13.4: The Motion of a Pendulum

In this section, we show how and when the motion of a pendulum can be described as simple harmonic motion. Consider the simple pendulum that is constructed from a mass-less string of length, L, attached to a fixed point on one end and to a point mass m on the other, as illustrated in Figure $13.4.1$.

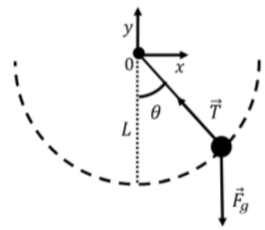

Figure 13.4.1: A simple pendulum which oscillates in a vertical plane.

The pendulum can swing in the vertical plane, and we have shown our choice of coordinate system (the z axis, not shown, is out of the page). The only two forces on the mass are the tension from the string and its weight. We can describe the position of the mass by the angle, $\theta(t)$, that the string makes with the vertical. We can model the dynamics of the simple pendulum by considering the net torque and angular acceleration about the axis of rotation that is perpendicular to the plane of the page and that goes through the point on the string that is fixed.

The force of tension cannot create a torque on the mass about the axis of rotation, as it is anti-parallel to the vector from the point of rotation to the mass. The net torque is thus the torque from the force of gravity:

$$\begin{aligned}\vec\tau^{net} &= \vec\tau_g \\ &= \vec r \times \vec F_g = (L\sin\theta\hat x - L\cos\theta\hat y) \times (-mg\hat y) \\ &= -mgL\sin\theta\hat z\end{aligned}$$

where L is the magnitude of the vector, $\vec r$, from the axis of rotation to where the force of gravity is exerted. The net torque is equal to the angular acceleration, α, multiplied by the moment of inertia, I, of the mass:

$$\begin{aligned}\vec\tau^{net} &= I\vec\alpha \\ -mgL\sin\theta\hat z &= mL^2\vec\alpha \\ -g\sin\theta\hat z &= L\vec\alpha\end{aligned}$$

where $I = ML^2$ is the moment of inertia for a point mass a distance L away from the axis of rotation. For the position illustrated in Figure $13.4.1$, the angular acceleration of the pendulum is in the negative z direction (into the page) and corresponds to a clockwise motion for the pendulum, as we would expect. The angular acceleration is the second time derivative of the angle, θ:

$$\alpha = \frac{d^2\theta}{dt^2}$$

We can thus re-write the equation that we obtained from the rotational dynamics version of Newton’s Second Law as:

$$\begin{aligned}-g\sin\theta\hat z &= L\vec\alpha \\ \frac{d^2\theta}{dt^2} &= -\frac{g}{L}\sin\theta\end{aligned}$$

where we only used the magnitudes in the second equation, since all of the angular quantities are in the z direction. This equation of motion for $\theta(t)$ almost looks like the equation for simple harmonic oscillation for the angle θ (except that we have $\sin\theta$ instead of θ). However, consider the "the small angle approximation"[1] for the sine function:

$$\sin\theta \approx \theta$$

If the oscillations of the pendulum are "small", such that the small angle approximation is valid, then the equation of motion for the pendulum is:

$$\begin{aligned} \frac{d^2\theta}{dt^2} &= -\frac{g}{L}\sin\theta \approx -\frac{g}{L}\theta \\ \therefore \frac{d^2\theta}{dt^2} &= -\frac{g}{L}\theta \quad \text{(for small } \theta) \end{aligned}$$

and the angle that the pendulum makes with the vertical is described by the equation for simple harmonic oscillation with angular frequency:

$$\omega = \sqrt{\frac{g}{L}}$$

The angle, θ, as a function of time is thus described by the function:

$$\theta(t) = \theta_{max}\cos(\omega t + \phi)$$

where θ_{max} is the maximal amplitude of the oscillations and ϕ is a phase that depends on when we choose to define $t = 0$.

Exercise 13.4.1

Kaiden built a grandfather clock using a simple pendulum, but he found that the period was twice as large as as he wanted it to be. In order to halve the period of the pendulum, he can

A. change the mass.
B. halve the length of the string.
C. quarter the length of the string.
D. double the length of the string.
E. quadruple the length of the string.

Answer

The physical pendulum

A physical pendulum is defined as any object that is allowed to rotate in the vertical plane about some axis that goes through the object, as illustrated in Figure 13.4.2.

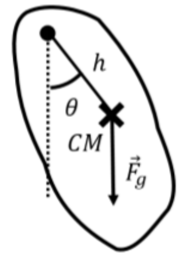

Figure 13.4.2: A physical pendulum which oscillates in a vertical plane about an axis through the object.

The only forces exerted on the pendulum are its weight (exerted at its center of mass) and a contact force exerted at the axis of rotation. The physical pendulum can be modeled in exactly the same way as the simple pendulum, except that we use the moment of inertia of the object about the axis of rotation. Only the weight results in a torque about the rotation axis, since the contact force is exerted at the rotation axis:

$$\tau^{net} = \tau_g = I\alpha$$
$$-mgh\sin\theta = I\alpha = I\frac{d^2\theta}{dt^2}$$

where h is the distance from the axis of rotation to the center of mass. In the small angle approximation, this becomes:

$$\frac{d^2\theta}{dt^2} = -\frac{mgh}{I}\theta \quad \text{(for small } \theta\text{)}$$

and we find that the physical pendulum oscillates with an angular frequency:

$$\omega = \sqrt{\frac{mgh}{I}}$$

Footnotes

1. Look up the Maclaurin/Taylor series for the sine function!
