Statistics II: Introduction to Inference

Problem set 1

In all the following problems, $X_1, \ldots, X_n$ represent a random sample of size n from a distribution $F_{\boldsymbol{\theta}}$. The specific form of the family $F_{\boldsymbol{\theta}}$ will be provided in each individual problem.

1. Let $F_{\boldsymbol{\theta}}$ be $\texttt{Binomial}(m, \rho)$ where $\theta = \rho$.

   (a) Consider the class of linear estimators of ρ, i.e., the estimators of the form $T_{\mathbf{l}}(\mathbf{X}) = \sum_{j=1}^{n} l_j X_j$, $l_j \in \mathbb{R}$, $j = 1, \ldots, n$. Find conditions on $\mathbf{l} = (l_1, \ldots, l_n)'$ under which $T_{\mathbf{l}}(\mathbf{X})$ is an unbiased estimator of ρ.

   [*Let* $\mathbf{l}^\star$ *be a choice of* $\mathbf{l}$ *which satisfies the condition obtained in part (a). Then the estimator obtained by replacing* $\mathbf{l}$ *by* $\mathbf{l}^\star$ *in* $T_{\mathbf{l}}(\mathbf{X})$ *is called a* linear unbiased estimator *of* ρ.]

   (b) Find the variance of $T_{\mathbf{l}}(\mathbf{X})$. Denote it by $\sigma^2_{\mathbf{l}}$.

   (c) Minimize $\sigma^2_{\mathbf{l}}$ with respect to $\mathbf{l}$ subject to the constraint obtained in part (a).

   [*Let the solution obtained in part (c) be* $\mathbf{l}^\star$. *The estimator obtained by replacing* $\mathbf{l}$ *by* $\mathbf{l}^\star$ *in* $T_{\mathbf{l}}(\mathbf{X})$ *is called the* Best Liner Unbiased Estimator (BLUE).]

   (d) Is the BLUE same as the UMVUE of ρ?

2. Let $F_{\boldsymbol{\theta}}$ be some distribution with mean μ and variance σ^2 (i.e., $E(X_1) = \mu$ and $\mathrm{var}(X_1) = \sigma^2$), and $\boldsymbol{\theta} = (\mu, \sigma^2)$. Find the BLUE for μ.

3. Let $F_{\boldsymbol{\theta}}$ be $\texttt{exponential}(\lambda)$ distribution with the pdf

   $$f_\lambda(x) = \lambda \exp\{-\lambda x\}, \qquad \lambda > 0,\ x > 0,$$

   and $\theta = \lambda$.

   (a) Find the UMVUE of the population mean $\psi(\theta) = \theta^{-1}$.

   (b) Is it an efficient estimator?

4. Let $F_{\boldsymbol{\theta}}$ be $\texttt{Poisson}(\lambda)$ distribution and $\theta = \lambda$.

   (a) Find the UMVUE of θ.

   (b) Is it an efficient estimator?

5. Let $F_{\boldsymbol{\theta}}$ be $\texttt{uniform}(0, \theta)$ distribution. The highest order statistics $X_{(n)} = \max\{X_1, \ldots, X_n\}$ is a complete and sufficient statistic (CSS) for this class of distributions. Can you identify the UMVUE of θ?

6. Let $F_{\boldsymbol{\theta}}$ be $\texttt{normal}(\mu, \sigma^2)$ and $\boldsymbol{\theta} = (\mu, \sigma^2)$. Consider the class of estimators of the form

   $$S^2_{a_n} = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}{a_n}, \quad a_n > 0.$$

   Find the estimator which minimizes the MSE in estimating σ^2 in this class. What is the bias associated with the best estimator?

   [*Let* $\chi = \sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. *You may use the fact that* $E(\chi) = (n-1)\sigma^2$ *and* $\mathrm{var}(\chi) = 2(n-1)\sigma^4$]

7. Let $F_{\boldsymbol{\theta}}$ be $\texttt{normal}(\mu, \sigma^2)$ and $\boldsymbol{\theta} = (\mu, \sigma^2)$.
   (a) Find the UMVUE of μ.
   (b) Find the UMVUE of σ^2.

8. Let $F_{\boldsymbol{\theta}}$ be $\texttt{normal}(\mu, 1)$ and $\theta = \mu$. Consider the class of linear estimators $T_{\mathbf{l}}(\mathbf{X})$ for μ. Show that minimizing the MSE of $T_{\mathbf{l}}(\mathbf{X})$ with respect to $\mathbf{l}$ does not lead to a valid estimator.

9. The waiting time (in whole minutes) for a bus is distributed as a Poisson(λ) distribution. We are interested in estimating the probability that the waiting time is at least one minute, i.e.,

$$\psi(\lambda) = P(X \geq 1) = 1 - \exp(-\lambda).$$

   To estimate this probability, we propose collecting the waiting times of n individuals, denoted as $X_1, \ldots, X_n$. Assuming $X_1, \ldots, X_n$ are independently distributed, find an unbiased estimator of $\psi(\lambda)$.

10. Suppose $n = 10$ individuals throw a biased die independently, continuing until they roll a *six*. The number of tosses for each person are as follows:

$$15 \quad 12 \quad 9 \quad 34 \quad 5 \quad 1 \quad 15 \quad 44 \quad 16 \quad 5.$$

   We are interested in estimating the probability of rolling a *six*, denoted as p. Find the UMVUE of p.