Complex Numbers

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I. MCQs with One or More than One Correct

1) If ω is an imaginary cube root of unity, then $(1+\omega-\omega^2)^7$ equals (1998-2 Marks)
   a. 128ω
   b. 128ω
   c. $128\omega^2$
   d. $-128\omega^2$
2) The value of the sum $\sum_{n=1}^{13}\left(i^n+i^{n+1}\right)$ where $i=\sqrt{-1}$, equals:
   (1998 - 2 Marks)
   a. i b. $i-1$ c. $-i$ d. 0
3) If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = \text{x + iy},$
   then (1998 - 2 Marks)
   a. x=3,y=4
   b. x=1,y=3
   c. x=0,y=4
   d. x=0,y=0
4) Let z_1 and z_2 be two distinct complex numbers and let $z=(1-t)z_1+tz_2$ for some real number t with $0<t<1$. If Arg(w) denotes the principal argument of a non-zero complex number w, then:
   (2010)
   a. $|z-z_1|+|z-z_2|=|z_1-z_2|$
   b. $\text{Arg}(z-z_1)=\text{Arg}(z-z_2)$
   c. $\begin{vmatrix} z-z_1 & \overline{z}-\overline{z_1} \\ z_2-z_1 & \overline{z_2}-\overline{z_1} \end{vmatrix}$
   d. $\text{Arg}(z-z_1)=\text{Arg}(z_2-z_1)$
5) Let $w=\frac{\sqrt{3}+i}{2}$ and $P=\{w^n : n=1,2,3,\ldots\}$. Further, let $H_1=\left\{z\in\mathbb{C} \mid \text{Re}(z)>\frac{1}{2}\right\}$ and $H_2=\left\{z\in\mathbb{C} \mid \text{Re}(z)<-\frac{1}{2}\right\}$, where $\mathbb{C}$ is the set of all complex numbers. If $z_1\in H_1$, $z_2\in H_2$, and O represents the origin, then the angle $\angle z_1Oz_2$ is: (JEE Adv. 2013)
   a. $\frac{p}{2}$ b. $\frac{p}{6}$ c. $\frac{2p}{3}$ d. $\frac{5p}{6}$
6) Let $a,b\in\mathbb{R}$ and $a^2+b^2\neq 0$. Suppose $S=\left\{z\in\mathbb{C} \mid z=\frac{1}{a+ibt},\ t\neq 0\right\}$, where $i=\sqrt{-1}$. If $z=x+iy$ and $z\in S$, then (x,y) lies on:
   (JEE Adv. 2016)
   a. the circle with radius $\frac{1}{2a}$ and center $\left(\frac{1}{2a},0\right)$ for $a>0,\ b\neq 0$.
   b. the circle with radius $\frac{1}{2a}$ and center $\left(\frac{-1}{2a},0\right)$ for $a<0,\ b\neq 0$.
   c. the x-axis for $a\neq 0,\ b=0$.
   d. the y-axis for $a=0,\ b\neq 0$.
7) Let $a,b,x,$ and y be real numbers such that $a-b=1$ and $y\neq 0$. If the complex number $z=x+iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right)=y$, then which of the following is (are) possible value(s) of x?
   a. $-1+\sqrt{1-y^2}$
   b. $-1-\sqrt{1-y^2}$
   c. $1+\sqrt{1+y^2}$
   d. $1-\sqrt{1+y^2}$
8) For a non-zero complex number z , let $\arg(z)$ denote the principal argument with $-\pi<\arg(z)\leq\pi$. Then,which of the following statement(s) is (are) FALSE?
   (JEE Adv. 2018)
   a. $\arg(-1-i)=\frac{\pi}{4}, where i=\sqrt{-1}$
   b. The function $f:\mathbb{R}\to(-\pi,\pi]$, defined by $f(t)=arg(-1+it)$ for all $t\in\mathbb{R}$, is continuous at all points of $\mathbb{R}$, where $i=\sqrt{-1}$
   c. For any two complex numbers z_1 and z_2 , $\arg(\frac{z_1}{z_2})-\arg(z_1)+\arg(z_2)$ is an integer multiple of 2π
   d. For any three given distinct complex numbers z_1,z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z-z_2)}{(z-z_3)(z_2-z_1)}\right)=\pi$,lies on a straight line
9) Let s,t,r be non-zero complex numbers and L be the set of solutions $z=x+iy$ $\left(x,y,\in\mathbb{R},i=\sqrt{-1}\right)$ of the equation $sz+t\overline{z}+r=0$, where $\overline{z}=x-iy$.Then, which of the following statements(s) is (are) TRUE?
   (JEE Adv. 2018)
   a. Let L has exactly one element, then $|s|\neq|t|$

b. If $|s| = |t|$,then L has infinitely many elements
c. The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2
d. If L has more than one element, then L has infinitely many elements

II. Subjective Problems

1) Express $\frac{1}{1-\cos\theta+2i\sin\theta}$ in the form $x + iy$.
(1978)
2) If $x = a + b$, $y = a\gamma + b\beta$ and $z = a\beta + b\gamma$ where γ and β are the complex cube roots of unity, show that $xyz = a^3 + b^3$
(1978)
3) If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, prove that $\left(x^2 + y^2\right)^2 = \frac{a^2+b^2}{c^2+d^2}$
(1979)
4) Find real values of x and y for which the following equation is satisfied $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$
(1980)
5) Let the complex numbers $z_1, z_2 and z_3$ be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.
(1981 - 4 Marks)
6) Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1 z_2 = 0$.
(1983 - 3 Marks)