ANSWER ANY THREE QUESTIONS. ALL QUESTIONS CARRY EQUAL MARKS.

Q1.

(a) Consider a causal linear time-invariant (LTI) system with the following linear constant coefficient difference equation relating the input $x(n)$ and output $y(n)$ of this system:

$$y(n) = 2x(n) - 2.4x(n-1) - 0.4x(n-2) + 0.3y(n-1) + 0.4y(n-2).$$

i) Determine the transfer function $H(z)$ of this system in the z-domain.

(3 marks)

ii) Determine the poles of $H(z)$. **(2 marks)**

iii) Determine the impulse response, $h(n)$, (i.e. the response of the system to the unit impulse sequence $\delta(n)$) of this system.

(6 marks)

iv) Is the system stable? If the above system is now non-causal and has the same transfer function $H(z)$, is it still stable? Explain.

(4 marks)

v) Determine the z-transform of $x(n) = e^{j(n\omega_0)}u(n)$, where $u(n)$ is the unit step sequence.

(2 marks)

vi) Determine the output $y(n)$ of the system to $x(n) = e^{j(n\omega_0)}u(n)$. Identify the transient and steady state responses in your solution.

(6 marks)

(b)

i) Determine the impulse response $h(n)$ of a highpass filter with the following ideal discrete-time frequency response:

$$H_{hp}(e^{j\omega}) = \begin{cases} 0 & 0 \le |\omega| < \omega_c \\ e^{-j\omega M/2} & \omega_c \le |\omega| \le \pi. \end{cases}$$ **(5 marks)**

ii) What is the delay, in samples, of $H_{hp}(e^{j\omega})$ above? **(1 marks)**

iii) Determine the discrete time Fourier transform (DTFT) of the sequence $h(n)(-1)^n$ in terms of $H_{hp}(e^{j\omega})$ and sketch its frequency response.

(4 marks)

[The DTFT of the sequence $x(n)$ is $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega}$.

The inverse discrete time Fourier transform of a function $X(e^{j\omega})$ is

$x(n) = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{jn\omega}d\omega$.]

Q2.

(a) Consider the discrete-time processing system in Figure Q2-1. It is required to perform an equivalent filtering in the continuous time domain with the following specifications:

1. The gain $|H_{eff}(j\Omega)|$ should be within ± 0.01 of unity (zero dB) in the frequency band $0 \le \Omega \le 2\pi(2250)\ Hz$.
2. The gain should be no greater than ± 0.001 in the frequency band $2\pi(2750)\ Hz \le \Omega$.

Ω is the continuous time radian frequency in radians.

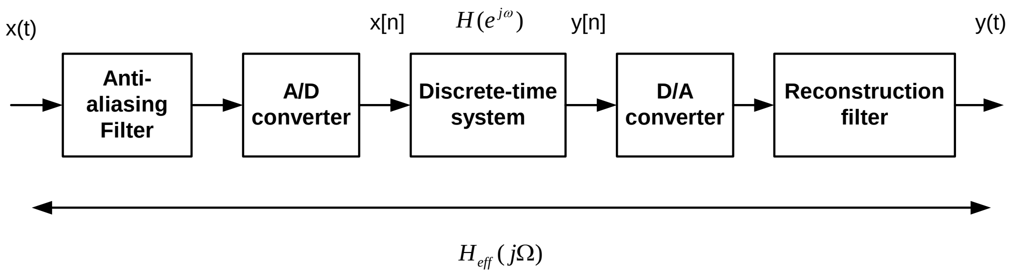

Figure Q2-1.

i) If the sampling rate is 10^4 samples per second, determine the equivalent specifications of the discrete-time system.

(4 marks)

ii) Using the following formulae, estimate the parameters M and β of the Kaiser window required to satisfy the specifications.

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50; \\ 0.0 & 0 \le A < 21 \end{cases}$$

$$M = (A-8)/(2.285 \cdot \Delta\omega),$$

where $A = -20\log_{10}\delta$ (dB), $\Delta\omega = |\omega_s - \omega_p|$, δ is the minimum amplitude of the passband and stopband ripples, ω_s is the stopband cutoff frequency, and ω_p is the passband cutoff frequency.

(4 marks)

iii) Use the following formula to estimate the order of the linear-phase filter if the McClellan-Parks design algorithm is used.

$$N = \frac{-20\log_{10}\sqrt{\delta_1\delta_2} - 13}{14.6\Delta f},$$

where δ_1, δ_2 are respectively the maximum amplitude of the passband and stopband ripples and $\Delta f = \Delta\omega / 2\pi$. Comment on the two algorithms.

(5 marks)

(b) It is required to design a diamond-shape filter as shown in Figure Q2-2, by transforming the 1-D prototype obtained in (a).

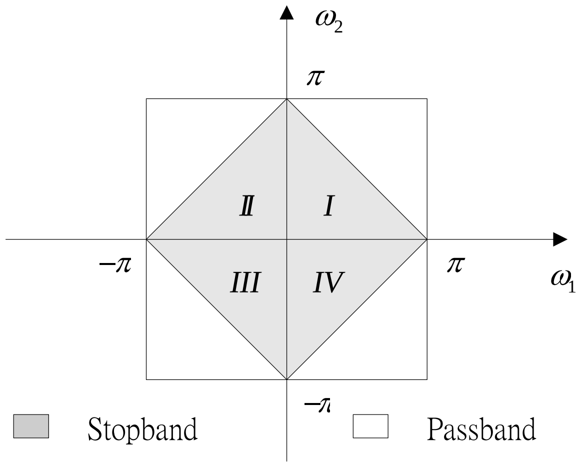

Figure Q2-2.

Suppose that the 1D prototype can be written as:

$$H(\omega) = h(0) + \sum_{n=1}^{N} h(n)[\exp(-j\omega n) + \exp(j\omega n)]$$
$$= \sum_{n=0}^{N} a(n)\cos(n\omega) = \sum_{n=0}^{N} a(n)T_n[\cos(\omega)] \qquad (*)$$

where $a(n) = \begin{cases} h(0), n = 0 \\ 2h(n), n > 0 \end{cases}$ and $T_n(x)$ is the *n-th* order Chebyshev polynomial.

The 2D frequency response is obtained by substituting $\cos\omega$ in equation (*) by $F(\omega_1, \omega_2)$ to obtain:

$$H(\omega_1, \omega_2) = \sum_{n=0}^{N} a(n)T_n[F(\omega_1, \omega_2)].$$

Let the transformation function be

$$F(\omega_1, \omega_2) = \frac{1}{2}(\cos\omega_1 + \cos\omega_2).$$

Show that:

i) $H(\pm\omega_2, \pm\omega_1) = H(\omega_1, \omega_2)$. **(2 marks)**

ii) The transformation is positive in the regions I, II, III and IV and negative in the remaining area as indicated in Figure Q2-2.

(4 marks)

iii) The values of the prototype filter in the interval $0 \le \omega \le 0.5\pi$ are mapped to the regions I, II, III and IV and the values of the prototype filter in the interval $0.5\pi \le \omega \le \pi$ are mapped to the remaining area, i.e. the passband.

(4 marks)

(c)

It is desired to design a 2D FIR filter by second order cone programming (SOCP) method, which can be written as:

$$\begin{aligned} &\min_{\boldsymbol{x}} && \boldsymbol{c}^T \boldsymbol{x} \\ &\text{subject to} && \|\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{b}_i\|_2 \le \boldsymbol{c}_i^T \boldsymbol{x} + d_i, \quad i = 1, \ldots, M, \\ & && \boldsymbol{F}\boldsymbol{x} = \boldsymbol{g}, \end{aligned}$$

where $\boldsymbol{x} \in \Re^N$ is the optimization variable, $\boldsymbol{A}_i \in \Re^{N_i \times N}$, $\boldsymbol{F} \in \Re^{P \times N}$, $\boldsymbol{b}_i \in \Re^{N_i}$, $\boldsymbol{c}, \boldsymbol{c}_i \in \Re^N$, $\boldsymbol{g} \in \Re^P$, $d_i \in \Re$ and $\|\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{b}_i\|_2 \le \boldsymbol{c}_i^T \boldsymbol{x} + d_i$ for $i = 1, \ldots, M$ are second-order cone constraints.

Consider a 2D FIR filter with frequency response: $H(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} h(n_1, n_2) e^{-jn_1\omega_1} e^{-jn_2\omega_2}$.

i) Show that $H(e^{j\omega_1}, e^{j\omega_2}) = \boldsymbol{h}^T (\boldsymbol{c}(\boldsymbol{\omega}) - j\boldsymbol{s}(\boldsymbol{\omega}))$

where $\boldsymbol{\omega} = [\omega_1, \omega_2]^T$, $\boldsymbol{h} = [h(0,0), h(0,1), ..., h(N_1 - 1, N_2 - 2), h(N_1 - 1, N_2 - 1)]^T$, $\boldsymbol{c}(\boldsymbol{\omega}) = [1, \cos(\omega_2), ..., \cos((N_1 - 1)\omega_1 + (N_2 - 1)\omega_2)]^T$ and $\boldsymbol{s}(\boldsymbol{\omega}) = [0, ..., \sin((N_1 - 1)\omega_1 + (N_2 - 1)\omega_2)]^T$.

(2 marks)

ii) Let the desired response in the passband $\omega_1, \omega_2 \in \Omega_p$ and stopband $\omega_1, \omega_2 \in \Omega_s$ be $H_d(e^{j\omega_1}, e^{j\omega_2})$. It is desired to design the FIR filter using the weighted least squares criterion subject to a set of peak magnitude error constraints around the band edges $\omega_1, \omega_2 \in \Omega_E$. The corresponding filter design problem can be written as:

$$\min_{\boldsymbol{h}} \int_{\Omega_p \cup \Omega_s} W(\boldsymbol{\omega}) \,|\, H(e^{j\omega_1}, e^{j\omega_2}) - H_d(e^{j\omega_1}, e^{j\omega_2}) \,|^2 d\boldsymbol{\omega},$$

subject to $|\, H(e^{j\omega_1}, e^{j\omega_2}) - H_d(e^{j\omega_1}, e^{j\omega_2}) \,| \le \delta_p$, $\omega_1, \omega_2 \in \Omega_E$, where $W(\boldsymbol{\omega})$ is a real positive weighting function and δ_p is a prescribed peak ripple. Show that the WLS function is given by

$$\mathrm{WLS}(\boldsymbol{h}) = \int_{\Omega_p \cup \Omega_s} W(\boldsymbol{\omega}) \,|\, H(e^{j\omega_1}, e^{j\omega_2}) - H_d(e^{j\omega_1}, e^{j\omega_2}) \,|^2 d\boldsymbol{\omega}$$

$$= \boldsymbol{h}^T \boldsymbol{Q} \boldsymbol{h} - 2\boldsymbol{h}^T \boldsymbol{g} + c\,,$$

where $\boldsymbol{Q} = \int_{\Omega_p \cup \Omega_s} W(\boldsymbol{\omega})(\boldsymbol{c}(\boldsymbol{\omega}) - j\boldsymbol{s}(\boldsymbol{\omega}))(\boldsymbol{c}(\boldsymbol{\omega}) - j\boldsymbol{s}(\boldsymbol{\omega}))^H \, d\boldsymbol{\omega}$,

$$\boldsymbol{g} = \int_{\Omega_p \cup \Omega_s} W(\boldsymbol{\omega}) \,\mathrm{Re}\{(\boldsymbol{c}(\boldsymbol{\omega}) - j\boldsymbol{s}(\boldsymbol{\omega}))H_d^*(e^{j\omega_1}, e^{j\omega_2})\} d\boldsymbol{\omega}\,,$$

$c = \int_{\Omega_p \cup \Omega_s} W(\boldsymbol{\omega}) \,|\, H_d(e^{j\omega_1}, e^{j\omega_2}) \,|^2 \, d\boldsymbol{\omega}$ and superscript ^{H} stands for conjugate transpose.

[Hint:

$$|\, H(e^{j\omega_1}, e^{j\omega_2}) - H_d(e^{j\omega_1}, e^{j\omega_2}) \,|^2$$
$$= (H(e^{j\omega_1}, e^{j\omega_2}) - H_d(e^{j\omega_1}, e^{j\omega_2}))(H(e^{j\omega_1}, e^{j\omega_2}) - H_d(e^{j\omega_1}, e^{j\omega_2}))^* \,]$$

(3 marks)

iii) Suppose that $\boldsymbol{Q}$ can be factorized as $\boldsymbol{Q} = \boldsymbol{L}\boldsymbol{L}^T$ where $\boldsymbol{L}$ is some nonsingular upper triangular matrix. Show that

$$\min_{\boldsymbol{h}} \ \mathrm{WLS}(\boldsymbol{h}) = \boldsymbol{h}^T \boldsymbol{Q} \boldsymbol{h} - 2\boldsymbol{h}^T \boldsymbol{g} + c$$

is equivalent to

$$\min_{\boldsymbol{h}} \| \boldsymbol{L}^T \boldsymbol{h} - \boldsymbol{L}^{-1} \boldsymbol{g} \|_2^2,$$

where the $L2$ norm of a vector $\boldsymbol{x}$ is given by $\| \boldsymbol{x} \|_2^2 = \boldsymbol{x}^T \boldsymbol{x}$.

(2 marks)

iv) By discretizing ω_1, ω_2 uniformly in $\omega_1, \omega_2 \in \Omega_E$ into M points, $\boldsymbol{\omega}_i$ for i=1,…,M, show that the above problem can be written as:

$$\begin{aligned}
&\min_{\boldsymbol{h},\delta} && \delta \\
&\text{subject to} && \delta_p - \{\alpha_R^2(\boldsymbol{\omega}_i) + \alpha_I^2(\boldsymbol{\omega}_i)\}^{1/2} \geq 0, \quad i = 1, \ldots, M, \\
& && \| \boldsymbol{L}^T \boldsymbol{h} - \boldsymbol{L}^{-1} \boldsymbol{g} \|_2 \leq \delta
\end{aligned}$$

$$\alpha_R(\boldsymbol{\omega}) = W(\boldsymbol{\omega}) \cdot \{\boldsymbol{h}^T \boldsymbol{c}(\boldsymbol{\omega}) - \mathrm{Re}[H_d(e^{j\omega_1}, e^{j\omega_2})]\},$$

$$\alpha_I(\boldsymbol{\omega}) = W(\boldsymbol{\omega}) \cdot \{\boldsymbol{h}^T \boldsymbol{s}(\boldsymbol{\omega}) + \mathrm{Im}[H_d(e^{j\omega_1}, e^{j\omega_2})]\}.$$

(3 marks)

Q3.

a) Let $x(n)$ be a wide-sense stationary random process with mean μ_x and autocorrelation $R_{xx}(k)$. Let $y(n)$ be the output obtained by applying $x(n)$ as input to a stable linear time-invariant system with impulse response $h(n)$ and transfer function $H(e^{j\omega})$.

i) Show that $y(n)$ is also a wide-sense stationary process with mean μ_y and autocorrelation $R_{yy}(k)$ given by

$$\mu_y = H(1) \cdot \mu_x ,$$

and $R_{yy}(k) = R_{xx}(k) * h(k) * h(-k)$,

where * denotes the discrete-time convolution operation.

(4 marks)

ii) Hence, show that the Power Spectral Density (PSD) of $y(n)$ is

$$S_{yy}(e^{j\omega}) = S_{xx}(e^{j\omega}) \left| H(e^{j\omega}) \right|^2 .$$ **(3 marks)**

b) Figure Q3-1 shows the signal flow graph of a causal stable second-order direct form II system with system function $H(z)$ and impulse response $h(n)$. Suppose that the results of the multiplications { $a_1 \cdot w(n-1)$, $a_2 \cdot w(n-2)$ }, and { $b_0 \cdot w(n)$, $b_1 \cdot w(n-1)$, $b_2 \cdot w(n-1)$ } are rounded to B_a+1 and B_b+1 bits, respectively and let $e_{a_i}(n)$, i=1,2, and $e_{b_k}(n)$, k=0,1,2, be the corresponding quantization noises. Let $\xi_{a_i}(n)$, i=1,2, be the quantization noises at the system output generated respectively by $e_{a_i}(n)$, i=1,2. Assume that:

1) $e_{a_i}(n)$ and $e_{b_k}(n)$ are zero-mean wide-sense stationary white-noises, and are uncorrelated with each other.
2) Each noise source has a uniform distribution of amplitudes over one quantization interval.
3) $e_{a_i}(n)$ and $e_{b_k}(n)$, are uncorrelated with the input of the corresponding quantizers, and the input to the system.

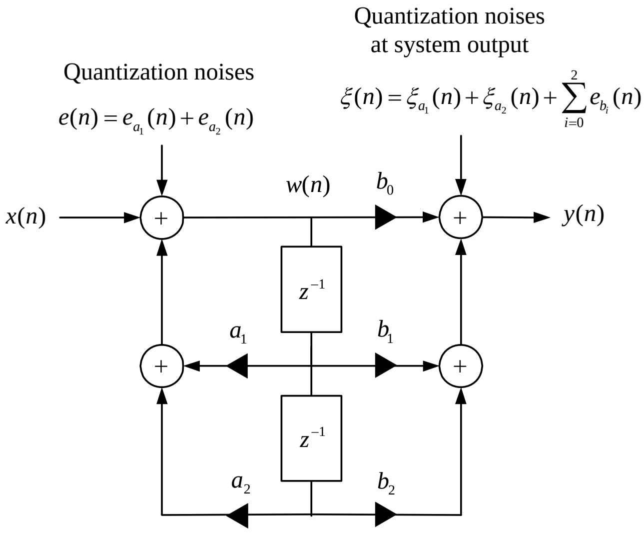

Figure Q3-1.

i) Determine the transfer function $H(z) = Y(z)/X(z)$. **(3 marks)**

ii) Show that

$$E[\{\xi_{a_1}(n) + \xi_{a_2}(n) + \sum_{i=0}^{2} e_{b_i}(n)\}^2] = E[\xi_{a_1}^2(n)] + E[\xi_{a_2}^2(n)] + \sum_{i=0}^{2} E[e_{b_i}^2(n)].$$

(5 marks)

iii) If the variances of $e_{a_i}(n)$ and $e_{b_j}(n)$ are given by $\sigma_{e_{a_i}}^2 = \frac{2^{-2B_a}}{12}$ and $\sigma_{e_{b_i}}^2 = \frac{2^{-2B_b}}{12}$ respectively, determine the power spectral densities, $P_{a_i}(\omega)$ and $P_{b_j}(\omega)$, of $e_{a_i}(n)$ and $e_{b_j}(n)$.

[Hint: $\sigma_x^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(\omega)d\omega$, where σ_x^2 and $P_x(\omega)$ are respectively the variance and power spectral density of a random process $x(n)$.] **(3 marks)**

iv) Show that the total noise variance at the output is

$$\sigma_\xi^2 = 3\frac{2^{-2B_b}}{12} + 2\frac{2^{-2B_a}}{12}\sum_{n=-\infty}^{\infty} |h(n)|^2 .$$

[Hint: You might use the Parseval's theorem

$$\sum_{n=-\infty}^{\infty} |h(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega,$$

or evaluate the result in b(ii) directly.] **(5 marks)**

v) Suppose that $x(n)$ is in fix-point representation using the signed magnitude representation. Moreover, assume the fix point is on the left of the most significant bit (MSB), i.e. $x(n)$ is less than 1. We wish to determine the worse case integer part of the fix-point representation of the quantities $w(n)$ and $y(n)$ so that no overflow will occur. We now determine the worse case integer part of $y(n)$ and that of $w(n)$ can be done similarly.

Using the relationship $y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$, show that

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)|, \text{ for all } n.$$ **(2 marks)**

Hence, show that the worse case integer part of $y(n)$ has a length equal to

$$\max\left(\left\lceil \log_2\left(\sum_{k=-\infty}^{\infty} |h(k)|\right)\right\rceil, 0\right) \text{ bits,}$$

where the operator $\lceil u \rceil$ gives the nearest integer which is larger or equal to u and the operator $\max(.)$ returns the maximum value of the set inside the bracket.

.

(3 marks)

vi) Suppose that the required accuracy of the output $y(n)$ due to signal round off noise is B+1 bits. Moreover, suppose that $B_a = 2B_b$, suggest a method to estimate the minimum B_a in order to satisfy the given output accuracy.

(5 marks)

Q4.

a) It is required to design an FIR Wiener filter $\{w_i, i = 0,..., L-1\}$ as shown in Figure Q4-1 to approximate a desired signal $d(n)$ from its input signal $x(n)$. The Wiener filter is of L taps and its output $y(n)$ is given by:

$$y(n) = \sum_{k=0}^{L-1} w_k x(n-k) = \boldsymbol{W}^T \boldsymbol{X}_n,$$

where $\boldsymbol{W} = [w_0 \quad w_1 \quad . \quad . \quad w_{L-1}]^T$ is the weight vector, and

$\boldsymbol{X}_n = [x(n) \quad x(n-1) \quad . \quad . \quad x(n-L+1)]^T$ is the input signal vector.

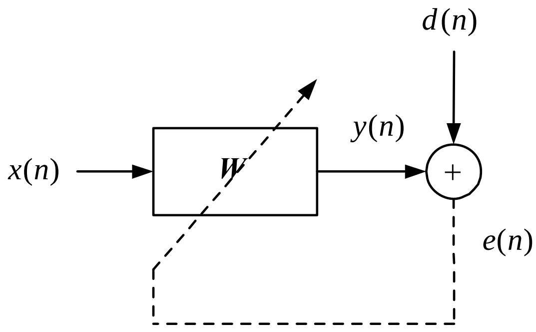

Figure Q4-1.

Assume that all the signals are *stationary*.

i) Let $e(n) = d(n) - y(n)$ be the approximation error. Show that the Mean Squared Error (MSE)

$$\xi = E[e^2(n)]$$

can be written as:

$$\xi = r_{dd}(0) + \boldsymbol{W}^T \boldsymbol{R} \boldsymbol{W} - 2\boldsymbol{P}^T \boldsymbol{W} ,$$

where $r_{dd}(0) = E[d^2(n)]$,

$\boldsymbol{R} = E[\boldsymbol{X}_n \boldsymbol{X}_n^T] = \begin{bmatrix} r_{xx}(0) & r_{xx}(1) & \dots & r_{xx}(L-1) \\ r_{xx}(1) & r_{xx}(0) & \dots & r_{xx}(L-2) \\ : & : & : & : \\ r_{xx}(L-1) & r_{xx}(L-2) & \dots & r_{xx}(0) \end{bmatrix}$ is the autocorrelation matrix of the input signal vector, and

$\boldsymbol{P} = E[d(n)\boldsymbol{X}_n] = \begin{bmatrix} r_{xd}(0) & r_{xd}(1) & \dots & r_{xd}(L-1) \end{bmatrix}^T$ is the cross correlation vector, and E and T denote the expectation operator and matrix transposition, respectively.

$r_{xd}(n)$ denotes the cross-correlation between signals $x(n)$ and $d(n)$ and $r_{xx}(n)$ denotes the autocorrelation of signal $x(n)$.

(4 marks)

ii) Using the result in part (i), show that the optimum weight vector $\boldsymbol{W}^*$ that minimizes the MSE is

$$\boldsymbol{W}^* = \boldsymbol{R}^{-1}\boldsymbol{P} ..$$

(3 marks)

iii) In recursive least squares (RLS) adaptive filtering algorithm, the correlation matrix $\boldsymbol{R}(n)$ and correlation vector $\boldsymbol{P}(n)$ at the *n-th* time instant are respectively estimated recursively as:

$$\boldsymbol{R}(n) = \lambda \boldsymbol{R}(n-1) + \boldsymbol{X}_n \boldsymbol{X}_n^T \text{ and } \boldsymbol{P}(n) = \lambda \boldsymbol{P}(n-1) + d(n)\boldsymbol{X}_n ,$$

where $0 < \lambda < 1$ is a forgetting factor, $\boldsymbol{R}(0) = \delta \boldsymbol{I}$ and $\boldsymbol{P}(0) = \boldsymbol{0}$ with δ a positive number. Using the matrix inversion formula

$$(\boldsymbol{A} + \beta \boldsymbol{x}\boldsymbol{y}^T)^{-1} = \boldsymbol{A}^{-1} - \frac{\beta \boldsymbol{A}^{-1}\boldsymbol{x}\boldsymbol{y}^T \boldsymbol{A}^{-1}}{1 + \beta \cdot \boldsymbol{y}^T \boldsymbol{A}^{-1}\boldsymbol{x}}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are column vectors of length L, $\boldsymbol{A}$ is any (LxL) nonsingular matrix, and β is a constant, one can show that

(*) $$\boldsymbol{R}^{-1}(n) = \lambda^{-1}(\boldsymbol{I} - \boldsymbol{K}(n)\boldsymbol{X}_n^T) \cdot \boldsymbol{R}^{-1}(n-1),$$

where $\boldsymbol{K}(n) = \dfrac{\boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n}{\lambda + \boldsymbol{X}_n^T \boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n} = \dfrac{\boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n}{\lambda + \mu(n)}$ is the Kalman gain vector and $\mu(n) = \boldsymbol{X}_n^T \boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n$.

Using (*) above, show that

$$\boldsymbol{R}^{-1}(n)\boldsymbol{X}_n = \boldsymbol{K}(n).$$ **(3 marks)**

iv) Using the results in (iii) and (*) above, show that the optimal weight vector of the RLS algorithm at the *n-th* time instance is given by

$$\boldsymbol{R}^{-1}(n)\boldsymbol{P}(n) = \boldsymbol{W}(n) = \boldsymbol{W}(n-1) + \boldsymbol{K}(n)e_p(n),$$

where $e_p(n) = d(n) - \boldsymbol{X}_n^T \boldsymbol{W}(n-1)$.

[Hint: note $\boldsymbol{P}(n) = \lambda \boldsymbol{P}(n-1) + d(n)\boldsymbol{X}_n$ and $d(n) = e_p(n) + \boldsymbol{X}_n^T \boldsymbol{W}(n-1)$]

(3 marks)

b) In the least mean squares (LMS) algorithm, the weight vector is updated recursively as follows

$$\boldsymbol{W}_{n+1} = \boldsymbol{W}_n + 2\mu e(n)\boldsymbol{X}_n,$$

where $e(n) = d(n) - \boldsymbol{W}_n^T \boldsymbol{X}_n$ and μ is an appropriately chosen stepsize parameter to ensure convergence of the algorithm. Briefly explain the derivation of this equation. Suggest a method to determine the stepsize parameter without having to compute the eigenvalues of $\boldsymbol{R}$. Comment on the selection of the stepsize parameter in stationary and time-varying environment.

(6 marks)

c) Consider the linear predictor for a wide-sense stationary process $x(n)$ as shown in Figure Q4-2.

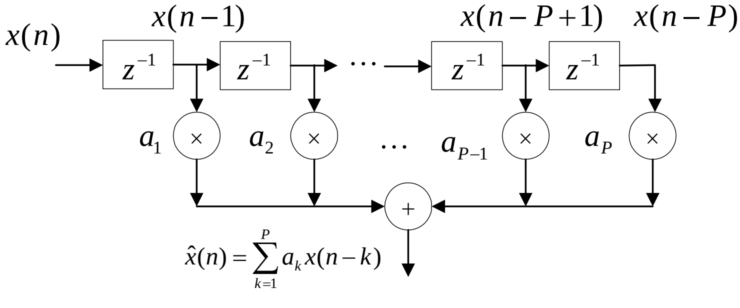

Figure. Q4-2 : *P-th* order Linear Predictor.

Let $e(n) = x(n) - \hat{x}(n)$ be the prediction error and *MSE* be the mean squared error given by:

$$MSE = E[e^2(n)] = E[(x(n) - \hat{x}(n))^2].$$

i) Show that:

$$MSE = r_{xx}(0) - 2\boldsymbol{a}^T \boldsymbol{r}_{xx} + \boldsymbol{a}^T \boldsymbol{R}_{xx} \boldsymbol{a},$$

where $\boldsymbol{R}_{xx} = E[\boldsymbol{x}\boldsymbol{x}^T]$ is the *autocorrelation matrix* of the input signal,

$\boldsymbol{r}_{xx} = E[x(n)\boldsymbol{x}]$ is the *autocorrelation* vector of the input signals,

$\boldsymbol{a}^T = [a_1, \quad a_2, \quad ..., \quad a_P]$ and $\boldsymbol{x}^T = [x(n-1) \quad x(n-2), \quad ..., \quad x(n-P)]$.

(5 marks)

ii) By differentiating *MSE* with respect to a_i, show that the optimal linear predictor coefficients with minimum *MSE* is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_P \end{bmatrix} = \begin{bmatrix} r_{xx}(0) & r_{xx}(1) & r_{xx}(2) & \cdots & r_{xx}(P-1) \\ r_{xx}(1) & r_{xx}(0) & r_{xx}(1) & \cdots & r_{xx}(P-2) \\ r_{xx}(2) & r_{xx}(1) & & \cdots & r_{xx}(P-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{xx}(P-1) & r_{xx}(P-2) & r_{xx}(P-3) & \cdots & r_{xx}(0) \end{bmatrix}^{-1} \begin{bmatrix} r_{xx}(1) \\ r_{xx}(2) \\ r_{xx}(3) \\ \vdots \\ r_{xx}(P) \end{bmatrix}.$$

(3 marks)

iii) Show that

$$S_e(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 S_x(e^{j\omega}),$$

where $S_e(e^{j\omega})$ and $S_x(e^{j\omega})$ are respectively the power spectral densities of $e(n)$ and $x(n)$, and

$$H(z) = 1 - \sum_{i=1}^{P} a_i z^{-i}.$$ **(3 marks)**

iv) Suppose that $e(n)$ is white Gaussian distributed with zero mean and variance σ_e^2 and $H(z)$ is minimum phase, i.e. $H^{-1}(z)$ is causal stable, determine the PSD of $x(n)$, i.e. $S_x(e^{j\omega})$.

(3 marks)

***** END OF PAPER *****

SOLUTION

Q1.

(a)

Consider a causal linear time-invariant (LTI) system with the linear constant coefficient difference equation:

y(n)=2x(n)-2.4x(n-1)-0.4x(n-2)+0.3y(n-1)+0.4y(n-2).

(i) [**3 marks**] $H(z) = \dfrac{2 - 2.4z^{-1} - 0.4z^{-2}}{(1 - 0.3z^{-1} - 0.4z^{-2})} = \dfrac{2 - 2.4z^{-1} - 0.4z^{-2}}{(1 + 0.5z^{-1})(1 - 0.8z^{-1})}$.

(ii) [**2 marks**] The poles are $p_1 = -0.5$, $p_2 = 0.8$.

(iii) [**6 marks**] The impulse response is the inverse z-transform of $H(z)$. First, we express $H(z)$ as its partial fraction expansion,

$$H(z) = 1 + \frac{1 - 2.1z^{-1}}{(1 + 0.5z^{-1})(1 - 0.8z^{-1})} = 1 + \frac{2}{(1 + 0.5z^{-1})} - \frac{1}{(1 - 0.8z^{-1})}.$$

Taking z-transform and noting that the system is causal, one gets

$$h(n) = \delta(n) + 2(-0.5)^n u(n) - (0.8)^n u(n).$$

(iv) [**4 mark**s] Yes, because all the poles are inside the unit circle. If the system is non-causal, then the region of convergence of any one of the poles 0.5 and 0.8 will extend inwards, and the ROC cannot cover the unit circle. Therefore, the system will become unstable.

(iv) [**2 marks**] $x(n) = e^{j(n\omega_0)} u(n)$. The z-transform is: $X(z) = \dfrac{1}{1 - e^{j\omega_0} z^{-1}}$, |z|>1.

(v) [**6 marks**] $Y(z) = [1 + \dfrac{2}{(1 + 0.5z^{-1})} - \dfrac{1}{(1 - 0.8z^{-1})}]\dfrac{1}{1 - e^{j\omega_0} z^{-1}}$, ROC $|z| > 1$

$$= [1 + \frac{2}{(1 + 0.5e^{-j\omega_0})} - \frac{1}{(1 - 0.8e^{-j\omega_0})}]\frac{1}{1 - e^{j\omega_0} z^{-1}}$$

$$+ \frac{1}{(1 + (0.5)^{-1} e^{j\omega_0})}\frac{2}{(1 + 0.5z^{-1})} - \frac{1}{(1 - (0.8)^{-1} e^{j\omega_0})}\frac{1}{(1 - 0.8z^{-1})}$$

$$y(n) = H(e^{j\omega_0}) e^{jn\omega_0} u(n) + \left[\frac{2}{(1 + (0.5)^{-1} e^{j\omega_0})} (-0.5)^n u(n) - \frac{1}{(1 - (0.8)^{-1} e^{j\omega_0})} (0.8)^n u(n) \right].$$

The first term on the RHS is the steady state response whereas the term inside the bracket is

(b)

i) [**5 marks**] The required impulse response is equal to the inverse DT-FT of

$$H_{hp}(e^{j\omega}) = \begin{cases} 0 & 0 \le |\omega| \le \omega_c \\ e^{-j\omega M/2} & \omega_c \le |\omega| \le \pi. \end{cases}$$

Hence,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{hp}(e^{j\omega})e^{jn\omega}d\omega = \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega[(M/2)-n]}d\omega + \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega[(M/2)-n]}d\omega$$

$$= \frac{-1}{2\pi j[(M/2)-n]}\{[e^{-j[(M/2)-n]\pi} - e^{-j[(M/2)-n]\omega_c}] + [e^{j[(M/2)-n]\omega_c} - e^{j[(M/2)-n]\pi}]\}$$

$$= \frac{\sin \pi[(M/2)-n]}{\pi[(M/2)-n]} - \frac{\sin \omega_c[(M/2)-n]}{\pi[(M/2)-n]}.$$

ii) [**1 marks**] What is the delay, in samples, of $H_{hp}(e^{j\omega})$ above?

The sample delay is $M/2$,

iii) **[4 marks]** Determine the discrete time Fourier transform (DTFT) of the sequence $h(n)(-1)^n$ in terms of $H_{hp}(e^{j\omega})$ and sketch its frequency response.

The DTFT of the sequence $h_{lp}(n)e^{jn\omega_0}$ is

$$H_{\omega_0}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_{hp}(n)e^{jn\omega_0}e^{-jn\omega} = \sum_{n=-\infty}^{\infty} h_{hp}(n)e^{-jn(\omega-\omega_0)} = H_{hp}(e^{j(\omega-\omega_0)}).$$

Now, $e^{jn\omega_0} = (-1)^n$ and $\omega_0 = \pi$.

Hence, the stopband is now shifted to $\omega_0 = \pi$, which gives rise to a low filter. This is called frequency transformation in designing lowpass to highpass or highpass to lowpass filters, and vice versa.

Q2.

(i) [**4 marks**] The discrete-time specifications are

$$0.99 \le \left|H(e^{j\omega})\right| \le 1.01, \qquad 0 \le \omega \le 0.45\pi .$$

$$\left|H(e^{j\omega})\right| \le 0.001, \qquad 0.55\pi \le \omega \le \pi .$$

(ii) [**4 marks**] Use the minimum specifications, we have

$$\delta = 0.001$$

$$\Delta\omega = 0.1\pi$$

$$A = -20 \cdot \log_{10} \delta = 60 \text{ dB}$$

$$\beta = 0.1102(A - 8.7) = 5.65326$$

$$M = \frac{A-8}{2.285\Delta\omega} = 72.438 .$$ Hence choose M to be 73.

iii) [**5 marks**] The required filter length from the McClellan-Parks algorithm is

$$N = \frac{-20\log_{10}\sqrt{0.01\cdot 0.001} - 13}{14.6 \cdot 0.05} = 50.685$$ Hence, choose N to be 51.

The McClellan-Parks design algorithm is more efficient and leads to a lower filter length. The design of the Kaiser window is very simple.

(b)

i) **[2 marks]** Now

$$F(\omega_1, \omega_2) = \frac{1}{2}(\cos\omega_1 + \cos\omega_2)$$

$$H(\omega) = \sum_{n=0}^{N} a(n) T_n[\cos(\omega)]$$

$$H(\omega_1, \omega_2) = \sum_{n=0}^{N} a(n) T_n[F(\omega_1, \omega_2)].$$

Therefore $F(\omega_1, \omega_2) = F(\pm\omega_2, \pm\omega_1)$,

$$H(\omega_1, \omega_2) = \sum_{n=0}^{N} a(n) T_n[F(\pm\omega_2, \pm\omega_1)] = H(\pm\omega_2, \pm\omega_1).$$

ii) **[4 marks]** $F(\omega_1, \omega_2) = \cos\left(\frac{\omega_1 + \omega_2}{2}\right)\cos\left(\frac{\omega_1 - \omega_2}{2}\right)$

Since $F(\omega_1, \omega_2) = F(\pm\omega_2, \pm\omega_1)$, it is necessary to consider region I.

For region I, we have $\left(\omega_1+\omega_2\right)\le\pi$. Both $\cos\left(\frac{\omega_1+\omega_2}{2}\right)$ and $\cos\left(\frac{\omega_1-\omega_2}{2}\right)$ will be positive and $F(\omega_1,\omega_2)$ is positive. For the stopband, $\left(\omega_1+\omega_2\right)>\pi$, $\cos\left(\frac{\omega_1+\omega_2}{2}\right)$ will be negative while $\cos\left(\frac{\omega_1-\omega_2}{2}\right)$ will be positive. Hence $F(\omega_1,\omega_2)$ is negative.

iii) **[4 marks]** Since $\cos\omega=F(\omega_1,\omega_2)$ and $\cos\omega$ is positive in the range $(0,0.5\pi)$, the values of the prototype filter in the range $(0,0.5\pi)$ is mapped to the stopband (regions I to IV). Also, as $\cos\omega$ is negative in the range $(0.5\pi,\pi)$, the values of the prototype filter in this range will be mapped to the passband.

(c)

i) **[2 marks]**

$$H(e^{j\omega_1},e^{j\omega_2})=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}h(n_1,n_2)e^{-jn_1\omega_1}e^{-jn_2\omega_2}$$

$$=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}h(n_1,n_2)\cos(n_1\omega_1+n_2\omega_2)-j\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}h(n_1,n_2)\sin(n_1\omega_1+n_2\omega_2)$$

$$=\boldsymbol{h}^T[\boldsymbol{c}(\boldsymbol{\omega})-j\boldsymbol{s}(\boldsymbol{\omega})],$$

where $\boldsymbol{\omega}=[\omega_1,\omega_2]^T$, $\boldsymbol{h}=[h(0,0),h(0,1),\ldots,h(N_1-1,N_2-2),h(N_1-1,N_2-1)]^T$, $\boldsymbol{c}(\boldsymbol{\omega})=[1,\cos(\omega_2),\ldots,\cos((N_1-1)\omega_1+(N_2-1)\omega_2)]^T$ and $\boldsymbol{s}(\boldsymbol{\omega})=[0,\ldots,\sin((N_1-1)\omega_1+(N_2-1)\omega_2)]^T$.

ii) **[3 marks]** Show that the WLS function is given by

$$\mathrm{WLS}(\boldsymbol{h})=\int_{\Omega_p\cup\Omega_s}W(\boldsymbol{\omega})\,|H(e^{j\omega_1},e^{j\omega_2})-H_d(e^{j\omega_1},e^{j\omega_2})|^2d\boldsymbol{\omega}$$

$$=\boldsymbol{h}^T\boldsymbol{Q}\boldsymbol{h}-2\boldsymbol{h}^T\boldsymbol{g}+c,$$

where $\boldsymbol{Q}=\int_{\Omega_p\cup\Omega_s}W(\boldsymbol{\omega})(\boldsymbol{c}(\boldsymbol{\omega})-j\boldsymbol{s}(\boldsymbol{\omega}))(\boldsymbol{c}(\boldsymbol{\omega})-j\boldsymbol{s}(\boldsymbol{\omega}))^H\,d\boldsymbol{\omega}$,

$$\boldsymbol{g}=\int_{\Omega_p\cup\Omega_s}W(\boldsymbol{\omega})\,\mathrm{Re}\{(\boldsymbol{c}(\boldsymbol{\omega})-j\boldsymbol{s}(\boldsymbol{\omega}))H_d^*(e^{j\omega_1},e^{j\omega_2})\}d\boldsymbol{\omega},$$

$c=\int_{\Omega_p\cup\Omega_s}W(\boldsymbol{\omega})\,|H_d(e^{j\omega_1},e^{j\omega_2})|^2\,d\boldsymbol{\omega}$ and superscript ^{H} stands for conjugate transpose.

[Hint:

$$| H(e^{j\omega_1}, e^{j\omega_2}) - H_d(e^{j\omega_1}, e^{j\omega_2}) |^2$$
$$= (H(e^{j\omega_1}, e^{j\omega_2}) - H_d(e^{j\omega_1}, e^{j\omega_2}))(H(e^{j\omega_1}, e^{j\omega_2}) - H_d(e^{j\omega_1}, e^{j\omega_2}))^*]$$

WLS($\boldsymbol{h}$)

$$= \int_{\Omega_p \cup \Omega_s} W(\boldsymbol{\omega})(H(e^{j\omega_1}, e^{j\omega_2}) - H_d(e^{j\omega_1}, e^{j\omega_2}))(H(e^{j\omega_1}, e^{j\omega_2}) - H_d(e^{j\omega_1}, e^{j\omega_2}))^* d\boldsymbol{\omega}$$

$$= \int_{\Omega_p \cup \Omega_s} W(\boldsymbol{\omega})(| H(e^{j\omega_1}, e^{j\omega_2}) |^2 - 2\operatorname{Re}\{H(e^{j\omega_1}, e^{j\omega_2}) H_d^*(e^{j\omega_1}, e^{j\omega_2})\} + | H_d(e^{j\omega_1}, e^{j\omega_2}) |^2) d\boldsymbol{\omega}$$

$$= \boldsymbol{h}^T [\int_{\Omega_p \cup \Omega_s} W(\boldsymbol{\omega})(\boldsymbol{c}(\boldsymbol{\omega}) - j\boldsymbol{s}(\boldsymbol{\omega}))(\boldsymbol{c}(\boldsymbol{\omega}) - j\boldsymbol{s}(\boldsymbol{\omega}))^H d\boldsymbol{\omega}]\boldsymbol{h}$$

$$- 2\boldsymbol{h}^T [\int_{\Omega_p \cup \Omega_s} W(\boldsymbol{\omega}) \operatorname{Re}\{(\boldsymbol{c}(\boldsymbol{\omega}) - j\boldsymbol{s}(\boldsymbol{\omega})) H_d^*(e^{j\omega_1}, e^{j\omega_2})\} d\boldsymbol{\omega}]$$

$$+ \int_{\Omega_p \cup \Omega_s} W(\boldsymbol{\omega}) | H_d(e^{j\omega_1}, e^{j\omega_2}) |^2 \, d\boldsymbol{\omega}$$

$$= \boldsymbol{h}^T \boldsymbol{Q} \boldsymbol{h} - 2\boldsymbol{h}^T \boldsymbol{g} + c \, .$$

iii) **[2 marks]** Show that $\min_{\boldsymbol{h}}$ WLS($\boldsymbol{h}$) $= \boldsymbol{h}^T \boldsymbol{Q} \boldsymbol{h} - 2\boldsymbol{h}^T \boldsymbol{g} + c$ is equivalent to $\min_{\boldsymbol{h}} \| \boldsymbol{L}^T \boldsymbol{h} - \boldsymbol{L}^{-1} \boldsymbol{g} \|_2^2$.

Since

$$\| \boldsymbol{L}^T \boldsymbol{h} - \boldsymbol{L}^{-1} \boldsymbol{g} \|_2^2 = \boldsymbol{h}^T \boldsymbol{L}\boldsymbol{L}^T \boldsymbol{h} - 2\boldsymbol{h}^T \boldsymbol{L}\boldsymbol{L}^{-1} \boldsymbol{g} + \boldsymbol{g}^T \boldsymbol{L}^{-T} \boldsymbol{L}^{-1} \boldsymbol{g} = \boldsymbol{h}^T \boldsymbol{Q} \boldsymbol{h} - 2\boldsymbol{h}^T \boldsymbol{g} + \| \boldsymbol{L}^{-1} \boldsymbol{g} \|_2^2$$

and both $\| \boldsymbol{L}^{-1} \boldsymbol{g} \|_2^2$ and c are constants independent of $\boldsymbol{h}$, hence

$\min_{\boldsymbol{h}}$ WLS($\boldsymbol{h}$) $= \boldsymbol{h}^T \boldsymbol{Q} \boldsymbol{h} - 2\boldsymbol{h}^T \boldsymbol{g} + c$ is equivalent to $\min_{\boldsymbol{h}} \| \boldsymbol{L}^T \boldsymbol{h} - \boldsymbol{L}^{-1} \boldsymbol{g} \|_2^2$.

iv) **[3 marks]** Show that the above problem can be written as:

$$\begin{aligned}
&\min_{\boldsymbol{h},\delta} && \delta \\
&\text{subject to} && \delta_p - \{\alpha_R^2(\boldsymbol{\omega}_i) + \alpha_I^2(\boldsymbol{\omega}_i)\}^{1/2} \geq 0, \quad i = 1, \ldots, M, \\
& && \| \boldsymbol{L}^T \boldsymbol{h} - \boldsymbol{L}^{-1} \boldsymbol{g} \|_2 \leq \delta
\end{aligned}$$

$$\alpha_R(\boldsymbol{\omega}) = W(\boldsymbol{\omega}) \cdot \{\boldsymbol{h}^T \boldsymbol{c}(\boldsymbol{\omega}) - \operatorname{Re}[H_d(e^{j\omega_1}, e^{j\omega_2})]\},$$

$$\alpha_I(\boldsymbol{\omega}) = W(\boldsymbol{\omega}) \cdot \{\boldsymbol{h}^T \boldsymbol{s}(\boldsymbol{\omega}) + \operatorname{Im}[H_d(e^{j\omega_1}, e^{j\omega_2})]\}.$$

Let δ^2 be an upper bound of WLS($\boldsymbol{h}$). The WLS minimization can be achieved by minimizing δ such that $\|\boldsymbol{L}^T\boldsymbol{h}-\boldsymbol{L}^{-1}\boldsymbol{g}\|_2 \le \delta$. Together with the other peak magnitude constraints, one gets:

$$\min_{\boldsymbol{h},\delta} \delta$$

$$\text{subject to } W(\boldsymbol{\omega})\,|H(e^{j\omega_1},e^{j\omega_2})-H_d(e^{j\omega_1},e^{j\omega_2})|\le \delta_p, \text{ for } \omega\in\Omega_E$$

$$\|\boldsymbol{L}^T\boldsymbol{h}-\boldsymbol{L}^{-1}\boldsymbol{g}\|_2 \le \delta .$$

By discretizing $\boldsymbol{\omega}$ uniformly into M points in the passband and stopband edges $\omega\in\Omega_E$**,** $\boldsymbol{\omega}_i$ for i=1,…,M, the above problem can be written as

$$\min_{\boldsymbol{h},\delta} \delta$$

$$\text{subject to } W(\boldsymbol{\omega}_i)\,|H(e^{j\boldsymbol{\omega}_i})-H_d(e^{j\boldsymbol{\omega}_i})|\le \delta_p,\ i=1,\ldots,M.$$

$$\|\boldsymbol{L}^T\boldsymbol{h}-\boldsymbol{L}^{-1}\boldsymbol{g}\|_2 \le \delta$$

Now $W(\boldsymbol{\omega}_i)[H(e^{j\boldsymbol{\omega}_i})-H_d(e^{j\boldsymbol{\omega}_i})]=W(\boldsymbol{\omega}_i)[\boldsymbol{h}^T[\boldsymbol{c}(\boldsymbol{\omega}_i)-j\boldsymbol{s}(\boldsymbol{\omega}_i)]-H_d(e^{j\boldsymbol{\omega}_i})]$
$=\alpha_R(\boldsymbol{\omega}_i)+j\alpha_I(\boldsymbol{\omega}_i)$,

where $\alpha_R(\boldsymbol{\omega})=W(\boldsymbol{\omega})\cdot\{\boldsymbol{h}^T\boldsymbol{c}(\boldsymbol{\omega})-\mathrm{Re}[H_d(e^{j\boldsymbol{\omega}_i})]\}$,

and $\alpha_I(\boldsymbol{\omega})=W(\boldsymbol{\omega})\cdot\{\boldsymbol{h}^T\boldsymbol{s}(\boldsymbol{\omega})+\mathrm{Im}[H_d(e^{j\boldsymbol{\omega}_i})]\}$. Therefore, the constraints become

$W(\boldsymbol{\omega}_i)\,|H(e^{j\boldsymbol{\omega}_i})-H_d(e^{j\boldsymbol{\omega}_i})|\le\delta_p \Leftrightarrow \delta_p-\{\alpha_R^2(\boldsymbol{\omega}_i)+\alpha_I^2(\boldsymbol{\omega}_i)\}^{1/2}\ge 0$. Hence, the problem becomes

$$\min_{\boldsymbol{h},\delta} \quad \delta$$

$$\text{subject to}\quad \delta_p-\{\alpha_R^2(\boldsymbol{\omega}_i)+\alpha_I^2(\boldsymbol{\omega}_i)\}^{1/2}\ge 0,\quad i=1,\ldots,M,$$

$$\|\boldsymbol{L}^T\boldsymbol{h}-\boldsymbol{L}^{-1}\boldsymbol{g}\|_2 \le \delta ,$$

$$\alpha_R(\boldsymbol{\omega})=W(\boldsymbol{\omega})\cdot\{\boldsymbol{h}^T\boldsymbol{c}(\boldsymbol{\omega})-\mathrm{Re}[H_d(e^{j\boldsymbol{\omega}})]\},\ \alpha_I(\boldsymbol{\omega})=W(\boldsymbol{\omega})\cdot\{\boldsymbol{h}^T\boldsymbol{s}(\boldsymbol{\omega})+\mathrm{Im}[H_d(e^{j\boldsymbol{\omega}})]\}.$$

Q.3

a)

i) [**4 marks**]

The output of the system is given by

$$y(n)=\sum_{k=-\infty}^{\infty}h(k)x(n-k).$$

Taking expectation on both sides, one gets

$$E[y(n)] = E\left[\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} h(k)E[x(n-k)] = \mu_x \cdot \sum_{k=-\infty}^{\infty} h(k) = \mu_x \cdot H(1),$$

because $x(n)$ be a wide-sense stationary random process with mean μ_x.

$$R_{yy}(k) = E[y(n)y(n+k)] = E\left[\sum_{i=-\infty}^{\infty} h(i)x(n-i) \cdot \sum_{l=-\infty}^{\infty} h(l)x(n+k-l)\right]$$

$$= \sum_{i=-\infty}^{\infty} h(i) \sum_{l=-\infty}^{\infty} h(l)E[x(n-i)x(n+k-l)]$$

$$= \sum_{i=-\infty}^{\infty} h(i) \sum_{l=-\infty}^{\infty} h(l)R_{xx}(k-l+i)$$

$$= \sum_{i=-\infty}^{\infty} h(-i)(R_{xx} * h)(k-i) = R_{xx}(k) * h(k) * h(-k).$$

Hence, $y(n)$ is also a wide-sense stationary process with mean μ_y and autocorrelation $R_{yy}(k)$ given by,

$$\mu_y = H(1) \cdot \mu_x,$$

and $$R_{yy}(k) = R_{xx}(k) * h(k) * h(-k).$$

ii) [**3 marks**]

Taking the DTFT of $R_{yy}(k)$, we get the the Power Spectral Density (PSD) of $y(n)$ as

$$S_{yy}(e^{j\omega}) = DTFT[R_{yy}(k)] = DTFT[R_{xx}(k) * h(k) * h(-k)].$$

Using the convolution theorem, we have

$$S_{yy}(e^{j\omega}) = DTFT[R_{xx}(k) * h(k) * h(-k)]$$

$$= S_{xx}(e^{j\omega})H(e^{j\omega})H^*(e^{j\omega}) = S_{xx}(e^{j\omega})\left|H(e^{j\omega})\right|^2.$$

b)

i) [**3 marks]**

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

ii) [**5 marks**] The transfer function between $\xi_{a_i}(n)$ and $e_{a_i}(n)$ is $H(z)$.

Let its impulse response be $h(n)$. Hence

$$\xi_{a_i}(n) = e_{a_i}(n) * h(n) = \sum_{k=0}^{\infty} e_{a_i}(n-k)h(k) . \qquad (3\text{-}1a)$$

Consider

$$E[\{\xi_{a_1}(n) + \xi_{a_1}(n) + \sum_{j=0}^{2} e_{b_j}(n)\}^2] = E[\xi_{a_1}^2(n)] + E[\xi_{a_2}^2(n)] + \sum_{j=0}^{2} E[e_{b_j}^2(n)]$$
$$+ 2E[\xi_{a_1}(n)\xi_{a_2}(n)] + 2E[\sum_{j=0}^{2} \xi_{a_1}(n)e_{b_j}(n)] + 2E[\sum_{j=0}^{2} \xi_{a_2}(n)e_{b_j}(n)] + 2E[\sum_{0 \le j \ne k \le 2} e_{b_j}(n)e_{b_k}(n)] ,$$

Since $e_{a_i}(n)$ and $e_{b_i}(n)$ are independent and are of zero means, the terms $\xi_{a_i}(n)$ and $e_{b_i}(n)$ are independent and are of zero means. The terms $E[\sum_{j=0}^{2} \xi_{a_1}(n)e_{b_j}(n)]$, $E[\sum_{j=0}^{2} \xi_{a_2}(n)e_{b_j}(n)]$, and $E[\sum_{0 \le j \ne k \le 2} e_{b_j}(n)e_{b_k}(n)]$ are equal to zero. To show that $E[\xi_{a_1}(n)\xi_{a_2}(n)] = 0$, we have after using (3-1) the following

$$E[\xi_{a_1}(n)\xi_{a_2}(n)] = E\left[\sum_{k=0}^{\infty} e_{a_1}(n-k)h(k) \cdot \sum_{m=0}^{\infty} e_{a_2}(n-m)h(n)\right]$$
$$= \left[\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} E[e_{a_1}(n-k)e_{a_2}(n-m)]h(k)h(n)\right] = 0$$

(because $e_{a_1}(n)$ and $e_{a_2}(n)$ are uncorrelated).

iii) [**3 marks**] The variances of $e_{a_i}(n)$ is given by

$$\sigma_{e_{a_i}}^2 = \int_{-2^{-(B+1)}}^{2^{-(B+1)}} x^2 \cdot p_{a_i}(x)dx ,$$

where $p_{a_i}(x)$ is the probability density function of $e_{a_i}(n)$ in the interval $[-\frac{1}{2}2^{-B_a}, \frac{1}{2}2^{-B_a}]$. Since the quantization error is uniformly distributed in the interval $[-\frac{1}{2}2^{-B_a}, \frac{1}{2}2^{-B_a}]$, we have $p_{a_i}(x) = \frac{1}{2^{-B_a}}$. Hence

$$\sigma_{e_{a_i}}^2 = \frac{1}{2^{-B_a}} \int_{-2^{-(B_a+1)}}^{2^{-(B_a+1)}} x^2 dx = \frac{1}{3 \cdot 2^{-B_a}} x^3 \Big|_{-2^{-(B_a+1)}}^{2^{-(B_a+1)}}$$

$$= \frac{2 \cdot 2^{-3(B_a+1)}}{3 \cdot 2^{-B_a}} = \frac{2^{-2B_a}}{12}.$$

Similarly, we have $\sigma_{e_{b_i}}^2 = \frac{2^{-2B_b}}{12}$.

Since $\sigma_{e_{a_i}}^2 = \frac{2^{-2B_a}}{12}; \sigma_{e_{b_j}}^2 = \frac{2^{-2B_b}}{12}$ and the quantization errors are white, $P_{a_i}(\omega) = P_a; P_{b_j}(\omega) = P_b$ are constants. Hence,

$$\sigma_{e_{a_i}}^2 = = \frac{2^{-2B_a}}{12} \frac{1}{2\pi} \int_{-\pi}^{\pi} P_a d\omega = P_a$$

$$\sigma_{e_{b_j}}^2 = \frac{2^{-2B_b}}{12} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_b d\omega = P_b.$$

Thus, we have $P_{a_i}(\omega) = \frac{2^{-2B_a}}{12}$. $P_{b_j}(\omega) = \frac{2^{-2B_b}}{12}$

iv) [**5 marks**] The power spectral density of $\xi_{a_i}(n)$ is given by

$$P_{\xi_a}(\omega) = P_{a_i}(\omega) \left| H(e^{j\omega}) \right|^2$$

$$= \frac{2^{-2B_a}}{12} \left| H_e(e^{j\omega}) \right|^2.$$

Similarly, we have

$$P_{\xi_{b_j}}(\omega) = \frac{2^{-2B_b}}{12}.$$

The variances of $\xi_{a_i}(n)$ are thus

$$\sigma_{\xi_{a_i}}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2^{-2B_a}}{12} \left| H(e^{j\omega}) \right|^2 d\omega.$$

Similar,

$$\sigma^2_{\xi_{b_j}} = \frac{2^{-2B_b}}{12}.$$

Using the Parseval's theorem

$$\sum_{n=-\infty}^{\infty} |h(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega,$$

we have

$$\sigma^2_{\xi_{a_i}} = \frac{2^{-2B_a}}{12} \sum_{n=-\infty}^{\infty} |h(n)|^2 .$$

Finally, we have

$$\sigma^2_{\xi} = 3\frac{2^{-2B_b}}{12} + 2\frac{2^{-2B_a}}{12} \sum_{n=-\infty}^{\infty} |h[n]|^2 .$$

vi) **[5 marks]**

Using the relationship $y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$, we have

$$|y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right| \le \sum_{k=-\infty}^{\infty} |h(k)x(n-k)| \le \max|x(n)| \cdot \sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-\infty}^{\infty} |h(k)|,$$

for all n, since $x(n)$ is always less than one.

(2 marks)

Hence,

$$\log_2 |y(n)| \le \log_2 \left(\sum_{k=-\infty}^{\infty} |h(k)|\right)$$

is the worse case increase in fractional bits of $y(n)$. One should choose the next larger integer for the required worse case integer part, i.e. $\left\lceil \log_2 \left(\sum_{k=-\infty}^{\infty} |h(k)|\right) \right\rceil$.

As the quantity can be negative, thus, the worse case integer part of $y(n)$ has a length equal to

$$\max\left(\left\lceil \log_2 \left(\sum_{k=-\infty}^{\infty} |h(k)|\right) \right\rceil, 0\right) \text{ bits},$$

where the operator $\lceil u \rceil$ gives the nearest integer which is larger or equal to u.

(3 marks)

vi) [**5 marks**] Suppose that the required accuracy of the output $y(n)$ due to signal round off noise is B bits. Moreover, suppose that $B_a = 2B_b$, determine the minimum B_a in order to satisfy the given output accuracy.

The output noise variance is given by

$$\sigma_{\xi}^2 = 3\frac{2^{-2B_b}}{12} + 2\frac{2^{-2B_a}}{12}\sum_{n=-\infty}^{\infty}|h[n]|^2 .$$

For B bits accuracy, the variance $\sigma_{\xi_{a_i}}^2$ should be less than or equal to $\frac{2^{-2B}}{12}$. Hence

$$3\frac{2^{-2B_b}}{12} + 2\frac{2^{-2B_a}}{12}\sum_{n=-\infty}^{\infty}|h[n]|^2 \le \frac{2^{-2B}}{12} .$$

Now $B_a = 2B_b$, we then have

$$\left(3\frac{x}{12} + 2\frac{x^2}{12}\sum_{n=-\infty}^{\infty}|h[n]|^2\right) \le \frac{2^{-2B}}{12} \quad \text{where } 2^{-2B_b} = x .$$

The minimum occurs when the inequality becomes an equality which given a quadratic equation in terms of x as follows:

$$\left(3\frac{x}{12} + 2\frac{x^2}{12}\sum_{n=-\infty}^{\infty}|h[n]|^2\right) = \frac{2^{-2B}}{12}$$

The solution is thus $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where

$a = \frac{2}{12}\sum_{n=-\infty}^{\infty}|h[n]|^2$, $b = 3/12$ and $c = -\frac{2^{-2B}}{12}$.

As $0 < 2^{-2B_b} = x \le 1$, we shall choose the positive sign which yields

$$-2B_b = -B_a = \log_2\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \text{ bits}$$

Hence $B_a = -\log_2\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)$ bits.

Note, this is a real number and hence it needs to be rounded to the nearest larger integer. Due to the real-valued relaxation, the result will be close only when Ba is sufficient large.

Alternatively, one can start with Ba = 1 and increases its one bit at a time so that error variance is just less than $\frac{2^{-2B}}{12}$. This will give the optimal integer value.

Q.4

i) **[4 marks]** The MSE is

$$\begin{aligned}\xi &= E[e^2(n)] = E[(d(n)-y(n))^2] \\ &= E[(d(n)-W^T\boldsymbol{X}_n)^2] \\ &= E[d^2(n)] + E[\boldsymbol{W}^T\boldsymbol{X}_n\boldsymbol{X}_n^T\boldsymbol{W}] - 2E[d(n)\boldsymbol{X}_n^T\boldsymbol{W}] \\ &= E[d^2(n)] + \boldsymbol{W}^T E[\boldsymbol{X}_n\boldsymbol{X}_n^T]\boldsymbol{W} - 2E[d(n)\boldsymbol{X}_n^T]\boldsymbol{W} \\ &= r_{dd}(0) + \boldsymbol{W}^T\boldsymbol{R}\boldsymbol{W} - 2\boldsymbol{P}^T\boldsymbol{W},\end{aligned}$$

ii) **[3 marks]** Differentiating ξ with respect to the weight vector, we obtain

$$\nabla = \frac{\partial \xi}{\partial \boldsymbol{W}} = 2\boldsymbol{R}\boldsymbol{W} - 2\boldsymbol{P}.$$

The optimal weight vector $\boldsymbol{W}^*$ is obtained by setting the gradient to zero. This yields the following normal equation

$$\boldsymbol{R}\boldsymbol{W}^* = \boldsymbol{P},$$

and $$\boldsymbol{W}^* = \boldsymbol{R}^{-1}\boldsymbol{P}.$$

iii) **[3 marks]** In recursive least squares (RLS) adaptive filtering algorithm, the correlation matrix $\boldsymbol{R}(n)$ and correlation vector $\boldsymbol{P}(n)$ at the *n-th* time instant are respectively estimated recursively as:

$$\boldsymbol{R}(n) = \lambda\boldsymbol{R}(n-1) + \boldsymbol{X}_n\boldsymbol{X}_n^T \text{ and } \boldsymbol{P}(n) = \lambda\boldsymbol{P}(n-1) + d(n)\boldsymbol{X}_n,$$

where $0 < \lambda < 1$ is a forgetting factor, $\boldsymbol{R}(0) = \delta\boldsymbol{I}$ and $\boldsymbol{P}(0) = \boldsymbol{0}$ with δ a positive number. Given

(*) $$\boldsymbol{R}^{-1}(n) = \lambda^{-1}(\boldsymbol{I} - \boldsymbol{K}(n)\boldsymbol{X}_n^T)\cdot\boldsymbol{R}^{-1}(n-1),$$

where $\boldsymbol{K}(n) = \frac{\boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n}{\lambda + \boldsymbol{X}_n^T\boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n} = \frac{\boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n}{\lambda + \mu(n)}$ is the Kalman gain vector and $\mu(n) = \boldsymbol{X}_n^T\boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n$.

Using (*) above, we have

$$\begin{aligned}\boldsymbol{R}^{-1}(n)\boldsymbol{X}_n &= \lambda^{-1}(\boldsymbol{I}-\boldsymbol{K}(n)\boldsymbol{X}_n^T)\cdot\boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n \\ &= \lambda^{-1}\boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n - \lambda^{-1}\boldsymbol{K}(n)\boldsymbol{X}_n^T\boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n \\ &= \lambda^{-1}\boldsymbol{R}^{-1}(n-1)\boldsymbol{X}_n - \lambda^{-1}\boldsymbol{K}(n)\mu(n) \\ &= \lambda^{-1}\boldsymbol{K}(n)(\lambda+\mu(n)) - \lambda^{-1}\boldsymbol{K}(n)\mu(n) = \boldsymbol{K}(n)\,.\end{aligned}$$

iv) **[3 marks]** Using the result in (iii) and (*) above, show that the optimal weight vector of the RLS algorithm at the *n-th* time instance is given by

$$\boldsymbol{R}^{-1}(n)\boldsymbol{P}(n) = \boldsymbol{W}(n) = \boldsymbol{W}(n-1) + \boldsymbol{K}(n)e_p(n)\,,$$

where $e_p(n) = d(n) - \boldsymbol{X}_n^T\boldsymbol{W}(n-1)$.

[Hint: note $\boldsymbol{P}(n) = \lambda\boldsymbol{P}(n-1) + d(n)\boldsymbol{X}_n$ and $d(n) = e_p(n) + \boldsymbol{X}_n^T\boldsymbol{W}(n-1)$]

By definition,

$$\begin{aligned}\boldsymbol{W}(n) &= \boldsymbol{R}^{-1}(n)\boldsymbol{P}(n) = \lambda\boldsymbol{R}^{-1}(n)\boldsymbol{P}(n-1) + d(n)\boldsymbol{R}^{-1}(n)\boldsymbol{X}_n \\ &= \lambda\boldsymbol{R}^{-1}(n)\boldsymbol{P}(n-1) + \boldsymbol{K}(n)(e_p(n) + \boldsymbol{X}_n^T\boldsymbol{W}(n-1))\end{aligned}$$

(Using hint and results (iii))

$$= (\boldsymbol{I}-\boldsymbol{K}(n)\boldsymbol{X}_n^T)\boldsymbol{R}^{-1}(n-1)\boldsymbol{P}(n-1) + \boldsymbol{K}(n)(e_p(n) + \boldsymbol{X}_n^T\boldsymbol{W}(n-1))$$

(Using (*))

$$= (\boldsymbol{I}-\boldsymbol{K}(n)\boldsymbol{X}_n^T)\boldsymbol{W}(n-1) + \boldsymbol{K}(n)(e_p(n) + \boldsymbol{X}_n^T\boldsymbol{W}(n-1))$$

(Using (iii) again)

$$= \boldsymbol{W}(n-1) + \boldsymbol{K}(n)e_p(n)$$

b) **[6 marks]** In the LMS algorithm, the weight vector is updated in the negative direction of the gradient vector ∇:

$$\boldsymbol{W}_{n+1} = \boldsymbol{W}_n - \mu\nabla$$

where μ is a stepsize parameter, which is used to average out the effect of additive noise and the noise generated by approximating the gradient. The gradient is approximated by the instantaneous gradient by replacing ξ by $e(n)$, its instantaneous value:

$$\hat{\nabla} = \frac{\partial e^2(n)}{\partial \boldsymbol{W}} = -2e(n)\boldsymbol{X}\,.$$

Putting them together, one gets the LMS update as follows:

$$\boldsymbol{W}_{n+1} = \boldsymbol{W}_n + 2\mu e(n)\boldsymbol{X}\,.$$

It can be shown that the LMS algorithm converges in the mean when the stepsize satisfies

$$|1-\mu\lambda_k|<1.$$

Hence, the range of values of μ that the LMS algorithm converges in the mean is

$$0<\mu<\frac{2}{\lambda_{\max}},$$

where $\lambda_{\max}$ is the largest eigenvalue of $\boldsymbol{R}_{XX}$. Since $\boldsymbol{R}_{XX}$ is an autocorrelation matrix, its eigenvalues are *nonnegative*. Hence an upper bound on $\lambda_{\max}$ is

$$\lambda_{\max}<\sum_{k=0}^{M-1}\lambda_k=trace(\boldsymbol{R}_{XX})=L\cdot r_{xx}(0),$$

where $r_{XX}(0)$ is the input signal power that is easily estimated, and $trace(\boldsymbol{A})=\sum_{k=0}^{L-1}a_{kk}$ for any $(L\times L)$ matrix *A*. Therefore, an upper bound on the step size μ is

$$2/(L\cdot r_{xx}(0)).$$

The smaller the value of $|1-\mu\lambda_k|$, the faster is its convergence rate. Even if we choose the stepsize to be

$$\mu=\frac{1}{\lambda_{\max}}.$$

The convergence rate of the LMS algorithm will however depend on the decay of the mode corresponding to the *smallest eigenvalue* $\lambda_{\min}$ at a rate

$$\left[\overline{W}(n)\right]_{\lambda_{\min}}\approx C\left(1-\frac{\lambda_{\min}}{\lambda_{\max}}\right)^n u(n),$$

where *C* is a constant and $u(n)$ is the unit step sequence.

Consequently, *the ratio* $(\lambda_{\min}/\lambda_{\max})$ *determines the convergence rate*. *If* $(\lambda_{\min}/\lambda_{\max})$ *is much smaller than unity, the convergence will be very slow*. *If* $(\lambda_{\min}/\lambda_{\max})$ *is close to unity, the convergence rate of the algorithm is fast*.

The total MSE at the output of the adaptive filter is

$$MSE=MSE_{\min}+J_{\mu},$$

where $MSE_{\min}$ is the minimum MSE , and J_{μ} is called the *excess error due to the noisy gradient estimate* (J_{Δ}) *and tracking errors* (J_l). The tracking errors result from the lag in track slowly time-variant signal statistics and it decreases with the stepsize, whereas the error due to the noisy graduate increases with increasing stepsizes. The optimal stepsize is obtained when the excess error due to the noisy gradient estimate is equal to that of the lag error.

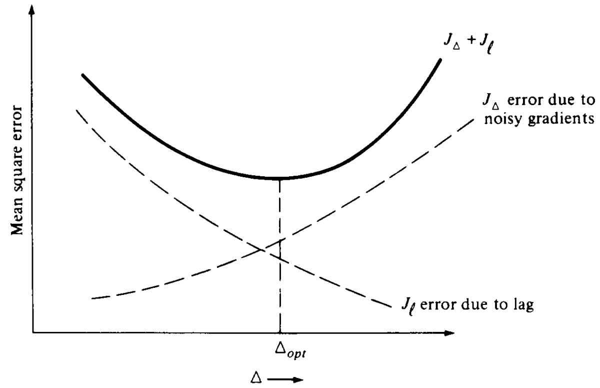

FIGURE 6.20 Excess mean-square error J_Δ and lag error J, as a function of the step size Δ.

c)

i) **[5 marks]** The mean squared error (MSE) can be written as,

$$MSE = E[\varepsilon_n^2] = E[(x(n) - y(n))^2] = E[(x(n) - \sum_{k=1}^{P} a_i x(n-k))^2]$$

$$= E\left[x^2(n)\right] - 2\sum_{k=1}^{P} a_k E\left[x(n)x(n-k)\right] + \sum_{k=1}^{P} a_k \sum_{j=1}^{P} a_j E\left[x(n-k)x(n-j)\right]$$

$$= r_{xx}(0) - 2\boldsymbol{a}^T \boldsymbol{r}_{xx} + \boldsymbol{a}^T \boldsymbol{R}_{xx} \boldsymbol{a}\,.$$

ii) **[3 marks]** The gradient of the mean squared error function with respect to the filter coefficient vector is given by:

$$\frac{\partial}{\partial \boldsymbol{a}} E\left[e^2(n)\right] = -2\boldsymbol{r}_{xx}^T + 2\boldsymbol{a}^T \boldsymbol{R}_{xx}\,,$$

where the gradient vector is defined as

$$\frac{\partial}{\partial \boldsymbol{a}} = \left[\frac{\partial}{\partial a_1}, \quad \frac{\partial}{\partial a_2}, \quad \frac{\partial}{\partial a_3}, \quad \ldots, \quad \frac{\partial}{\partial a_P}\right]^T.$$

The minimum is obtained by setting the gradient to zero as

$$\boldsymbol{R}_{xx}\boldsymbol{a} = \boldsymbol{r}_{xx}\,,$$

or equivalently,

$$\boldsymbol{a} = (\boldsymbol{R}_{xx})^{-1}\boldsymbol{r}_{xx}.$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_P \end{bmatrix} = \begin{bmatrix} r_{xx}(0) & r_{xx}(1) & r_{xx}(2) & \cdots & r_{xx}(P-1) \\ r_{xx}(1) & r_{xx}(0) & r_{xx}(1) & \cdots & r_{xx}(P-2) \\ r_{xx}(2) & r_{xx}(1) & & \cdots & r_{xx}(P-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{xx}(P-1) & r_{xx}(P-2) & r_{xx}(P-3) & \cdots & r_{xx}(0) \end{bmatrix}^{-1} \begin{bmatrix} r_{xx}(1) \\ r_{xx}(2) \\ r_{xx}(3) \\ \vdots \\ r_{xx}(P) \end{bmatrix}$$

iii) **[3 marks]** By definition, $e(n)$ and $x(n)$ are related by,

$$e(n) = x(n) - \sum_{i=1}^{P} a_i x(n-i) = \sum_{i=0}^{P} h_i x(n-i),$$

where $h_i = \begin{cases} 1 & i = 0 \\ -a_i & i = 1,\dots,P \end{cases}$.

Using the result in part (a), we have the desired result,

$$S_e(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 S_x(e^{j\omega}).$$

iv) **[3 marks]** Since $e(n)$ is white Gaussian distributed with zero mean and variance σ_e^2, $S_e(e^{j\omega}) = \sigma_e^2$. Since $H^{-1}(z)$ is causal stable and hence we can take its inverse to obtain

$$S_x(e^{j\omega}) = \frac{\sigma_e^2}{\left|H(e^{j\omega})\right|^2}.$$