SESSION 14

13 March 2025 12:09

Weak and strong duality (Ref: Sec. 5.2.2)

Let $d^* = \sup_{\lambda,\nu} g(\lambda,\nu) = \sup_{\lambda,\nu} \inf_{x} L(x,\lambda,\nu)$

$\updownarrow$

$p^* = \inf_{x} \sup_{\lambda,\nu} L(x,\lambda,\nu)$

weak duality : $\boxed{d^* \leq p^*}$

Strong duality : $\boxed{d^* = p^*}$, duality gap $= 0 \; [d^* - p^*]$

Condition for strong duality

↳ This holds only for some situations

- Slater condition (Ref: 5.2.3)

$f_i(x) < 0 \quad \forall \; i = 1(1)m$

then strong duality holds

$x^* \Rightarrow$ primal optimal point

$(\lambda^*, \nu^*) \Rightarrow$ is the dual optimal point.

KKT Conditions

$f_i(x) \leq 0 \; \forall \; i = 1(1)m$

$h_i(x) = 0 \; \forall \; i = 1(1)p$

$\lambda_i^* \geq 0 \; \forall \; i = 1(1)m$

$$\lambda_i^* f_i(x^*) = 0 \quad \forall \; i = 1(1)m$$

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda^* \nabla f_i(x^*) + \sum \nu_i^* \nabla h_i(x^*) = 0$$

Under strong duality

$$f_0(x^*) = g(\lambda^*, \nu^*)$$

$$\Rightarrow f_0(x^*) = \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \right)$$

$$f_0(x^*) \le f_0(x^*) + \cancelto{0}{\left[\sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*) \right]}$$

$$f_0(x^*) \le f_0(x^*)$$

Conclusion $\sum \lambda_i^* f_i(x^*) = 0$

since each term non positive

$$\lambda_i^* f_i(x^*) = 0 \quad \forall \; i = 1(1)m$$

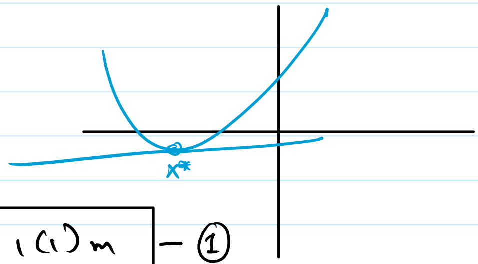
This condition is known as complimentary slackness (ref: 5.2.2)

Supporting hyperplane interpretation of KKT condition

minimise $f_0(x)$
subject to $f_i(x) \le 0$, $i = 1(1)m$

Let, f_0 is convex, f_i, $i = 1(1)m$ are convex let $x^* \in \mathbb{R}^n$, $\lambda^* \in \mathbb{R}^m$ satisfy KKT condition

Then show that $\nabla f_0(x^*)^T (x - x^*) \geq 0 \quad \forall$ feasible x.



x be a feasible solution

$$f_i(x) \leq 0$$

$$\boxed{0 \geq f_i(x) \geq f_i(x^*) + \nabla f_i(x^*)(x - x^*), \quad i = 1(1)m} \quad - (1)$$

(Give reason for the above (see chapter 3 and 4))

Since $\lambda_i^* \geq 0$, $\quad 0 \geq \sum_{i=1}^{m} \lambda_i^* \left(f_i(x^*) + \nabla f_i(x^*)(x - x^*) \right)$

(from the above condition (1))

$$R.H.S = \sum \lambda_i^* f_i(x^*) + \sum \lambda_i^* \nabla f_i(x^*)(x - x^*)$$

$$= - \nabla f_0(x^*)^T (x - x^*)$$

Since, 5^{th} condition of KKT is

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda^* f_i(x^*) = 0$$

$$\Rightarrow \sum_{i=1}^{m} \lambda^* f_i(x^*) = -\nabla f_0(x^*)$$

$$\Rightarrow \sum_{i=1}^{m} \lambda^* \nabla f_i(x^*)(x - x^*) = -\nabla f_0(x^*)^T (x - x^*)$$

$$\therefore \quad -\nabla f_0(x^*)^T (x - x^*) \leq 0$$

$$\Rightarrow \nabla f_0(x^*)^T (x - x^*) \geq 0 \quad \forall \text{ feasible } x$$