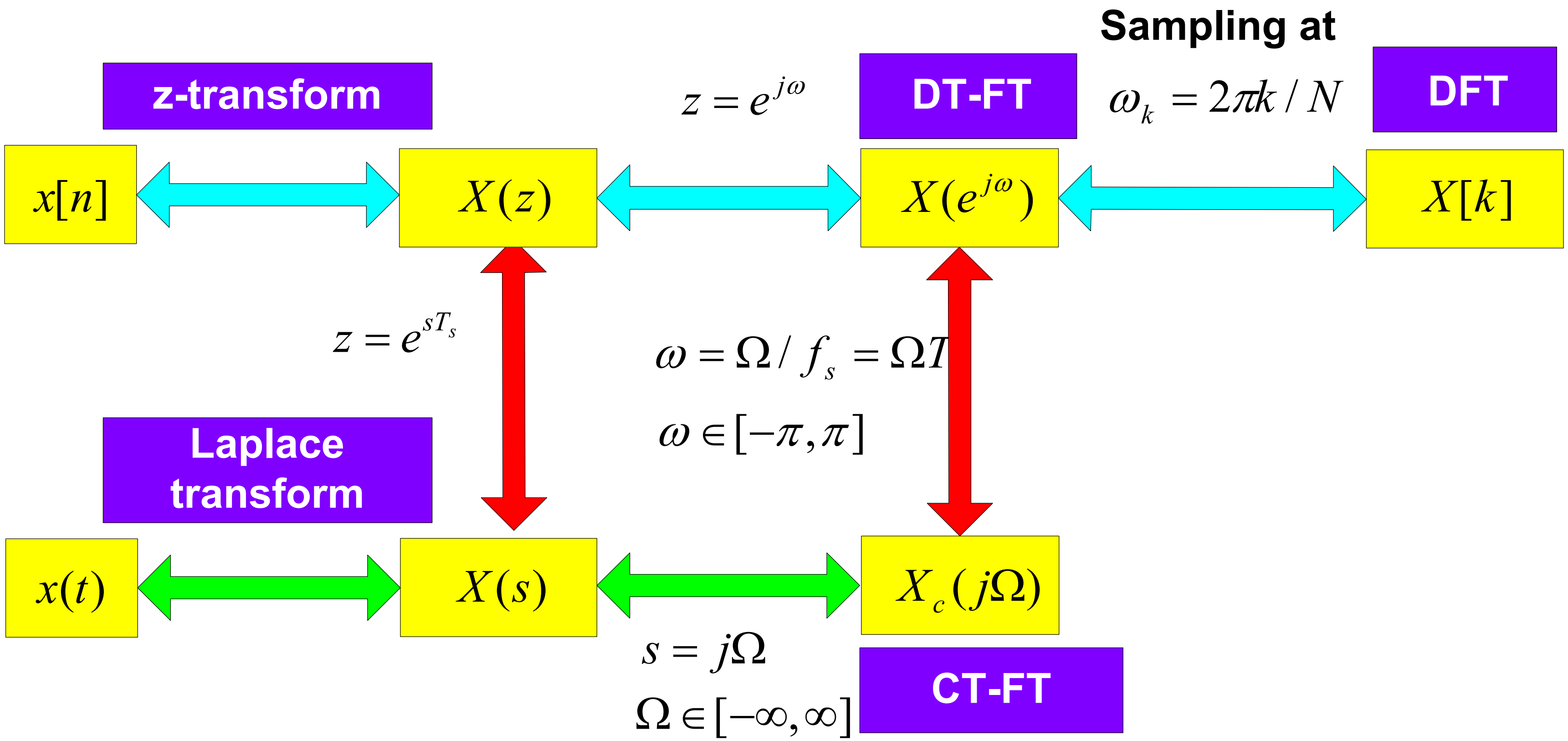
RELATIONSHIP BETWEEN THE TRANSFROMATIONS



If x(t) is bandlimited and the sampling rate is greater than the Nyquist rate, then x(t) can be recovered from x[n]. Anti-aliasing filter with bandwidth fmax has to be applied to x(t) to avoid aliasing.

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \qquad \textbf{(10.1)}$$

Cascade form of LTI system

Factor the numerator and denominator polynomials of $H(z)$ as

$$H(z) = A\frac{\prod_{k=1}^{M_1}(1-g_k z^{-1})\prod_{k=1}^{M_2}(1-h_k z^{-1})(1-h_k^* z^{-1})}{\prod_{k=1}^{N_1}(1-c_k z^{-1})\prod_{k=1}^{N_2}(1-d_k z^{-1})(1-d_k^* z^{-1})}$$

The first-order factors represent real zeros at g_k and c_k real poles, and the second-order factors represent complex conjugate pairs of zeros at h_k and h_k^* and complex conjugate pairs of poles at d_k and d_k^*.

A modular structure is usually preferred and is obtained by combing pairs of real factors and complex conjugate pairs onto second-order factors.

$$H(z) = \prod_{k=1}^{N_S}\frac{b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2}}{1-a_{1k}z^{-1}-a_{2k}z^{-2}}$$

where $N_s = \lfloor (N+1)/2 \rfloor$ is the largest integer contained in $(N+1)/2$.

We can implement a cascade structure with a minimum number of multiplications and a minimum number of delay elements if we use the direct form II structure (interchange the feedforward and feedback parts of the 2nd order section and combine the delays) for each second-order section.

Interchange the feedback and feedforward parts give the direct form I that we have discussed before.

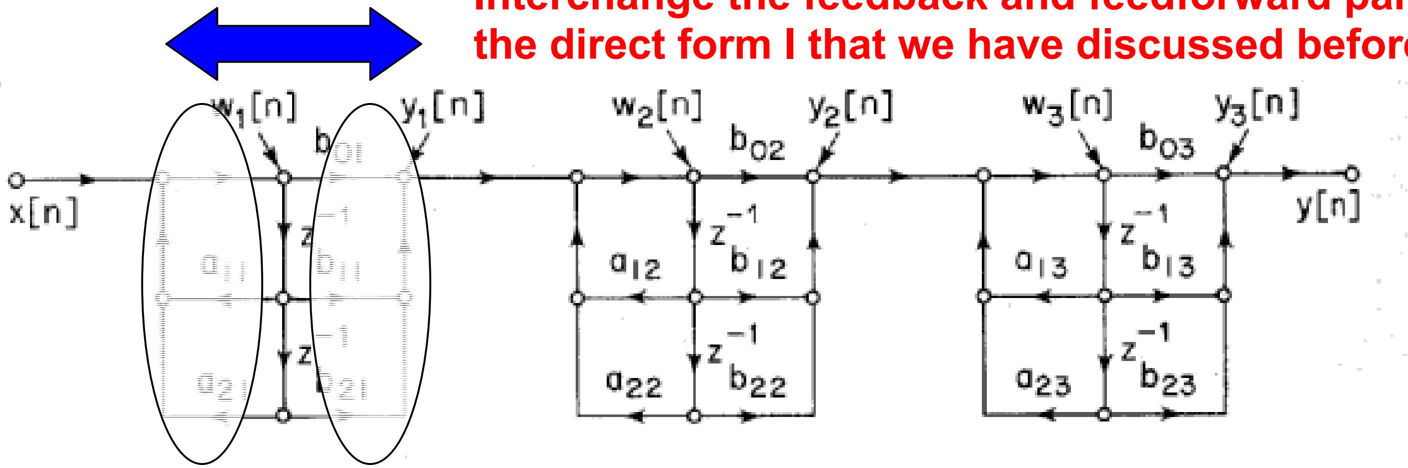


Figure 6.14 Cascade structure for a sixth-order system with a direct form II realization of each second-order subsystem.

A variety of equivalent systems can be obtained by pairing the poles and zeros in different ways and by ordering the second-order sections in different ways.

THE DISCRETE-TIME AND DISCRETE FOURIER TRANSFORMS

CONTENTS

9. Frequency response of LTI systems

The response of an LTI system to a sinusoidal input is sinusoidal with the same frequency and the amplitude and phase are determined by the LTI system.

Consider a sinusoidal input sequence: $x[n] = e^{j\omega n}$. The output of an LTI system with an impulse response $h[n]$ is

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} = e^{j\omega n}\left[\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}\right] = e^{j\omega n} H(e^{j\omega})$$

where

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \qquad \text{(9.1)}$$

is called the frequency response of the system.

- **(9.1) is the discrete-time Fourier transform (DT-FT) of *h*[*n*].**

- **The DT-FT is obtained by substituting $z = e^{j\omega}$ in *H*(*z*), i.e. evaluating the z-transform at the unit circle.**

- **A sufficient condition for convergence of the frequency response (DT-FT of *h*[*n*]) is that *h*[*n*] is absolutely summable. In other words, the system must be stable (c.f. (6.1)). Consequently, the ROC of *H*(*z*) covers the unit circle.**

In general,

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \qquad \textbf{(8.7)}$$

Example: Transient and steady-state responses.

Consider a LTI system defined by

$$y[n] = a_1 y[n-1] + b_0 x[n]. \qquad \textbf{(e1)}$$

causal用以确定ROC

Taking the z-transform on both sides, one gets

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z). \qquad \textbf{(e2)}$$

The transfer function is then given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 - a_1 z^{-1}}. \text{ ROC } |z| > |a_1|. \qquad \textbf{(e3)}$$

Taking the inverse z-transform and noting that the system (e1) is causal, one gets the impulse response h[n] as follows

$$h[n] = b_0 a_1^n u[n]. \qquad \textbf{(e4)}$$

Assuming $|a_1| < 1$, so that the frequency response of the system exists (the system is stable). It is then obtained by evaluating the z-transform in (e3) on the unit circle:

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{b_0}{1 - a_1 e^{-j\omega}}. \qquad \textbf{(e5)}$$

Suppose that the input to the system is a sinusoids starting at n=0:

$$x[n] = e^{j\omega_0 n}u[n]. \qquad \textbf{(e6)}$$

We want to determine the output of the system. Taking the z-transform of (e6), one gets

$$X(z) = \frac{1}{1 - e^{j\omega_0} z^{-1}}. \quad \textbf{ROC } |z| > 1. \qquad \textbf{(e7)}$$

Using the convolution theorem, the z-transform of the output is the product of H(z) and X(z):

$$Y(z) = H(z)X(z) = \frac{b_0}{1 - a_1 z^{-1}} \cdot \frac{1}{1 - e^{j\omega_0} z^{-1}}. \quad \textbf{ROC } |z| > 1. \qquad \textbf{(e8)}$$

Using partial fraction expansion, we have

$$Y(z) = \frac{A_0}{1 - a_1 z^{-1}} + \frac{A_1}{1 - e^{j\omega_0} z^{-1}}. \quad \textbf{ROC } |z| > 1, \qquad \textbf{(e9)}$$

where $A_0 = \dfrac{b_0 a_1}{a_1 - e^{j\omega_0}}$ and $A_1 = \dfrac{b_0}{1 - a_1 e^{-j\omega_0}}$. Taking the inverse z-transform, one get the desired output:

$$y[n] = \underbrace{\overset{\text{正比于系统冲击响应}}{A_0 a_1^n u[n]}}_{\text{transient response}} + \underbrace{\overset{\text{正比于输入响应}}{A_1 e^{jn\omega_0} u[n]}}_{\text{steady state response}} . \qquad \text{(e10)}$$

The first term is proportional to the impulse response *h*[*n*], which is solely determined by the poles of the systems. For stable system, it dies down as n tends to infinity. Therefore it is called the transient response. The second term is proportional to the input complex exponential and $A_1 = \dfrac{b_0}{1 - a_1 e^{-j\omega_0}}$ is the frequency response of the system at $\omega = \omega_0$. It is called the steady-state component of the output.

Exercise:

If b_0=5, a_1=-0.8, and $\omega_0 = 2\pi/10$. Determine the transient and steady state component of the above example.

[Answer:

Transient: $$y_t[n] = \left(\frac{-4}{-0.8 - e^{j0.2\pi}}\right)(-0.8)^n u[n].$$

Steady-state: $$y_s[n] = \left(\frac{5}{1 + 0.8e^{-j0.2\pi}}\right)e^{j0.2n} u[n].$$ **]**

9.1 Magnitude and Phase responses

In general, $H(e^{j\Omega})$ is complex and it can be expressed in terms of its real and imaginary parts as:

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}). \tag{9.2}$$

or in terms of magnitude and phase as

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\angle H(e^{j\omega})}. \tag{9.3}$$

$\left|H(e^{j\omega})\right|$ is the magnitude response or the gain of the system and $\angle H(e^{j\omega})$ is the phase response or phase shift of the system.

9.1.1 Magnitude and Phase responses

The magnitude and phase responses can also be expressed in terms of the poles and zeros as follows:

$$H(z)\Big|_{z=e^{j\Omega}} = \left.\frac{b_0\prod_{k=1}^{M}(1-c_k z^{-1})}{a_0\prod_{k=1}^{N}(1-d_k z^{-1})}\right|_{z=e^{j\Omega}} = \frac{b_0\prod_{k=1}^{M}(1-c_k e^{-j\omega})}{a_0\prod_{k=1}^{N}(1-d_k e^{-j\omega})} \qquad \textbf{(9.4)}$$

c_k **and** d_k **are the zeros and poles of H(z).**

- **The magnitude-squared function is**

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1-c_k e^{-j\omega})(1-c_k^* e^{j\omega})}{\prod_{k=1}^{N}(1-d_k e^{-j\omega})(1-d_k^* e^{j\omega})} \qquad \textbf{(9.5)}$$

Log magnitude of $H(e^{j\omega})$ **(in decibels dB) or gain in dB as follows:**

$$20\log_{10}\left|H(e^{j\omega})\right| = 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}\left|1 - c_k e^{-j\omega}\right|$$

$$- \sum_{k=1}^{N} 20\log_{10}\left|1 - d_k e^{-j\omega}\right| \qquad \textbf{(9.6)}$$

Zero dB corresponds to $\left|H(e^{j\omega})\right| = 1$ while $\left|H(e^{j\omega})\right| = 10^m$ is 20m dB.

$20\log_{10}\left|H(e^{j\omega})\right|$ **is negative when** $\left|H(e^{j\omega})\right| < 1$.

- **The phase response is**

$$\angle H(e^{j\omega}) = \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M} \angle\left(1 - c_k e^{-j\omega}\right) - \sum_{k=1}^{N} \angle\left(1 - d_k e^{-j\omega}\right) \qquad \textbf{(9.7)}$$

The principal value of the phase function can be computed as:

$$\mathrm{ARG}\left[H(e^{j\omega})\right] = \arctan\left[\frac{H_R(e^{j\omega})}{H_I(e^{j\omega})}\right] \qquad \textbf{(9.8)}$$

Since

$$-\pi < \mathrm{ARG}[H(e^{j\omega})] < \pi \qquad \textbf{(9.9)}$$

$$\angle H(e^{j\omega}) = \mathrm{ARG}[H(e^{j\omega})] + 2\pi \cdot r(\omega), \qquad \textbf{(9.10)}$$

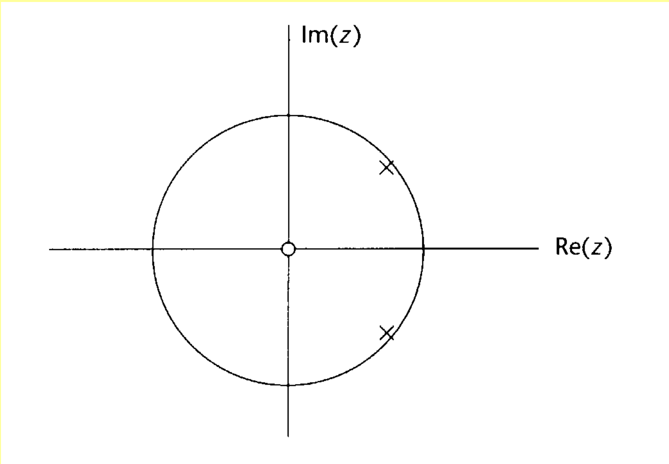
where $r(\omega)$ is a positive or negative integer that can be different at each value of ω. Alternatively, $\mathrm{ARG}[H(e^{j\omega})]$ can be obtained from taking the principal value of each term in (5.3.5):

$$\mathrm{ARG}\left[H(e^{j\omega})\right] = \mathrm{ARG}\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \mathrm{ARG}\left[1 - c_k e^{-j\omega}\right] - \sum_{k=1}^{N} \mathrm{ARG}\left[1 - d_k e^{-j\omega}\right] + 2\pi \cdot r(\omega) \qquad \textbf{(9.11)}$$

Except at the discontinuities of $\mathrm{ARG}[H(e^{j\omega})]$ corresponding to jumps between $+\pi$ and $-\pi$.

Examples

Frequency response of a LTI system with poles at $0.95\angle\pm45^\circ$ and a zero at the origin. Sampling period is 1ms (sampling rate 1kHz).



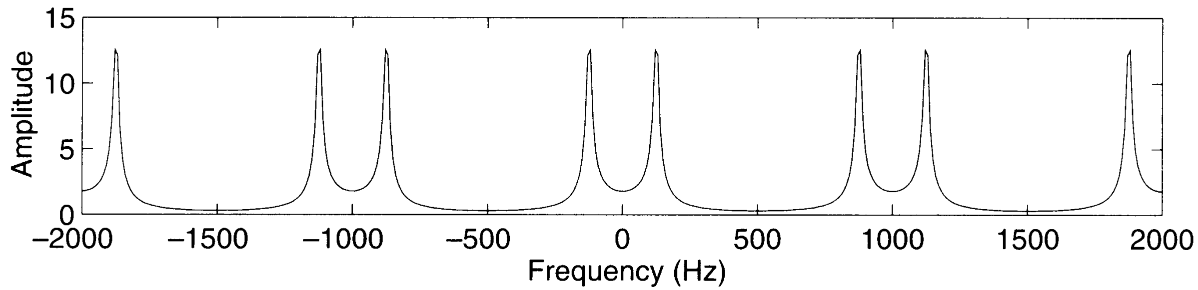
Pole-zero plot

X – poles

O – zero

Note the peaks in the frequency response near the poles.

Fig. 14.19
Predicted amplitude response



$\omega = -\pi$ $\omega = \pi$

Examples 14.7 and 14.8 (in textbook)

$H(z) = 1/(1-0.9z^{-1})$. **The sampling frequency is 10 kHz.**

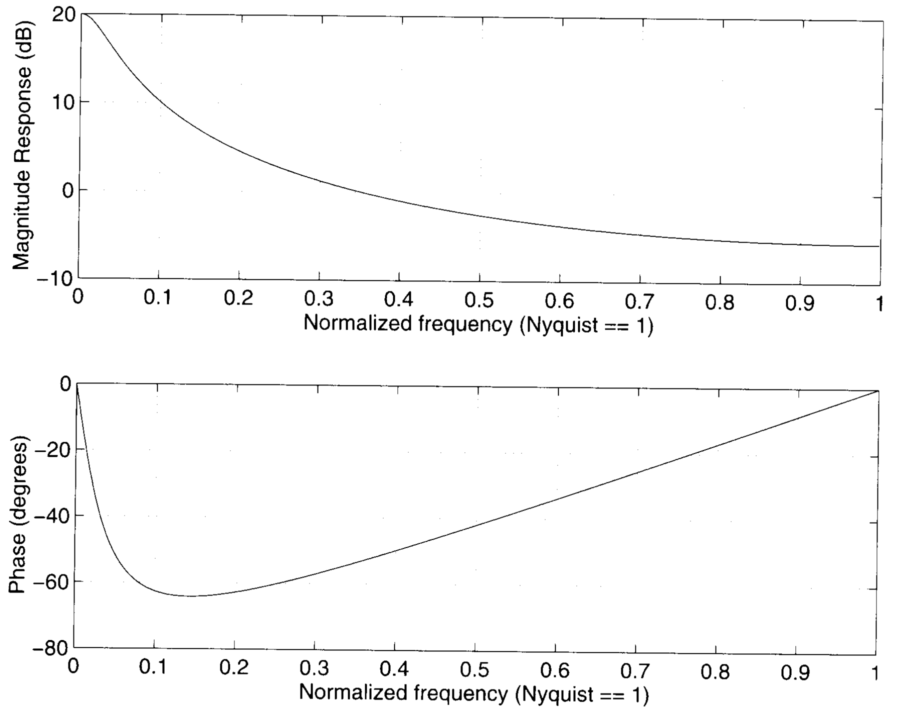


Fig. 14.28 Frequency response plot using MATLAB

MATLAB COMMANDS

```
b=[1 0];
a=[1 –0.9];
subplot(2,2,1);
zplane(b,a)
freqz(b,a)
```

$\omega = \pi$

$$\Omega = 2\pi \times 5000$$

9.3 Discrete-time Fourier transform (DT-FT)

The discrete-time Fourier transform pair is defined as follows:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad \text{(DT-FT)} \qquad (9.12)$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad \text{(Inverse DT-FT)} \qquad (9.13)$$

- **The DT-FT represents the frequency components of *x*[*n*] at digital radian frequency ω.**
- **The inverse DT-FT synthesizes the sequence $x[n]$ from infinitesimally small complex sinusoids of form.**

$$\frac{1}{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

To show that they are inverses of each other, substitute (9.13) into (9.12), we obtain

$$\tilde{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right]e^{j\omega n}d\omega = \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega\right] \qquad \textbf{(9.14)}$$

Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} = \delta[n-m],$$

we obtain the desired result.

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] = x[n]$$

Relationship between continuous time Fourier transform and DT-FT (see section 10, eqn. (10.1)).

9.4 Discrete Fourier transform (DFT)

Consider the DT-FT of a finite length sequence: x[n], n=0,…,N-1.

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

(DT-FT)

Sampling Ω regularly at $\omega_k = 2\pi k / N$ (spacing $2\pi / N$), k=0,1,…,N-1, we obtain the discrete Fourier transform (DFT).

$$X[k] = X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi nk/N)} \quad (9.15)$$

(DFT)

Examples: Compute the DFT of the following sequence

$$x[n] = \begin{cases} 1 & \mathrm{n} = 0,\ldots,4 \\ 0 & \mathrm{n} = 5,\ldots,\mathrm{N}-1 \end{cases} \quad , \quad X[k] = \sum_{n=0}^{4} e^{-j(2\pi nk/N)} = \frac{1-e^{-j5(2\pi k/N)}}{1-e^{-j(2\pi k/N)}}$$

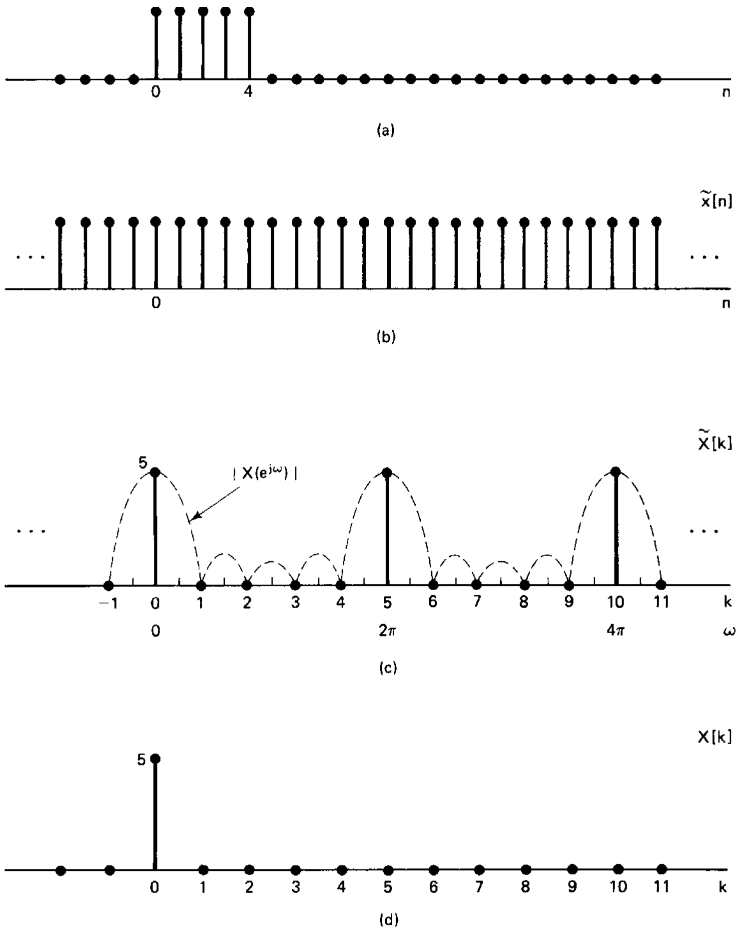


Figure 8.10 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 5$. (c) Fourier series coefficients $\tilde{X}[k]$ for $\tilde{x}[n]$. To emphasize that the Fourier series coefficients are samples of the Fourier transform, $|X(e^{j\omega})|$ is also shown. (d) DFT of $x[n]$.

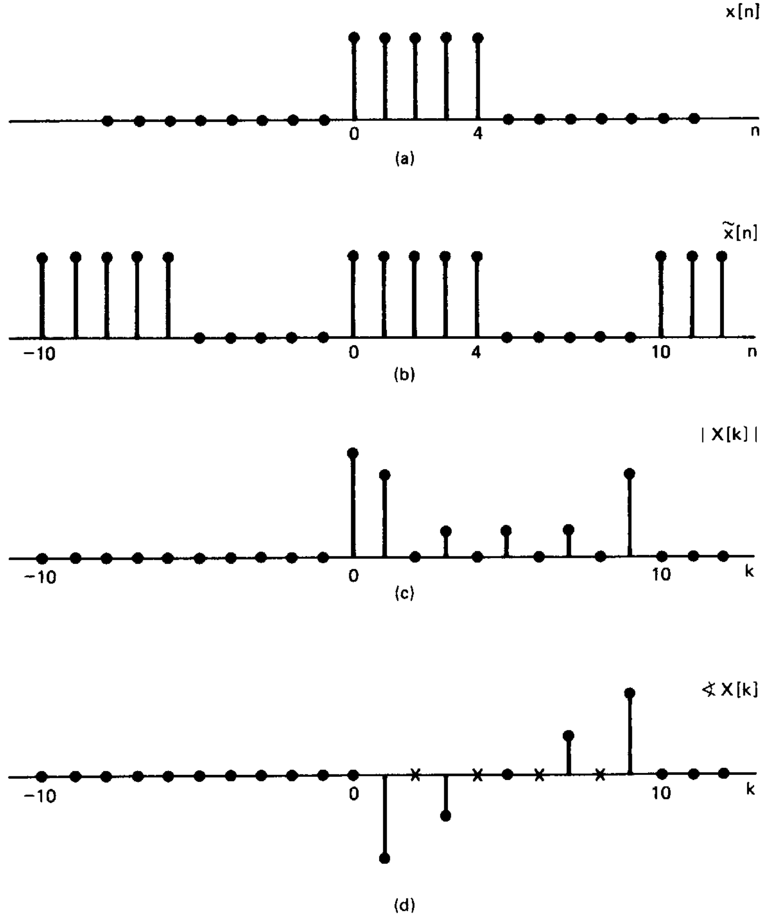


Figure 8.11 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 10$. (c) DFT magnitude. (d) DFT phase. (x's indicate indeterminate value.)

N=5 **N=11**

- **DFT as uniform samples of the DT-FT in the frequency domain (**spacing $2\pi / N$**).**

9.4.1 Inverse Discrete Fourier transform (IDFT)

DFT is an orthogonal transformation and it has a simple inversion formula (the inverse DFT):

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j(2\pi nk/N)} \qquad (9.16)$$

- It is identical to the DFT, except for the scaling factor (1/N) and the kernel $e^{j(2\pi nk/N)}$ (note the negative sign in the DFT).
- DFT supports a kind of convolution called "circular convolution" and it can be used to compute discrete-time convolution (i.e. real-time filtering using FIR filters). Fast algorithms for DFT called fast Fourier transform (FFT) with order O($N\log_2 N$) arithmetic complexity are available and they found many applications. (MATLAB COMMAND: X=fft(x, N), N – transform length).

Exercises:

1. **By substituting (9.15) into (9.16), verify that (9.16) is the inverse of the DFT [Hint: use the identity:** $x[n] = \sum_{k=0}^{N-1} e^{j(2\pi k/N)(n-m)} = N \cdot \delta[n-m]$**].**
2. **Verify (9.16) and compute its inverse DFT.**
3. **Show that a finite length sequence of length N can be represented by its uniform samples of DT-FT at** $\omega_k = 2\pi k / N$**, k=0,1,…,N-1. [Hint: the DFT is reversible with the inverse transformation given by the IDFT.]**