac{b}{a} = b \cdot a^{-1}$. In particular, $\frac{1}{a} = a^{-1}$, $\frac{b}{1} = b$.

(T10) If $a \neq 0$, then $(a^{-1})^{-1} = a$

(T11) If $ab = 0$, then $a = 0$ or $b = 0$ (inclusive)

(T12) $(-a) \cdot b = a \cdot (-b) = -ab$ and $(-a)(-b) = ab$

(T13) $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ if $b \neq 0$ and $d \neq 0$.

(T14) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

(T15) $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$ if $b \neq 0, d \neq 0$ and $c \neq 0$.

Remark: properties T(1) to T(15) follow from A(1) - A(6) then (A1) - (A6) can be stated as axioms called 'field axioms', and then (T1) - (T15) can be proved as theorems.

*Ref { Apostol, calculus Vol 1)

* $\exists$ a subset $\mathbb{R}^+ \subseteq \mathbb{R}$ of positive real numbers satisfying the following:

(A7) If $x, y \in \mathbb{R}^+$, then $x+y \in \mathbb{R}^+$ and $xy \in \mathbb{R}^+$

(A8) $\forall x \in \mathbb{R}-\{0\}$, either $x \in \mathbb{R}^+$ or $(-x) \in \mathbb{R}^+$ but not both.

(A9) $0 \notin \mathbb{R}^+$

Defn's:- For $x, y, z \in \mathbb{R}$,

→ $x < y$ means $y - x \in \mathbb{R}^+$;

→ $y > x$ means $x < y$;

→ $x \leq y$ mens $x < y$ or $x = y$
exclusive.

→ $y \geq x$ means $x \leq y$

→ In particular, $x > 0$ means $x \in \mathbb{R}^+$
(x is positive).
$x < 0$ means $-x \in \mathbb{R}^+$
(x is negative)

→ $x < y < z$ means $x < y$ and $y < z$; similarly, define $x \leq y \leq z$, $x < y \leq z$ and $x \leq y < z$.

(T16) Trichotomy Law:- $\forall a, b \in \mathbb{R}$, exactly one of the three relations $a<b$, $a=b$, $a>b$ hold.

(T17) Transitive law:- $\forall a, b, c \in \mathbb{R}$, if $a<b$ and $b<c$ then $a<c$.

(T18) $\forall a, b, c \in \mathbb{R}$ if $a<b$, then $a+c < b+c$

(T19) $\forall a, b, c \in \mathbb{R}$, where $c>0$, if $a<b$ then $ac < bc$

(T20) If $a \in \mathbb{R}-\{0\}$, then $a^2 > 0$

(T21) $1 > 0$

Exc 7):- Show that (T19) ⇒ (T20)

(T22) $\forall a, b, c \in \mathbb{R}$ with $c<0$, $a<b \Rightarrow ac > bc$

(T23) Suppose $a, b \in \mathbb{R}$. If $a<b$, then $-a > -b$. In particular $a<0 \Rightarrow -a>0$

(T24) Suppose $a, b \in \mathbb{R}$. Then $ab>0$ iff either both a, b are positive or negative.
(exclusive)

(T25) Suppose $a, b, c, d \in \mathbb{R}$. If $a<c$ and $b<d$, then $a+b < c+d$.

Remarks:- ① properties A7 - A9 are stated as axioms (called "order axioms") and T16-T25 can then be proved as theorems.

② Axioms A1 - A9 do not define $\mathbb{R}$ uniquely. For eg. $\mathbb{Q}$ satisfies all of these axioms. with $\mathbb{R}^+$ replaced as $\mathbb{Q}^+$.
We need one more axiom for $\mathbb{R}$.

Defn:- Suppose $S \subseteq \mathbb{R}$.

1) S is called bounded above (bdd) if $\exists B \subseteq \mathbb{R}$ s.t
$x \leq B \ \forall x \in S$.
In this case, we say that S is bdd above by B and B is called an upper bound for S.

2) If an upper bound B of S also belongs to S, then B is called the max (element) of S and we write $B = \max S$.
(Thus $B = \max S$, iff $B \in S$ and $x \leq B \ \forall x \in S$)

3) A set with no upper bound is said to be unbdd above.

Exc:-8) Define the following terminologies for a set $S \subseteq \mathbb{R}$: (1) bdd below, lower bound, (2) minimum (element), (3) unbdd below.

Eg:- (1) $S = \mathbb{N}$ is not bdd above but is bounded below. In fact, $1 = \min$ of S.

(2) $S = [0,1]$ is bdd below and above, $0 = \min$ of S, $1 = \max$ of S

(3) $S = (0,1)$ is bdd below and above but it does not admit to have a min or a max element.

(4) $S = [0,1)$ is bdd below and above is said to be just bounded. $0 = \min$ of S but S doesn't have a max.

Exc:- 9) Find whether the following sets are bounded below and above. also find if they have min and max and compute them.

(i) $S = \{2 - \frac{1}{n} : n \in \mathbb{N}\}$ (iii) $S = \{x \in \mathbb{Q} : x^2 < 2\}$

(ii) $S = (0,1]$ (iv) $S = \{x \in \mathbb{Q}\} \cap (0,1)$

Exc-3) $A_i = [i, \infty)$ $\forall\, i \in \mathbb{N}$

To show $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

A) Archimedian property

let $x > 0$ in $\mathbb{R}$. then $\exists\, n \in \mathbb{N}$ such that ~~$n \in \mathbb{N}$~~ $x < n$.

~~Exc~~ Let $x \in \mathbb{R}$.

Then from AMP, $\exists\, n_0 \in \mathbb{N}$ s.t $x < n_0$

$$x \notin A_{n_0} = [n_0, \infty)$$

$\Rightarrow x \notin \bigcap_{n \in \mathbb{N}} A_n$ $\Rightarrow \bigcap_{n \in \mathbb{N}} A_n = \emptyset$

Exc-4) $\forall\, a, b \in \mathbb{R}$. $a+1 < b$.

A) $\bigcap_{n \in \mathbb{N}} (a, b + \frac{1}{n}) = (a, b]$ B) $\bigcup_{n \in \mathbb{N}} (a, b - \frac{1}{n}) = (a, b)$

Sol:- A) $\underbrace{\bigcap_{n \in \mathbb{N}} (a, b + \frac{1}{n})}_{=C} = \underbrace{(a, b]}_{=D}$

let $x \in$ ~~$\bigcap_{n \in \mathbb{N}} (a, b+\frac{1}{n})$~~ $(a, b]$

Then $a < x \le b$. 1) $a < x$ 2) $x \le b < b + \frac{1}{n}$ $\forall\, n \in \mathbb{N}$

$\Rightarrow x \in (a, b + \frac{1}{n})$ $\forall\, n \in \mathbb{N}$. $\Rightarrow x < b + \frac{1}{n}$ $\forall\, n \in \mathbb{N}$

$\Rightarrow x \in \bigcap_{n \in \mathbb{N}} (a, b + \frac{1}{n})$ $\Rightarrow D \subseteq C$.

To prove $C \subseteq D$ $\Rightarrow$ $D^c \subseteq C^c$

let $x \in D^c = (-\infty, a] \cup (b, \infty)$

(a) Let $x \in (-\infty, a]$ $\Rightarrow x \le a$.

$x \notin (a, b + \frac{1}{n})$ $\Rightarrow x \notin \bigcap_{n \in \mathbb{N}} (a, b + \frac{1}{n}) = C$

$x \subseteq C^c$

~~B) $\bigcup_{n \in \mathbb{N}} (a, b - \frac{1}{n}) = (a, b)$ (To show)~~

(b) let $x \in (b, \infty)$ $\Rightarrow b < x$

$\Rightarrow x - b > 0$. $\Rightarrow \frac{1}{x-b} > 0$.

Then $\exists\, n_0 \in \mathbb{N}$ s.T $\frac{1}{x-b} < n_0$

$\Rightarrow \frac{1}{n_0} < x - b$.

$\Rightarrow b + \frac{1}{n_0} < x$ $\Rightarrow$ ~~$x \notin (a$~~

$\Rightarrow x \notin (a, b + \frac{1}{n_0})$.

$x \notin \bigcap_{n \in \mathbb{N}} (a, b + \frac{1}{n}) = C$.

$\Rightarrow x \in C^c$.

c) Let $x \in \bigcup_{n\in\mathbb{N}} (a, b-\frac{1}{n}]$. (a,b).

$\Rightarrow x \in (a, b-\frac{1}{n_0}]$ for some $n_0 \in \mathbb{N}$

$a < x \le b-\frac{1}{n_0} < b$.

$a < x < b$.

$\Rightarrow x \in (a,b)$.

$\Rightarrow \bigcup_{n\in\mathbb{N}} (a, b-\frac{1}{n}] \subseteq (a,b)$.

let $y \in (a,b)$

$a < y < b$.

$b-y > 0$.

$y \le b-\frac{1}{n}$

$\frac{1}{n} \le b-y$

$\frac{1}{b-y} \le n$

$\Rightarrow \frac{1}{b-y} > 0$ S.T $\exists$ $n \in \mathbb{N}$ S.T $\frac{1}{b-y} \le n_0$

bcz nothing changes

$\frac{1}{n_0} \le b-y \Rightarrow y \le b-\frac{1}{n_0}$

Also $a<y$ Hence $a < y \le b-\frac{1}{n_0}$

$\Rightarrow y \in (a, b-\frac{1}{n}]$

$y \in \bigcup_{n\in\mathbb{N}} (a, b-\frac{1}{n}]$

Exc-5) i) $|A\cup B| = |A|+|B| - |A\cap B|$

Proof: $A\cup B = (A/B) \dot\cup B$.

* $\sqcup$ or $\dot\cup$ is disjoint union.

$|A\cup B| = |A/B| + |B|$

$A = (A/B) \cup (A\cap B)$ as A/B, $A\cap B$ are disjoint sets

$|A| = |A/B| + |A\cap B|$.

$\Rightarrow |A/B| = |A| - |A\cap B|$

$\Rightarrow |A\cup B| = |A| - |A\cap B| + |B|$

$|A\cup B| = |A|+|B| - |A\cap B|$.

ii) $|A\cup B\cup C| = |A|+|B|+|C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$

$|A| + |B\cup C| - |A\cap(B\cup C)| = |A| + |B| + |C| - |B\cap C| - |(A\cap B)\cup(A\cap C)|$

$= |A|+|B|+|C| - |B\cap C| - (|A\cap B| + |A\cap C| - |(A\cap B)\cap(A\cap C)|)$

$= |A|+|B|+|C| - |A\cap B| - |B\cap C| - |A\cap C| + |A\cap B\cap C|$.

Exc-6) i) $A = (A\cap B) \cup (A\cap B^c)$

$B \cup B^c = \Omega$

$A \cap (B\cup B^c) = A\cap\Omega = A$.

$(A\cap B) \cup (A\cap B^c) = A$. //.

ii) $A\subseteq C, B\subseteq C \Rightarrow A\cup B \subseteq C$.

Let $x \in A\cup B$ the $x\in A$ or $x \in B$.

since $A\subseteq C$, $B\subseteq C \Rightarrow x \in C$.

$\Rightarrow A\cup B \subseteq C$.

iii) $A\subseteq B \Rightarrow B^c \subseteq A^c$

let $x \in B^c$ then $x \notin B$.

To prove $x \notin A$.

if $x\in A$, then $A\subseteq B$ will give $x\in B$.

Hence $x\notin A$ i.e $x \in A^c$ Hence $B^c \subseteq A^c$

Exc-4) (B) $\underbrace{\bigcap_{n\in\mathbb{N}} (a-\frac{1}{n}, b)}_{A} = \underbrace{[a,b)}_{B}$ $(a+1<b)$.

let $x \in B \Rightarrow x$ $a \le x < b$.

$a-\frac{1}{n} < a \le x < b$.

$a-\frac{1}{n} < x < b$.

$x \in (a-\frac{1}{n}, b)$.

$\Rightarrow B \subseteq A$.

$y > a-\frac{1}{n}$

Let $y \in B^c$ ~~$\Rightarrow a-\frac{1}{n} < y < b$.~~

$y \in (-\infty, a) \cup [b, \infty)$

① $y<a \Rightarrow a-y>0 \Rightarrow \frac{1}{a-y} > 0$ $\exists n_0 \in \mathbb{N}$ S.T $\frac{1}{a-y} < n_0$

$y < a-\frac{1}{n_0} \notin A \Rightarrow B^c \subseteq A^c$.

① $y \geq b \; \therefore \; \notin (a-\frac{1}{n}, b]$

$B^c \subseteq A^c$

As $B \subseteq A$, $B^c \subseteq A^c$ $A = B$

Proof of T6: $\forall a \in \mathbb{R}$ $a \cdot 0 = 0 \cdot a = 0$

W.K.T $a \cdot b = b \cdot a$ (Axiom 1)

Thus $a \cdot 0 = 0 \cdot a$

Thus To prove $a \cdot 0 = 0$.

(A4) $\exists \, 0 \in \mathbb{R}$ s.t $x + 0 = x$. $\forall x \in \mathbb{R}$

That is $0 + 0 = 0$.

$a \cdot 0 = a(0+0)$

(A3) $a \cdot 0 = a \cdot 0 + a \cdot 0$.

(A4) put $x = a \cdot 0$ in A_4.

$a \cdot 0 + 0 = a \cdot 0$.

$\Rightarrow a \cdot 0 + 0 = a \cdot 0 = a \cdot 0 + a \cdot 0$.

$\Rightarrow a \cdot 0 \cdot 0$. from (T_1)

Proof of T11: If $ab = 0$ then $a = 0$ or $b = 0$ (inclusive)

To prove $a = 0$ or $b = 0$.

Assume $a = 0$ then we are done.

Assume $a \neq 0$, from (A6) ($\forall x \neq 0$ in $\mathbb{R}$ $\exists \, y \in \mathbb{R}$ s.t $yx = 1$)

$\exists \, c \in \mathbb{R}$ s.t $ac = 1$.

Then $c(ab) = c \cdot 0$.

from A_3 $(ac)b = c \cdot 0$

$\therefore 1 \cdot b = 0$ (from T_6)

from A_4 $b = 0$

Proof of T6: Trichotomy laws:

let $a + x = b$ then $x = b - a$. $\forall a, b \in \mathbb{R}, x \in \mathbb{R}$.

$\mathbb{R}$, notation not $\mathbb{R}$

Proof: (T_2) possibility of substitution, $x = b - a$.

$-x = -(b-a) = a - b$.

→ If $x = 0$, $b - a = 0$ and $a - b = 0$. i.e $a = b$.

From A9) $0 \notin \mathbb{R}^+$

$\Rightarrow b - a$ = not posive, $a - b \neq +ve$

$a < b$ not hold, $b \nless a$ not hold.

→ If $x \neq 0$

A8) $\forall x \neq 0$ either $x \in \mathbb{R}^+$ or $(-x) \in \mathbb{R}^+$ but not both.

ie either $b - a \in \mathbb{R}^+$ or $-(b-a) \in \mathbb{R}^+$ but not both.

Hence $a < b$ or $a > b$ but not both.

T25: If $a < c$ and $b < d$ then $a + b < c + d$

Proof: $a < c \Rightarrow c - a \in \mathbb{R}^+$

$b < d \Rightarrow d - b \in \mathbb{R}^+$

To prove $(c+d) - (a+b) \in \mathbb{R}^+$

Axiom 7: $x \in \mathbb{R}^+, y \in \mathbb{R}^+ \Rightarrow x + y \in \mathbb{R}^+$

Thus $(c-a) + (d-b) \in \mathbb{R}^+$

(A1) $(c+d) + (-(a+b)) \in \mathbb{R}^+$

we have to prove $(-a) + (-b) = -(a+b)$.

By T_2 $\cancel{0 - a + 0 - b} = a + -a = 0$

$b + -b = 0$

$0 + 0 = 0$ (A6)

$a + (-a) + b + (-b) = 0$.

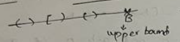
$(-a) + (-b) + (a+b) = 0$.

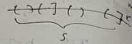
$(-a) + (-b) = -(a+b)$.

Thus $(c+d) - (a+b) \in \mathbb{R}^+$.

$a + b < c + d$.

28/8/24 Lecture-4





$S=[0,1)$ 1 = most optimal upper bound = least upper bound.

Def^n:- Suppose $\phi \neq S \subseteq \mathbb{R}$. A number $s \in \mathbb{R}$ is called a least upper bound (LUB or lub) or a supremum of S if

(a) s is an upper bound for S.

(b) No no. strictly less than s is an upper bound for S.

Notation :- $s = \sup S$.

Remark:- ① If S has a max element, then the max element is also a supremum of S, i.e, $\max S = \sup S$, if $\max S$ exists.

e.g:- $S=(0,1]$, then $1=\max S = \sup S$.

② Part (b) of above def^n can be restated as follows:- $\forall\ \varepsilon>0$, $s-\varepsilon$ is not an upper bound for S, which is equivalent to the following. Statement ★ $\forall\ \varepsilon>0$, $\exists\ x = x(\varepsilon) \in S$ (depends on ε) such that $x > s-\varepsilon$.

③ If $s = \sup S$ and B is any upper bound for S, then $s \leq B$ (s: least upper bound, B: any upper bound).

④ From ③, it follows that $\sup S$, if exists, is unique.

[Suppose, if possible, $s_1 = \sup S$ and $s_2 = \sup S$. Then by ③ $s_1 \leq s_2$ (least ub, ub) and also $s_2 \leq s_1$ (l.u.b, u.b).] Hence $s_1 = s_2$

Eg:- $S=[0,1)$ $\sup S = 1$.
(will check soon)

Some more properties of real numbers:

(A10) Every non empty set S of real numbers which is bdd above has a supremum, i.e, $\exists\ s \in \mathbb{R}$ s.t $s = \sup S$.

Remark:- (A10) ensures that $\mathbb{R}$ has "no gaps."
That's why (A10) is called the completeness axiom/ least upper bound axiom.

Exc:- Suppose $\phi \neq S \subseteq \mathbb{R}$. Define greatest lower bound (GLB or glb) or infimum of S: write down remarks ①-④ analogous to the ones for the supremum.

(T26) Every nonempty set S of real numbers that is bounded below has an infimum, i.e $\exists\ r \in \mathbb{R}$ s.t $r = \inf S$:= infimum of S.

Proof (sketch):- Consider $T = \{-x : x \in S\}$.

Exc:- s.t T is non empty and bdd above.

Exc:- Use A10 to get $t = \sup T \in \mathbb{R}$. show that $r := -t = \inf S$. ($r \in \mathbb{R}$.)

Qn: ~~why~~ How does A10 ensure $\mathbb{R}$ has no gap?

A:- eg, $\mathbb{Q}$ $\{x \in \mathbb{Q} : x > 0 \text{ and } x^2 < 2\}$

$\phi \neq S \subseteq \mathbb{Q}$ and S is bdd above in $\mathbb{Q}$, i.e $2 \in \mathbb{Q}$ and 2 is an upper bound for S. But

S does not have a supremem in $\mathbb{Q}$. This is bcz $\sup S = \sqrt{2} \notin \mathbb{Q}$.

eg:) let $\mathbb{R}-\{0\}$

$\phi \neq S = [-1,0) \subseteq \mathbb{R}-\{0\}$,

This is bounded above (by 0.01 say) in $\mathbb{R}-\{0\}$

But S has no supremum in $\mathbb{R}-\{0\}$ bcz $\sup S = 0 \notin \mathbb{R}-\{0\}$.

<u>Remarks</u>: ① properties (T1) - (T26) follow from (A1) - (A10). Hence (A1) to (A10) can be stated as axioms from which (T1) - (T26) follows as theorems.

(Apostol calculus Vol I)

② Axioms (A1) - (A10) unique define $\mathbb{R}$ as an "ordered field satisfying the completeness axiom".

<u>Properties of Sup and Inf</u>:-

<u>Thm</u>:- Suppose $\phi \neq S \subseteq \mathbb{R}$. Then $\delta = \sup S$ iff

(a) $x \leq \delta \ \forall x \in S$, and

(b) $\forall \varepsilon > 0, \exists x = x(\varepsilon) \in S$ S.T $x > \delta - \varepsilon$.

x dependi on ε.

<u>Proof</u>:- Use remark ② right after the def^n of sup.

eg: $S = [0,1)$ & $\sup S = 1$

<u>Proof</u>: To show: (a) $x \leq 1 \ \forall x \in [0,1)$. obvious.

(b) $\forall \varepsilon > 0, \exists x = x(\varepsilon) \in S = [0,1)$ S.T

$x > 1-\varepsilon$

$\varepsilon > 0 \Rightarrow \frac{\varepsilon}{2} > 0 \Rightarrow \varepsilon > \frac{\varepsilon}{2}$

$-\varepsilon < -\frac{\varepsilon}{2}$

$1-\varepsilon < 1-\frac{\varepsilon}{2}$.

$x(\varepsilon) = 1-\frac{\varepsilon}{2}$ if $\varepsilon < 1$.

$x = x(\varepsilon) := 1-\frac{\varepsilon}{2} > 1-\varepsilon$.

$\in S$

If $1-\frac{\varepsilon}{2} < 0$. then choose a smaller ε.

Also $1-\frac{\varepsilon}{2} < 1$.

If $\varepsilon \geq 1$, then $1-\varepsilon \leq 0$.

So any $x(\varepsilon) \in (0,1)$ will work.

eg:- $x(\varepsilon) = \frac{1}{2}$.

Finally $x(\varepsilon) := \begin{cases} 1-\frac{\varepsilon}{2} & \text{if } \varepsilon < 1 \\ \frac{1}{2} & \text{if } \varepsilon \geq 1 \end{cases}$

<u>Exc</u>:- $S = \{2-\frac{1}{n} : n \in \mathbb{N}\}$. find $\sup S$.

non empty & bdd above (by 12)

<u>Exc</u>:- Write down an analogous thm (as the last one) for inf and prove it. Use it to compute inf of (1) $S = (9,1]$ and (2) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$

<u>Exc</u>:- Find if the following sets are bdd above and below. Also find sup S and inf S whenever they exist. $\boxed{\forall a,b \in \mathbb{R}, \exists r \in \mathbb{Q} \text{ S.T with } a<b, \text{ S.T } (a<r<b)}$

(i) $S = (0,\infty)$ $\mathbb{Q}$ is dense in $\mathbb{R}$. (ii) $S = (-\infty,1)$. (iii) $S = \{x \in \mathbb{Q} : x^2 < 2\}$,

(iv) $S = (0,1)$. (v) $S = \mathbb{Q} \cap (7,\infty)$ (vi) $S = \mathbb{Q} \cap (-\infty,9)$

(vii) $S = \{12 - \frac{4}{n} : n \in \mathbb{N}\}$ (viii) $S = \mathbb{N} \cup \{0\}$,

(ix) $S = \{2^n : n \in \mathbb{N}\}$, (x) $S = \{2^{-n} : n \in \mathbb{N}\}$

Exc-7 $T_{19} \to T_{20}$

T_{19}) $\forall a,b,c \in \mathbb{R}$, where $c>0$ if $a<b$ then $ac<bc$

T_{20}) If $a \in \mathbb{R}-\{0\}$, then $a^2>0$.

Proof: To prove $x \in \mathbb{R}-\{0\}$ then $x^2>0$.

A8) $x \in \mathbb{R}^+$ or $-x \in \mathbb{R}^+$

Case 1) $x \in \mathbb{R}^+$

i.e $x>0$ i.e $\underset{a}{0}<\underset{b}{x}$

Take $c=x, a=0, b=x$

(T_{19}) Then $ac<bc$

$0<x^2$ (T_6)

$\Rightarrow x^2>0$

Case 2) $-x \in \mathbb{R}^+$

$-x>0 \Rightarrow 0<-x$

Take $c=-x$, $a=0$, $b=-x$

(T_{19}) Then $ac<bc$

(T_6) $0<-x \cdot -x$

$0<x^2$

$\Rightarrow x^2>0$

Exc-9 i) $S=\{2-\frac{1}{n} : n \in \mathbb{N}\}$

$S \geq [1,2)$

$n \geq 1$

$\frac{1}{n} \leq 1$

$\frac{-1}{n} \geq -1$

$2-\frac{1}{n} \geq 1$

$n>0$

$\frac{1}{n}>0$

$\frac{-1}{n} \leq 0$

$2-\frac{1}{n} \leq 2$

$\Rightarrow 1 \leq 2-\frac{1}{n} \not\leq 2$

i.e S is bdd below by 1

~~$2-\frac{1}{n}=1$~~ for $n=1$

$1=n$. $2-\frac{1}{n}=1 \in S$.

1 is the min of S.

i.e S is bdd above by 2.

~~To show 2 for $n=2$ $2-\frac{1}{n}=1.5 \notin S$~~

To prove max S exists we have to show an upper bdd belong to S.

Thus to prove max S D.N.E we have to show $\forall$ B an upper bd of S $b \notin S$.

Let B be an ub for S.

Then, $2-\frac{1}{n} \leq B$ $\forall n \in \mathbb{N}$.

To prove $B \notin S$

Suppose $B \in S$

then $B=2-\frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$.

$n_0+1>n_0$.

$\frac{1}{n_0+1} \not< \frac{1}{n_0}$

$\frac{-1}{n_0+1} > \frac{-1}{n_0}$

$2-\frac{1}{n_0+1} > 2-\frac{1}{n_0} = B$.

i.e $\exists x=2-\frac{1}{n_0+1} \in S$ s.t $B<x$, a contradiction

Hence $B \notin S$.

Hence S not having a max.

ii) $S=\{x \in \mathbb{Q} : x^2<2\}$.

$\Rightarrow (-\sqrt{2},\sqrt{2}) \cap \mathbb{Q}$

$\sqrt{2}$ is ub, $-\sqrt{2}$ is an lb for S.

max d.n.e: Suppose B is an ub for S.

the $x \leq B$ $\forall x \in S$

$\Rightarrow x^2 \leq B^2$

To prove $B \notin S$. (To prove $B^2 \geq 2$)

Suppose $B^2<2$.

$2-B^2>0$.

$-\sqrt{2}<B<\sqrt{2}$

$\sqrt{2}-B>0$ by AMP

~~$\frac{1}{2-B^2}>0$. By Arch $\exists n_0$ s.t $\frac{1}{2-B^2}<n_0$ $\Rightarrow \frac{1}{n_0}<2-B^2 \Rightarrow B^2<2-\frac{1}{n_0}$~~

$\frac{1}{\sqrt{2}-B} < n_0$ (by AMP)

$\frac{1}{n_0} < \sqrt{2}-B \Rightarrow B < \sqrt{2} - \frac{1}{n_0} < \sqrt{2}$

$\Rightarrow B + \frac{1}{n_0} < \sqrt{2}$.

$(B+\frac{1}{n_0})^2 < 2$.

So up d.N.E.

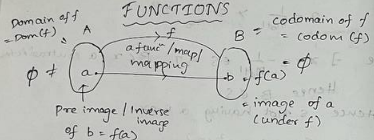
2/9/24 — Lecture-5

Exc:- Show that the following sets are bdd and find their sup and inf:

(i) $S_1 = \{37 + \frac{24}{n} : n \in \mathbb{N}\}$

(ii) $S_2 = \{\frac{1}{m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$

FUNCTIONS



$\therefore$ Range $(f) = \{b \in B : b = f(a)$ for som $a \in A\}$
$\subseteq$ codom $(f) = B$

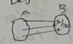
Def^n: ① $f: A \to B$ is called onto / Surjective if Range $(f) = B$.

② $f: A \to B$ is called 1–1 or injective if $a_1, a_2 \in A$, $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$.

This means $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

③ If $f: A \to B$ is one-to-one and onto then we say that f is a bijection func^n / bijective map.
f is also called one-to-one correspondence.

Eg:- ① Constant functions :- Fix $b_0 \in B$
Define $f: A \to B$ by $f(a) = b_0$ $\forall a \in A$.



This is constant function at b_0.
Notation :- $f \equiv b_0$

② Identity functions :- $f: A \to A$ defined by $f(a) = a$ $\forall a \in A$. is identity map on A.
Notation: $f = I_A = id_A$
It is always a bijection.

For the rest of examples, $A \subseteq \mathbb{R}$, $B = \mathbb{R}$.

③ Polynomial functions :- Fix $n \in \mathbb{N} \cup \{0\}$ and fix $a_0, a_1, \ldots a_n \in \mathbb{R}$ with $a_n \neq 0$. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = a_0 + a_1 x + \cdots a_n x^n$, $x \in \mathbb{R}$.
$n :=$ degree of f = deg (f)
If $n = 0$, then we get $f \equiv a_0$ (Const func^n)
If $n = 1$, $a_0 = 0$ and $a_1 = 1$ then $f = I_{\mathbb{R}}$.

Exc:- If deg $(f) = 1$ (So $a_1 \neq 0$), then S.T $f: \mathbb{R} \to \mathbb{R}$ is a bijection.

④ Trignometric Functions :- $f_i : A_i (\subseteq \mathbb{R}) \to \mathbb{R}$, $i = 1, 2, \ldots 6$. ("Suitably chosen")

$f_1(x) = \sin x$, $f_4(x) = \operatorname{cosec} x$
$f_2(x) = \cos x$, $f_5(x) = \sec x$
$f_3(x) = \tan x$, $f_6(x) = \cot x$.

Exc :- ① Find domains of $f_1, f_2, \cdots f_6$.
② Draw the graphs of $f_1, f_2, \cdots f_6$.

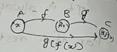
⑤ Exponential Functions :- Define $f: \mathbb{R} \to \mathbb{R}$ by
$f(x) = \exp\{x\} = e^x := 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$, $x \in \mathbb{R}$

It can be shown that the above series converges $\forall\, x \in \mathbb{R}$ and Range $(f) = (0, \infty)$.

⑥ Logarithmic function :- Define $g: (0,\infty) \to \mathbb{R}$ by
$g(y) = \log_e y = \ln y$, $y \in (0, \infty)$

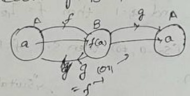
Exc :- Draw the graphs of f in ⑤ and g in ⑥

Def^n :- Suppose $f: A \to B$ and $g: B \to C$ are two maps so that



Range $(f) \subseteq B = \mathrm{dom}\,(g)$. Then the composition map $g\circ f : A \to C$ is defined as
$$g \circ f(x) = g(f(x)) \;\forall\, x \in A.$$

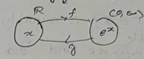
Thm :- If $f: A \to B$ is a bijection, then $\exists$ a unique function $g: B \to A$ ST $g \circ f = I_A$, $f \circ g = I_B$.



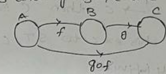
Def^n :- In the above setup, g is called the inverse map of f and is denoted by f^{-1}.

Remarks :
① f^{-1} is also a bijection
② $f = g^{-1}$, i.e., $(f^{-1})^{-1} = f$.

Eg :- $A = \mathbb{R}$, $B = (0,\infty)$. $f(x) = e^x$, $x \in \mathbb{R}$
$g(y) := \log_e y$, $y \in (0,\infty)$, Then $g = f^{-1}$; $f = g^{-1}$.



Exc :- If $f: A \to B$ and $g: B \to C$ are bijections then show that $g \circ f : A \to C$ is also a bijection.



COUNTABLE AND UNCOUNTABLE SETS

Def^n :- ① A set C is called countably (ctbly) infinite if $\exists$ a bijection $f: \mathbb{N} \to C$. This means that all the elements of C can be listed / enumerated as follows :
$$C = \{ f(1), f(2), \cdots \}.$$

② A set is called countable if it is either finite or countably infinite.

Eg :- ⊖1 Any finite set is countable (ctble)
⓪ $\mathbb{N}$ is ctbly infinite. $(f = 1_{\mathbb{N}})$ $f: \mathbb{N} \to \mathbb{N}$, $f(n) = n\ \forall n$
and hence countable.

① $\mathbb{Z}$ is ctbly infinite.
$$\left[\begin{array}{l} \mathbb{N} = \{1, 2, 3, 4, \cdots \\ \mathbb{Z} = \{0, +1, -1, +2, -2, \cdots\} \end{array} \right]$$
i.e , $f: \mathbb{N} \to \mathbb{Z}$ is defined as

$$f(i) = \begin{cases} 0 & \text{if } i=1 \\ \frac{i}{2} & i=2,4,6,8,\dots \\ -\frac{i-1}{2} & i=3,5,7,9,\dots \end{cases}$$

Exc: Check that $f: \mathbb{N} \to \mathbb{Z}$ defined above is indeed a bijection. Therefore $\mathbb{Z}$ is ctbly infinite

eg: ② $C = \{7K+3 : K \in \mathbb{Z}\}$ is countbly infinte.

Exc: If C is ctbly infinite (ctble) and $\exists$ a bijection $g: C' \to C$, then C is also ctbly infinite (ctble).

Use this exc to S.T $C = \{7K+3 : K \in \mathbb{Z}\}$ is ctbly infinite.

Def^n: A set U is called uncountable if it is not ctble. i.e if U is neither finite nor ctbly infinite.

Properties:- ① Any subset of a ctble set is ctble. Therefore any superset of an unctble set is unctble.

② If C is ctbly infinite, then the set $P(C)$ (Power set of C) of all subsets of C is unctble.

③ $(0,1)$ is unctble $\Rightarrow \mathbb{R}$ is also unctble.

④ $\mathbb{N} \times \mathbb{N} = \{(i,j) : i \in \mathbb{N}, j \in \mathbb{N}\}$ is countably infinite.

Sketch of Proof:- $(1,1)\ (1,2)\ (1,3)\ (1,4)\ \dots$

$(2,1)\ (2,2)\ \dots$

⑤ If $A_1, A_2, \dots$ are ctble, then so is $\bigcup_{i \in \mathbb{N}} A_i$. (ctble union of ctble sets is ctble)

⑥ $\mathbb{Q}$ is ctbly infinite. (and yet dense).

Exc-9) (i) $S = \{2 - \frac{1}{n} : n \in \mathbb{N}\}$

max S DNE

min S = 1.

Sup S, inf S

inf S = 1 sup S = 2

Let B be an upper bd for S

To prove $2 \le B$.

suppose $2 > B$.

$\Rightarrow \frac{1}{2-B} > 0$. So by AMP. $\exists n_0 \in \mathbb{N}$ s.t $\frac{1}{2-B} < n_0$

$\Rightarrow B < 2 - \frac{1}{n_0}$, a $\times$ n. to the fact that B is an upper bd.

Thus $2 \le B$ i.e $\sup S = 2$

(ii) $S = \{x \in \mathbb{Q} : x^2 < 2\}$

1. $\mathbb{Q}$ is dense in $\mathbb{R}$

(X is dense in Y $\forall$ open set in Y, $\exists x \in X$ s.t $x \in U$)

Pf: Let $a, b \in \mathbb{R}$ s.t $a < b$

$\frac{1}{b-a} > 0$. Then by AMP

$\frac{1}{b-a} < n_0 \Rightarrow 1 < bn_0 - an_0$

$\exists n \in \mathbb{N}$ s.t $an_0 < m < bn_0$ $\quad \frac{m}{n_0} = r \in \mathbb{Q}$

$\Rightarrow a < \frac{m}{n_0} < b$

(ii) $S = \{x \in \mathbb{Q} : x^2 < 2\}$

$\Rightarrow (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$

Let $B = \max S$ (if possible)

then B = upper bd & $B \in S$

$x \leq B \quad \forall x \in S$

and $B^2 < 2$.

$-\sqrt{2} < B < \sqrt{2}$

By dense thm, $\exists q \in \mathbb{Q}$ s.t $B < q < \sqrt{2}$

$\Rightarrow -\sqrt{2} < B < q < \sqrt{2} \Rightarrow -\sqrt{2} < q < \sqrt{2}$

$\Rightarrow q^2 < 2 \quad \Rightarrow q \in S$

But $B < q$, a $\not\equiv$n to fact that B is an upp bd of S. Hence max S DNE.

min S DNE.

Let $B = \min S$ (if possible)

$b \leq x \quad \forall x \in S$ and $b \in S \Rightarrow b^2 < 2$

$\Rightarrow -\sqrt{2} < b < \sqrt{2}$

By density thm for $\mathbb{Q}$, $\exists q \in \mathbb{Q}$ s.t

$-\sqrt{2} < q < b < \sqrt{2} \Rightarrow q^2 < 2 \Rightarrow q \in S$,

a$\not\equiv$n to that fact that 'q' is a lower bound.

To Find Sup S

Pf: $\sup S = \sqrt{2}$ Let $\varepsilon > 0$

case ①: $\sqrt{2} - \varepsilon > 0$.

$-\sqrt{2} < 0 < \sqrt{2} - \varepsilon < \sqrt{2}$.

By density thm $\exists q \in \mathbb{Q}$ s.t

$-\sqrt{2} < \sqrt{2} - \varepsilon < q < \sqrt{2}$

$\Rightarrow q^2 < 2 \Rightarrow q \in S$

and $\sqrt{2} - \varepsilon < q$

$\in S$

$-\sqrt{2} \quad \sqrt{2}-\varepsilon \quad x \quad \sqrt{2} = \sup S$

case ②: $\sqrt{2} - \varepsilon \leq 0$.

$q = 1 \quad q^2 < 2 \Rightarrow q \in S$

$\sqrt{2} - q \leq 0 \leq 1$

($\therefore$ To prove $a = \sup S$, try to prove $\forall \varepsilon > 0 \; \exists x \in S$ s.t $a - \varepsilon < x$.)

$\inf S = -\sqrt{2}$

HW

Q) $\phi \neq A \subseteq B \subseteq \mathbb{R}$

(i) if B bdd below, then so is A and $\inf B \leq \inf A$.

(ii) if B is bdd above, then so is A and $\sup A \leq \sup B$.

Solⁿ: (ii) $\forall b \in B$, $b \leq M$, M is an upper bd of B

$\forall a \in A$, $A \subseteq B \Rightarrow a \in B$

$\Rightarrow a \leq M$

Since $\sup A$ is least among all upper bd of A $\quad \sup A \leq \sup B$

(i) Any lower bd of B is a l bd of A.

$\Rightarrow m \leq a \quad \forall a \in A$

$\Rightarrow \inf B \leq a \quad \forall a \in A$

$\Rightarrow \inf B \leq \inf A$.

$\phi \neq S \subseteq R$ $\qquad$ $\phi \neq T \subseteq R$

S.T $\quad s \leq t \quad \forall\, s \in S, \forall\, t \in T$

then, $\sup S \leq \inf T$

Fix $t \in T$, then t is an upper bd of S

$\Rightarrow \sup S \leq t$

i.e $\forall\, t \in T$, $\sup S \leq t$

i.e $\sup S$ is a lower bd of T.

$\Rightarrow \inf T$ exists as $\sup S \leq \inf(T)$.

Any lower bound of B is lower bd of $\underline{A}$

$\Rightarrow m \leq a \quad \forall\, a \in A$

$\Rightarrow \inf(B) \leq a \quad \forall\, a \in \underline{A}$

$\Rightarrow \inf(B) \leq \inf(A)$

4/9/24 $\qquad$ **Lecture-6**

Thm:- Principle of Mathematical Induction.

Let $S \subseteq \mathbb{N}$ S.T (a) $1 \in S$ and

(b) $k \in S \Rightarrow k+1 \in S$. Then $S = \mathbb{N}$

Thm:- well-ordering principle

If $\phi \neq S \subseteq \mathbb{N}$, then S has a min element.

Exc:- (i) S.T for a 2x2 table of $\mathbb{R}$ the sum of row sums = Sum of column sums.

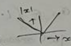
a	b	a+b
c	d	c+d
a+c	b+d	a+b+c+d

*(ii) Show the same for any $m \times n$ table of real numbers where $m, n \in \mathbb{N}$, $m, n \geq 2$.

***Exc:-** S.T (well ordering principle) $S \subseteq \mathbb{N}$ is ctble.

Absolute Value / Modulus.

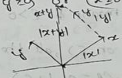
For $x \in \mathbb{R}$, $|x| = \begin{cases} +x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$



Properties:- (1) If $a \geq 0$, then $|x| \leq a$ iff $-a \leq x \leq a$ i.e, $x \in [-a, a]$.

(2) **Triangle inequality** (TI) :- $\forall\, x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$

Moreover equality holds in TI iff either ($x \geq 0$ and $y \geq 0$) or ($x \leq 0$ and $y \leq 0$).



$|x+y| \leq |x| + |y|$

Sequences of Real Numbers and Their Limits:-

Def^n:- A fun^n $f: \mathbb{N} \to \mathbb{R}$ is called an (infinite) sequence of $\mathbb{R}$, For each $n \in \mathbb{N}$, $f(n)$ = the n^{th} term of the sequence. We typically a_n, b_n, etc instead of $f(n)$.

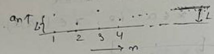
A sequence will be written as $a_1, a_2, a_3, \ldots$ (*) where $a_n = f(n)$. $\forall\, n \in \mathbb{N}$.

Notations:- (*) will be denoted by $\{a_n\}_{n \in \mathbb{N}}$ or $\{a_n\}_{n \geq 1}$ or $\{a_n\}$

Eg:- $a_n = (-1)^n$, $n \in \mathbb{N}$ would mean $-1, +1, -1, +1, \ldots$

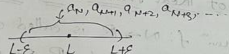
Def^n: A seq $\{a_n\}$ of $\mathbb{R}$ is said to converge to $L \in \mathbb{R}$ if $\forall\, \varepsilon > 0$, $\exists\, N = N(\varepsilon) \in \mathbb{N}$ s.t
$|a_n - L| < \varepsilon \;\forall\, n \geq N$, i.e
$a_n \in (L-\varepsilon, L+\varepsilon) \;\forall\, n \geq N$.
(This just means that a_n becomes closer and closer to L as n becomes larger and larger.)

L can be



~~Remark:- a_n.~~
Notation:- "$\lim\limits_{n\to\infty} a_n = L$" or "$\lim\limits_{n\to\infty} a_n = L$" or "$a_n \to L$ as $n \to \infty$" or "$a_n \to L$"

Remark: $a_n \to L\ (\in \mathbb{R})$ (converges to) means $\forall\, \varepsilon > 0$ after some stage (that depends on ε), all the terms of seq a_n comes within $(L-\varepsilon, L+\varepsilon)$



Eg:- (0) **Constant Seq:-** $a_n = c \;\forall\, n \in \mathbb{N}$ for some $c \in \mathbb{R}$ (Notation:- $\{a_n\} \equiv c$ or $a_n \equiv c$)
$\Rightarrow a_n \to c$ as $n \to \infty$

(1) $a_n := \frac{1}{n} \;\forall\, n \in \mathbb{N}$ $\quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
Claim: $a_n = \frac{1}{n} \to 0 = L$ as $n \to \infty$

Proof:- Fix an $\varepsilon > 0$. To show $\exists\, N = N(\varepsilon) \in \mathbb{N}$ s.t $|a_n - L| = |\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon \;\forall\, n \geq N$,
i.e to show $n\varepsilon > 1 \;\forall\, n \geq N$.
Use the Arch prop to get $\boxed{N = N(\varepsilon) \in \mathbb{N} \text{ s.t } N\varepsilon > 1}$ $\quad N > \frac{1}{\varepsilon}$
Then $\forall\, n \geq N$, $n\varepsilon \geq N\varepsilon > 1$
$\Rightarrow n\varepsilon > 1 \;\forall\, n \geq N$.
$\Rightarrow |\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon \Rightarrow a_n = \frac{1}{n} \to 0$ as $n \to \infty$

$N(\varepsilon)$ = smallest +ve integer $> \frac{1}{\varepsilon}$
$= [\frac{1}{\varepsilon}] + 1 \longleftarrow [\frac{1}{\varepsilon}]$ = Greatest integer $\leq \frac{1}{\varepsilon}$

$\varepsilon = 0.01 \Rightarrow N(0.01) = [\frac{1}{0.01}] + 1 = 101$
$\varepsilon = 0.0001 \Rightarrow N(0.0001) = [\frac{1}{0.0001}] + 1 = 10001$
$\Rightarrow n \geq 10001 \Rightarrow |\frac{1}{n} - 0| < 0.0001$

Exc: (i) $a_n = 2 - \frac{1}{n} \to 2$
(ii) $a_n = 2 + (-1)^n \frac{1}{n} \to 2.$ $\Big\}$ Prove them from def^n

Def^n:- If $\exists\, L \in \mathbb{R}$ s.t $a_n \to L$, then we say that $\{a_n\}$ converges / $\{a_n\}$ is convergent (in $\mathbb{R}$).
L is Limit of $\{a_n\}$.

Q) will Lim of seq always exist $\Rightarrow$ No (Example)
Q) will Lim if exists, be unique $\Rightarrow$ Yes (Proof) $\to$ Triangular inequality

Def^n:- If $\nexists\, L \in \mathbb{R}$ s.t $a_n \to L$, then we say $\{a_n\}$ does not converge / $\{a_n\}$ is not a convergent seq.

Exc:- S.T $a_n = (-1)^n$, $n \in \mathbb{N}$ does not converge.

Thm:- Limit, if exists, is unique:-

Proof:-

Proof: Suppose $a_n \to L$ and $a_n \to L'$

Shall show: $L = L'$. To this end, fix $\varepsilon > 0$

$a_n \to L \Rightarrow \exists N = N(\varepsilon) \in \mathbb{N}$ s.t $|a_n - L| < \varepsilon$ $\forall n \geq N$.

$a_n \to L' \Rightarrow \exists N' = N'(\varepsilon) \in \mathbb{N}$ s.t $|a_n - L'| < \varepsilon$ $\forall n \geq N'$

Take $M = \max\{N, N'\}$ $(\in \mathbb{N}) \geq N, N'$

$\Rightarrow |a_M - L| < \varepsilon$ and $|a_M - L'| < \varepsilon$

Therefore $|L - L'| = |(a_M - L) - (a_M - L')|$

$\leq |a_M - L| + |a_M - L'|$ (Δ inequality)

$< \varepsilon + \varepsilon < 2\varepsilon$

$\Rightarrow |L - L'| < 2\varepsilon$ $\forall$ arbitary $\varepsilon > 0$

Since $\varepsilon > 0$ is arbitary, it follows that $L = L'$.

$2\varepsilon_0 = \frac{|L-L'|}{2}$, 0, $|L-L'|$

[Suppose if possible $|L - L'| > 0$, then take $\varepsilon_0 = \frac{|L-L'|}{4}$

i.e $2\varepsilon_0 = \frac{|L-L'|}{2} < |L - L'|$.

Which is contradiction.]

* Countable union of countable sets is Countable.

$\bigcup_{i=1}^{\infty} A_n = A$

A_n is countable then A is countable.

$\mathbb{Q}$ is countable.

$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$

Define $\forall n \in \mathbb{N}$, the set $A_n := \{\frac{m}{n} : m \in \mathbb{Z}\}$

Then $\bigcup_{n=1}^{\infty} A_n = \mathbb{Q}$

Pf:- Let $\frac{m}{n} \in \mathbb{Q}$. Then $n \neq 0$ and $m \in \mathbb{Z}$

$\Rightarrow \frac{m}{n} \in A_n$

$\Rightarrow \frac{m}{n} \in \bigcup_{n=1}^{\infty} A_n$

Converse suppose, $a \in \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow a \in A_n$ for some $n \in \mathbb{N}$

$\Rightarrow a = \frac{m}{n}$, $m \in \mathbb{Z}$

$\Rightarrow a \in \mathbb{Q}$

$\forall n \in \mathbb{N}$ A_n = countable

$f : \mathbb{N} \to A_n$ Rrd

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \frac{0}{n} & \frac{1}{n} & -\frac{1}{n} & \frac{2}{n} & -\frac{2}{n} & \frac{3}{n} \end{array}$$

$$f(m) = \begin{cases} 0 & : m = 0 \\ \frac{m}{2n} & : m = \text{even} \\ -\frac{m-1}{2n} & : m = \text{odd}, \end{cases}$$

explicit representation of bijection.

Prove this is a bijection

Q) $(-1)^n = a_n$ does not converge.

Pf:- Suppose $a_n \to a$

case 1: $a = 1$

$|(-1)^i - a| > 0$

Choose $\varepsilon = \frac{|(-1)-a|}{2} > 0$

$\exists N \in \mathbb{N}$ s.t $|a_n - a| < \varepsilon$ $\forall n \geq N$.

In particular, $|a_{2N+1} - a| < \frac{|-1-a|}{2}$

$|-1-a| < \frac{|-1-a|}{2}$, $a \# n$.

case 2: $a \neq 1$

$\varepsilon = \frac{|1-a|}{2} > 0$.

Since $a_n \to a$, $\exists N \in \mathbb{N}$ s.t $\forall n \geq N$, $|a_n - a| < \varepsilon = \frac{|1-a|}{2}$

in part, $|a_{2N} - a| < \varepsilon$

$\Rightarrow |1-a| < \frac{|1-a|}{2}$,

$a \# n$.

Thm: Suppose $a_n \to a$ in $\mathbb{R}$, $b_n \to b$ in $\mathbb{R}$, then,

(i) $a_n + b_n \to a+b$

(ii) $a_n - b_n \to a-b$

(iii) $a_n b_n \to ab$

(iv) If $b_n \neq 0$, $b \neq 0$ then $\frac{a_n}{b_n} \to \frac{a}{b}$

Q) $\underset{n\to\infty}{Lt} \frac{n^2+27n-275}{n^2}$

$$c_n = \frac{n^2+27n-275}{n^2} = \underset{a_n}{1} + \underset{b_n}{\frac{27}{n}} - \underset{d_n}{\frac{275}{n^2}}$$

$Lt\ a_n = 1$

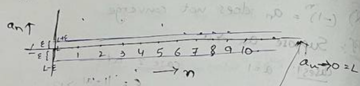
$Lt\ b_n = 0$ $\Rightarrow Lt\ c_n = 1$

$Lt\ d_n = 0$

9/9/24

Lecture 7

Def^n: $a_n \to L$ mean $\forall \varepsilon > 0$, $\exists N = N(\varepsilon) \in \mathbb{N}$, $a_n \in (L-\varepsilon, L+\varepsilon)$ $\forall n \geq N$



After some stage, the seq entirely comes within this band.

Thm: Convergent sequences are bdd, i.e. the Set $\{a_n : n \in \mathbb{N}\}$ is bdd above and below.

Proof:

Corollary: The unbounded seq does not converge.

e.g:- $a_n = n$ $\forall n \in \mathbb{N}$

Proof of Thm:-

Suppose $a_n \to L \in \mathbb{R}$

Take $\varepsilon = 1$. Then $\exists N \in \mathbb{N}$ ST $|a_n - L| < 1$ $\forall n \geq N$,

which implies $|a_n| = |(a_n - L) + L| \leq |a_n - L| + |L|$
$\leq 1 + |L|$ $\forall n \geq N$

therefore, if we take $M = \max\{|a_1|, |a_2|, \dots |a_{N-1}|, 1+|L|\}$,

then $|a_n| \leq M$ $\forall n \in \mathbb{N}$

$-M \leq a_n < M$ $\forall n \in \mathbb{N}$.

$\Rightarrow \{a_n\}_{n \geq 1}$ is bdd.

For sequences

$$\boxed{\text{bdd}} \;\overset{\longrightarrow}{\underset{\not\longleftarrow}{}}\; \boxed{\text{Conv}}$$

ex: $a_n = \sin n$, $n \in \mathbb{N}$

$a_n = (-1)^n$, $n \in \mathbb{N}$

Thm: Suppose $a_n \to a \in \mathbb{R}$ and $b_n \to b \in \mathbb{R}$. Then we have

(i) $a_n + b_n \to a+b$

(ii) $a_n - b_n \to a-b$

(iii) $a_n b_n \to ab$

(iv) $\frac{a_n}{b_n} \to \frac{a}{b}$ if $b_n \neq 0$. $\forall n \in \mathbb{N}$ and $b \neq 0$

Steps of proof: ① Prove (i) using Δ^{ular} ineq.

② prove $b_n \to b \Rightarrow -b_n \to -b$ from def^n

③ ① & ② $\Rightarrow$ (ii) ④ $a_n \to a \Rightarrow a_n^2 \to a^2$

$[|a_n^2 - a^2| = |a_n + a||a_n - a|]$

bdd (small for large n) handle

⑤ $a_n \cdot b_n = \frac{(a_n+b_n)^2 - (a_n-b_n)^2}{4}$

$\to \frac{(a+b)^2 - (a-b)^2}{4} \to ab.$

⑥ $b_n \neq 0 \ \forall n$, $b_n \to b \neq 0 \Rightarrow \frac{1}{b_n} \to \frac{1}{b}$.

⑦ use ⑥+⑤ to get part (ii).

Defs:-

① A seq $\{a_n\}_{n \geq 1}$ is called increasing/non decreasing if $a_1 \leq a_2 \leq a_3 \leq \cdots$

② $\{a_n\}$ is called decreasing / non incr. if

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots$$

③ $\{a_n\}$ is called monotone if it is either ~~first two~~ $\uparrow$ or $\downarrow$ing.

inclusive (const seq)

Thm:- A bounded and monotone sequence converges. In fact ① an $\uparrow$ing seq that is bdd above is convergent.

② a $\downarrow$ing seq i.e bdd below is convergent.

Exc: solve ① suppose $\{A_n\}$ is $\uparrow$ing and bdd above, i.e; the set $S = \{a_n : n \in \mathbb{N}\}$ is bdd above. Then S.T $a_n \to \delta := \sup S$

Notation: In this situation, we write

$\boxed{a_n \uparrow \delta}$ or $\boxed{a_n \nearrow \delta}$

Similarly $\boxed{b_n \downarrow \gamma}$ or $\boxed{b_n \searrow \gamma}$ means $b_1 \geq b_2 \geq b_3 \ldots$ and $b_n \to \gamma$ as $n \to \infty$.

Exc: Suppose $\{b_n\}$ is decreasing and bdd below i.e, $T = \{b_n : n \in \mathbb{N}\}$ is bdd below. Then S.T b_n decrease to $\gamma := \inf T$.

$b_n \downarrow \gamma := \inf T$.

Eg:- ① $\frac{1}{n} \searrow 0 = \inf\{\frac{1}{n} : n \in \mathbb{N}\}$.

② $2 - \frac{1}{n} \nearrow 2 = \sup\{2 - \frac{1}{n} : n \in \mathbb{N}\}$.

Thm:- If $a_n \to a \in \mathbb{R}$, then $\frac{a_1 + a_2 + \cdots a_n}{n} \to a$

Cor:- $\frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}}{n} \to 0$. as $n \to \infty$.

Thm:- (Sandwich Theorem for Limits):
Squeeze

If $\{a_n\}, \{b_n\}, \{c_n\}$ are three sequences of real numbers S.T $a_n \leq b_n \leq c_n \ \forall n \in \mathbb{N}$, and $a_n \to L \in \mathbb{R}$, $c_n \to L \in \mathbb{R}$, then $b_n \to L$.

Cor:- $\frac{1}{(n!)^{1/n}} \to 0$ as $n \to \infty$.

Proof:- By A.M $\geq$ G.M, we get

$$\underset{a_n}{0} \leq \underset{b_n}{\frac{1}{(n!)^{1/n}}} = \left(\frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n}\right)^{1/n} \leq \underset{c_n}{\frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}}{n}}$$

$a_n \leq b_n \leq c_n \ \forall \ n \geq 1$.

Also $a_n \to 0$, $c_n \to 0$ (cor).

$\overset{\text{by}}{\Rightarrow}$ Sandwich Thm $b_n \to 0$ $\Rightarrow \frac{1}{(n!)^{1/n}} \to 0$..

Exc:- Use Sandwich Thm to prove that

$$a_n := \frac{\sin n}{n} \to 0 \text{ as } n \to \infty.$$

More generally, S.T if $a_n = b_n c_n \ \forall \ n \in \mathbb{N}$, where $\{b_n\}$ is bdd seq and $c_n \to 0$ then $a_n \to 0$.

Thm:- (a) $\frac{1}{n^{\alpha}} \to 0$ for each $\alpha \in (0,\infty)$.

(b) $x^n \to 0$ for each $x \in (-1,1)$.

(c) $\frac{(\log n)^a}{(n)^b} \to 0$ for each $a,b \in (0,\infty)$

(d) $n^{1/n} \to 1$

(e) $(1+\frac{a}{n})^n \to e^a$ for each $a \in \mathbb{R}$.

Exc:- Compute the limit of following sequences

(i) $a_n = \frac{2n}{n^2+1} + (-1)^n\left(\frac{9}{10}\right)^n + \left(1+\frac{7}{n}\right)^n\left(\frac{n+27}{n+2}\right)$

(ii) $a_n = \frac{n}{n+1} + \frac{(\log n)^{2024}}{\sqrt{n}}$; (iii) $a_n = \frac{n^{2/3}\sin(n!)}{n+1}$

Observation:- Suppose $a_n \to L$. Then $\forall\ \varepsilon > 0$, $\exists\ N = N(\frac{\varepsilon}{2}) \in \mathbb{N}$ S.T $|a_n - L| < \frac{\varepsilon}{2}\ \forall\ n \geq N$.

This means $\forall\ m,n \geq N$, $|a_m - a_n| = |(a_m - L) - (a_n - L)|$
$\leq |a_m - L| + |a_n - L|$ [By Δular inequality]
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$.
$\Rightarrow |a_m - a_n| < \varepsilon$

Def^n:- A seq $\{a_n\}$ is called cauchy if $\forall\ \varepsilon > 0$ $\exists\ N = N(\varepsilon) \in \mathbb{N}$ S.T $|a_m - a_n| < \varepsilon\ \forall\ m,n \geq N$.

(cauchy means after some stage, any two terms seq become very close to each other.)

- S.T cauchy seq are bdd.

We have shown:-

Thm: If $\{a_n\}$ is convergent, then it is cauchy.

$$\begin{tikzcd} \boxed{\text{Conv}} \arrow[rr, shift left] \arrow[dr] & & \boxed{\text{Cauchy}} \arrow[ll, shift left, "\text{Thm below}"] \arrow[dl] \\ & \boxed{\text{Bdd}} \arrow[ul, "\times", "a_n=(-1)^n"'] \arrow[ur, "\times"', "a_n=(-1)^n"] & \end{tikzcd}$$

Exc:- S.T $a_n = (-1)^n$ is not cauchy.

Thm: If $\{a_n\}_{n\geq 1}$ is cauchy seq of $\mathbb{R}$, then $\{a_n\}$ is convergent. i.e, $\exists\ L \in \mathbb{R}$ S.T $a_n \to L$.
(Completeness prop of $\mathbb{R}$)

Exc:- S.T $\mathbb{R}-\{0\}$ is not complete, i.e $\exists$ a cauchy seq $\{x_n\}_{n\geq 1}$ S.T each $x_n \in \mathbb{R}-\{0\}$ but x_n doesn not converge to any element of $\mathbb{R}-\{0\}$.

Q. Let $\{a_n\}$ is an increasing sequence and bdd above. Prove that $a_n \to \underset{S}{\underset{\shortparallel}{\sup\{a_n : n \in \mathbb{N}\}}}$.

Sol^n:- $S = \sup\{a_n : n \in \mathbb{N}\}$ exists

Let $\varepsilon > 0$. Then $\exists\ N \in \mathbb{N}$ S.T $S - \varepsilon < a_N \leq a_n$ $\forall\ n > N$

$S - a_n < \varepsilon\ \forall\ n \geq N$.

$|a_n - S| = S - a_n < \varepsilon\ \forall\ n \geq N$

$\Rightarrow a_n \to S$.

Q. Use Sandwich theorem to prove $\frac{\sin n}{n} \to 0$.

Pf:- Let $\{b_n\}$ be bdd and $C_n \to 0$. Then $a_n := b_n C_n \to 0$

- $C_n \to 0 \Leftrightarrow |C_n| \to 0$

Proof: Let $C_n \to 0$. Let $\varepsilon > 0$ then $\exists\ N \in \mathbb{N}$ S.T

$|c_n - 0| < \varepsilon \quad \forall n \geq N$

$|c_n| < \varepsilon$

$\downarrow$

$||c_n| - 0| < \varepsilon \Rightarrow |c_n| \to 0$

Conversly suppose $|c_n| \to 0$

To prove $c_n \to 0$.

let $\varepsilon > 0$, since $|c_n| \to 0$ $\exists N \in \mathbb{N}$ s.t

$||c_n| - 0| < \varepsilon \quad \forall n \geq N$

$\Downarrow$

$|c_n| \Rightarrow |c_n - 0|$

$\Rightarrow c_n \to 0$ Proof of ②

$\exists m, B$ s.t $m \leq b_n \leq B \ \forall n \in \mathbb{N}$

$M = \max\{B, -m\}$

$B \leq M$

$-m \leq M$

$-M \leq m$

$-M \leq m \leq b_n \leq B \leq M$

$\Rightarrow |b_n| \leq M$

$\Rightarrow |b_n||c_n| \leq |c_n| M$

$0 \leq |a_n| \leq |c_n| M$

$M|c_n| \to 0$

$= M|c_n| < \varepsilon$

$|c_n| < \frac{\varepsilon}{M}$

(observe $c_n \to 0 \Rightarrow |c_n| \to 0$
$M|c_n| \to 0$)

Sandwichthm! $\Rightarrow |a_n| \to 0$

$\Rightarrow a_n \to 0$ //

$\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$

Q) $a_n = \frac{2n}{n^2+1} + (-1)^n \left(\frac{9}{10}\right)^n + \left(1+\frac{7}{n}\right)^n \left(\frac{n+27}{n+2}\right)$

A) $\frac{2}{n+\frac{1}{n}} \to 0$ $\qquad$ $(-1)^n \to$ bdd, $\frac{9}{10} \to 0$, $\left(\frac{9}{10}\right)^n \to 0$ $\qquad$ $\left(1+\frac{7}{n}\right)^n \to e^7$ $\qquad$ $\frac{1+\frac{27}{n}}{1+\frac{2}{n}} \to 1$

$\Rightarrow e^7$.

Cauchy Sequence :-

In $\mathbb{R}$ any cauchy sequence will be converged.

a, a_1, a_2, a_3

If $\mathbb{R} - \{0\}$ $\qquad$ 0, 0.25, 0.5, 1 $\rightarrow$ Not convergent.

Q) Cauchy $\Rightarrow$ bdd

Suppose $\{x_n\}$ is cauchy. Then corresponding to $\varepsilon = 1$, $\exists N \in \mathbb{N}$ s.t $|x_n - x_m| < 1$

$\forall m > n \geq N$

In part, for $n = N \Rightarrow |x_N - x_m| < 1 \quad \forall m \geq N$.

$|x_m| = |x_N - x_m - x_N| \leq |x_N - x_m| + |x_N|$

$\leq 1 + |x_N|$

$M = \max\{|x_1|, |x_2|, |x_3|, \ldots |x_m|, 1 + |x_N|\}$

$|x_m| \leq 1 + |x_N|$

$\Rightarrow |x_m| \leq M \ \forall m \in \mathbb{N}$

11/9/24 **Lecture-8**

$\left.\begin{array}{l} a_n \to a,\ b_n \to b \\ a, b \in \mathbb{R} \end{array}\right\} \Rightarrow \begin{array}{l} a_n \pm b_n \to a \pm b \\ a_n b_n \to ab \\ a_n/b_n \longrightarrow a/b \end{array}$ if all $b_n \neq 0$ and $b \neq 0$

Exc: Suppose f is a polynomial function. Then S.T $a_n \to a \Rightarrow f(a_n) \to f(a)$.

Remark: $\boxed{b_n = n^{1/n} \quad \log b_n = \frac{\log n}{n} \to 0 \quad \text{expfun}^n \nearrow \Rightarrow b_n \to 1}$

In fact, for any continous function $f: \mathbb{R} \to \mathbb{R}$, $a_n \to a \Rightarrow f(a_n) \longrightarrow f(a)$

* $\exists L \in \mathbb{R}$ S.T $a_n \to L$ $\boxed{\text{Conv}} \underset{(-1)^n}{\overset{\longrightarrow}{\not\longleftarrow}}$ (Bdd) $|a_n| \le M\ \forall\ n \in \mathbb{N}$, $\exists M$ S.T

$\boxed{\text{Conv}} \rightleftarrows \boxed{\text{Cauchy}}$, (Bdd) $\xleftarrow{} \boxed{\text{Cauchy}}$, (Bdd) $\not\longrightarrow \boxed{\text{Cauchy}}$ $(-1)^n$

$\boxed{\text{Cauchy}}$ $\forall\ \varepsilon > 0,\ \exists\ N = N(\varepsilon)$ S.T $|a_m - a_n| < \varepsilon\ \forall\ m, n \ge N$

Thm: A seq $\{a_n\}_{n \ge 1}$ of real numbers is convergent iff it is cauchy.

Exc: Let $\{x_n\}$ be a seq of real numbers, defined as follows: $x_1 = 1$, $\forall\ n \in \mathbb{N}$, $x_{n+1} = \frac{3+2x_n}{3+x_n}$

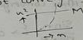
(a) S.T $\exists\ \lambda \in (0,1)$ S.T $\forall\ n \ge 2$, $|x_{n+1} - x_n| \le \lambda |x_n - x_{n-1}|$

(b) prove that $\{x_n\}$ converges and find its limit.

Exc: Suppose $\emptyset \neq S \subseteq \mathbb{R}$ and bdd above. Let s be the $\sup S \in \mathbb{R}$. S.T $\exists$ a seq $\{s_n\}_{n \ge 1}$ S.T $s_n \in S\ \forall\ n \in \mathbb{N}$ and $s_n \uparrow s$.

Exc: Suppose $\emptyset \neq T \subseteq \mathbb{R}$ and bdd below. Let $t := \inf T \in \mathbb{R}$. S.T that $\exists$ a seq $\{t_n\}_{n \ge 1}$ S.T $t_n \in T\ \forall\ n \in \mathbb{N}$ and $t_n \downarrow t$.

Eg: $a_n = n^2\ \forall\ n \in \mathbb{N}$ is not convergent, bcz not bdd (above) hence ?



Def^n: A seq $\{a_n\}$ of real numbers is said to diverge to $+\infty$ (or ∞) if $\forall\ M \in \mathbb{R}$, $\exists\ n_0 = n_0(M) \in \mathbb{N}$ S.T $a_n > M\ \forall\ \boxed{n \ge n_0}$

Notation: $\lim_{n \to \infty} a_n = \infty$ or a_n diverges to ∞; $\lim_{n \to \infty} a_n = \infty$ or $a_n \to \infty$

Remark: $a_n \to \infty$ means given any level M, the sequence will cross that level after some stage. $a_n \xrightarrow[\text{dive}]{} \infty$

$\Rightarrow a_n$ is not bdd above

$\Rightarrow a_n$ is not bdd

$\Rightarrow a_n$ is not conv.

Def^n: $a_n \to -\infty$ (a_n diverges to $-\infty$) if $\forall\ M \in \mathbb{R}$, $\exists\ n_1 = n_1(M) \in \mathbb{N}$ S.T $a_n < M\ \forall\ \boxed{n \ge n_1}$.

Remarks:

① $a_n \xrightarrow{d} -\infty$ means given any level M, after some stage, the seq is going to go below M.

② a_n may be neither convergent nor divergent.

Def^n: a_n diverges means either $a_n \to +\infty$ or $a_n \to -\infty$.

eg: $a_n = \sin n$, $a_n = (-1)^n$

Exc: (1) $a_n \to +\infty \Rightarrow \frac{1}{a_n} \to 0$

(2) $a_n \to -\infty \Rightarrow \frac{1}{a_n} \to 0$

(3) $a_n \to 0$, $a_n > 0\ \forall\ n \in \mathbb{N} \Rightarrow \frac{1}{a_n} \to +\infty$.

(4) $a_n \to 0$, $a_n < 0\ \forall\ n \in \mathbb{N} \Rightarrow \frac{1}{a_n} \to -\infty$.

eg: ⓐ $a_n = \frac{(-1)^n}{n}$ $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \dots$

$\frac{1}{a_n}$ = $-1, 2, -3, 4, -5, \dots$

⇒ Neither converge nor diverge.

ⓑ $a_n = \frac{\sin n}{n} \xrightarrow{c} 0$

Exc: Recall that $\forall x \in \mathbb{R}$, $[x] :=$ G.I.F of x.

⇒ $\sup\{n \in \mathbb{Z} : n \le x\} = x$

⇒ $x = \max\{n \in \mathbb{Z} ; n \le x\}$.

In particular, $x-1 < [x] \le x$.

eg: $[2.75] = 2$, $[\pi] = 3$, $[-2.75] = -3$, $[-\pi] = -4$

Take any $b \in \mathbb{R}$. Define a seq of rational numbers $r_n := \frac{[10^n b]}{10^n}$, $n \in \mathbb{N}$. $\in \mathbb{Q}$

(0) Compute r_1, r_2, r_3, r_4, r_5 when (A) $b = \pi$, (B) $b = -\pi$.

(1) S.T (show) $r_n \le b$ $\forall n \in \mathbb{N}$.

(2) Using sandwich thm, show that $r_n \to b$ as $n \to \infty$. (ie) any real number can be approximated by (a seq) of rationals.

(3) Using (2) show that $\forall a, b \in \mathbb{R}$ with $a < b$, $(a,b) \cap \mathbb{Q} \ne \phi$

*(4) Show that $r_n \uparrow b$. (First show this for $b > 0$.)

Series and its sum :- (Not in M-I)

Let $\{a_n\}_{n \ge 1}$ be a seq of real numbers.

Q: What is meaning of ctbly infinite sum $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$?

A: Consider the partial sum

$$A_N := \sum_{n=1}^{N} a_n, \quad N \in \mathbb{N}.$$

And then take Limit (if exists) as $N \to \infty$.

If this limit exists ie if $A_N \to A$ as $N \to \infty$ then we say that the series $\sum_{n=1}^{\infty} a_n$ converges.

and define $\sum_{n=1}^{\infty} a_n := \lim_{N \to \infty} \sum_{n=1}^{N} a_n = A$.

otherwise, i.e if limit $n \to \infty$ does not exist, then we say that the series $\sum_{n=1}^{\infty} a_n$ does not converges.

Eg: Suppose $a_n = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$, $n \in \mathbb{N}$. Then $\sum_{n=1}^{N} a_n$

$$= \sum_{n=1}^{N} \left(\frac{1}{2}\right)^n = 1 - \left(\frac{1}{2}\right)^N \quad \frac{-\frac{1}{2}\left(\left(\frac{1}{2}\right)^N - 1\right)}{\frac{1}{2}}$$

$\to 1$ as $N \to \infty$.

⇒ $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

11/9/24

Q) $x_1 = 1$, $x_{n+1} = \frac{3 + 2x_n}{3 + x_n}$

a) We have to prove if

$$|x_{n+1} - x_n| \le \lambda |x_n - x_{n-1}| \text{ for some}$$

$\lambda \in (0,1)$ then $\{x_n\}$ converge.

1 photo (Tuto) after this.

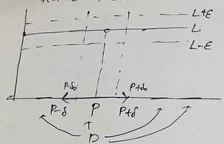
18/9/24 __Limit and Continuity of a function__

We have studied limit of a sequence. Now we shall learn about limit of a func^n.

Def^n: Suppose $f: D \to \mathbb{R}$ is a func^n, where $D \subseteq \mathbb{R}$ and $L \in \mathbb{R}$. Let $p \in \mathbb{R}$ be such that $\exists\, \delta_0 > 0$ with $(p-\delta_0, p+\delta_0) - \{p\} \subseteq D$.

Then $\underset{x\to p}{Lt} f(x) = L$ (or $\underset{x\to p}{Lim} f(x) = L$) or $f(x) \to L$ as $x \to p$) means $\forall\, \varepsilon > 0\ \exists\ \underset{=\delta(\varepsilon)}{\delta_\varepsilon} \in (0, \delta_0)$ s.t.

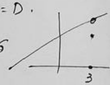
$\underset{f(x)\in(L-\varepsilon, L+\varepsilon)}{\underbrace{|f(x) - L| < \varepsilon}}$ $\forall\ x \in (p-\delta, p+\delta) - \{p\}$ i.e. $0 < |x-p| < \delta$



L: Lim of $f(x)$

Ex: $f(x) = \dfrac{x^2-9}{x-3}$, $x \in \mathbb{R} - \{3\} = D$.

Then $\underset{x\to 3}{Lt} f(x) = \underset{x\to 3}{Lt}(x+3) = 6$



Ex: $f(x) = \begin{cases} x+3 & \text{if } x \in \mathbb{R} - \{3\} \\ 3 & \text{if } x = 3 \end{cases}$ in this case

$D = \mathbb{R}$ but $\underset{x\to 3}{Lt} f(x) = 6$

Thm: Limit, if exists, is unique.

Proof: similar to proof of uniqueness of lim of seq.

Warning ① Limit may not exist.

Exc: Define $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \mathbb{R} - \{0\}, \\ 0 & \text{if } x = 0 \end{cases}$

Show that $\underset{x\to 0}{Lt} f(x)$ D.N.E.

Two approaches to solve.

Exc ① If $\underset{x\to p}{Lt} f(x) = L \in \mathbb{R}$, then S.T. $\exists\, \delta > 0$ s.t. $f(x)$ is bdd on $(p-\delta, p+\delta) - \{p\}$, i.e., the set $\{f(x) : p-\delta < x < p+\delta,\ x \neq p\} = \{f(x) : 0 < |x-p| < \delta\}$ is bdd.

Exc: If $\underset{x\to p}{Lt} f(x) = L \in \mathbb{R}$, then S.T. for any seq $\{x_n\}_{n\geq 1}$ of real no. such that $x_n \neq p\ \forall\ n \in \mathbb{N}$ and $x_n \to p$, we have $f(x_n) \to L$.

Warning ②: Limit $\underset{x\to p}{Lt} f(x)$ may exist even if $f(p)$ is not defined (i.e, $p \notin D$). Even $f(p)$ is defined (i.e $p \in D$), it may happen $\underset{x\to p}{Lt} f(x)$ exists but $\underset{x\to p}{Lt} f(x) \neq f(p)$.

__Left hand Limit and Right hand limit:__

① $\underset{x\to p^-}{Lim} f(x) = L$ mean $\forall\ \varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon) > 0$ s.t. $p - \delta < x < p \Rightarrow |f(x) - L| < \varepsilon$.

(L.H.L of f as $x \to p$ from the left side)

(2) $\underset{x\to p^+}{Lt} f(x)=L$ means $\forall\, \epsilon>0,\ \exists\, \delta=\delta(\epsilon)>0$ S.T $p<x<p+\delta \Rightarrow |f(x)-L|<\epsilon$

[R.H.L of f as $x\to p$ from the right side.]

Thm: $\underset{x\to p}{Lt} f(x)=L$ iff $\underset{x\to p^-}{Lt} f(x) = \underset{x\to p^+}{Lt} f(x) = L$.

Remark: even if LHL, RHL are equal, $f(p)$ may not exist or even when it exists, $f(p)$ may not be equal to L. In these situations we say that f is discontinuous at p. When $\underset{x\to p}{Lt} f(x) = f(p)$, then we say that f is cont at p.

both exist (in part $p\in D$)

Thm: Suppose $f: D\to\mathbb{R}$ and $g: D\to\mathbb{R}$ are two func^n (here $D\subseteq\mathbb{R}$) and $p\in\mathbb{R}$ is S.T $\exists\ \delta_0>0$ with $(p-\delta_0, p+\delta_0)-\{p\}\subseteq D$. Suppose $\underset{x\to p}{Lt} f(x)=A$ and $\underset{x\to p}{Lt} g(x)=B$. Then we have,

(i) $\underset{x\to p}{Lt} [f(x)+g(x)] = A+B$,

(ii) $\underset{x\to p}{Lt} [f(x)-g(x)] = A-B$,

(iii) $\underset{x\to p}{Lt} [f(x)\cdot g(x)] = AB$, and

(iv) $\underset{x\to p}{Lt} \dfrac{f(x)}{g(x)} = \dfrac{A}{B}$ provided $g\neq 0$ on D and $B\neq 0$.

Thm: (sandwich / squeezing principle) suppose $f(x)\le g(x)\le h(x)\ \forall\ x\in(p-\delta_0, p+\delta_0)-\{p\}$ for some $\delta_0>0$. Suppose also that $\underset{x\to p}{Lt} f(x) = \underset{x\to p}{Lt} h(x) = L$. then we have

$\Rightarrow \underset{x\to p}{Lt} g(x) = L$.

A few known limits:

Thm: (a) $\underset{x\to 0}{Lt} |x|^\alpha = 0$ for each $\alpha\in(0,\infty)$.

(b) $\underset{x\to 0}{Lt} \dfrac{\sin x}{x} = 1$

(c) $\underset{x\to 0}{Lt} \dfrac{\log(1+x)}{x} = 1$

(d) $\underset{x\to 0}{Lt} \dfrac{e^x-1}{x} = 1$

(e) $\underset{x\to 0}{Lt} (\log_e|x|)^a |x|^b = 0$ for each $a,b\in(0,\infty)$

(f) $\underset{x\to 0}{Lt} \dfrac{1-\cos x}{x} = 0$.

Exe: Find following Lts. whenever they exist. If any of them d.N.E give reasons.

(i) $\underset{x\to 0}{Lt} \dfrac{|x|}{x}$, $\underset{x\to 0^-}{Lt} \dfrac{|x|}{x}$, $\underset{x\to 0^+}{Lt} \dfrac{|x|}{x}$.

(ii) $\underset{x\to 0}{Lt} \left[\dfrac{\sin 2x}{x} + \dfrac{1-\cos 7x}{x} + \dfrac{(e^{2x}-1)\log_e(1+x)}{x^2}\right]$

(iii) $\underset{x\to 0}{Lt} \left[\dfrac{2x^2-3x}{x} + (\log_e|x|)^7 x^2 \dfrac{1-\cos 7x}{x}\right]$

(iv) $\underset{x\to 0}{Lt} \left[\dfrac{\sin 5x}{\sin x} + \dfrac{e^{7x}-1}{x}\right]$.

(v) $\underset{x\to 5}{Lt} [x]$, $\underset{x\to -7}{Lt} [x]$, $\underset{x\to \pi}{Lt} [x]$

Def^n:- Suppose $f: D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$. We say that f is cont at $p \in D$ if $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t. $x \in (p-\delta, p+\delta) \cap D \Rightarrow |f(x) - f(p)| < \varepsilon$.
f is called cont func^n if it is cont at every $p \in D$.

Thm: $f: D \to \mathbb{R}$ is cont at p iff $\lim_{x \to p} f(x) =$

$\lim_{x \to p^-} f(x) = \lim_{x \to p^+} f(x) = f(p)$ iff

$\lim_{x \to p} f(x) = f(p)$.

Eg: ① polynomial, and exponential are cont on $\mathbb{R}$.
② $f(x) = \sin x$ and $f(x) = \cos x$ are cont on $\mathbb{R}$.
③ $f(x) = \log_e x$ is cont on $(0, \infty)$

Exc:- Find all pts of continuity for the following func^ns (i) $f(x) = |x|$, $x \in \mathbb{R}$ (ii) $f(x) = [x], x \in \mathbb{R}$.
(iii) trignometric func^n other that $\sin x, \cos x$,

Properties of Continous functions:-

Thm:- Suppose $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ are cont func^ns $(D \subseteq \mathbb{R})$ and $p \in D$ is a continuity point of both f, g. Then we have,

(i) $f+g, f-g, fg$ are all cont at p,

(ii) f/g is cont at p provided $g \neq 0$ on D.

In part, sum, diff, product of 2 cont func are cont. The ratio of cont func^n and a non zero cont func^n is also cont.

***Thm**:- Suppose $f: D \to \mathbb{R}$ $(D \subseteq \mathbb{R})$ is a map, and $p \in D$. Then f is cont at p iff $\forall$ seq $\{x_n\}_{n \geq 1}$ s.t. $x_n \in D \ \forall n \in \mathbb{N}$ and $x_n \to p$, we have $f(x_n) \to f(p)$. (sequential continuity)

$\to$ $r_n = \frac{[10^n b]}{10^n}$ $n \in \mathbb{N}$. = Rational

$b = \pi$
$= 3.14592$ $\quad r_1 = \frac{[10^1 \times 3.14]}{10^1} = 3.1$

$r_2 = \frac{[10^2 \times 3.14]}{10^2} = 3.14$

$b = -\pi$
$= -3.1415$ $\quad r_1 = \frac{[10^1 \times -3.1415]}{10^1} = -3.2$

$r_2 = \frac{[10^2 \times -3.1415]}{10^2} = -3.15$

Hence r_n converges to b.

Proof:- $10^n b - 1 < [10^n b] \leq 10^n b$.

$\Rightarrow b - \frac{1}{10^n} < r_n \leq b$.

as $\lim_{n \to \infty} b - \frac{1}{10^n} \to b \Rightarrow r_n \to b$.

* $\forall a, b \in \mathbb{R}$ and $a < b$ the $(a,b) \cap \mathbb{Q} \neq \phi$

as $r_n \to c \Rightarrow \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$
s.t. $|r_n - c| < \varepsilon \ \forall n \geq n_0$.

Hence there exist infinite no. of $\mathbb{Q}$ on the neighbourhood of c.

~~$b = c + \varepsilon$~~
$\varepsilon = b - c$

Proof:- let $c = \frac{a+b}{2}$, $\varepsilon = \frac{b-a}{2}$

$r_n \to c = \frac{a+b}{2}$ $\Rightarrow \left| r_{n_0} - \frac{a+b}{2} \right| < \frac{b-a}{2}$

$\frac{a-b}{2} < r_{n_0} - \left(\frac{a+b}{2}\right) < \frac{b-a}{2}$.

Ex ① $\underset{x \to p}{Lt}\, f(x) = L \in \mathbb{R}$. Show that $\exists$ δ s.t in some deleted neighbourhood of p func^c is bdd. $M>0$ s.T $|f(x)| \le M$ $\forall$ $x \in \underbrace{(p-\delta, p+\delta)}_{N'_\delta(p)}$

Proof: Lim def^n

$\forall$ $\varepsilon > 0$, $\exists$ $\delta > 0$ s.T $|f(x) - L| < \varepsilon$ $\forall$ x with $0 < |x - p| < \delta$. $\to$ ①.

let $\varepsilon = 1$, then $\exists$ a $\delta > 0$ s.T ① holds

$|f(x) - L| < 1$ $\forall$ ~~$0 < |x-p| < 1$~~ $x \in N'_\delta(p)$.

$-1 < f(x) - L < 1$.

Hence it is bdd.

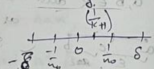
* $f(x) = \begin{cases} \frac{1}{x}, & x \in \mathbb{R} \setminus \{0\} \\ 0, & x = 0 \end{cases}$ S. That $\underset{x \to 0}{Lt}\, f(x)$ d.N.E.

Proof: Suppose limit exists, $\exists$ a $\delta > 0$ S.T $f(x)$ is bdd on $0 < |x| < \delta$.

So $\exists$ $M > 0$ S.T $|f(x)| < M$ $\forall$ x with $0 < |x| < \delta$.

as δ is small. $\frac{1}{\delta}$ = big $\Rightarrow$ $\exists$ $n_0 > \frac{1}{\delta}$.

$\Rightarrow$ $\frac{1}{n_0} < \delta$.



let $K := \max\{M, n_0\}$.

$\Rightarrow$ $K+1 > M$, $K+1 > n_0$

$\Rightarrow$ $\frac{1}{K+1} < \frac{1}{n_0}$

$f\left(\frac{1}{K+1}\right) = K+1 > M$ as $|f(x)| < M$.

So this is a contradiction Hence $\underset{x \to 0}{Lt}\, f(x)$ d.N.E.

23/9/24

LECTURE - 10

thm :- $f : \underset{(\subseteq \mathbb{R})}{D} \to \mathbb{R}$ is a map, $p \in D$. Then f is cont at p if and only if $\forall$ seq $\{x_n\}_{n \ge 1}$ s.t $x_n \in D$ $\forall$ $n \in \mathbb{N}$ and $x_n \to p$, we have $f(x_n) \to f(p)$

Sequential continuity.

Proof: (only if part) Suppose f is cont at p. Take $x_n \in D$ $\forall$ $n \in \mathbb{N}$ S.T $x_n \to p$. To show : $f(x_n) \to f(p)$

Fix an $\varepsilon > 0$, since f is cont at p, get $\delta = \delta(\varepsilon) > 0$, S.T $|x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$. $[x \in D]$ $p \in D$

Since $x_n \to p$, get $N = N(\delta(\varepsilon)) \in \mathbb{N}$ S.T $\forall$ $n \ge N$ $\Rightarrow$ $|x_n - p| < \delta$. Therefore, $\forall$ $n \ge N$, we have $|x_n - \delta| < \delta$, $x_n \in D$. $\Rightarrow$ $|f(x_n) - f(p)| < \varepsilon$.

Hence $f(x_n) \to f(p)$ as $n \to \infty$.

If part : Suppose $f(x_n) \to f(p)$ whenever $x_n \in D$ $\forall$ $n \in \mathbb{N}$ and $x_n \to p$.

To show :- f is cont at p.

Suppose f is not cont at p.

This means $\exists$ $\varepsilon_0 > 0$ S.T $\forall$ $\delta > 0$, $\exists$ $x = x(\delta)$ $\in (p-\delta, p+\delta) \cap D$ but $|f(x) - f(p)| \ge \varepsilon_0$

$\therefore$ For each $n \in \mathbb{N}$, choose $\delta = \frac{1}{n}$ to get $x_n = x\left(\frac{1}{n}\right)$ $\in \left(p - \frac{1}{n}, p + \frac{1}{n}\right) \cap D$ satisfying $|f(x_n) - f(p)| \ge \varepsilon_0$.

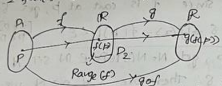
Then $x_n \in D$ $\forall$ $n \in \mathbb{N}$ and $p - \frac{1}{n} < x_n < p + \frac{1}{n}$ and hence by sandwich thm, $x \to p$.

However $f(x_n) \not\to f(p)$ bcz. $|f(x_n) - f(p)| \geq$ ... $\forall n \in \mathbb{N}$. This contradicts our hypothesis.
and hence f must be continuous at p.

Exc:- Use the last thm to show $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ not cont at $x = 0$. [Use $x_n = \frac{1}{n} \to 0$]

Exc:- S.T $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is a cont func^n on $\mathbb{R}$. [Use Sandwich Thm, next thm].

Thm:- Let $f : D_1 \to \mathbb{R}$ $(\subseteq \mathbb{R})$ and $g : D_2 \to \mathbb{R}$ $(\subseteq \mathbb{R})$ be the 2 func^{n}s S.T Range $(f) \subseteq D_2 \subseteq \mathbb{R}$.

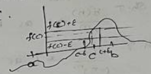


In particular, $g \circ f : D_1 \to \mathbb{R}$ is well defined.
Suppose $p \in D_1$ is a continuity pt of f and $f(p) \in D_2$ is a continuity pt of $g \circ f$.
As a consequence, it follows that composition of two cont func^n is also cont.

Cor:- All of these are cont func^{n}s
(i) $f(x) = \sin(3x^2 + 7x + 8) + e^{\cos x}(x-7)^{10}$, $x \in \mathbb{R}$.
(ii) $f(x) = e^{\sin x + \cos x + 3x^2} + \log_e(x^2+1)$, $x \in \mathbb{R}$.
(iii) $f(x) = \sin(e^{x^2+2}) \cos(e^{x^3+23})$, $x \in \mathbb{R}$.

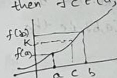
Thm:- (Sign-preserving property of cont Fun^{n}s)
Let $a < b$ and $f : (a,b) \to \mathbb{R}$ be a cont func^n.
If $f(c) \neq 0$ for some $c \in (a,b)$, then $\exists \, \delta > 0$ S.T f has same sign as $f(c)$ in the interval $(c-\delta, c+\delta) \subseteq (a,b)$.

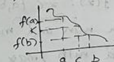
Proof:- suppose $f(c) > 0$. Then $\forall \varepsilon > 0$
$\exists \, \delta = \delta(\varepsilon) > 0$ S.T $x \in (c-\delta, c+\delta) \subseteq (a,b)$
$\Rightarrow f(c) - \varepsilon < f(x) < f(c) + \varepsilon$.
$\Rightarrow |f(x) - f(c)| < \varepsilon$.



take $\varepsilon = \frac{f(c)}{2}$ then
$\exists \, \delta = \delta\left(\frac{f(x)}{2}\right) > 0$ S.T
$x \in (c-\delta, c+\delta) \subseteq (a,b) \Rightarrow f(c) - \frac{f(c)}{2} < f(x) < f(c) + \frac{f(c)}{2}$
$\Rightarrow f(x) > \frac{f(c)}{2} > 0$. $\forall x \in (c-\delta, c+\delta) \subseteq (a,b)$

Thm:- (Intermediate Value thm for cont Func^{n}s)
Suppose $a < b$ and $f : [a,b] \to \mathbb{R}$ is cont func^n.
if $f(a) \neq f(b)$, then f takes every value btw $f(a)$ and $f(b)$, i.e if $\min(f(a), f(b)) < K < \max(f(a), f(b))$.
then $\exists \, c \in (a,b)$ S.T $f(c) = K$.





Cor:- (Bolzano's thm) Suppose $a < b$ and $f : [a,b] \xrightarrow{c} \mathbb{R}$ is a cont func^n. If $f(a) f(b) < 0$ (if $f(a), f(b)$ have diff signs and $\neq 0$).
then $\exists \, c \in (a,b)$ S.T $f(c) = 0$.

Exc: Suppose $f: [0,1] \to [0,1]$ is a cont func. ShT f has a fixed pt, i.e, $\exists x_0 \in [0,1]$ s.t $f(x_0) = x_0$. [Such an x_0 is called fixed pt of f.]

Exc: Suppose $f(x)$ is a monic poly func^n of odd degree, i.e $\forall x \in \mathbb{R}$, $f(x) = x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots + a_0$, where $a_0, a_1, \dots a_{m-1} \in \mathbb{R}$ and m is odd.

(a) Sh.T $\frac{f(n)}{n^m} \to 1$ as $n \to \infty$.

(b) Sh.T $\frac{f(-n)}{(-n)^m} \to 1$ as $n \to \infty$

(c) Using (a), (b) and Bolzano's Thm sh. that $\exists p \in \mathbb{R}$ s.t $f(p) = 0$.

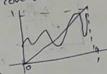
(d) Using (c) sh.T any polynomial func^n (with real coeffs) of odd degree has a root in $\mathbb{R}$

Thm:- Suppose $a<b$ and $f: [a,b] \to \mathbb{R}$ is a cont func. Then f is bdd on $[a,b]$: i.e, the set $S :=$
$S := \underbrace{\{f(x) : x \in [a,b]\}}_{= Range(f)}$ is bdd. Also f attains both min and max on $[a,b]$, i.e the set S has a min, max element

Warning: $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $f(x) = \tan x$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is cont. This f is neither bdd below nor bdd above. It doesnot admit min or max.

Q) If $f: [0,1] \to [0,1]$

$g(x) = f(x) - x$ on $[0,1]$. Since diff of const func^ns is cont, g is cont.



$g(0) = f(0)$
$g(1) = f(1) - 1$

If $f(0) = 0$, we are done take $x_0 = 0$
If $f(1) = 1$, we are done take $x_0 = 1$.
Now suppose $f(0) \neq 0$, $f(1) \neq 1$ $\quad f: [0,1] \to [0,1]$
$\Rightarrow f(0) > 0 \quad f(1) < 1 \quad 1 \geq f(x) \geq 0$.
$g(0) > 0 \quad g(1) < 0$

By Boltzman's Intermediate value theorem, $\exists x_0 \in [0,1]$ s.t $g(x_0) = 0 \Rightarrow f(x_0) = x_0$.

Q) $f(x) = x^m + a_{m-1}x^{m-1} + \dots a_0$ $m \to$ odd.

a) $\frac{f(n)}{n^m} = 1 + \underbrace{\frac{a_{m-1}}{n}}_{\to 0} + \dots + \underbrace{\frac{a_0}{n^m}}_{\to 0} \to 1$ as $n \to \infty$

b) $\frac{f(-n)}{n^m} = 1 - \underbrace{\frac{a_{m-1}}{n}}_{\to 0} + \dots - \underbrace{\frac{a_0}{n^m}}_{\to 0} \to 1$.

c) $\exists n_1 \in \mathbb{N}$ s.t $\frac{f(n_1)}{n_1^m} > 0 \Rightarrow f(n_1) > 0$

lly $\exists n_2 \in \mathbb{N}$ s.t $\frac{f(n_2)}{(-n_2)^m} > 0 \Rightarrow f(-n_2) < 0$

by B.I.T $\Rightarrow \exists$ some pt at which $f(p) = 0$.

Hence $p \in \mathbb{R}$.

(d) $g(x) = \alpha f(x)$ $\quad$ f is monic $\Rightarrow$ (i) $\alpha > 0$. (ii) $\alpha < 0$

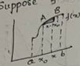
(i) $g(-n_2) = \alpha f(-n_2) < 0$
$g(n_1) = \alpha f(n_1) > 0$

25/9/24

Lecture-11

$-\infty \le a < b \le \infty$

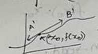
* Suppose $f: (a,b) \in \mathbb{R}$. Take $x, x_0 \in (a,b)$.



slope of $\overline{AB} = \dfrac{f(x)-f(x_0)}{x-x_0}$

where $A = (x_0, f(x_0))$,
$B = (x, f(x))$.

therefore, $\displaystyle\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ (if exists) and is finite $=$ slope of the tangent to the graph $y = f(x)$ at the pt $x = x_0$ or at the pt $(x_0, f(x_0))$.



$\Rightarrow$ Slope of $\overleftrightarrow{AB}$.

Def^n: ① Suppose $f: (a,b) \in \mathbb{R}$ is a function. Take a $x_0 \in (a,b)$. We say that f is differentiable at x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} \text{ exists}$$

and is finite. then we say limit is diff and is finite.

② In this case, the above limit is denoted by $f'(x_0)$ or by $\left.\dfrac{d(f(x))}{dx}\right]_{x=x_0}$ or $\left\{\dfrac{d}{dx} f(x)\right\}_{x=0}$ as called derivative of f at $x = x_0$.

③ If f is diff at each $x_0 \in (a,b)$, then we say that f is diff on (a,b).

④ By a diff func^n, we mean a function that is diff at each pt x_0 in its domain.

Thm: Suppose $f: (a,b) \to \mathbb{R}$ is diff at a pt $x_0 \in (a,b)$. $-\infty \le a < b \le \infty$. Then f is cont at x_0. In part, diff func^{n}s are also cont func^n.

Proof: $\displaystyle\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left[[f(x) - f(x_0)] + f(x_0) \right]$

$$= f(x_0) + \lim_{x \to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0} \times (x - x_0) \right].$$

$$= f(x_0) + \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \times \lim_{x \to x_0} (x - x_0).$$

$$= f(x_0) + f'(x_0) \cdot 0 = f(x_0).$$

$\Rightarrow f$ is cont at x_0.

$\boxed{\text{Diff at } x_0} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{\text{cont at } x_0}$

Counter. eg: $f(x) = |x|$ at $x = 0$

$$\left.\begin{aligned} Rf'(0) &= \lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = 1 \\ Lf'(0) &= \lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = -1 \end{aligned}\right\} \neq \Rightarrow f'(0) \text{ d.N.E}$$

Eg: Polynomial, trignometric, logarithmic, and exponential func^{n}s are all diff'ble.

* $\dfrac{d}{dx}(\underset{\text{const}}{c}) = 0$, $\dfrac{d}{dx}(x^n) = n x^{n-1}$, $n \in \mathbb{N}, \mathbb{R}$

$\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$ $(x>0)$, $\dfrac{d}{dx}(e^x) = e^x$, $\dfrac{d}{dx}(\sin x) = \cos x$,

$\dfrac{d}{dx}(\cos x) = -\sin x$, $\dfrac{d}{dx}(\tan x) = \sec^2 x$, $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$,

$\dfrac{d}{dx}(\sec x) = \sec x \tan x$, $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$.

(Valid on their respective domains.)

Thm: Suppose $f:(a,b)\to\mathbb{R}$, $g:(a,b)\to\mathbb{R}$ are two func^n's, $c_1,c_2,c_3\in\mathbb{R}$ and $x_0\in(a,b)$ are such that both f and g are diff at $x=x_0$. Then $c_1f+c_2g+c_3$ = diff. (in part $f+g$, $f-g$, c_1f+c_3), fg are diff at $x=x_0$. Also if $g\neq 0$ on (a,b) ~~and $g'(x_0)\neq 0$~~, then $\frac{f}{g}$ is also diff at $x=x_0$. More over, we have:

(i) $(c_1f+c_2g+c_3)'(x_0) = c_1f'(x_0)+c_2g'(x_0)$

(In part $(f\pm g)'(x_0) = f'(x_0)\pm g'(x_0)$

$(c_1f+c_3)'(x_0) = c_1f'(x_0)$.)

(ii) $(fg)'(x_0) = f'(x_0)g(x_0)+f(x_0)g'(x_0)$

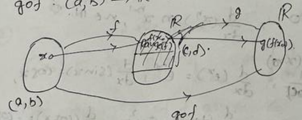
(iii) $\left(\frac{f}{g}\right)'(x_0) = \dfrac{g(x_0)f'(x_0)-f(x_0)g'(x_0)}{(g(x_0))^2}$ if

$g\neq 0$ on (a,b) and $g(x_0)\neq 0$.

g is non zero in an interval around x_0.

(chain rule)

Thm: Suppose $f:(a,b)\to\mathbb{R}$, $g:(c,d)\to\mathbb{R}$ are func^n's s.t Range $(f)\subseteq(c,d)$, so that $g\circ f:(a,b)\to\mathbb{R}$ is well-defined.



Suppose $x_0\in$ ~~(a,b)~~ ~~is a pt~~ f is diff at $x_0\in(a,b)$ and g is diff at $f(x_0)\in(c,d)$

Then $g\circ f$ is diff at $x=x_0$ and

$(g\circ f)'(x_0) =$ ~~$g'(f(x_0))+g(f...)$~~

$(g\circ f)'(x_0) = g'(f(x_0)).f'(x_0)$.

[Think $\left.\frac{d}{dx}g(f(x))\right|_{x=x_0} = \left.\frac{d\,g(f(x))}{d\,f(x)}\right|_{f(x)=f(x_0)}\cdot\left.\frac{df(x)}{dx}\right|_{x=x_0}$

$= \left.\frac{d\,g(y)}{dy}\right|_{y=f(x)}\cdot\left.\frac{df(x)}{dx}\right|_{x=x_0}$.]

In part, composition of diff func^n is diff

Exc: Show that the following fun^c are diff on their respective domain of def. and find their derivatives:

(i) $\sin(e^x)\log x + x^3$. (iii) $\cos(e^x + e^{-x})$

(ii) $e^{\cos x} + e^{\sin x} + 27$

Q. Find Limit if it exists, if not, give reason:

(i) $\lim\limits_{x\to 5}[x]$ (ii) $\lim\limits_{x\to -7}[x]$ (iii) $\lim\limits_{x\to\pi}[x]$

A) (i) Let $\varepsilon=\frac{1}{2}$. Then $\forall\,\delta>0$, $(5-\delta,5+\delta)$.

$5-\delta<5<5+\delta$

$x=5-\frac{\delta}{2}\in(5-\delta,5+\delta)$ but

$|f(5-\frac{\delta}{2})-f(5)|\geq|4-5|\geq 1\not<\varepsilon$

So Lim d.N.E.

$x\to 5^-$
$f(x)=4\to 4$
$x\to 5^+$
$f(x)=5\to 5$

(iii) $\lim\limits_{x\to\pi}[x]$

Let $\varepsilon>0$. Choose $\delta=0.1$

Then $\forall\,x\in(\pi-\delta,\pi+\delta)=(3.04..,3.24..)$

$|f(x)-3| = |3-3| = 0<\varepsilon$.

$\Rightarrow$ Limit exists and is equal to 3.

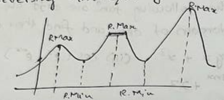
14/10/24

Lecture - 12

Relative Max (or) **Local Max**: $f: \underset{(\subseteq \mathbb{R})}{S} \to \mathbb{R}$ is said to have a relative (local) max at a pt c in S if $\exists$ an open interval I containing c such that

$$f(x) \leq f(c) \quad \forall x \in I \cap S.$$

Similarly, we can define local or relative min by reversing this inequality. i.e $f(x) \geq f(c)$ $\forall x \in I \cap S$.



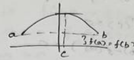
Def^n: A number c at which a relative max or min is obtained is called an (local) extreme value (or) an local extremum of f.

Thm: let f be defined on an open interval I, and assume that f has a relative max/min at an interior pt c ~~$\in$~~ I. If the func^n f is cont on I and is diff on I, then $f'(c) = 0$.

Sketch: Uses sign preserving property of cont functions, the def^n of derivative [Ref: Apostol's Calculus]

Thm: (Rolle's Thm)

Let $a<b$, $a,b \in \mathbb{R}$. let $f: [a,b] \to \mathbb{R}$ be a cont func^n i.e diff on (a,b). Also assume that $f(a) = f(b)$. Then $\exists$ atleast one pt $c \in (a,b)$ s.t $f'(c) = 0$.



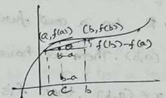
Thm: (Mean Value Thm) **MVT**:-

let $a<b$, $a,b \in \mathbb{R}$. let f be a cont func^n that is diff on (a,b). Then $\exists$ at least one pt 'c' in the (a,b) s.t

$$f(b) - f(a) = f'(c)\,(b-a)$$

Remark: MVT ~~[illegible]~~ $\to$ Rolle's Thm

Proof:



$$\tan\theta = \frac{f(b)-f(a)}{b-a}$$

MVT means $\exists$ a pt c in (a,b) s.t the slope of the tangent to the graph of f at c = slope of chord joining the pts $(a,f(a)), (b,f(b))$.

Proof of MVT using Rolle's thm:-

Define $h: [a,b] \to \mathbb{R}$ by

$$h(x) = f(x)\,(b-a) - x\,(f(b)-f(a))$$

- h is cont on $[a,b]$.
- h is diff on (a,b).
- $h(a) = h(b) = bf(a) - af(b)$

check. (Exc)

By Rolle's Thm applied to h, we get that

$\exists\ c \in (a,b)$ s.T $h'(c) = 0$

$\Rightarrow f'(c)(b-a) - (f(b) - f(a)) = 0.$

$\Rightarrow f'(c) = \dfrac{f(b)-f(a)}{b-a}$

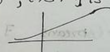
Warning:- $f(x) = |x|,\ x \in [-1,1]$

$f(-1) = f(1)$ but $\nexists\ c \in (-1,1)$ st $f'(c) = 0$

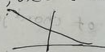
Hence MVT / Rolle's Thm ~~may~~ will fail if the function is not diff'ble at one point.

Thm:- Let $f : [a,b] \to \mathbb{R}$, $a<b$, be a cont func^n, s.t f is diff on (a,b). Then we have:

(i) If $f'(x) > 0\ \forall\ x \in (a,b)$, then f is strictly increasing on $[a,b]$.



(ii) If $f'(x) < 0\ \forall\ x \in (a,b)$, then f is strictly decreasing on $[a,b]$.



(iii) If $f'(x) = 0\ \forall\ x \in (a,b)$, then f must be a constant function on $[a,b]$.

Proof (i):- To show $f(x) < f(y)$ whenever $a \le x < y \le b$.

~~Apply MVT on interval x,y~~

Apply MVT on f restricted to $[x,y]$, we get $\exists\ c \in (x,y)$ s.T $f(y) - f(x) = f'(c)(y-x)$

as $f'(c) > 0$, $y - x > 0$ then $f(y) - f(x) > 0$

$\Rightarrow f(y) > f(x)$.

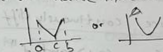
This shows f is strictly $\uparrow$ on $[a,b]$ when $f'(x) > 0$.

Excer:- Prove (ii), (iii) using MVT.

Thm:- Let $a, b \in \mathbb{R}$ s.T $a<b$. let $f : [a,b] \to \mathbb{R}$ be a cont func^n s.T f is diff on (a,b), except possibly at a pt $c \in (a,b)$.

(i) If $f'(x) > 0\ \forall\ x \in (a,c)$ and $f'(x) < 0\ \forall\ x \in (c,b)$ then f has a local or relative max at c.

(ii) If $f'(x) < 0\ \forall\ x \in (a,c)$ and $f'(x) > 0\ \forall\ x \in (c,b)$ then f has a local or relative min at c.



Exc:- Prove this thm using MVT.

Thm:- Let $f : [a,b] \to \mathbb{R}$ be a cont func^n s.T f is twice diff on (a,b). Let $c \in (a,b)$ be a critical pt of f so that $f'(c) = 0$. Then we have the following:
extremum

(i) If $f''(x) < 0$, $\forall\ x \in (a,b)$, then f has a local max at c.

(ii) If $f''(x) > 0$, $\forall\ x \in (a,b)$ then f has a local min at c.

21/10/24

Lecture-13

Exc: (i) Suppose $a<b$ and $f:[a,b]\to\mathbb{R}$ is a cont fun^n s.t f is diff on (a,b). Suppose $\exists c\in\mathbb{R}$ s.t $f'(x)=c$ $\forall$ $x\in(a,b)$. Using MVT, show that $\exists d\in\mathbb{R}$ s.t $f(x)=cx+d$ $\forall$ $x\in[a,b]$.

(ii) Suppose $a<b$ and $g:[a,b]\to(0,\infty)$ is a cont $func^n$ s.t g is diff on (a,b). Suppose $\exists c\in\mathbb{R}$ s.t $g'(x)=c\,g(x)$ $\forall$ $x\in(a,b)$. Using (i) show that $\exists A\in(0,\infty)$ s.t $g(x)=Ae^{cx}$ $\forall$ $x\in[a,b]$.

[Hint: Use $f(x)=\log_e g(x)$ in (i)].

Thm: Let $f:[a,b]\to\mathbb{R}$ be a cont $func^n$ s.t f is twice continuosly diff on (a,b) (i.e, f is twice diff on (a,b) and f'' is cont on (a,b).) Let $c\in(a,b)$ be a critical pt, i.e $f'(c)=0$. Then we have:

(i) If $f''(c)<0$, then f has a local or a relative max at c.

(ii) If $f''(c)>0$, then f has a local min at c.

Remark: If $f''(c)=0$, then nothing can be said about max or min. at c.

$f(x)=x^3$, $x\in[-1,1]$

$f'(0)=3x^2=0$

$f''(0)=6x=0$

So Neither max nor minimum

Exc: Find all local maxima and minima for the following $func^n$s.

(i) $f(x)=x^3-9x^2+24x$ (iii) $f(x)=3x^4-4x^3$

(ii) $f(x)=2+3x-x^3$ (iv) $f(x)=x^4-2x^2+3$

(v) $f(x)=\cos 2x-x$

Taylor's thm: Let $k\in\mathbb{N}$, and let the $func^n$ $f:\mathbb{R}\to\mathbb{R}$ be k times diff at $a\in\mathbb{R}$. Then $\exists$ a $func^n$ $R_k:\mathbb{R}\to\mathbb{R}$ s.t $f(x)=f(a)+f'(a)(x-a)$

$$f(x)=\frac{f(a)}{0!}+\frac{f'(a)(x-a)}{1!}+\frac{f''(a)}{2!}(x-a)^2+\dots\frac{f^k(a)}{k!}(x-a)^k+R_k(x) \quad (*)$$

where $\lim_{x\to a}\frac{R_k(x)}{(x-a)^k}=0$.

Moreover, if f is $(k+1)$ times diff on the open interval joining a, x and $f^{(k)}(x)$ is cont on closed interval joining a, x, then $R_k(x)=\frac{f^{(k+1)}(\eta)}{(k+1)!}(x-a)^{k+1}$ $x\neq a$

for some η strictly btw a and x.

Remarks: ① Think of Taylor's thm as a generalisation of M.V.T.

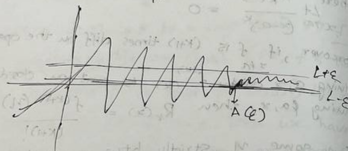
② It is useful in approximation theory and ~~numbe~~ numerical analysis.

③ RHS of $(*)$, without the remainder/error term $R_k(x)$, is called the k^{th} Taylor polynomial for $f(x)$ around a.

Exc:- Let f be twice contly diff func^n [illegible]. $f(0) = f'(0) = 0$. If $|f''(x)| \le 1$ $\forall x \in \mathbb{R}$, then show that $|f(x)| \le \frac{1}{2}$ $\forall x \in [-1, 1]$.

L'Hôpital's Rule: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a map. We know the meaning of $\underset{x \to c}{Lt} f(x) = L$ when $c, L \in \mathbb{R}$. What if L and/or c is either $+\infty$ or $-\infty$?

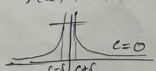
Def^n: Let $L \in \mathbb{R}$ and $c = \infty$, then $\underset{x \to \infty}{Lt} f(x) = L$ means $\forall \varepsilon > 0$, $\exists A = A(\varepsilon) \in \mathbb{R}$ such that $x > A \Rightarrow |f(x) - L| < \varepsilon$.



Similarly, one can define $\underset{x \to -\infty}{Lt} f(x) = L$.

Def^n: Let $c \in \mathbb{R}$ and $L = \infty$. Then ~~[illegible]~~

$\underset{x \to c}{Lt} f(x) = \infty$ means that given any $A \in \mathbb{R}$, $\exists \delta = \delta(A) \in (0, \infty)$ s.t. $0 < |x - c| < \delta$ $\Rightarrow f(x) > A$.

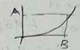


$f(x) = \frac{1}{|x|}$, $x \ne 0$

$c = 0$

Similarly, one can define $\underset{x \to c}{Lt} f(x) = -\infty$.

Def^n: Let $c = \infty$, $L = \infty$. Then $\underset{x \to \infty}{Lim} f(x) = \infty$ means given any $A \in \mathbb{R}$, $\exists B = B(A) \in \mathbb{R}$ such that $x > B \Rightarrow f(x) > A$.



$f(x) = x^2$

Similarly, one can define $\underset{x \to \infty}{Lt} f(x) = -\infty$, $\underset{x \to -\infty}{Lt} f(x) = \infty$ or $\underset{x \to -\infty}{Lt} f(x) = -\infty$.

* Suppose $c, L \in \mathbb{R} \cup \{-\infty, \infty\}$. This means that each of c, L is either a real number or $\pm\infty$. By a deleted nbhd I of c, we mean:

- $(c - \delta, c + \delta) - \{c\}$ for some $\delta > 0$ if $c \in \mathbb{R}$;
- (A, ∞) for some $A \in \mathbb{R}$ if $c = \infty$;
- $(-\infty, B)$ for some $B \in \mathbb{R}$ if $c = -\infty$

Thm:- (L'Hôpital's Rule / Bernoulli's Rule)

Suppose f, g are two real valued diff func^{n}s defined on a deleted nbhd I of $c \in \mathbb{R} \cup \{-\infty, \infty\}$. Assume that $g'(x) \ne 0$ $\forall x \in I$. Also assume that $\underset{x \to c}{Lt} f(x) = \underset{x \to c}{Lt} g(x) \in \{0, \infty, -\infty\}$ and $\cancel{\frac{f(x)}{g(x)}}$

$\underset{x \to c}{Lt} \frac{f'(x)}{g'(x)} = L$ (exists). $L \in \mathbb{R} \cup \{-\infty, \infty\}$. Then $\underset{x \to c}{Lt} \frac{f(x)}{g(x)}$ also exists $= L$

Eg:- $\displaystyle\lim_{x\to 0} \frac{e^x-1}{x^2+x} = ?$

A) $\displaystyle\lim_{x\to 0} \frac{e^x-1}{x^2+x}$ Note that $\displaystyle\lim_{x\to 0} \frac{\frac{d}{dx}(e^x-1)}{\frac{d}{dx}(x^2+x)}$

$$= \lim_{x\to 0} \frac{e^x}{2x+1} = 1.$$

Therefore, all the condⁿ of L'Hôpital's Rule are satisfied. Hence by L'Hôpital's Rule

$\displaystyle\lim_{x\to 0} \frac{e^x-1}{x^2+x} = 1$ (exists).

Exc:- Use Bernoulli's Rule to evaluate following Lim

(i) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$, More generally show by induction on $n \in \mathbb{N}$ that $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$. Fix $n \in \mathbb{N}$.

(ii) $\displaystyle\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x}$ (iv) $\displaystyle\lim_{x\to 0} \frac{x}{1-e^{2x}}$

(iii) $\displaystyle\lim_{x\to \frac{\pi}{2}} \frac{\tan x - 5}{\sec x + 4}$ (v) $\displaystyle\lim_{x\to 3} \frac{x^2-4x+3}{2x^2-13x+21}$

Q) Let $f: [a,b] \to \mathbb{R}$ be cont s.t f is diff on (a,b)

(ii) If $f'(x) < 0$ $\forall x \in (a,b)$, then f is strictly decreasing on $[a,b]$.

Proof:- Let $x_1, x_2 \in [a,b]$ with $x_1 < x_2$. To prove $f(x_1) > f(x_2)$.

Consider $f: [x_1, x_2] \to \mathbb{R}$. Since $f: [a,b] \to \mathbb{R}$ is cont $f: [x_1, x_2] \to \mathbb{R}$ is also cont. And since f is diff on (a,b), f is diff on (x_1, x_2) also.

By M.V.T. $\exists\, x \in (x_1, x_2)$ s.t

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x) < 0.$$

Since $x_1 < x_2$, $x_2 - x_1 > 0 \Rightarrow f(x_2) - f(x_1) < 0$

$f(x_2) < f(x_1)$.

Hence f is decreasing.

(iii) If $f'(x) = 0$ $\forall x \in (a,b)$, then f must be a const funcⁿ on $[a,b]$.

Pf:- Let $x \in (a,b]$ with then $a < x$. Will prove $f(x) = f(a)$.

Consider $f: [a,x] \to \mathbb{R}$ (Similarly like before)

by MVT. $\exists\, c \in (a,x)$ s.t

$$\frac{f(x)-f(a)}{x-a} = f'(c) = 0.$$

$\Rightarrow f(x) = f(a)$.

Hence f is const.

Q) Let $f: [a,b] \to \mathbb{R}$ be cont s.t f is diff on (a,b) except possibly at a pt $c \in (a,b)$.

(i) If $f'(x) > 0$ $\forall x \in (a,c)$ and $f'(x) < 0$ $\forall x \in (c,b)$ then f has local max at c.

Pf:- $f: [a,c] \to \mathbb{R}$, and apply previous thm (i), then we get f is strictly ↑ing on $[a,c]$. i.e,

$\forall x \in [a,c)$, $f(x) < f(c)$. → (1)

$f: [c,b] \to \mathbb{R}$, and apply previous thm (ii), then we get f is strictly ↓ing on $[c,b]$ i.e,

$\forall x \in (c,b]$, $f(x) \not< f(c)$ → (2).

Take $I = (a,b)$, then clearly $c \in I$. Then $\forall\, x \in I \cap [a,b]$, we saw $f(x) \le f(c)$.

$\Rightarrow f$ has local max at C.

(ii) If $f'(x) < 0$ $\forall x \in (a,c)$, $f'(x) > 0$ $\forall x \in (c,b)$ then f has local min at C.

Pf:- $f: [a,c] \to \mathbb{R}$, apply (ii).
f is strictly $\downarrow$ing on $[a,c]$.
i.e $\forall x \in [a,c)$, $f(x) > f(c)$

$f: [c,b] \to \mathbb{R}$, apply (i).
f is strictly $\uparrow$ing on $[c,b]$
i.e $\forall x \in (c,b]$, $f(x) > f(c)$.

Take $I = (a,b)$, then clearly $c \in I$.
Then $\forall x \in I \cap [a,b]$, we saw $f(x) \ge f(c)$.
$\Rightarrow f$ has local min at C.

Q) $f: [a,b] \to \mathbb{R}$ is a cont func^n s.t f is diff on (a,b). Suppose $\exists c \in \mathbb{R}$ s.t $f'(x) = c$ $\forall x \in (a,b)$. Using MVT show that $\exists d \in \mathbb{R}$ s.t $f(x) = cx + d$ $\forall x \in [a,b]$

Sol^n:- Fix $x \in (a,b]$. Then $f: [a,x] \to \mathbb{R}$ is cont and diff on (a,x). By MVT, $\exists x_0 \in (a,x)$ s.t $\dfrac{f(x) - f(a)}{x-a} = f'(x_0) = c$

$\Rightarrow f(x) = c(x-a) + f(a)$
$= cx + \underbrace{f(a) - ca}_{d}$

We know $\forall x > a$, $f(x) = cx + d$.

$f(a) = \lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^+} |cx + d| = ca + d$

i.e $\forall x \in [a,b]$, $f(x) = cx + d$.

Q) Local max, min
(ii) $f(x) = 2 + 3x - x^3$
A) $f'(x) = 3 - 3x^2 = 0$.
$3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

$f''(x) = -6x$
$f''(1) < 0$ $\quad f''(-1) > 0$.

1 is local max, -1 is local min.

If c is a local max or local min pt of f, then $f'(c) = 0$.

$f'(x) = 3 - 3x^2 = 0 \Rightarrow x = \pm 1$.
(i) $c = 1$ $\quad$ (ii) $c = -1$

If $f''(x) < 0$, then x is a local max pt.
If $f''(x) > 0$, then x is a local min pt
$f''(1) < 0$ $\quad f''(-1) > 0$.
$c = 1$ is a local max pt for f.
$c = -1$ is a local min pt for f.

Q) $f: \mathbb{R} \to \mathbb{R}$ is twice contly diff func^n with $f(0) = f'(0) = 0$. If $|f''(x)| \le 1$ $\forall x \in \mathbb{R}$. Then s.t $|f(x)| \le \frac{1}{2}$ $\forall x \in [-1,1]$.

A) For $K=1$, apply Taylor's thm at $a=0$.

$f(x) = f(0) + \dfrac{f'(0)}{1!}(x-0) + R_1 x$.

$R_1 x = \dfrac{f''(\eta)}{2!}(x-0)^{1+1} = \dfrac{f''(\eta)}{2} \times x^2$

as f is twice diff.

$f(x) = \frac{f''(\eta)}{2} \cdot x^2$

$|f(x)| \leq \frac{1}{2}|x^2|$

and on interval $[-1,1]$, $|x^2| \leq 1$.

$|f(x)| \leq \frac{1}{2}$.

as f is twice diff. $\exists \eta \in \mathbb{R}$ by moreover part of Taylor's thm

Exc:- Bernoulli's Rule

A) (i) $\lim_{x\to\infty} \frac{e^x}{x^2}$

Pf:- Take $f(x) = e^x$, $g(x) = x^2$; $f'(x) = e^x$, $g'(x) = 2x \neq 0$ on $(1,\infty)$. $(0,\infty)$ $(3,\infty)$ $(5,\infty)$

should be open interval

$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to\infty} g(x) = \infty$

$\Rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x)$.

Now Hence by Bernoulli's Rule

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{e^x}{2x}$$

let $f_1(x) = e^x$, $g_1(x) = 2x$.

$f_1'(x) = e^x$, $g_1'(x) = 2 \neq 0$ on $\mathbb{R}$.

$\lim_{x\to\infty} f_1(x) = \infty = \lim_{x\to\infty} g(x) = \infty$

By Bernoulli's Rule,

$$\lim_{x\to\infty} \frac{f_1(x)}{g_1(x)} = \lim_{x\to\infty} \frac{f_1'(x)}{g_1'(x)} = \frac{\cancel{e^x}}{2} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

Hence $\lim_{x\to\infty} \frac{e^x}{x} = \infty$.

$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty = P(n)$.

$P(1)$ is true.

Prove $\lim_{x\to\infty} \frac{e^x}{x} = \infty$

Assume $P(n-1)$ is true, $\forall n \geq 2$.

Prove $P(n)$ is true.

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{n x^{n-1}} = \frac{1}{n} \lim_{x\to\infty} \frac{e^x}{x^{n-1}}$$

$= \frac{1}{n} \cdot \infty = \infty$.

Hence proved by induction $\forall x \geq 1$, $P(x)$ is true.

23/10/24

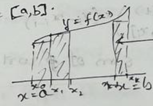
Lecture - 14

Riemann Integration:-

Fix $a, b \in \mathbb{R}$, $a < b$. Also fix a bdd func $f: [a,b] \to \mathbb{R}$.

ie, $\forall |f(x)| \leq M$, ie, $\exists M \in (0,\infty)$ s.t $|f(x)| \leq M$.

$\forall x \in [a,b]$.



Area $= \int_a^b f(x)\,dx$.

Def^n:- A partition P of $[a,b]$ is an ordered set of values $\{a = x_0 < x_1 < x_2 < \cdots < x_k = b\}$ for some $k \in \mathbb{N}$

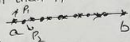
$\boxed{\|P\| = \text{mesh of the partition } P}$

$\|P\| = \max_{1 \le i \le K} |x_i - x_{i-1}|$

Def^n: Suppose P_1 & P_2 are two partions of $[a,b]$. We say that P_2 is a refinement of P_1 or P_2 is a finer partition than P_1 if as sets.

$P_1 \subseteq P_2$

(we say P_1 is a coarser Partion of P_2)



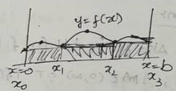
$(P_2 > P_1)$

Exc: If $P_1 \subseteq P_2$ then $\|P_2\| \le \|P_1\|$

(P_2 is refinement of P_1)

Def^n: Given a partition P of $[a,b]$, we define

$$\boxed{L(f,P) := \sum_{i=1}^{K} \left(\inf_{t \in [x_{i-1}, x_i]} f(t) \right) \times (x_i - x_{i-1}) \\ = \text{The lower sum of } f \text{ wrt } P.}$$



$$\boxed{U(f,P) := \sum_{i=1}^{K} \left(\sup_{t \in [x_{i-1}, x_i]} f(t) \right) \times (x_i - x_{i-1}) \\ = \text{The upper sum of } f \text{ wrt } P.}$$

Obs: $L(f,P) \le U(f,P)$ $\forall$ partition P of $[a,b]$.

Exc: Suppose take any partition $P = \{a = x_0 < x_1 < \cdots < x_K = b\}$ of $[a,b]$ and take pts $t_1 \in [x_0, x_1]$, $t_2 \in [x_1, x_2]$, $\ldots$ $t_K \in [x_{K-1}, x_K]$. Show that

$\left| \sum_{i=1}^{K} f(t_i)(x_i - x_{i-1}) \right| \le M(b-a)$, where if $|f(x)| \le M$ $\forall$ $x \in [a,b]$.

Therefore if f is cont, then $\cancel{L(f,P) \le U(f,P) \le M(b-a)}$,

$-M(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$

ST $\forall$ partition P of $[a,b]$.

In part, the sets

$\{L(f,P) : P$ is a partition of $[a,b]\}$

$\{U(f,P) : P$ " " " " $[a,b]\}$ are both

non empty and bdd.

Def^n: $\underline{\int_a^b} f(x)\,dx := \sup\{L(f,P) : P$ is a partition of $[a,b]\}$

= lower Riemann integral of f on/over $[a,b]$.

$\overline{\int_a^b} f(x)\,dx := \inf\{U(f,P) : P$ is a partition of $[a,b]\}$

= Upper Riemann Integral of f on/over $[a,b]$.

Exc: $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$

Def^n: f is called Riemann integrable on $[a,b]$ if

$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx =: I$ (say). In this case, we define I to be the Riemann Integral of f on/over $[a,b]$ and write

$$I = \int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx$$

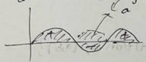
tsoa

Notation: $R[a,b]$ = the set of all Riemann integrable functions on $[a,b]$.

$f \in R[a,b]$ will mean f is Riemann integrable on $[a,b]$.

Proposition: $f \in R[a,b] \Rightarrow$ (i) $|f| \in R[a,b]$,

(ii) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx \quad \forall\, a \le b$



Exc: Show that $f:[0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$, is not a Riemann integrable $[0,1]$.

Thm: (i) If $f:[a,b] \to \mathbb{R}$ is a func^n with ctbly many discontinuity's then $f \in R[a,b]$. In part, any cont func^n of $[a,b]$ is Riemann integrable on $[a,b]$.

(ii) If $f \in R[a,b]$ and $g: \mathbb{R} \to \mathbb{R}$ is cont then

$$\begin{tikzcd} {[a,b]} \arrow[r, "f"', "\in R[a,b]"] \arrow[rr, bend right, "g \circ f \in R[a,b]"'] & \mathbb{R} \arrow[r, "g", "\text{cont}"'] & \mathbb{R} \end{tikzcd}$$

Def^n: Suppose I is a (bdd or unbdd) open interval and $f: I \to \mathbb{R}$ is a map. A diff func^n $F: I \to \mathbb{R}$ is called a primitive/an antiderivative of f on I if $F'(x) = f(x) \;\forall\, x \in I$.

Notation: $F(x) = \int f(x)\,dx$

Remark: If F is a primitive of f on I, then so is $F+c$ for any real constant c.

Exc: If F_1 and F_2 are two primitives of f on an open interval I, then $\exists\, c \in \mathbb{R}$ s.t. $F_1(x) = F_2(x) + c \;\forall\, x \in I$.

Thm: (Second Fundamental thm of Calculus) Suppose I is an open interval and $f: I \to \mathbb{R}$ is a cont func^n. Let $F: I \to \mathbb{R}$ be a primitive or antiderivative of f on I. Then $\forall\, c, x \in I$, we have $F(x) = F(c) + \int_c^x f(t)\,dt$.

i.e, $\int_c^x f(t)\,dt = F(x) - F(c)$

[We have defined $\int_a^b f(x)\,dx$ when $a<b$, f is Riemann integral $f \in R[a,b]$.] $\int_a^a f(x)\,dx = 0$.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \quad (\text{if } a<b)$$

Thm: (First Fundamental thm of Calculus) Suppose $f \in R[a,b]$. Fix $c \in [a,b]$ and define $F:[a,b] \to \mathbb{R}$ by $F(x) = \int_c^x f(t)\,dt$, $x \in [a,b]$

Then we have:

(i) F is cont on $[a,b]$.

(ii) If $x_0 \in (a,b)$ is a pt of continuity of f, then F is diff at x_0. and $F'(x_0) = f(x_0)$

Remark: Integration makes a function "smoother" than before.

Exc:- Prove the second fundamental thm using 1st.

28/10/24

Sketch of proof:-

(i) $f \in \mathcal{R}[a,b] \Rightarrow f$ bdd , Fix $c \in [a,b]$

Suppose $|f(x)| \leq M$ $\forall$ $x \in [a,b]$. To show:

$F: [a,b] \to \mathbb{R}$ defined by $F(x) = \int_c^x f(t)dt$, $x \in [a,b]$

is cont.

Fix $x_0 \in [a,b]$. To show F is cont at x_0.

Taking any $x \in [a,b]$, we look at $|F(x) - F(x_0)|$

Recall:- $\forall$ $a<b$, $g \in \mathcal{R}[a,b]$,

$$\left|\int_a^b g(t)\,dt\right| \leq \int_a^b |g(t)|\,dt \quad (a<b)$$

$$|F(x) - F(x_0)| = \left|\int_c^x f(t)dt - \int_c^{x_0} f(t)dt\right|$$

$$= \left|\int_{x_0}^x f(t)\,dt\right| = \left|\int_x^{x_0} f(t)\,dt\right|$$

$$= \begin{cases} \int_{x_0}^x |f(t)|\,dt, & x \geq x_0 \\ \int_x^{x_0} |f(t)|\,dt, & x < x_0 \end{cases}$$

$$\leq \begin{cases} M(x - x_0) & \text{if } x \geq x_0 \\ M(x_0 - x) & \text{if } x < x_0 \end{cases}$$

$$\leq M|x - x_0|$$

Hence $\forall$ $x \in [a,b]$, $F(x) - F(x_0)$ is bdd by M times abs $x - x_0$.

$$|F(x) - F(x_0)| \leq M|x - x_0|$$

Exc: Using (***) show that f is cont at x_0.

[Hint: For $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M}$]

(ii) If $x_0 \in (a,b)$ s.t f is cont at x_0. To show,

To show:- F is diff at x_0 and $F'(x_0) = f(x_0)$

Take h small enough in absolute value so that $x_0 + h \in (a,b)$.

$$F(x_0+h) - F(x_0) = \int_{x_0}^{x_0+h} f(t)\,dt$$

$$\Rightarrow F(x_0+h) - F(x_0) = h f(x_0) + \int_{x_0}^{x_0+h} (f(t) - f(x_0))\,dt$$

$$\Rightarrow \frac{F(x_0+h) - F(x_0)}{h} = f(x_0) + \underbrace{\frac{1}{h}\int_{x_0}^{x_0+h} f(t) - f(x_0)\,dt}_{G(h)}$$

If we show $\lim_{h \to 0} G(h) = 0$, then we will get

$$\lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h} = f(x_0)$$ proving F is diff at x_0

and $F'(x_0) = f(x_0)$

Using two cases i) $h \in (0,\delta)$ ii) $h \in (-\delta, 0)$

Using continuity of f at x_0, if can be shown that $\forall$ $\varepsilon > 0$ $\exists$ $\delta = \delta(\varepsilon) > 0$ s.T $0 < |h| < \delta \Rightarrow |G(h)| < \varepsilon$

establishing $\left(\lim_{h \to 0} G(h) = 0\right)$

Ref: Apostol's calculus Vol I.

Q^n: How to find a primitive of a Riemann Integrable function?

A^{n}s: • List of derivatives $\leftrightarrow$ List of primitives

• Method of substitution

• Integration by parts

Thm: (Substitution Thm for Riemann Integrals)
Assume that a real-valued f^n g has a cont derivative, g' on open interval I (bdd or unbdd). Let $J = \{g(x) : x \in I\}$. Assume that $f : J \to \mathbb{R}$ is another f^n which is cont on J. Then for each $x \in I$ and each $c \in I$, we have,

$$\int_c^x f(g(t)) \cdot g'(t)\, dt = \int_{g(c)}^{g(x)} f(u) \cdot du$$

Remember/Use: Put $u = g(t) \Rightarrow du = g'(t)\,dt$.

Exc:- $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = ?$ [Hint: Put $x = \sin\theta$]

Thm:- [Integration by Parts]
Suppose $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are two continuously diff func^n. Then

$$\int_a^b f(x) g'(x)\,dx = \left(f(b)g(b) - f(a)g(a)\right) - \int_a^b f'(x) g(x)\,dx$$

Exc: $\int_0^{\pi/2} x \cos x\,dx = ?$ [Use int by parts with $f, g : [0, \frac{\pi}{2}] \to \mathbb{R}$ defined by $g(x) = \sin x$ and $f(x) = x$]

Improper Riemann Integral

Question:

1) How to define Riemann integrals on unbdd integrals? $\to$ Improper Riemann Integral of 1st kind

2) How to define Riemann integrals for unbdd func^n on bdd intervals?
$\to$ Improper Riemann Integral of 2nd kind.

Ans to Q1: Fix $a \in \mathbb{R}$. Suppose $f : [a,\infty) \to \mathbb{R}$ is a bdd func^n s.t $f \in \mathcal{R}[a,b]$ $\forall\ b > a$.
If further, $\lim_{b\to\infty} \int_a^b f(x)\,dx = I \in \mathbb{R}$ exists and is finite, then we say that f is Riemann integrable on $[a,\infty)$ (and write $f \in \mathcal{R}[a,\infty)$) and define the improper integral of first kind as

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx = I$$

In this case, we also say (when $f \geq 0$) that integral $\int_a^\infty f(x)\,dx$ exists/converges. On other hand, if $f \geq 0$ and $f \in \mathcal{R}[a,b]$ $\forall\ b > a$ but

$$\lim_{b\to\infty} \int_a^b f(x)\,dx = \infty,$$ then we say that $\int_a^\infty f(x)\,dx$ diverges or DNE.

Similarly, fixing $b \in \mathbb{R}$ and taking limit (if exists and is finite) as $a \to -\infty$, we can define Riemann integrability on $(-\infty, b]$ and $\int_{-\infty}^b f(x)\,dx := \lim_{x\to-\infty} \int_a^b f(x)\,dx$

Provided lim exists and is finite.

Q^n:- How to define $\int_{-\infty}^{\infty} f(x)\,dx$?

$f(x) = x$, $\int_0^{\infty} x\,dx = \infty$, $\int_{-\infty}^{0} x\,dx = -\infty$.

Ans:- Suppose $f: \mathbb{R} \to \mathbb{R}$ is a fu^n such that

(*) ... $f \in \mathcal{R}(-\infty, c] \cap \mathcal{R}[c, \infty)$ for some $c \in \mathbb{R}$. Then we say that f is Riemann integrable on $\mathbb{R} = (-\infty, \infty)$ (and write $f \in \mathcal{R}(-\infty, \infty)$) and define

(**) ... $\int_{-\infty}^{\infty} f(x)\,dx = \int_{\mathbb{R}} f(x)\,dx \doteq \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$

Remark:- (**) is well defined. In other words if (*) holds for some $c \in \mathbb{R}$, then (*) holds for all $c \in \mathbb{R}$, and this value (**) does not depend on the choice c.

Ex:- $\int_0^{\infty} e^{-x}\,dx = ?$ $\forall\, b > 0$ $f(x) = e^{-x} \in \mathcal{R}[0,b]$

and $\int_0^b e^{-x}\,dx = \left[-e^{-x}\right]_0^b = 1 - e^{-b}$

$\Rightarrow \underset{b\to\infty}{Lt} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} (1 - e^{-b}) = 1$

$\Rightarrow \int_0^{\infty} e^{-x}\,dx = 1$

$\left(\int_0^{\infty} e^{-x}\,dx = \left[-e^{-x}\right]_0^{\infty} = 1 - 0 = 1 \right.$

Eg:- $\int_1^{\infty} \frac{dx}{x^2} = ?$

$f(x) = \frac{1}{x^2} \in \mathcal{R}[1, b]$ $\forall\, b > 1$

$\int_1^b \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^b$

$= \left[\frac{1}{x}\right]_b^1 = 1 - \frac{1}{b}$.

$\underset{b\to\infty}{Lt} \int_1^b \frac{dx}{x^2} = \underset{b\to\infty}{Lt} \left(1 - \frac{1}{b}\right) = 1$.

$\Rightarrow \int_1^{\infty} \frac{dx}{x^2} = 1$.

Exc:- Fix a number $s \in (0, \infty)$. Show that

$\int_1^{\infty} \frac{dx}{x^s} = \begin{cases} \text{converges iff } s > 1 \\ \text{diverges iff } s \le 1 \end{cases}$. Also find the value of integral when $s > 1$.

[Take case of $s = 1$ case seperately]

29/10/24

Q) $P_1 \subseteq P_2$ then $\|P_2\| < \|P_1\|$

(P_2 is refinement of P_1)

A) Let $P_1 = \{x_0, x_1, \dots x_n\}$, $P_2 = \{y_0, y_1, \dots y_k\}$

Then $\forall\, x_r = y_s$ for some s $(0 \le s \le k)$ $(0 \le r \le n)$

i.e. $\forall\, 0 \le r \le n$, $x_r \in P_1 \subseteq P_2$. Hence $\exists\, s$ st $0 \le s \le k$ and $x_r = y_s$.

Put $t \in \{1, 2, \dots n-1\}$ then $\exists\, s$ s.t $\{1, 2, 3, \dots k+1\}$

$[y_{t-1}, y_t] \subseteq [x_{s-1}, x_s]$

$\Rightarrow |y_t - y_{t-1}| \le |x_s - x_{s-1}|$ $1 \le s \le k$

$\le \max |x_s - x_{s-1}| = \|P_1\|$

i.e for every t inside $\{0, 1, 2, \dots n\}$,

$|y_t - y_{t-1}| \le \|P_1\|$, a const

$\Rightarrow \max_{1 \le t \le n} |y_t - y_{t-1}| \le \|P_1\| \Rightarrow \|P_2\| \le \|P_1\|$

Q) Take any partition $P = \{a = x_0 < x_1 < \dots x_k = b\}$ of $[a,b]$ and take pts $t_1 \in [x_0, x_1], \dots t_k \in [x_{k-1}, x_k]$

(i) S.T $\left| \sum_{i=1}^{k} f(t_i)(x_i - x_{i-1}) \right| \le M(b-a)$ if $|f(x)| \le M$ $\forall x \in [a,b]$

A) $$\left| \sum_{i=1}^{k} f(t_i)(x_i - x_{i-1}) \right| \le \sum_{i=1}^{k} |f(t_i)| (x_i - x_{i-1})$$
$$\le M \sum_{i=1}^{k} (x_i - x_{i-1})$$
$$= M(x_k - x_0) = M(b-a)$$

(ii) If f is cont on $[a,b]$. S.T $-M(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$ $\forall$ partition P of $[a,b]$ & for some $M \in \mathbb{R}$.

Pf:- Since f is cont on $[a,b]$, $\exists M$ s.t $|f(x)| \le M$, $\forall x \in [a,b]$. Also $\sup f$, $\inf f$ all exist on $[x_{i-1}, x_i]$ $\forall$ $1 \le i \le k$.

Now, $-M \le \inf_{x \in [x_{i-1}, x_i]} f(x) \le \sup_{x \in [x_{i-1}, x_i]} f(x) \le M$

$$\sum_{i=1}^{k} -M(x_i - x_{i-1}) \le \sum_{i=1}^{k} \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) \le \sum_{i=1}^{k} \sup_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1})$$
$$\le \sum_{i=1}^{k} M(x_i - x_{i-1})$$

$\Rightarrow$ $-M(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$

* $L(P,f) = L(f,P)$, $U(P,f) = U(f,P)$

Q) $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$

A:-

<u>Lemma</u>:- Let P, Q be any two partitions then
$$L(P,f) \le U(Q,f)$$

Pf:- $P_1 = P \cup \{y\}$, $y \in [x_{k-1}, x_k]$ be the refinement of P.

Note. $[x_{k-1}, x_k]$ got divided into $[x_{k-1}, y]$ & $[y, x_k]$

$$\inf_{x \in [x_{k-1}, x_k]} f(x)(x_k - x_{k-1}) = \inf_{x \in [x_{k-1}, x_k]} f(x)(x_k - y + y - x_{k-1})$$
$$\le \inf_{x \in [x_{k-1}, y]} f(x)(y - x_{k-1}) + \inf_{x \in [y, x_k]} f(x)(x_k - y) \longrightarrow (1)$$

$$(x_k - y + y - x_k) \inf_{x \in [x_{k-1}, x_k]} f(x) = (x_k - y) \inf_{x \in [x_{k-1}, x_k]} f(x) + (y - x_k) \inf_{x \in [x_{k-1}, x_k]} f(x)$$
$$\le (x_k - y) \inf_{x \in [y, x_k]} f(x) + (y - x_k) \inf_{x \in [x_{k-1}, y]} f(x)$$

Add $\sum_{i \ne k} \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1})$ to L.H.S & R.H.S of (1) to get $L(f,P) \le L(f,P_1)$

(Ex:- Show that $U(Q_1, f) \le U(Q,f)$, $Q_1 = Q \cup \{t\}$)

$P \cup \{q_1\} \cup \{q_2\} \dots = P \cup Q$.

$L(f,P) \le L(P \cup Q, f)$

$|||^{ly}$ from $Q \cup \{P_1\} \cup \{P_2\} \dots = P \cup Q$.

$U(Q \cup P, f) \le U(Q,f)$

For fixed partition
$L(\tilde{P},f) \le U(\tilde{P},f)$ use it for $\tilde{P} = P \cup Q$.

Fix P-Part $\Rightarrow L(P,f) \le \int_a^{\bar{b}} f(x)\,dx$

Now vary P to get $\underline{\int_a^b} \le \overline{\int_a^b}$

30/10/24

Lecture-16

Q? Suppose $a \in \mathbb{R}$, $f : [a,\infty) \to [0,\infty)$ is a function.
How to check if $f \in \mathcal{R}[a,\infty)$?

Directly evaluation or use Tests.

thm:- (Comparision Test) Suppose $a \in \mathbb{R}$ and $f : [a,\infty) \to [0,\infty)$ and $g : [a,\infty) \to [0,\infty)$ are two maps. Suppose that $f, g \in \mathcal{R}[a,b]$ $\forall\, b > a$. and $0 \le f(x) \le g(x)$ $\forall\, x \ge a$. Then we have:

(i) If $\int_a^\infty g(x)\,dx < \infty$, then $\int_a^\infty f(x)\,dx < \infty$.

(ii) If $\int_a^\infty f(x)\,dx = \infty$, then $\int_a^\infty g(x)\,dx = \infty$.

thm:- (Ratio Test of improper integrals of 1st kind)
Suppose $a \in \mathbb{R}$, and $f : [a,\infty) \to [0,\infty)$ and $g : [a,\infty) \to (0,\infty)$ are two f^{n}s s.t $f, g \in \mathcal{R}[a,b]$ $\forall\, b > a$. If

(*) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c \in (0,\infty)$, then either both $\int_a^\infty f(x)\,dx$, $\int_a^\infty g(x)\,dx$ converges or both diverges.

Moreover if the limit in (*) $c = 0$, then
$\int_a^\infty g(x)\,dx < \infty \Rightarrow \int_a^\infty f(x)\,dx < \infty$

Eg:- ① For any $s \in \mathbb{R}$, $\int_1^\infty x^s e^{-x}\,dx < \infty$.

Pf:- Note that $\lim_{x\to\infty} \frac{x^s e^{-x}}{x^{-2}} = \lim_{x\to\infty} \frac{x^{s+2}}{e^x} = 0$

Hence $\exists\, A > 1$ s.t $x \ge A \Rightarrow \frac{x^s e^{-x}}{x^{-2}} < 1$

i.e $x^s e^{-x} < x^{-2}$ $\forall\, x \ge A$

We know $\int_A^\infty x^{-2}\,dx \le \int_1^\infty x^{-2}\,dx < \infty$.

$\xrightarrow{\text{Comparision Test}}$ $\int_A^\infty x^s e^{-x}\,dx < \infty$

$\Rightarrow \int_1^\infty x^s e^{-x}\,dx < \infty$.

Eg:- ② $\int_1^\infty \frac{dx}{x^2 + x^4 + x^3} < \infty$

Pf:- $\lim_{x\to\infty} \frac{\frac{1}{x^2+x^3+x^4}}{\frac{1}{x^4}} = \lim_{x\to\infty} \frac{1}{\frac{1}{x^2} + \frac{1}{x} + 1} = 1 \in (0,\infty)$

Also, $\int_1^\infty \frac{dx}{x^4} < \infty$ (An Exc)

$\xrightarrow{\text{Ratio Test}}$ $\int_1^\infty \frac{dx}{x^2 + x^3 + x^4} < \infty$

Exc:- check if the following improper integrals converge? or diverge

(i) $\int_1^\infty \frac{dx}{\sqrt{x} + x^2}$ (ii) $\int_1^\infty \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$ (iii) $\int_1^\infty \frac{e^{-x}}{\sqrt{x} + x^2}\,dx$

(use another excessise)

Sequence of Functions:- Def^n and Examples

If for each $n \in \mathbb{N}$, $f_n : S \to \mathbb{R}$ is a function, $(S \subseteq \mathbb{R})$ then we call $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions defined on S.

Eg:- 1) $S = [0,1]$. Define for each $n \in \mathbb{N}$,
$f_n(x) = x^n, \quad x \in [0,1]$.

Eg:- 2) $S = \mathbb{R}$. Define for each $n \in \mathbb{N}$,
$f_n(x) = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots \frac{x^n}{n!}, \quad x \in \mathbb{R}$.

Eg:- 1) $\lim_{n\to\infty} f_n(x) = g(x) = \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 1, & \text{if } x = 1 \end{cases}$

Pointwise limit → of $\{f_n\}_{n\in\mathbb{N}}$

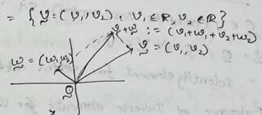
$f_n(x)$ is cont, but $g(x)$ not cont.

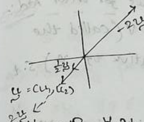
LINEAR ALGEBRA

Recall that $\mathbb{R}$ satisfies (A1) – (A6). Note that $\mathbb{C}$ also satisfies (A1) – (A6).

Motivating Example

$V = \mathbb{R}^2 = \{ \underset{\sim}{v} = (v_1, v_2) : v_1 \in \mathbb{R}, v_2 \in \mathbb{R} \}$





$\forall \alpha \in \mathbb{R}, \ \forall \underset{\sim}{u} \in V = \mathbb{R}^2, \quad \alpha \underset{\sim}{u} = \alpha(u_1, u_2) = (\alpha u_1, \alpha u_2)$
Scalar mult — componentwise multip

Note:- We have a binary operation
$V \times V \to V$ ← vector add
$(\underset{\sim}{v}, \underset{\sim}{w}) \mapsto (\underset{\sim}{v} + \underset{\sim}{w})$

called vector addition on $V = \mathbb{R}^2$ and another map, $\mathbb{R} \times V \to V$, $(\alpha, \underset{\sim}{u}) \longmapsto \alpha \underset{\sim}{u}$ called Scalar multiplication (of vectors) such that the following Properties hold: (that follow from (A1) to (A6))

(VS 1) Associativity of vector Addition:
$\forall \ \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V$

$(\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w} = \underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w})$, (VS2) Commutativity of vector addition:-

$\forall\ \underset{\sim}{u}, \underset{\sim}{v} \in V,\quad \underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$

(VS3) Existence of identity element for vector add:-

$\exists\ \underset{\sim}{0} \in V$ (called the zero vector) s.t

$\underset{\sim}{v} + \underset{\sim}{0} = \underset{\sim}{v}\quad \forall\ \underset{\sim}{v} \in V.$

$\underset{\sim}{0}$ = Identity element for vector add.

(VS4) Existence of Inverse elements for Vector Add:-

For each $\underset{\sim}{v} \in V$, $\exists\ -\underset{\sim}{v} \in V$ (called the (vector) additive Inverse / Negative of $\underset{\sim}{v}$) s.t

$\underset{\sim}{v} + (-\underset{\sim}{v}) = \underset{\sim}{0}$

(VS5) Compatibility of Scalar Multiplication with the ~~Vector addition~~ Multiplication of Real Numbers

$\forall\ \alpha, \beta \in \mathbb{R}$ and $\forall\ \underset{\sim}{v} \in V$, ~~$\alpha(\beta \underset{\sim}{x}) = \alpha\beta(\underset{\sim}{x})$~~

$\alpha(\beta \underset{\sim}{v}) = (\alpha\beta)\underset{\sim}{v}$

(VS6) Existence of Identity element for scalar multiplication:-

the multiplicative Identity element 1 of $\mathbb{R}$ satisfies

$\underset{\substack{\uparrow \\ \text{Identity element} \\ \text{for scalar Multiplication}}}{1} \underset{\sim}{v} = \underset{\sim}{v}\quad \forall\ \underset{\sim}{v} \in V$

(VS7) Distributivity of Scalar Multiplication over / w.r.t Vector Addition:

$\forall\ \alpha \in \mathbb{R}$ and $\forall\ \underset{\sim}{u}, \underset{\sim}{v} \in V$,

$\alpha(\underset{\sim}{u} + \underset{\sim}{v}) = \alpha \underset{\sim}{u} + \alpha \underset{\sim}{v}.$

(VS8) Distributivity of Scalar Multiplication over / w.r.t Add of Real Numbers

$\forall\ \alpha, \beta \in \mathbb{R}$, $\forall\ \underset{\sim}{v} \in V$,

$(\alpha + \beta)\ \underset{\substack{\uparrow \\ v\ \text{tilde}}}{\underset{\sim}{v}} = \alpha \underset{\sim}{v} + \beta \underset{\sim}{v}$

Exc: Prove / verify (VS1) to (VS8) for $V = \mathbb{R}^2$.

(L-14)

Exc: S.T $f: [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ is not a R.I on $[0,1]$.

A: For any partition P of $[0,1]$, $P = \{0, x_1, x_2, \ldots x_{n-1}\}$

$\forall\ x > y$, $\exists\ q \in (x,y)$ as well as $r \in (x,y)$

where $q \in \mathbb{Q}$, $r \notin \mathbb{Q}$

Hence $\inf_{x \in [x_{i-1}, x_i]} f(x) = 0$, $\sup_{x \in [x_{i-1}, x_i]} f(x) = 1$

$L(f,P) = 0$

Illy $U(f,P) = 1(x_1 - x_0) + 1(x_2 - x_1) + \cdots 1(x_n - x_{n-1})$
$= 1$

Since this is true for any part P,

$\int_0^1 f(x)\,dx \neq \int_0^1 f(x)\,dx = 0.$

Exc:- F_1, F_2 are 2 Primitives

$F_1(x) = F_2(x) + C\quad \forall\ x \in \mathbb{R}.$

P.

~~Pf: Define $G(x) = F_1(x) - F_2(x)$ Since F_1, F_2 are primitives they are diff. Hence G is diff~~

Pf:- Define $G(x) = F_1(x) - F_2(x)$

Since F_1, F_2 are primitives they are diff.

Hence G is diff

$$G'(x) = F_1'(x) - F_2'(x)$$
$$= f(x) - f(x) = 0$$

i.e $G' = 0$ on $\mathbb{R}$

(already proved if $g' = 0$, then $g = \text{const}$)

$\Rightarrow G(x) = c$ for some $c \in \mathbb{R}$

$\Rightarrow F_1(x) = F_2(x) + c$

Q) Prove 2nd F.T.C

Statement:- Suppose I - open interval $f: I \to \mathbb{R}$ is a cont func. Let $F: I \to \mathbb{R}$ be a primitive of f on I. Then $\forall\, c, x \in I$, we have

$$F(x) = F(c) + \int_c^x f(t)\,dt$$

Pf:- Let $A(x) = \int_c^x f(t)\,dt$

Since f is cont, it is R.I. So A is well-defined

And by 1st FTC, $A'(x) = f(x) \quad \forall\, x \in I$

i.e, A is a primitive of f.

Also F is given to be the primitive of f by

Previous Q, $F(x) - A(x) = K$, a const,

Since $A(c) = \int_c^c f(t)\,dt = 0$, $K = F(c)$

i.e, $F(x) - A(x) = K = F(c)$

$\Rightarrow F(x) = A(x) + F(c)$

$\Rightarrow F(c) + \int_c^x f(t)\,dt$

14/11/24

LECTURE-17

* Fix $n \in \mathbb{N}$.

$V = \mathbb{R}^n = \{\underset{\sim}{v} = (v_1, v_2, \dots v_n)\}$ and define vector addition component wise, i.e,

$\underset{\sim}{v} + \underset{\sim}{w} := \{v_1 + \omega_1, v_2 + \omega_2, \dots v_n + \omega_n\}$

where $\underset{\sim}{v} = (v_1, v_2, \dots v_n)$ and $\underset{\sim}{w} = (\omega_1, \omega_2, \dots \omega_n) \in V = \mathbb{R}^n$

Also define scalar multiplication

$\alpha \underset{\sim}{v} = \alpha(v_1, v_2, \dots v_n) = (\alpha v_1, \alpha v_2, \dots \alpha v_n)$

where $\alpha \in \mathbb{R} \Rightarrow$ Scalar, $\underset{\sim}{v} \in V = \mathbb{R}^n$.

It is easy to check that $V = \mathbb{R}^n$ also satisfies (VS1)-(VS8)

* Let $X = [0,1]$ and look at the space $V = \mathbb{R}^X$

$V = \mathbb{R}^X = \{f \mid f: X \to \mathbb{R} \text{ is a map}\}$ of all real valued f's defined on $X = [0,1]$.

Eg: Suppose $0 < |A| < \infty$, $0 < |B| < \infty$, A, B are sets.

then How many maps from $B \to A$? $\Rightarrow |A|^{|B|}$.

$A^B - \{f : f: B \to A \text{ is a map}\}$

* Define vector addition and scalar multiplication pointwise, i.e, $\forall\, \alpha \in \mathbb{R}$ and $\forall\, f, g \in V$, define $f+g \in V$ and $\alpha f \in V$ as follows:

$(f+g)(x) = f(x) + g(x)$, $x \in X$

$(\alpha f)(x) = \alpha f(x)$, $x \in X$

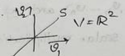
It can be checked that V also satisfies (VS1) to (VS8)

With the zero function $\underset{\sim}{0}$ (i.e., $\underset{\sim}{0}(x) = 0 \;\forall\, x \in X$)

being the identity element for vector addition and the negative of a f^n $f \in V$ (i.e, $(-f)(x) := -f(x)$ $\forall x \in X$) being the vector additive inverse of f.

Ex:- Define $S = \{(v_1, v_2) \in V = \mathbb{R}^2 : v_1 = v_2\} \subseteq V = \mathbb{R}^2$

Let S borrow the vector addition and scalar multiplication from $V = \mathbb{R}^2$. One can check that S also satisfies (VS1)-(VS8) with these borrowed operation.



Vector Add	Scalar Multi
$S \times S \to S$	$\mathbb{R} \times S \to S$
$(\underset{\sim}{v}, \underset{\sim}{w}) \mapsto (\underset{\sim}{v} + \underset{\sim}{w})$	$(\alpha, \underset{\sim}{v}) \mapsto \alpha \underset{\sim}{v}$

Def^n: Let V be a non-empty set with a binary operation called vector addition on V, i.e, a map $V^2 = V \times V \to V$ defined by

$$(\underset{\sim}{u}, \underset{\sim}{v}) \mapsto \underset{\sim}{u} + \underset{\sim}{v} \quad \forall (\underset{\sim}{u}, \underset{\sim}{v}) \in V^2$$

and a scalar multiplication by real numbers, i.e, a map $\mathbb{R} \times V \to V$ defined by

$$(\alpha, \underset{\sim}{v}) \mapsto \alpha \underset{\sim}{v} \quad \forall \underset{\sim}{v} \in V$$

Such that the axioms (VS1)-(VS8) are satisfied. then we say that V is a vector space / linear space over $\mathbb{R}$.

Remarks: ① Elements of $V \to$ vectors
" of $\mathbb{R} \to$ scalars

② One can define a vector space over $\mathbb{C}$ or more generally over any "field" F (Satisfies (A1) to (A6) axioms).

For this course $F = \mathbb{R}$.

Eg:- 1) $V = \mathbb{R}^n$, $n \in \mathbb{N}$ with component wise vector addition and component wise scalar multiplication, is a vector space over $\mathbb{R}$. We shall define $\mathbb{R}^0 := \{\underset{\sim}{0}\}$.

2) $X = [0,1]$, $V = \{f \mid f : X \to \mathbb{R}$ is a map.$\}$

Define both vector addition and scalar multiplication pointwise. Then V is a vector space over $\mathbb{R}$.

Thm:- Suppose V is a vector space over $\mathbb{R}$. Then $\forall$ $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V$ and $\forall \alpha, \beta, \gamma \in \mathbb{R}$, we have

(i) $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{w} \Rightarrow \underset{\sim}{v} = \underset{\sim}{w}$
(Cancellation law for vector addition)

(ii) $\underset{\sim}{0}$ is unique $\Rightarrow$ Identity element for Vector Add

(iii) $-\underset{\sim}{v}$ is unique (given a $\underset{\sim}{v}$).

(iv) $\underset{\sim}{v} + \underset{\sim}{v} = \underset{\sim}{v} \Rightarrow \underset{\sim}{v} = \underset{\sim}{0}$

(v) $0 \underset{\sim}{v} = \underset{\sim}{0}$

(vi) $\alpha \underset{\sim}{0} = \underset{\sim}{0}$

(vii) $\alpha \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow \alpha = 0$ or $\underset{\sim}{v} = \underset{\sim}{0}$

i.e $\alpha \underset{\sim}{v} = \underset{\sim}{0}$, $\alpha \in \mathbb{R} - \{0\} \Rightarrow \underset{\sim}{v} = \underset{\sim}{0}$

(Cancellation law for scalar Multiplic)

(viii) $(-\alpha) \underset{\sim}{v} = -(\alpha \underset{\sim}{v})$. In particular, $(-1)\underset{\sim}{v} = -\underset{\sim}{v}$

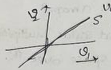
Exc:- (optional) Prove (i) - (viii) above in generality

Mandatory:- Prove them for $V = \mathbb{R}^n$.

Exc:- (Atleast for $\mathbb{R}^n$)

$(\alpha + \beta)\underset{\sim}{0} = (\alpha + \gamma)\underset{\sim}{v} \Rightarrow \underset{\sim}{v} = 0$ or $\beta = \gamma$ $\forall$ $\alpha, \beta, \gamma \in \mathbb{R}$ and $\forall$ $\underset{\sim}{v} \in V$.

Eg:- (Before one) $\underset{\substack{\uparrow \\ \text{vector space}}}{S} = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 = v_2\} \subseteq \underset{\substack{\uparrow \\ \text{Vector space}}}{V = \mathbb{R}^2}$



Def^n:- If V is a vector space over $\mathbb{R}$ and $\phi \neq S \subseteq V$, then S is called a (linear) Subspace / (vector) Subspace of V if S, with vector addition and scalar multiplication borrowed from V, forms a vector space. Notation: $\underset{(\text{Subspace})}{S \subseteq V}$

Eg:- ① $V = \mathbb{R}^2$, $S = \{\underset{\sim}{v} = (v_1, v_2) \in \mathbb{R}^2 : v_1 = v_2\}$. Then S is a linear Subspace of V.

Exc:- ② $V = \mathbb{R}^3$, show that $S := \{\underset{\sim}{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} v_i = 0\}$ is a linear Subspace of V.

Eg:- ③ Let $X = [0,1]$. $V = \mathbb{R}^X = \{f \mid f : X \to \mathbb{R}$ is a map$\}$

V forms a vector space with operations defined point wise.

Exc:- Define $S = \{f \in V : f$ is cont$\}$
$= \{f \mid f : [0,1] \to \mathbb{R}$ is a cont map$\}$.

Show that S is an vector subspace of V.

Remark:- In Ex ③ above, if we define $S_1 = \mathbb{Q}[0,1]$, then it can be shown that S_1 is also a linear Subspace of V. In fact,

$\underset{(\text{Subsp})}{S \subseteq} S_1 \underset{(\text{Subspace})}{\subseteq} V$

thm:- Let V be any vector space (over $\mathbb{R}$). Let S be A non-empty subset $S \subseteq V$ is a (linear / Vector) Subspace of V iff S is closed under taking linear combinations, i.e; $\underbrace{\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}}_{\text{a linear combination}} \in S$ $\forall$ $\alpha, \beta \in \mathbb{R}$, $\forall$ $\underset{\sim}{u}, \underset{\sim}{v} \in S$.

of vectors $\underset{\sim}{u}, \underset{\sim}{v} \in V$ with scalar coeff $\alpha, \beta \in \mathbb{R}$.

Remark:- Closed under taking linear combinations $\Leftrightarrow$ closed under both vector add, scalar multip

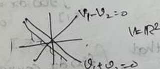
Cor:- (I) $\underset{(\text{Subsp})}{S \subseteq V} \Rightarrow \underset{\sim}{0} \in S$

(ii) $\underset{\substack{\uparrow \\ \text{Smallest possible subspace}}}{S_0} = \{\underset{\sim}{0}\} \underset{(S.S)}{\subseteq} V$ — trivial S.S of V.

(iii) $\underset{(S.S)}{V \subseteq V}$ (largest)

Any other S.S S, i.e, $\{\underset{\sim}{0}\} \subsetneq S \subsetneq V$, is called nontrivial S.S of V.

Exc:



$S = \{(v_1, v_2) \in \mathbb{R}^2 : (v_1 + v_2)(v_1 - v_2) = 0\}$

Show ~~that~~ $S \subseteq V$ but S is not a linear subspace of V.

Eg:- Let $X = [0,1]$, $V = \mathbb{R}^X = \{f \mid f : [0,1] \to \mathbb{R}\}$

Define $\mathcal{P} = \{f \in V : f \text{ is a polynomial } f^n\}$.

Exc: Show that $\underset{(S.S)}{\mathcal{P} \subseteq V}$.

Fix a $n \in \mathbb{N}$. Define $\mathcal{P}_n = \{f \in \mathcal{P} : \deg(f) \leq n-1\}$

In part, $\mathcal{P}_1 = \{f \in \mathcal{P} : f$ is a const poly or fn$\}$

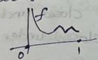
Exc: ~~show that~~ Fix $n \in \mathbb{N}$. Show that
$$\mathcal{P}_n \underset{(S.S)}{\subseteq} \mathcal{P} \underset{(S.S)}{\subseteq} V$$

5/11/24

Improper Riemann Integral of Second Kind:

We specialize the following case:

Suppose $f : (0,1] \to \mathbb{R}$ is an unbdd func^n that "only blows up near 0".



i.e $\forall\, \varepsilon \in (0,1)$, $f : [\varepsilon, 1] \to \mathbb{R}$ is bdd. and $f \in \mathcal{R}([\varepsilon,1])$

Ex: $f : (0,1] \to \mathbb{R}$, defined by $f(x) = \frac{1}{\sqrt{x}}$; $x \in (0,1]$

In this situation, if the limit
$$I := \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 f(x)\,dx \in \mathbb{R} \quad \text{exists and is finite,}$$
then we say that f is Riemann Integrable on $(0,1]$ (and write $f \in \mathcal{R}(0,1]$) and define $\int_{(0,1]} f(x)\,dx = \int_0^1 f(x)\,dx := I$ $\}$

In this case we also say that $\int_{(0,1]} f(x)\,dx$ converges, if $f \geq 0$, and write $\int_0^1 f(x)\,dx < \infty$. ~~if~~ finite.

If for $f \geq 0$, $\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 f(x)\,dx = \infty$, we say that $\int_{(0,1]} f(x)\,dx$ diverges and we write $\int_0^1 f(x)\,dx = \infty$

Eg: $f(x) = \frac{1}{\sqrt{x}}$; $x \in (0,1]$. Note that $\forall\, \varepsilon \in (0,1)$, $f \in \mathcal{R}([\varepsilon,1])$.
$$\int_\varepsilon^1 f(x)\,dx = \int_\varepsilon^1 x^{-1/2}\,dx = \left[\frac{x^{-1/2+1}}{-\frac{1}{2}+1}\right]_\varepsilon^1 = \left[2x^{1/2}\right]_\varepsilon^1$$
$$= 2 - 2\sqrt{\varepsilon}$$

Hence, $I := \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 f(x)\,dx = \lim_{\varepsilon \to 0^+} 2 - 2\sqrt{\varepsilon} = 2 < \infty$

$\Rightarrow f \in \mathcal{R}(0,1]$, and $\int_{(0,1]} f(x)\,dx = \int_{(0,1]} \frac{1}{\sqrt{x}}\,dx = 2$.

Exc: Define $\forall\, s \in (0,\infty)$, $f(x) = \frac{1}{x^s}$, $x \in (0,1]$.
Then $\int_{(0,1]} f(x)\,dx = \int_{(0,1]} \frac{1}{x^s}\,dx = \begin{cases} \text{converges} & \text{if } s < 1 \\ \text{diverges} & \text{is } s \geq 1 \end{cases}$

Q: What if the direct evaluation test fails?

Thm: (Comparision Test): Suppose $f : (0,1] \to [0,\infty)$ and $g : (0,1] \to [0,\infty)$ are two unbdd func^s st $\forall\, \varepsilon \in (0,1)$, $f, g \in \mathcal{R}[\varepsilon,1]$. If further $0 \leq f(x) \leq g(x)$ $\forall\, 0 < x \leq 1$ then we have:

(i) $\int_{(0,1]} g(x) < \infty \Rightarrow \int_{(0,1]} f(x)\,dx < \infty$

(ii) $\int_{(0,1]} f(x)\,dx = \infty \Rightarrow \int_{(0,1]} g(x)\,dx = \infty$

Thm: (Ratio Test): Suppose $f : (0,1] \to [0,\infty)$ and $g : (0,1] \to (0,\infty)$ are two contd f^n's s.t. $\forall \, \varepsilon \in (0,1)$ $f, g \in R[\varepsilon, 1]$. If further $\underset{x \to 0^+}{Lt} \frac{f(x)}{g(x)} = c \in (0,\infty)$ ①

then either both $\int_{(0,1]} f(x)\,dx$, $\int_{(0,1]} g(x)\,dx$ converges or both diverges.

Moreover, if $c = 0$ in ① then $\int_{(0,1]} g(x)\,dx < \infty$

$\Rightarrow \int_{(0,1]} f(x)\,dx < \infty$ but vice versa may not be true.

6/11/24

LECTURE - 18

Goal: To understand $V = \mathbb{R}^n$ and its linear Subspaces

Defn:- Let V be a vector space (over $\mathbb{R}$). We take $k \in \mathbb{N}$ and vector $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k \in V$. Any vector of the form

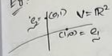
$\underset{\sim}{v} = \alpha_1 \underset{\sim}{v}_1 + \alpha_2 \underset{\sim}{v}_2 + \dots \alpha_k \underset{\sim}{v}_k =: \sum \alpha_i \underset{\sim}{v}_i$,

where $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$, is called a linear combination of $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots \underset{\sim}{v}_k \in V$

Exc:- Let V be a vector space and $\phi \neq S \subseteq V$. Then show that $S \underset{(S.S)}{\subseteq} V$ iff $\forall \, k \in \mathbb{N}$ and $\forall$ $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$ and $\forall$ $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots \underset{\sim}{v}_k \in S$,

$\sum_{i=1}^{k} \alpha_i \underset{\sim}{v}_i \in S$. [Hint:- If part $\to$ just use $k = 2$ + Thm
only if part $\to$ induction on k]

Eg:- $V = \mathbb{R}^2$ and $B = \{ \underset{\sim}{e}_1 := (1,0), \underset{\sim}{e}_2 := (0,1) \}$



Note that any vector $\underset{(v_1, v_2)}{\underset{\sim}{v}} \in V = \mathbb{R}$ can be written as

$\underset{\sim}{v} = (v_1, v_2) = (v_1, 0) + (0, v_2) = v_1(1,0) + v_2(0,1)$
$= v_1 \underset{\sim}{e}_1 + v_2 \underset{\sim}{e}_2 =: \sum_{i=1}^{2} v_i \underset{\sim}{e}_i$

We shall say that V is the "linear span"

$\bar{B} = Sp(B)$ of B.

Defn:- Let V be any vector space. For a nonempty finite set $B = \{ \underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots \underset{\sim}{v}_k \} \subseteq V$ of vectors, we define the linear span of B as the set

$\bar{B} = Sp(B) := \{ \sum \alpha_i v_i : \alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R} \}$ of all possible linear combinations of the elements of B

We also define $\bar{\phi} = Sp\{\phi\} = \{ \underset{\sim}{0} \}$.

Exc:- Let V be a vector space. $B \subseteq V$ is any finite subset. Show that linear span of $\bar{B} \underset{(S.S)}{\subseteq} V$.

* In the above example, $V = \mathbb{R}^2$, $B = \{ \underset{\sim}{e}_1 := (1,0), \underset{\sim}{e}_2 = (0,1) \}$ and $\bar{B} = V = \mathbb{R}^2$ ($\bar{B} \subseteq V$ is obvious, $V \subseteq \bar{B}$ has been shown)

Defn:- In the above setup, whenever $S = \underset{\text{Span}}{\bar{B}} = Sp\{ \underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots \underset{\sim}{v}_k \}$, we say that the vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots \underset{\sim}{v}_k$ generate S or simply say B generates S.

Eg: If $V = \mathbb{R}^2$, then $B = \{ \underset{\sim}{e}_1, \underset{\sim}{e}_2 \}$ generates V.

More generally, if $V = \mathbb{R}^n$, then $B = \{ \underset{\sim}{e}_1, \underset{\sim}{e}_2, \dots \underset{\sim}{e}_n \}$, (where $\underset{\sim}{e_1} = (1,0,\dots,0) \in \mathbb{R}^n$

$\underset{\sim}{e_2} = (0,1,0,\dots,0) \in \mathbb{R}^n$

$\vdots$

$\underset{\sim}{e_n} = (0,0,\dots,0,1) \in \mathbb{R}^n$)

generates V. Bcz (linear span) of $\bar{B} \subseteq V$. To show $V \subseteq \bar{B}$ observe that any $\underset{\sim}{v} = (v_1, v_2, \dots v_n) \in V = \mathbb{R}^n$ is of the form

$$\underset{\sim}{v} = \sum_{i=1}^{n} v_i e_i = \begin{pmatrix} (v_1,0,0,\dots 0) + (0,v_2,0,\dots 0) \\ + \dots + (0,0,\dots, v_n) \end{pmatrix} = v_1 \underset{\sim}{e_1} + v_2 \underset{\sim}{e_2} + \dots v_n \underset{\sim}{e_n} \in \bar{B}$$

Claim:- If $V = \mathbb{R}^2$, then $B_1 = \{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_1} + \underset{\sim}{e_2}\} = \{(1,0), (0,1), (1,1)\}$ also generates V.

Proof:- Clearly $\bar{B}_1 \subseteq V$. To show $V \subseteq \bar{B}_1$. Take any vector $\underset{\sim}{v} = (v_1, v_2) = V = \mathbb{R}^2$. then

$$\underset{\sim}{v} = v_1 e_1 + v_2 \underset{\sim}{e_2} + 0(\underset{\sim}{e_1} + \underset{\sim}{e_2}) \in \bar{B}_1$$

Therefore $\bar{B}_1 = V$.

However $(1,1) = \underset{\sim}{e_1} + \underset{\sim}{e_2}$ is redundant (unnecessary) because $\underset{\sim}{e_1}$ and $\underset{\sim}{e_2}$ generate V anyway! This is happening bcz

$$1(\underset{\sim}{e_1} + \underset{\sim}{e_2}) + (-1)\underset{\sim}{e_1} + (-1)\underset{\sim}{e_2} = \underset{\sim}{0},$$

i.e, $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_1} + \underset{\sim}{e_2}$ are "linearly dependent".

Exc: Show that $\underset{\sim}{e_1} = (1,0)$, $\underset{\sim}{e_1} + \underset{\sim}{e_2} = (1,1)$ also generate $V = \mathbb{R}^2$

Q?: Can we remove one of these and yet generate $V = \mathbb{R}^2$?

Ans: NO! (bcz $\underset{\sim}{e_1}$ and $\underset{\sim}{e_1} + \underset{\sim}{e_2}$ are "linearly independent")

Def^n: ① For any vector space V, a finite non empty collection $C = \{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_k}\} \neq \phi \subseteq V$ of vectors is called "linearly independent" (or $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_k}$ are called "linearly independent") if

$$\sum_{i=1}^{k} \alpha_i \underset{\sim}{v_i} = \underset{\sim}{0}, \quad \alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$$

$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_k = 0$. Trivial linear comb.

In other words, no nontrivial linear combination of $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_k} = \underset{\sim}{0}$.

Eg.: (1) $V = \mathbb{R}^2$. $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_1} + \underset{\sim}{e_2}$ are not linearly independent (i.e linearly dependent) because

$$1(\underset{\sim}{e_1} + \underset{\sim}{e_2}) + (-1)\underset{\sim}{e_1} + (-1)\underset{\sim}{e_2} = \underset{\sim}{0}$$

Non trivial linear comb of $\underset{\sim}{e_1} + \underset{\sim}{e_2}, \underset{\sim}{e_1} + \underset{\sim}{e_2}$

Def^n: ② In the setup of ①, if $\exists\, \alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$ s.t $(\alpha_1, \alpha_2, \dots \alpha_k) \neq (0, 0, \dots 0)$ (i.e atleast one $\neq 0$). and $\sum_{i=1}^{k} \alpha_i \underset{\sim}{v_i} = \underset{\sim}{0}$ (non-trivial linear comb) then $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_k}$ are called linearly dependent or $C = \{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_k}\}$ is called linearly dependent. ** If $C = \phi$, then we define C to be linearly independent.

Eg: (2) $V = \mathbb{R}^n$. Fix $m \in \{1, 2, \dots n\}$. Let $C = \{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots \underset{\sim}{e_m}\}$

Then C is linearly independent

Ans:- Suppose $\underbrace{\sum_{i=1}^{m} \alpha_i \underset{\sim}{e_i}}_{\mathbb{I}} = \underset{\sim}{0}$

$\Rightarrow (\alpha_1, \alpha_2, \alpha_3, \dots \alpha_m, 0, 0, \dots 0) = \underset{\sim}{0}$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \dots \alpha_m = 0.$

$\Rightarrow \underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots \underset{\sim}{e_m}$ are lin independent

Exc: $V = \mathbb{R}^n$. Show that $C = \{\underset{\sim}{e_1}, \underset{\sim}{e_1}+\underset{\sim}{e_2}, \underset{\sim}{e_1}+\underset{\sim}{e_2}+\underset{\sim}{e_3}, \dots, \underset{\sim}{e_1}+\underset{\sim}{e_2}+\dots \underset{\sim}{e_n}\}$ is linearly independent.

[Hint: $C = \{(1,0,0\dots0), (1,1,0,0,\dots0), \dots (1,1,1,1,\dots1)\}$]

Exc:- $V = \mathbb{R}^4$. Show that $\underset{\sim}{e_1}, \underset{\sim}{e_1}+2\underset{\sim}{e_2}, \underset{\sim}{e_1}+\underset{\sim}{e_2}, \underset{\sim}{e_3}, \underset{\sim}{e_4}$ are linearly dependent.

Thm:- Let V be any vector space. Then we have:

(i) If $B (\subseteq V)$ generates V (as v.s.), then so does any finite superset of B which is also a subset of V.

(ii) If $C (\subseteq V)$ is linearly ind, then so is any subset of C.

(iii) If $\underset{\sim}{0} \in C$, then C is always lin dep.

(iv) $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_k} \in V$ are linearly dep iff at least one of these vectors can be written as a linear comb of other vectors. In particular, for $k=2$, $\underset{\sim}{v_1}, \underset{\sim}{v_2}$ are lin dep iff one of them is scalar multiple of other.

Proof:- If part: Suppose WLOG (without loss of generality) that

$\underset{\sim}{v_1} = \sum_{i=2}^{k} \alpha_i \underset{\sim}{v_i}$, where $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$.

Then $\underbrace{1}_{\neq 0}\underset{\sim}{v_1} + (-\alpha_2)\underset{\sim}{v_2} + (-\alpha_3)\underset{\sim}{v_3} + \dots + (-\alpha_k)\underset{\sim}{v_k} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_k}$ are lin dep.

Only if part: Suppose $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_k}$ are linearly dep $\Rightarrow \sum_{i=1}^{k} \alpha_i \underset{\sim}{v_i} = \underset{\sim}{0}$ for some $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$ s.t. $(\alpha_1, \alpha_2, \dots \alpha_k) \neq (0,0,\dots0)$. Let $\alpha_j \neq 0$. Then

$$\alpha_j \underset{\sim}{v_j} = \sum_{\substack{i=1 \\ i \neq j}}^{n} (-\alpha_i)\underset{\sim}{v_i}$$

$$\Rightarrow \underset{\sim}{v_j} = \sum_{\substack{i=1 \\ i \neq j}}^{n} \underbrace{\left(-\frac{\alpha_i}{\alpha_j}\right)}_{\mathbb{R}} \underset{\sim}{v_i}$$

$\Rightarrow \underset{\sim}{v_j}$ is lin comb of rest.

* Note that $B = \{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots \underset{\sim}{e_n}\} \subseteq V = \mathbb{R}^n$ is a lin ind set that generates $V = \mathbb{R}^n$. We have a name for such a set.

The standard/canonical basis for $\mathbb{R}^n$.

Def^n: Let V be any vector space over $\mathbb{R}$. If

$$B = \begin{cases} \{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_k}\} & \text{if } k \in \mathbb{N} \\ \phi & \text{, if } k=0 \end{cases} \subseteq V \text{ (finite subset)}, \quad k \in \mathbb{N} \cup \{0\}$$

is lin indep. Such that $\overline{B} = V$, then we say that V is a finite dimensional vector space over $\mathbb{R}$ and B is called a basis of / for V.

(If $B = \phi$, then $V = \{\underset{\sim}{0}\}$.)

$B \neq \phi$, then V = non trivial vector space

8/11/24

Q^n: How to combine two types of improper Riemann Integrals?

Ans:- When $f:(0,\infty)\to\mathbb{R}$ is S.T it blows up near zero, and $f\in\mathcal{R}(0,1]$ and $f\in\mathcal{R}[1,\infty)$, then we say that f is Riemann Integrable on $(0,\infty)$ (and write $f\in\mathcal{R}(0,\infty)$), and define $\int_{(0,\infty)} f(x)\,dx$

$$\int_{(0,\infty)} f(x)\,dx = \int_0^\infty f(x)\,dx = \int_{(0,1]} f(x)\,dx + \int_{[1,\infty)} f(x)\,dx.$$

Eg:- For each $\alpha\in(0,\infty)$, define a map $f:(0,\infty)\to(0,\infty)$ by $f(x):=x^{\alpha-1}e^{-x}$, $x\in(0,\infty)$.

Clearly (from class) that $f\in\mathcal{R}[1,\infty)$

Q^n:- Is $\int_0^1 f(x)\,dx<\infty$?

Case 1):- $\alpha\geq 1$

Then $f(x)=x^{\alpha-1}e^{-x}$; $x\in[0,1]$.

This is a cont f^n on closed, bdd interval. hence f is $\mathcal{R}[0,1] \Rightarrow \int_0^1 f(x)\,dx<\infty$.

Hence if $\alpha\geq1$, then $f\in\mathcal{R}(0,\infty)$.

Case 2):- $\alpha<1$ means $\alpha\in(0,1)$.

Define $g(x)=\frac{1}{x^{1-\alpha}}$, $x\in(0,1]$, and look at f restricted to $(0,1]$, i.e $f|_{(0,1]}$, i.e

$f(x)=\frac{1}{x^{1-\alpha}}e^{-x}$; $x\in(0,1]$

$f(x)=\frac{1}{x^{1-\alpha}}\cdot e^{-x}$

$g(x)=\frac{1}{x^{1-\alpha}}$

NOTE:- $f,g\in\mathcal{R}[\varepsilon,1]$ $\forall\ \varepsilon\in(0,1)$ and both are $(0,\infty)$ - valued f^ns and $\lim_{x\to0^+}\frac{f(x)}{g(x)}=\lim_{x\to0^+}e^{-x}=1\in(0,\infty)$.

Moreover, $g(x)=\frac{1}{x^s}$, $x\in(0,1]$, where $s=1-\alpha\in(0,1)$ is done as an exc.

Hence $g\in\mathcal{R}(0,1]$ and hence $\int_0^1 g(x)\,dx<\infty$.

So by ratio test $\int_0^1 f(x)\,dx$ is finite. Now, combining everything we get that $\forall\ \alpha\in(0,\infty)$, $\int_0^\infty x^{\alpha-1}e^{-x}\,dx$

$$=\underbrace{\int_0^1 x^{\alpha-1}e^{-x}\,dx}_{<\infty}+\int_1^\infty x^{\alpha-1}e^{-x}\,dx.$$

Def^n:- $\forall\ \alpha\in(0,\infty)$, define $\Gamma(\alpha):=\int_0^\infty x^{\alpha-1}e^{-x}\,dx$

(Gamma Function)

Note that $\Gamma:(0,\infty)\to(0,\infty)$.

*** Exc: Show that (i) $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ $\forall\ \alpha\in(0,\infty)$.

(ii) $\forall\ n\in\mathbb{N}$, S.t $\Gamma(n)=(n-1)!$

(Hint: Use (i) and induction on n).

Exc:- S.T $\int_0^\infty \frac{1}{x^2+\sqrt{x}}\,dx<\infty$

(Hint:- Compare with $\frac{1}{\sqrt{x}}$ on $(0,1]$, with $\frac{1}{x^2}$ on $[1,\infty)$).

Breaking and comparision.

11/11/24

LECTURE-19

Thm:- Suppose V is a finite dimensional vector space over $\mathbb{R}$ (fdvs). Then TFAE (the following are equivalent). And $B \subseteq V$ (finite)

(i) $B = \{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_k}\}$ is a basis for V.

(ii) B is a minimum generating set for V, i.e no proper subset of B generates V.

(iii) B is a maximal linearly independent subset of V, i.e, $\forall\, \underset{\sim}{v} \in V \setminus B$, $B \cup \{\underset{\sim}{v}\}$ is linearly dependent.

(iv) For each vector $\underset{\sim}{w} \in V$ can be written uniquely as a linear combination of $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_k}$ i.e, for each $\underset{\sim}{w} \in V$, $\exists$ unique $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ s.t

$$\underset{\sim}{w} = \sum_{i=1}^{k} \alpha_i \underset{\sim}{v_i}.$$

Eg:- (1) $V = \mathbb{R}^n$

Exc:- Show that the following sets are all bases of $V = \mathbb{R}^n$

(i) $B = \{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_n}\}$ = standard basis for $\mathbb{R}^n$.

(ii) $B_1 = \{\underset{\sim}{e_1}, \underset{\sim}{e_1}+\underset{\sim}{e_2}, \underset{\sim}{e_1}+\underset{\sim}{e_2}+\underset{\sim}{e_3}, \dots, \underset{\sim}{e_1}+\underset{\sim}{e_2}+\dots+\underset{\sim}{e_n}\}$

(iii) $B_2 = \{\underset{\sim}{e_1}, \underset{\sim}{e_1}+\underset{\sim}{e_3}, \underset{\sim}{e_3}, \underset{\sim}{e_4}, \dots, \underset{\sim}{e_n}\}$

Thm:- Let V be fdvs over $\mathbb{R}$. Then any two basis for V have same size

Defn:- Let V be fdvs over $\mathbb{R}$ and B be any basis of V. Then we define the dimensional of V as

$\dim := |B|$

From the exc, it follows that $\dim(\mathbb{R}^n) = n$ $\forall\, n \in \mathbb{N}$. On the other hand $\dim(\{\underset{\sim}{0}\}) = 0$.

Therefore, it is customery to define

$\mathbb{R}^0 := \{\underset{\sim}{0}\}$.

Eg(2) $V = \mathbb{R}^3$. $S = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0\}$

It is possible to check that $S \underset{(S.S)}{\subseteq} V$.

Qn:- $\dim(S) = ?$

$$S = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_3 = -v_1 - v_2\}$$
$$= \{(v_1, v_2, -v_1 - v_2) : v_1, v_2 \in \mathbb{R}\}$$

$\Rightarrow$ any $\underset{\sim}{v} = (v_1, v_2, v_3) \in S$ is of the form

$$\underset{\sim}{v} = (v_1, v_2, -v_1 - v_2) = (v_1, 0, -v_1) + (0, v_2, -v_2)$$
$$\underset{\sim}{v} = v_1(1, 0, -1) + v_2(0, 1, -1)$$

Claim:- $B = \{(1,0,-1), (0,1,-1)\}$ is a basis for S.

Proof:- We just showed that $S \subseteq \bar{B}$.

$B \subseteq S \Rightarrow \bar{B} \subseteq S \Rightarrow S = \bar{B}$.

To show B in lin ind since $S \underset{(S.S)}{\subseteq} V$

Take $\alpha_1, \alpha_2 \in \mathbb{R}$ s.t $\alpha_1(1,0,-1) + \alpha_2(0,1,-1) = (0,0,0)$

$(\alpha_1, \alpha_2, -\alpha_1 - \alpha_2) = (0,0,0)$

$\Rightarrow \alpha_1 = 0,\ \alpha_2 = 0$

$\Rightarrow B$ is lin ind and $\bar{B} = S$.

$\Rightarrow B$ is a basis for S.

In part, dim of $S = 2 = |B|$

$\dim(S) = |B| = 2$.

Exc. $V = \mathbb{R}^3$. Define $S_1 := \{ \underset{\sim}{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 = v_2 = v_3 \}$

$S_2 := \{ \underset{\sim}{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 = v_2 \}$.

(i) Show that $S_1 \underset{(S.S)}{\subseteq} S_2 \underset{(S.S)}{\subseteq} V$.

(ii) Find a basis for S_1 and show that $\dim(S_1) = 1$.

(iii) Find a basis for S_2 and S.T $\dim(S_2) = 2$.

Thm: Suppose V is fdvs and $S \underset{(S.S)}{\subseteq} V$. Then any basis for S can be extended to a basis for V. In part, $\dim(S) \leq \dim(V)$

Def. Let V be a fdvs and S_1, S_2 are two subspaces of V. Define $S_1 + S_2$ to be

$$S_1 + S_2 := \{ \underset{\sim}{v_1} + \underset{\sim}{v_2} : \underset{\sim}{v_1} \in S_1, \underset{\sim}{v_2} \in S_2 \}$$

Exc: Show that $S_1 \cap S_2$ and $S_1 + S_2$ are both subspaces of V.

* Suppose $V = \mathbb{R}^2$. Let $S \underset{(S.S)}{\subseteq} V \Rightarrow \dim(S) \leq \dim(V) = 2$

$\Rightarrow \dim(S) \in \{0, 1, 2\}$

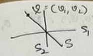
If $S = \{\underset{\sim}{0}\}$, then $\dim(S) = 0$

If $S = V = \mathbb{R}^2$, then $\dim(S) = 2$

For all other subspaces (i.e, nontrivial subspaces of V), $\dim(S) = 1$.

Take any $\underset{\sim}{v} \in S \setminus \{\underset{\sim}{0}\}$.

Then it can be shown that $S = \{\overline{\underset{\sim}{v}}\}$. This means that $S = \{ \alpha \underset{\sim}{v} = (\alpha v_1, \alpha v_2) : \alpha \in \mathbb{R} \}$



$\dim(S) = 1 \Rightarrow S$ is a st.line passing through $\underset{\sim}{0}$.

Take $S_1 = \{ (v_1, v_2) \in \mathbb{R}^2 : v_2 = 0 \}$
$= \{ (v_1, 0) : v_1 \in \mathbb{R} \}$

$S_2 = \{ (v_1, v_2) \in \mathbb{R}^2 : v_1 = 0 \}$
$= \{ (0, v_2) : v_2 \in \mathbb{R} \}$

Clearly $S_1 \cap S_2 = \{(0,0)\} = \{\underset{\sim}{0}\}$.

$S_1 + S_2 = V = \mathbb{R}^2$ (bcz Any vector $\underset{\sim}{v} = (v_1, v_2)$ can be written as $\underset{\sim}{v} = \underset{\in S_1}{(v_1, 0)} + \underset{\in S_2}{(0, v_2)} \Rightarrow S_1 + S_2 = \mathbb{R}^2$)

Thm: Let V be a fdvs with $\dim(V) = n \geq 2$. Let $S \underset{(S.S)}{\subseteq} V$. Then $\dim(S) = 0 \Rightarrow S = \{\underset{\sim}{0}\}$, ~~dim(S) = n~~

$\dim(S) = n \Rightarrow S = V$,

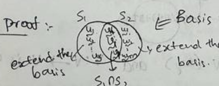
For all other subspaces S of V, i.e whenever $0 < \dim(S) < n$, we have $\{\underset{\sim}{0}\} \underset{\neq}{\subseteq} S \underset{\neq}{\subseteq} V$, i.e; S is a nontrivial subspace of V.

Thm: let V be a fdvs and $S_1 \underset{(S.S)}{\subseteq} V$, $S_2 \underset{(S.S)}{\subseteq} V$. Then

$$\dim(S_1 + S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$$

(Inclusion - Exclusion Formulae for Dimension)

Sketch of Proof :-



$\Leftarrow$ Basis Venn diagram

$\dim(S_1 \cap S_2) = l$, $\dim(S_1) = k + l$, $\dim(S_2) = l + m$

$\dim(S_1 + S_2) = k + l + m$

$\Rightarrow \dim(S_1 + S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$.

Application :- Let $V = \mathbb{R}^2$, $S_1 = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 - v_2 = 0\}$

$S_2 = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 + v_2 = 0\}$

Easy to check ① $S_1 \underset{(ss)}{\subseteq} V$

② $S_2 \underset{(ss)}{\subseteq} V$

③ $S_1 \cap S_2 = \{\underset{\sim}{0}\}$.

④ $\dim(S_1) = \dim(S_2) = 1$

Then what is $\dim(S_1 + S_2)$

$\Rightarrow \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2) = 1 + 1 - 0 = 2.$

$\Rightarrow \boxed{S_1 + S_2 = V = \mathbb{R}^2}$

(checking this directly is not very easy).

Exc:- $V = \mathbb{R}^3$, $S_1 = \{\underset{\sim}{v} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0\}$.

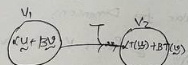
$S_2 = \{\underset{\sim}{v} \in \mathbb{R}^3 : v_1 = v_2 = v_3\}$

S.t $S_1 + S_2 = V$. (Show all steps)

Linear Map / Linear Transformation :-

Defn: ① Let V_1, V_2 be fdvs. A map $T : V_1 \to V_2$ is called a linear map / transformation if $\forall \alpha, \beta \in \mathbb{R}$, $\forall \underset{\sim}{u}, \underset{\sim}{v} \in V_1$

$$T(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) = \alpha T(\underset{\sim}{u}) + \beta T(\underset{\sim}{v})$$



② If $V_1 = V_2 = V$ (say) and $T : V \to V$ is a linear map, then T is called a linear operator.

③ If $V_2 = \mathbb{R}$ (also a fdvs over $\mathbb{R}$ with $\dim(V_2) = 1$) and $T : V_1 \to \mathbb{R}$ is a linear map, then T is called a linear f^nal.

④ If $T : V_1 \to V_2$ is a bijective linear map, then T is called a vectorspace isomorphism. and V_1, V_2 are called isomorphic vector spaces.

Notation :- $V_1 \cong V_2$.

13/11/24

LECTURE - 20

Thm: ⓐ If two fdvs V_1, V_2 are isomorphic (i.e, $V_1 \cong V_2$) then they have same dimension $\dim(V_1) = \dim(V_2)$

ⓑ If further $\underset{\in \mathbb{N} \cup \{0\}}{\dim(V_1) = n}$, then $V_1 \cong V_2 \cong \begin{cases} \{0\} & \text{if } n = 0 \\ \mathbb{R}^n & \text{if } n \in \mathbb{N} \end{cases}$

(Recall that $\mathbb{R}^0 := \{\underset{\sim}{0}\}$)

Eg:- (1) $V_1 = \mathbb{R}^2$, $V_2 = \mathbb{R}$

Define $T : V_1 \to V_2$ by $T(u_1, u_2) = 2u_1 - 3u_2$ $\forall (u_1, u_2) \in V_1$, Then T is a linear functional because $\forall \alpha, \beta \in \mathbb{R}$ and $\underset{\sim}{u}, \underset{\sim}{v} \in V_1 = \mathbb{R}^2$,

$$\begin{aligned} T(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) &= T(\alpha(u_1, u_2) + \beta(v_1, v_2)) \\ &= T(\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2) \\ &= 2(\alpha u_1 + \beta v_1) - 3(\alpha u_2 + \beta v_2) \\ &= \alpha(2u_1 - 3u_2) + \beta(2v_1 - 3v_2) \\ &= \alpha T(u_1, u_2) + \beta T(v_1, v_2) \\ &= \alpha T(\underset{\sim}{u}) + \beta T(\underset{\sim}{v}) \end{aligned}$$

Exc:

(2) $V_1 = \mathbb{R}^3$, $V_2 = \mathbb{R}^2$, ST $T : V_1 \to V_2$ defined by ~~Show that~~ $T(v_1, v_2, v_3) = (v_1 + v_2, v_2 + v_3)$; $(v_1, v_2, v_3) \in V_1$ is a linear transformation from V_1 to V_2.

(3) $V_1 = V_2 = \mathbb{R}^2$. S.T $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2)$, $(v_1, v_2) \in \mathbb{R}^2$ is a linear operator on $\mathbb{R}^2$.

(ii) Take $V_1 := \{\underbrace{(\omega_1,\omega_2,\omega_3)}_{\in \mathbb{R}^3} : \omega_1+\omega_2+\omega_3 = 0\} \underset{(S.S)}{\subseteq} \mathbb{R}^3$

$V_2 := \{(u_1,u_2)\in\mathbb{R}^2 : u_1+u_2 = 0\} \underset{(S.S)}{\subseteq} \mathbb{R}^2$

$V_3 := \mathbb{R}^2$

Exc:- Show the following

(a) $T_1 : V_1 \to V_2$ defined by $T_1(\omega_1,\omega_2,\omega_3) = (\omega_1+\omega_2, \omega_3)$ $\forall\ \omega_1,\omega_2,\omega_3 \in V_1$ is a linear map from $V_1 \to V_2$.

(b) $T_2 : V_1 \to V_3$ defined by $T_2(\omega_1,\omega_2,\omega_3) = (\omega_1,\omega_2)$ and $\omega_1,\omega_2,\omega_3 \in V_1$ is a vector space isomorphism: V_1 to V_3.

* W.K.T any non trivial finite dimensional vector space over $\mathbb{R}$ looks like $\mathbb{R}^n$, $n\in\mathbb{N}$ (as non-trivial)

Fix $m,n\in\mathbb{N}$

Q: What are all possible linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$?

From now on whenever we write a vector ($\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ etc) it is always a column vector. For example, $\underset{\sim}{u} \in \mathbb{R}^n$ means $\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \leftarrow n\times 1$ matrix or a column vector in $\mathbb{R}^n$.

To write a row vector, I shall use a transpose notation:

$\underset{\sim}{u}^T = (u_1, u_2, \dots, u_n) \leftarrow 1\times n$ matrix or row vector in $\mathbb{R}^n$
$(u_1\ u_2 \cdots u_n)$

Fix $m,n\in\mathbb{N}$

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map.

Let us denote by $\underbrace{\underset{\sim}{e}_1^{(n)}, \underset{\sim}{e}_2^{(n)}, \dots \underset{\sim}{e}_n^{(n)}}$ (column vectors $\in \mathbb{R}^n$) the standard basis elements for $\mathbb{R}^n$ and by $\underbrace{\underset{\sim}{e}_1^{(m)}, \underset{\sim}{e}_2^{(m)}, \dots \underset{\sim}{e}_m^{(m)}}_{\text{column vectors } \in \mathbb{R}^m}$, the standard basis elements for $\mathbb{R}^m$.

For each $j \in \{1,2,\dots n\}$,

$$T(\underbrace{\underset{\sim}{e}_j^{(n)}}_{\mathbb{R}^n})\ (\in \mathbb{R}^m) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \text{ (say)} \quad (\text{here } a_{ij}\in\mathbb{R}\ \forall\ i = 1,2,\dots m)$$

jth column of A

$$= a_{1j}\underset{\sim}{e}_1^{(m)} + a_{2j}\underset{\sim}{e}_2^{(m)} + \dots + a_{mj}\underset{\sim}{e}_m^{(m)}$$

$\left[\ \underset{\sim}{e}_1^{(m)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m,\ \underset{\sim}{e}_2^{(m)} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m \dots\ \right]$

Now take any column vector $\underset{\sim}{x} \in \mathbb{R}^n$.

Clearly $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{j=1}^{n} x_j \underset{\sim}{e}_j^{(n)}$.

Since $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, we have

$$\underbrace{T(\underset{\sim}{x})}_{\in \mathbb{R}^m} = T\left(\sum_{j=1}^{n} x_j \underset{\sim}{e}_j^{(n)}\right) \xrightarrow{\text{linearity}} \sum_{j=1}^{n} x_j\, T(\underset{\sim}{e}_j^{(n)}) \quad \text{※ vectors (not matrices)}$$

$$= \sum_{j=1}^{n} x_j \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \sum_{j=1}^{n} a_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{pmatrix} \leftarrow m\times 1 \text{ matrix}$$

$$= \underset{m\times n}{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}} \underset{n\times 1}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} = A\underset{\sim}{x}, \text{ where}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \leftarrow m\times n \text{ matrix with real entries}$$

$T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then

Summary: $\forall\, \underset{\sim}{x} \in \mathbb{R}^n$, $T(\underset{\sim}{x}) = A\underset{\sim}{x}$ for some $m\times n$ matrix with real entries.

Conversely, for any $m \times n$ matrix A with real entries the map $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $\underbrace{T(\underset{\sim}{x})}_{m\times 1} = \underbrace{\underset{m\times n}{A}\ \underset{n\times 1}{\underset{\sim}{x}}}_{m\times 1}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

This is bcz $\forall\, \underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$, $\forall\, \alpha, \beta \in \mathbb{R}$,

$A(\alpha \underset{\sim}{x} + \beta \underset{\sim}{y}) = \alpha A \underset{\sim}{x} + \beta A \underset{\sim}{y}$.

*check!!

We have thus proved :

Thm:- Fix $m, n \in \mathbb{N}$. $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if $\exists$ an ~~real~~ $m\times n$ matrix A with real entries s.t

$T(\underset{\sim}{x}) = A\underset{\sim}{x}$ $\forall\, \underset{\sim}{x} \in \mathbb{R}^n$
(Column Vector)

Remark :- This A is also unique

Def^n:- A is called the matrix ~~for~~ for the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ (wrt. the standard bases).

Eg:- ① If $\underset{m\times n}{A} = 0$ (zero matrix), then $T: \mathbb{R}^n \to \mathbb{R}^m$ is the zero map, i.e $T(\underset{\sim}{x}) = \underset{\sim}{0} \in \mathbb{R}^m$ $\forall\, \underset{\sim}{x} \in \mathbb{R}^n$

② If $m = n$ and

$\underset{n\times n}{A} = \underset{\text{Identity matrix of order } n}{I_n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$

(i.e $a_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$), then $T: \mathbb{R}^n \to \mathbb{R}^n$ is the identity map, i.e., $T(\underset{\sim}{x}) = \underset{\sim}{x}$ $\forall\, \underset{\sim}{x} \in \mathbb{R}^n$.

③ $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(\underset{\sim}{x}) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$ $\forall\, \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$.

It is easy to check that T is a linear map.

Q^n: What is the matrix for T?

Here $n=3$ and $m=2$ so the matrix for T would be a 2×3 matrix A.

$T(\underset{\sim}{e}^{(3)}_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1^{st}$ column of A.

$T(\underset{\sim}{e}^{(3)}_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2^{nd}$ column of A.

$T(\underset{\sim}{e}^{(3)}_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3^{rd}$ column of A.

$\therefore A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}_{2\times 3}$

[Sanity check:- $A\underset{\sim}{x} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$
$= T(\underset{\sim}{x})$ $\forall\, \underset{\sim}{x} \in \mathbb{R}^3$]

Exe:- ~~Find the~~ S.T $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 - x_2 \\ x_2 \end{pmatrix}$, $\forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ is a linear map

Then find the matrix for this linear combination?