INDIAN STATISTICAL INSTITUTE
Mathematics I: BSDS First Year
Semester I, Academic Year 2024-25
Practice Final Exam

Full Marks: 50 **Duration: 3 Hours**

- **This is just for your practice - no need to submit the solutions. However, writing down the detailed solutions in a time-bound manner will help you in the actual exam.**
- **In the actual exam, please show all your work and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **The actual exam is a closed-book exam. You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function such that $f(0) = f'(0) = 0$ and $|f''(x)| \leq 1$ for all $x \in \mathbb{R}$.

   (a) (6 marks) Prove that $|f(x)| \leq 1/2$ for all $x \in [-1, 1]$.

   (b) (1 + 3 =4 marks) Show that f is Riemann integrable on $[-1, 1]$ and

   $$\left| \int_{-1}^{1} f(x)dx \right| \leq 1.$$

2. (a) For each $s \in (0, \infty)$, define a function $\phi_s : (0, \infty) \to (0, \infty)$ by

   $$\phi_s(x) = x^{-s}, \quad x \in (0, \infty).$$

   Show the following:

   - (3 marks) ϕ_s is Riemann integrable on $(0, 1]$ if $s < 1$.
   - (3 marks) ϕ_s is Riemann integrable on $[1, \infty)$ if $s > 1$.

   (b) (4 marks) Show that the function $g : (0, \infty) \to (0, \infty)$ defined by

   $$g(x) = \frac{1}{x^3 + \sqrt[3]{x}}, \quad x \in (0, \infty)$$

   is Riemann intergrable on $(0, \infty)$.

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3. Define a map $T : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$T\big((x_1, x_2, x_3, x_4)^T\big) = (x_1 - 2x_2,\ x_2 + x_4,\ x_1 + 2x_4)^T$$

for all $(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$.

(a) (4 marks) Show that T is a linear transformation.

(b) (4 marks) Calculate the matrix A for the linear transformation T (with respect to the standard bases).

(c) (2 marks) Compute the rank of A, where A is as in 3(b) above.

4. Suppose $V := \mathbb{R}^4$ and define

$$S_1 := \{(v_1, v_2, v_3, v_4)^T \in \mathbb{R}^4 : v_1 - v_2 = v_3 - v_4 = 0\},$$
$$S_2 := \{(u_1, u_2, u_3, u_4)^T \in \mathbb{R}^4 : u_1 + u_2 = u_3 + u_4 = 0\}.$$

(a) (3+3=6 marks) Show that S_1 and S_2 are linear subspaces of V.

(b) (5 marks) Find, with full justification, a basis for S_1.

(c) (1 mark) Compute the diemnsion of S_1.

(d) (5 marks) Show that the map ψ defined by

$$\psi\big((v_1, v_2, v_3, v_4)^T\big) = (v_1,\ -v_2,\ v_3,\ -v_4)^T$$

for all $(v_1, v_2, v_3, v_4)^T \in S_1$ is a vector space isomorphism from S_1 onto S_2. Justify all the steps.

(e) (3 marks) Show that $S_1 + S_2 = V$.