THE Z-TRANSFORM

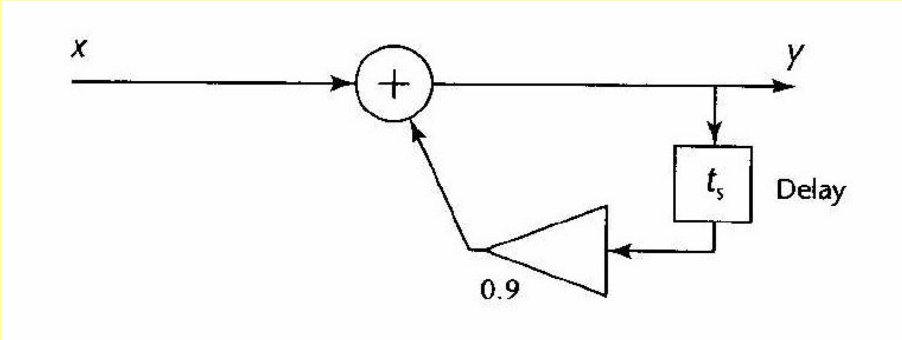
CONTENTS

7. s-plane and z-plane treatment of LTI systems

CONSIDER: $y[n] = a \cdot y[n-1] + x[n]$.

- **Treat it as an analogue network.**
- **The impulse response is** $h_c(t)$

$$h_c(t) = a \cdot h_c(t - t_s) + \delta(t). \qquad (7.1)$$



Structure (signal flow graph) of a simple recursive filter.

- **Take the Laplace transform on both sides of (7.1) gives:**

$$H_c(s) = 1 + a \cdot H_c(s) e^{-st_s} \quad \text{or} \quad H_c(s) = \frac{1}{1 - a \cdot e^{-st_s}} \quad \text{(CT Transfer function)}$$

- **It is possible to study the stability of the system in the s-domain. The transfer function becomes a rational function in the new variable *z* (easier to analyze):**

$$z \equiv e^{st_s} \qquad (7.2)$$

7.1 Transfer function

$$y(t) = -\sum_{k=1}^{N} a_k y(t - kt_s) + \sum_{k=0}^{M} b_k x(t - kt_s) \quad (7.3)$$

Feedback **Feedfoward**

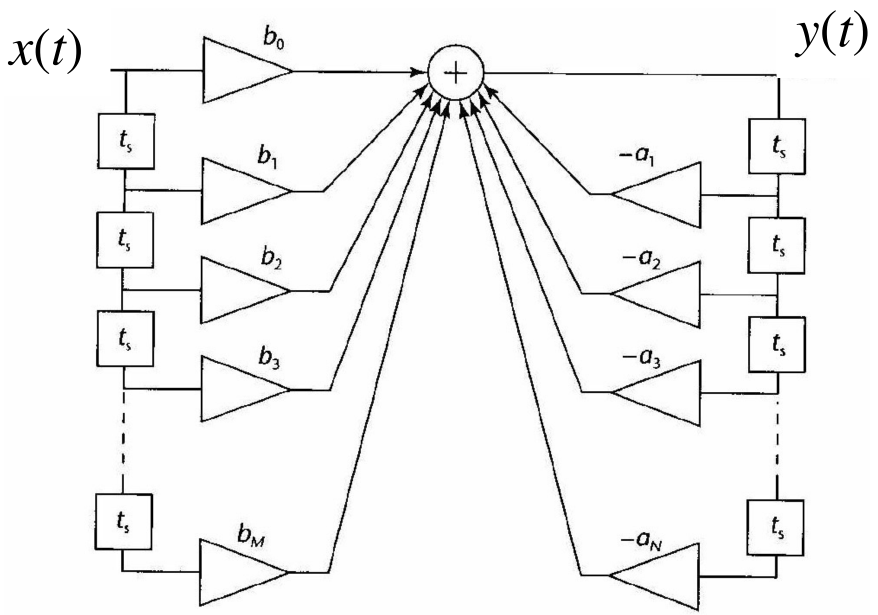
- **Laplace transform of (7.3):**

$$Y(s) = -\sum_{k=1}^{N} a_k Y(s) e^{-skt_s} + \sum_{k=0}^{M} b_k X(s) e^{-skt_s} \quad (7.4)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\Sigma_{k=0}^{M} b_k e^{-skt_s}}{1 + \Sigma_{k=1}^{N} a_k e^{-skt_s}} \quad (7.5)$$

- **Substituting $z \equiv e^{st_s}$ into (7.4):**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\Sigma_{k=0}^{M} b_k z^{-k}}{1 + \Sigma_{k=1}^{N} a_k z^{-k}} \quad (7.6)$$



Feedforward (non-recursive) part

Feedback (recursive) part

(IIR) digital filter

- ***H*(*z*), the transfer function, is a rational function.**

- **Assume that $a_k = 0,\ k = 1,....,M$, we obtain a causal FIR system (i.e. $h[k] = 0,\ k < 0$) with impulse response $h[k] = b_k$, *k*=0,…,*N*-1. The transfer function *H*(*z*) is then given by**

$$H(z) = \sum_{n=0}^{N-1} h[n] \cdot z^{-n}$$

- **If *h*[*n*] is considered as an infinite sequence, it is therefore natural to define the z-transform of a sequence *h*[*n*] as**

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} \qquad (7.7)$$

- **Thus, the transfer function of a LTI system is equal to the z-transform of its impulse response.**

7.1.1 Poles and zeros of a transfer function

Express the numerator and denominator of the transfer function

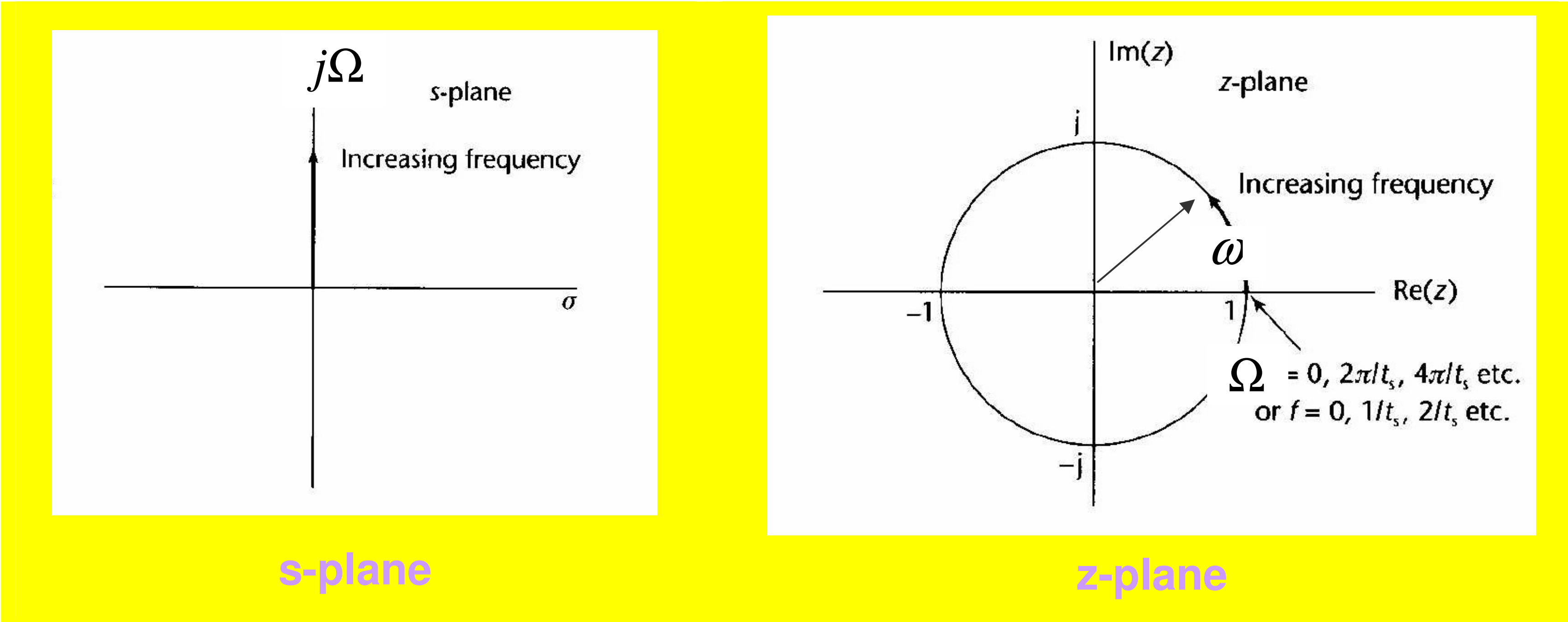
$$H(z) = \frac{\Sigma_{k=0}^{M} b_k z^{-k}}{1 + \Sigma_{k=1}^{N} a_k z^{-k}}$$

as polynomial in *z*, *H*(*z*) can be factored into the following form

$$H(z) = z^{-(M-N)} \cdot \frac{K(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)} \qquad (7.8)$$

- $z_1, z_2, ..., z_M$ **are the zeros of *H*(*z*) because** $H(z_i) = 0,\ i = 1, ..., M$ **.**
- $p_1, p_2, ..., p_N$ **are the poles of *H*(*z*) because** $H(p_i)$ is infinity for $i = 1, ..., N$ **.**
- **For transfer functions with real coefficients, poles and zeros, if complex-valued, occur in conjugates.**

7.2 Mapping from s- to z-plane



- **Substituting $j\Omega$ for *s* in $z \equiv e^{st_s}$ gives $z \equiv e^{j\Omega t_s}$.**
- **The imaginary axis in the s-plane is mapped to a unit circle in the z-plane.**

DIGITAL RADIAN FREQUENCY

- Let $\omega \equiv \Omega \cdot t_s \mod 2\pi$ be the digital radian frequency. The range of ω is $[-\pi, \pi]$.

- If the input is bandlimited to

$$-\pi f_s < \Omega < \pi f_s$$

(sampling theorem is satisfied),

there is a one-to-one mapping between each Ω in the desired signal spectrum and ω in the z-domain:

$$\begin{array}{ccc} \omega & \equiv & \Omega \cdot t_s \\ \omega \in [-\pi, \pi] & & -\pi f_s < \Omega < \pi f_s \end{array} \quad \text{(Note, } f_s = 1/t_s\text{)} \qquad (7.9)$$

- If the sampling rate is insufficient, the aliasing components will be mapped "repeatedly" onto the unit circle.

- **If the z-transform converges on the unit circle, it gives the Discrete-time Fourier transform, which describes the frequency selectivity of the system.**

$$H(z)|_{z=e^{j\omega}} = H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\omega} \qquad (7.10)$$

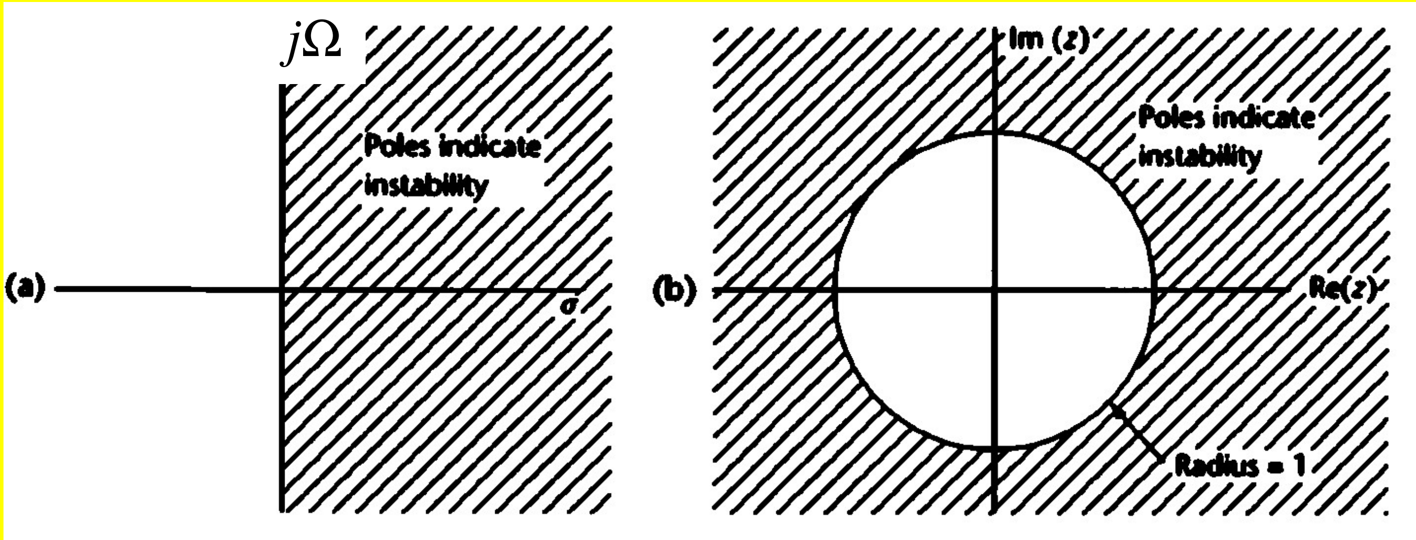
- **It is periodic in ω with period 2π.**

Substituting $s = \sigma + j\Omega$ into $z \equiv e^{st_s}$ gives

$$z \equiv e^{\sigma t_s} \cdot e^{j\Omega t_s} \qquad (7.11)$$

- **Right half plane (RHP) of the s-plane is mapped to the outside of the unit circle in z-plane ($\sigma > 0$).**
- **Left half plane (LHP) of the s-plane is mapped to the inside of the unit circle of the z-plane. ($\sigma \leq 0$).**

7.3 Stability analysis



Poles in shaded areas indicate in-stability for a **causal system**.

- **Since the RHP is mapped to the outside of the unit circle in the z-plane, poles of causal transfer functions, *H*(z), which are outside the unit circle lead to unstable systems.**
- **The poles of a causal-stable LTI system must lie inside the unit circle.**

8 z-transform and inverse z-transform

- **z-transform is an infinite sum. It might not converge for all values of z.**
- **The region of the z-plane where the sum (Laurent series) converges is called the region of convergence (ROC) and it represents an analytic function with all its derivatives being continuous functions of *z*.**

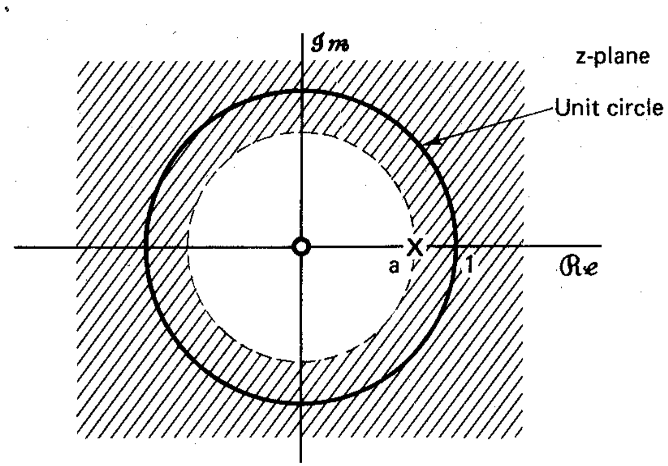
Example 1: z-transform of the signal $x[n] = a^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n .$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} \left| az^{-1} \right|^n < \infty ,$$

OR $\left| az^{-1} \right| < 1 \quad \Leftrightarrow \quad |z| > |a| .$



- **Inside the ROC, the series converges to** $X(z) = (1 - az^{-1})^{-1}$, **(ROC:** $|z| > |a|$**.) (8.1)**

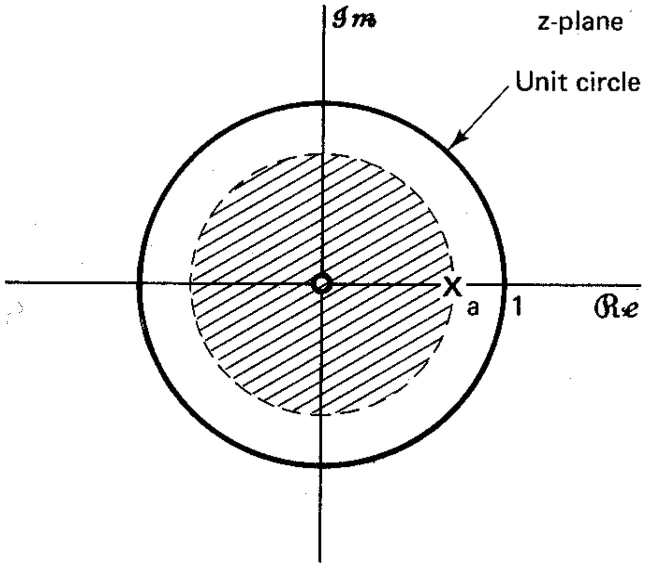
Example 2: z-transform of the signal $x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

If $\left|a^{-1}z\right| < 1$, or equivalently, $|z| < |a|$, the sum converges to

$$1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}}, \qquad |z| < |a| \qquad \textbf{(8.2)}$$



- The algebraic expressions for $X(z)$ and the corresponding pole-zero plots are identical in Examples 1 and 2. The z-transform differs only in the ROC. Therefore, it is important to specify both the algebraic expression and the ROC for the z-transform of a given sequence.

- A right-sided sequence is zero prior to some value of *n*, say N_1. A left-sided sequence is zero after some value of *n*, say N_2.

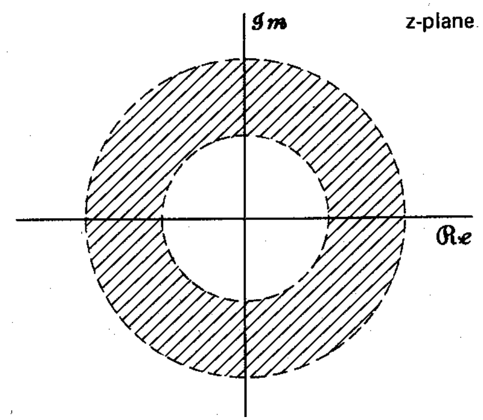
- Right-sided sequences have a ROC extending outward, while left-sided sequences have a ROC extending inward.

- The most important and useful z-transforms are those for which $X(z)$ is a *rational function* inside the region of convergence, i.e.

$$X(z) = P(z)/Q(z)$$

where $P(z)$ and $Q(z)$ are polynomials in *z*.

- It can be shown that the ROC of a rational function is in general an annular ring in the z-plane, bounded internally by right-sided poles (if any) and externally by left-sided poles (if any).



Exercise:

1. i) By considering the z-transform on a circle with radius ρ , i.e.

$$z = \rho e^{j\theta}, \; -\pi < \theta < \pi ,$$

show that the absolute convergent of the z-transform $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ does not dependent on θ , but on ρ and *x*[n]. Absolute convergent implies $\lim_{n\to\infty} \sum_{n=-\infty}^{\infty} | x[n] z^{-n} | < \infty$.

ii) Hence, show that a system is stable if the ROC covers the unit circle (use (6.1)).

iii) Using ii) and the fact that the impulse response of a causal LTI system is a right-sided sequence, show that a causal-stable LTI system has all its poles inside the unit circle.

2. Use the result in Examples 1 and 2, or directly from the definition, evaluate the z-transform of

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1].$$

(Answer: $X(z) = \dfrac{2z\left(z - \frac{1}{12}\right)}{\left(z + \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}$, ROC $\frac{1}{3} < |z| < \frac{1}{2}$).

TABLE 4.1 SOME COMMON *z*-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos \omega_0 n]u[n]$	$\dfrac{1-[\cos \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin \omega_0 n]u[n]$	$\dfrac{[\sin \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\dfrac{1-[r\cos \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\dfrac{[r\sin \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Exercise:

Evaluate the z-transform of sequences (11) and (13) in Table 4.1 on the left hand side.

8.1 Properties of z-transform

Notations:

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \quad \text{ROC} = R_x$$

where R_x represents a set of values such that $r_R < |z| < r_L$

$$x_1[n] \overset{Z}{\longleftrightarrow} X_1(z), \quad \text{ROC} = R_{x1}$$

$$x_2[n] \overset{Z}{\longleftrightarrow} X_2(z), \quad \text{ROC} = R_{x2}$$

8.1.1 Linearity

$$ax_1[n] + bx_2[n] \overset{Z}{\longleftrightarrow} aX_1(z) + bX_2(z), \ \text{ROC contains } R_{x1} \cap R_{x2} \qquad (8.3)$$

If $aX_1(z) + bX_2(z)$ contains all the poles of $X_1(z)$ and $X_2(z)$, i.e. there is no pole-zero cancellation, then the region of convergence is the intersection of the individual regions of convergence. If some zeros are introduced that cancel some of the poles, the region of convergence may be larger.

8.1.2 Time shifting

$$x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0} X(z),\ \text{ROC} = R_x \qquad \textbf{(8.4)}$$

(except for the possible addition or deletion of $z=0$ or $z=\infty$). n_0 is an integer.

EXERCISE :

8.1.3 Exponential multiplication property

$$z_0^n x[n] \overset{Z}{\longleftrightarrow} X(z/z_0), \quad \text{ROC} = |z_0| R_x \qquad \textbf{(8.5)}$$

$|z_0|R_x$ **denotes that the ROC is R_x scaled by $|z_0|$; i.e. if R_x is the set of values of *z* such that $r_R < |z| < r_L$, then $|z_0|R_x$ is the set of values of *z* such that $|z_0|r_R < |z| < |z_0|r_L$.**

Use the exponential multiplication property to evaluate the z-transform of

$$x[n] = r^n \cos(\omega_0 n)u[n] = \frac{1}{2}\left(re^{j\omega_0}\right)^n u[n] + \frac{1}{2}\left(re^{-j\omega_0}\right)^n u[n].$$

8.1.4 Convolution theorem

$$x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X_1(z)X_2(z), \text{ ROC contains } R_{x1} \cap R_{x2} \quad \textbf{(8.6)}$$

PROOF: **Assume that the z-transform is well defined so that we can interchange the order of summation.**

Consider the convolution of $x_1[n]$ and $x_2[n]$: $\quad y[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$.

The z-transform of *y*[*n*] is

$$Y(z) = \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right\} z^{-n}.$$

Interchange the order of summation yields

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}.$$

Changing the index of summation in the second sum from *n* to $m = n - k$**, gives**

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \left\{ \sum_{m=-\infty}^{\infty} x_2[m] z^{-m} \right\} z^{-k} = X_1(z) X_2(z)$$

where the ROC includes the intersection of the regions of convergence of $X_1(z)$ and $X_2(z)$.

EXERCISE

Consider a sinusoidal input $x[n] = e^{j\omega n}$ with frequency ω. Using (3.3), show that the output of a LTI system with impulse response $h[n]$ is

$$y[n] = e^{jn\omega} \cdot H(e^{j\omega})$$

where $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$ is the frequency response of the system. (c.f. (8.7)). What can we conclude from the above equation?

Implication in LTI systems

For a LTI system,

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

Using the convolution theorem, we have

$$h[n] * x[n] \xleftrightarrow{Z} H(z)X(z).$$

The convolution relationship becomes a product in the z-transform.

Since the discrete-time Fourier transforms (if they exist) of *h*[*n*] and *x*[*n*] are the z-transform evaluated on the unit circle (c.f. 7.10), we have

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \qquad (8.7)$$

By appropriately designing the frequency response of $H(e^{j\omega})$, we can amplify or attenuate precisely any frequency components of *x*[*n*].

TABLE 4.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	4.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	4.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x except for the possible addition or deletion of the origin or ∞
3	4.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0 \rvert R_x$
4	4.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x except for the possible addition or deletion of the origin or ∞
5	4.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	4.4.6	$x[-n]$	$X(1/z)$	$1/R_x$
9	4.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
10	4.4.8	Initial value theorem: $x[n] = 0,\quad n < 0 \qquad \lim_{z\to\infty} X(z) = x[0]$		
11	4.6	$x_1[n]x_2[n]$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2(z/v)v^{-1}\,dv$	Contains $R_{x_1}R_{x_2}$
12	4.7	Parseval's relation: $\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}\,dv$		

8.2. Inverse z-transform

8.2.1 Inspection method

It may be possible to express a given z-transform as a sum of terms so that they can be inversed individually, say from standard table.

8.2.2 Partial fraction expansion

For rational functions, we can obtain their partial fraction expansions as a sum of simple terms, which are then individually inversed.

Assume that $X(z)$ is expressed as a ratio of polynomial in z^{-1}

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

where c_k's are the zero of $X(z)$ and the d_k's are the nonzero poles of $X(z)$.

- **If $M < N$ and the poles are all first order, then $X(z)$ can be expressed as**

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}. \qquad \textbf{(8.1)}$$

Multiplying both sides by $(1 - d_k z^{-1})$ and evaluating the resulting identity at $z = d_k$, one gets the coefficients, A_k, as follows

$$A_k = (1 - d_k z^{-1}) X(z)\Big|_{z=d_k} \qquad \textbf{(8.2)}$$

- **If $M \geq N$, the complete partial fraction expansion is**

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \qquad \textbf{(8.3)}$$

where B_r's can be obtained by long division of the numerator by the denominator, with the division process terminating when the remainder is of lower degree than the denominator.

- If $X(z)$ has multiple-order, say an order *s* pole at $z = d_i$, and $M \geq N$, we have

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m} \qquad \textbf{(8.4)}$$

The coefficient C_m are obtained as follows:

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} \left[(1 - d_i w)^s X(w^{-1}) \right] \right\}_{w = d_i^{-1}} \qquad \textbf{(8.5)}$$

For each multiple-order pole, there will be a term like the third sum in (8.4).

The term $B_r z^{-r}$ corresponds to terms of form $B_r \delta[n-r]$. The fractional terms $\frac{A_k}{1 - d_k z^{-1}}$ corresponds to $(d_k)^n u[n]$ or $-(d_k)^n u[-n-1]$ depending on the ROC:

If $X(z)$ has only simple poles and the ROC is of the form $r_R < |z| < r_L$, then a given pole d_k will correspond to a right-sided exponential $(d_k)^n u[n]$ if $|d_k| < r_R$ and it will correspond to a left-sided exponential if $|d_k| > r_L$.

Example 4.5: Find the inverse z-transform of

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{1+2z^{-1}+z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)},$$ **with ROC** $|z|>1$.

Solution:

From the ROC, it is clear that the sequence is a right-sided sequence. Since $M = N = 2$ and the poles are all first order, $X(z)$ is of form

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

B_0 can be found by long division:

$$
\begin{array}{r}
2 \phantom{z^{-2}+2z^{-1}+1} \\
\tfrac{1}{2}z^{-2}-\tfrac{3}{2}z^{-1}+1 \big) \overline{z^{-2}+2z^{-1}+1} \\
\underline{z^{-2}-3z^{-1}+2} \\
5z^{-1}-1
\end{array}
$$

Thus, $X(z)$ can be expressed as

$$X(z) = 2 + \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}.$$

Coefficients A_1 and A_2 are found using (8.2) as

$$A_1 = \left.\frac{1+2z^{-1}+z^{-2}}{1-z^{-1}}\right|_{z^{-1}=2} = -9, \qquad A_2 = \left.\frac{1+2z^{-1}+z^{-2}}{1-\frac{1}{2}z^{-1}}\right|_{z^{-1}=1} = 8.$$

Therefore $X(z)$ is

$$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}, \ |z| > 1$$

From Table 4.1, we see that

$$2 \stackrel{Z}{\longleftrightarrow} 2\delta[n], \qquad \frac{1}{1-\frac{1}{2}z^{-1}} \stackrel{Z}{\longleftrightarrow} \left(\frac{1}{2}\right)^n u[n], \qquad \frac{1}{1-z^{-1}} \stackrel{Z}{\longleftrightarrow} u[n]$$

From the linearity of the z-transform, we have

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

EXERCISE:

Repeat the above example with ROC:

1) $|z| < 1/2$**,** **2)** $1/2 < |z| < 1$

Answers:

1) $x[n]$ **is a left-sided sequence and** $x[n] = 2\delta[n] + 9\left(\frac{1}{2}\right)^n u[-n-1] - 8u[-n-1]$

2) $x[n]$ **is a two-sided sequence and** $x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] - 8u[-n-1]$**.**