SESSION 2

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1) $f(x) = \max_{i=1,\dots,n} \{x_i\}$ is a convex function

for any $0 < \theta < 1$ $\boxed{f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)}$

$x = (-2, 1, 5)$
$y = (2, 3, 5)$

$\theta = \frac{1}{2}$

2)

Then objective function becomes $\rightarrow 1 - e^{-z} (c^T x + d)$

As $(c^T x + d) > 0$ and $(1 - e^{-z})$ is concave function of $z$

$\therefore$ Profit. (affine funct$^n$) is a concave function

[Section 3.5.2, pg → 105] → Product of two log concave functions is also log concave.

# Lagrange Dual function

objective function $\min_x f_0(x)$

constraint functions $f_i(x) \leq 0$, $i = 1, \dots, m$
$h_i(x) = 0$, $i = 1, \dots, p$

$D = D(f_i(x)) \wedge D(h_i(x))$

$$g(\lambda, \nu) = \inf_{x \in D} \underbrace{\left[ f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right]}_{L(x, \lambda, \nu)}$$

$g$ is a point wise infimum of affine function

If $p^*$ is a optimal value of $L(x, \lambda, \nu)$ then $g(\lambda, \nu) \leq p^*$

Q) Prove that $g$ is a concave function of $(\lambda, \nu)$

Example: formulate least square solutions of system of linear equations as Lagrange dual function.

Optimization problem for least squares is.

$$\begin{cases} \min_{x} \; x^T x & ; \; x \in \mathbb{R}^n \\ \text{subject to } Ax = b & , \; A \in R^{p \times n} \end{cases}$$

Note: there is no inequality constraints

No. of Equality constraint $= p$

Lagrangian

$$L(x, \nu) = x^T x + \nu^T (Ax - b)$$

$D$ = Domain $\mathbb{R}^n \times \mathbb{R}^p$

Dual function,

$$g(\nu) = \inf_{x \in D} L(x,\nu)$$

$L(x,\nu)$ is convex and quadratic function of $x$.

Minimise $L(x,\nu)$ using optimality condition.

$$\nabla_x L(x,\nu) = 2x + \nu^T A$$

Equate, $\nabla_x L(x,\nu) = 0$

$$x^* = \frac{1}{2} A^T \nu$$

Dual funct$^n$ $g(\nu) = L(x^*, \nu)$

$$= x^{*T} x^* + \nu(Ax^* - b)$$

$$= \underline{-\frac{1}{4}\nu^T A A^T \nu - b^T \nu}$$

↳ check it

By the lower bound property

$$g(\nu) \leq \inf\{x^T x \,/\, Ax = b\}$$

Model :⇒ $Y = Ax + \varepsilon$

Assumptions $\begin{cases} Y = \text{measurements} \in \mathbb{R}^m \\ \varepsilon \in \mathbb{R}^m, \text{ noise}, E(\varepsilon) = \underline{0}, \; E(\varepsilon\varepsilon^T) - \cancel{E(\varepsilon)E(\varepsilon^T)} = I_m \end{cases}$

$$\left.\begin{array}{l} E(\varepsilon_i \varepsilon_i) = 1 \\ E(\varepsilon_i \varepsilon_j) = 0 \\ \quad i \neq j \end{array}\right\} \varepsilon_i \text{ are independent} \qquad I_m = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}_{m \times m}$$

$A$ = matrix of order $(m \times n)$

$\underline{x}$ = unknow vector to be estimated.

Consider only linear estimate of $x$ from $\boxed{\hat{x} = Fy}$

formulate the problem of $\hat{x}$ as a concave optimization problem.

Solution $\Rightarrow$ An estimator $\hat{x}$ of '$x$' is defined if $E(\hat{x}) = x \ \forall \ x$.

$$\overset{m\times1}{Y} = \overset{m\times n}{A} \overset{n\times1}{x} + \varepsilon^{m\times1} \longrightarrow (1)$$

Show that under model (1) $\Rightarrow E(\hat{x}) = E(FY)$

$$= E(F(Ax + \varepsilon))$$

$$= E[FAx] + E[F\varepsilon]$$

$$= FAE[x] + FE[\varepsilon]$$

$$= FAE[x] + 0$$

$$= FA\,E[x]$$

$$\boxed{E(\hat{x}) = FAx}$$

So, $\boxed{E(\hat{x}) = x \quad \text{if } FA = I}$

Variance - Covariance matrix of $\hat{x}$

$$E\left[(\hat{x}-x)(\hat{x}-x)^T\right]$$

Hint:

$$= E\left[(Fy-x)(Fy-x)^T\right] \qquad Fy = F(Ax+\varepsilon)$$

$$= FAx + F\varepsilon$$

Convex Optimization Problem

$$\min \; FF^T$$

subject to $FA = I$