FILTER DESIGN

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REFERENCES

A.V. OPPENHEIM AND R.W. SCHAFER, DISCRETE-TIME SIGNAL PROCESSING. ENGLEWOOD CLIFFS, NJ: PRENTICE-HALL, INC., 1989.

Fourier Transform (FT) (appendix A)

The Fourier transform is the extension of Fourier Series to nonperiodic signals:

$$\Im[x(t)] = X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \qquad \text{(Fourier Transform)} \qquad \text{(A-1a)}$$

The inverse Fourier transform of X(f) is

$$\Im^{-1}[X(f)] = x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \qquad \text{(Inverse FT)} \qquad \text{(A-1b)}$$

The Fourier transform of a signal is called the spectrum of the signal and it is in general a complex function of *f*.

Properties

1. If x(t) is a real-valued signal, then X(f) satisfies the Hermitian symmetry:

$$X(-f) = X^*(f) \qquad \text{(A-2)}$$

2. Duality:

$$\Im[X(t)] = x(-f) \qquad \text{(A-3)}$$

3. Modulation: Multiplication by an exponential in the time domain corresponds to a frequency shift in the frequency domain

$$\Im[e^{j2\pi f_0 t} x(t)] = X(f - f_0)$$

$$\Im[x(t)\cos(2\pi f_0 t)] = \frac{1}{2}[X(f - f_0) + X(f + f_0)] \qquad \text{(A-4)}$$

4. Convolution : Convolution in the time domain is equivalent to multiplication in the frequency domain, and vice versa.

If $\Im[x(t)] = X(f)$ **and** $\Im[y(t)] = Y(f)$**, then**

$$\Im[x(t) * y(t)] = X(f)Y(f)$$

$$\Im[x(t)y(t)] = X(f) * Y(f).$$

(A-5)

5. Parseval's relation :

If $\Im[x(t)] = X(f)$ **and** $\Im[y(t)] = Y(f)$**, then**

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df .$$

(A-6)

Table 1.1 Table of Fourier transform pairs

$x(t)$	$X(f)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f t_0}$	$\delta(f-f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\delta(f-f_0)-\frac{1}{2j}\delta(f+f_0)$
$\Pi(t)$	$\mathrm{sinc}(f)$
$\mathrm{sinc}(t)$	$\Pi(f)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$
$\mathrm{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t}u_{-1}(t), \quad \alpha>0$	$\frac{1}{\alpha+j2\pi f}$
$te^{-\alpha t}u_{-1}(t), \quad \alpha>0$	$\frac{1}{(\alpha+j2\pi f)^2}$
$e^{-\alpha\|t\|}, \quad \alpha>0$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$u_{-1}(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)$

Sampling Theorem (Appendix B)

The sampling theorem says that a bandlimited signal – that is, a signal whose Fourier transform vanishes for $|f| > W$ for some W – can be completely described in terms of its sample values taken at intervals $T_s \le 1/(2W)$. $f_s = (2W)$ is called the Nyquist rate.

- **The signal x(t) can be reconstructed from the samples $x[n] = x(nT_s),\ n = -\infty, \ldots, \infty$, as**

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \sin c(2W(t - nT_s)) \quad \textbf{(B-1)}$$

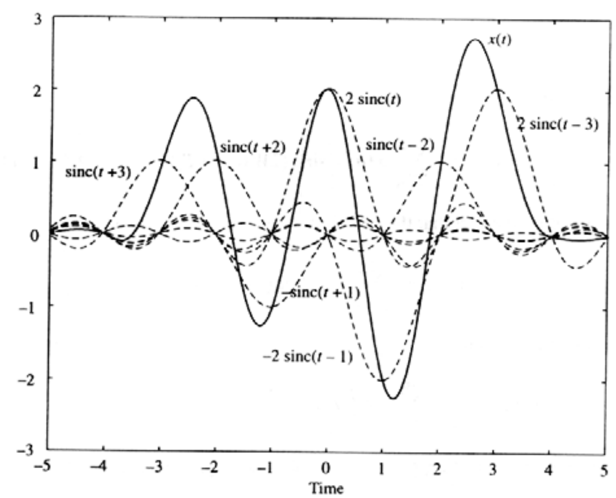


Figure 1.17 Representation of the sampling theorem

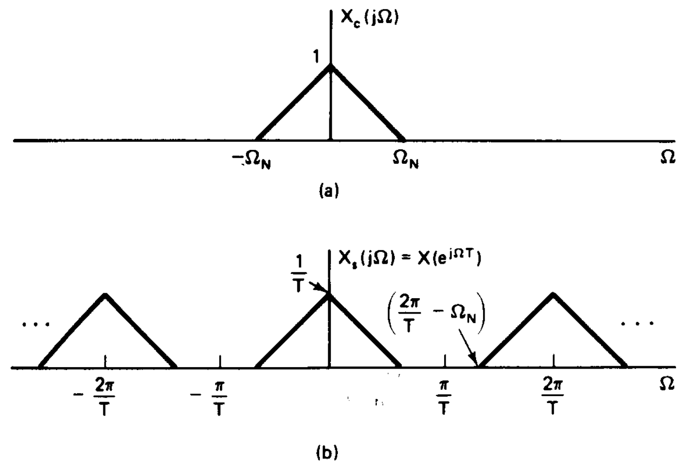
Proof:

The sampled waveform $x_\delta(t)$ can be written as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

has a Fourier transform given by

$$X_\delta(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s}) \ .$$



Passing $x_\delta(t)$ through a lowpass filter with a bandwidth of W and a gain of T_s in the passband will reproduce the original signal.

10. Structure of a digital signal processing system

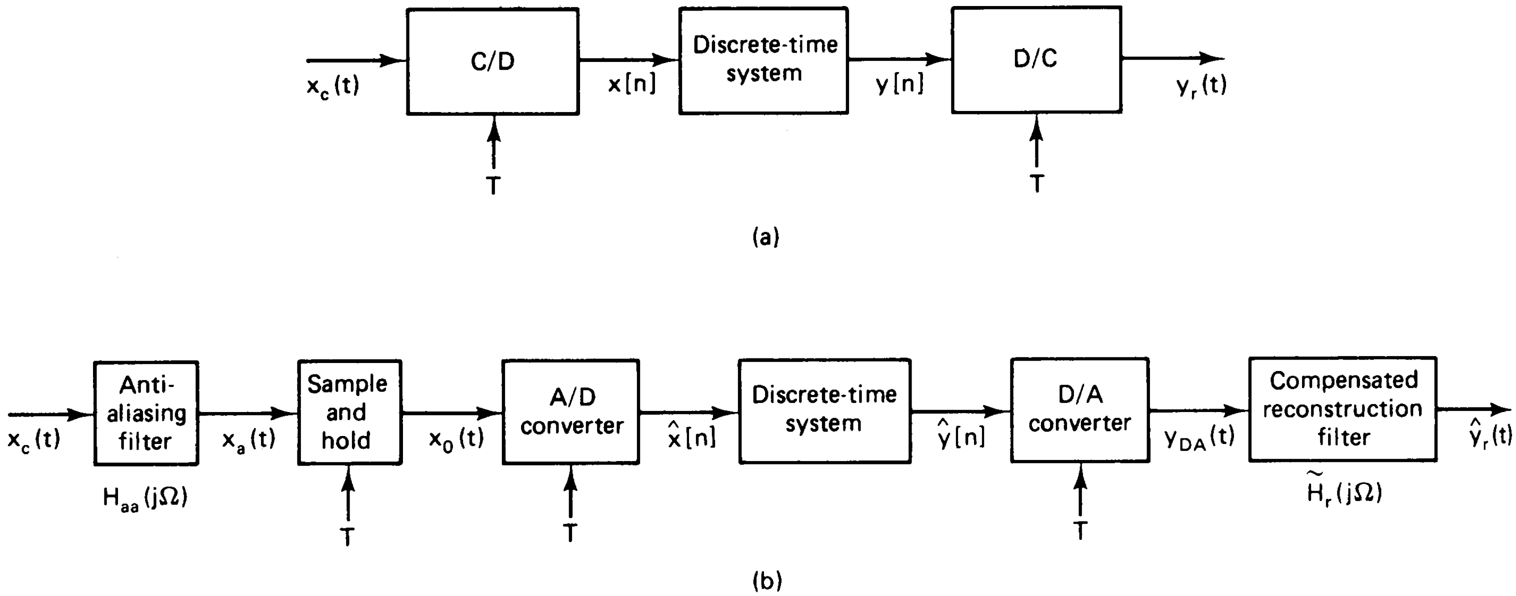


Fig. 10.1 (Fig. 3.26 in Oppenheim's book)

Relationship between continuous-time Fourier transform and DTFT:

$$x[n] = x_c(nT).$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \qquad (10.1)$$

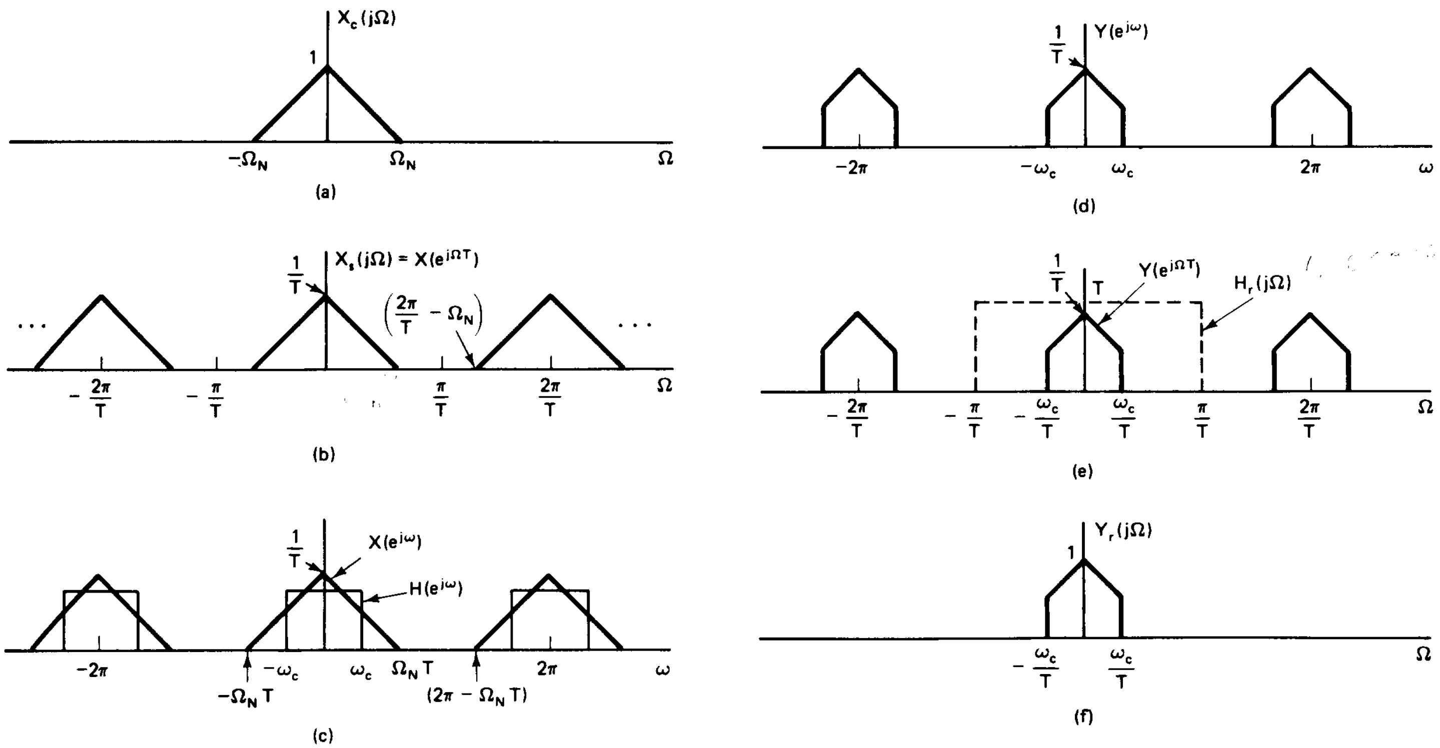


Figure 3.11 (a) Fourier transform of a bandlimited input signal. (b) Fourier transform of sampled input plotted as a function of continuous-time frequency Ω. (c) Fourier transform $X(e^{j\omega})$ of sequence of samples and frequency response $H(e^{j\omega})$ of discrete-time system plotted vs. ω. (d) Fourier transform of output of discrete-time system. (e) Fourier transform of output of discrete-time system and frequency response of ideal reconstruction filter plotted vs. Ω. (f) Fourier transform of output.

Signal reconstruction:

$$y_r[n] = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}.$$

$$Y_r(j\Omega) = H_r(j\Omega) Y(e^{i\Omega T}). \qquad \textbf{(10.2)}$$

After passing through a LTI filter with $H(e^{j\omega})$, the DT-FT of output y[n] is

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}). \qquad \textbf{(10.3)}$$

From (10.2) and (10.3), the continuous-time Fourier transform of output y(t) is

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}) \qquad \textbf{(10.4)}$$

If sampling theorem is satisfied (i.e $X_c(j\Omega) = 0, \text{ for } |\Omega| \geq \pi/\mathrm{T}$), then

$$H_r(j\Omega)X(e^{j\Omega T}) = X_c(j\Omega)$$

and (10.4) becomes

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(e^{j\Omega T}) & |\Omega| < \pi/T, \\ 0 & |\Omega| \geq \pi/T. \end{cases}$$

Thus, the equivalent analog filter of $H(e^{j\omega})$ is

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \pi/T, \\ 0 & |\Omega| \geq \pi/T. \end{cases} \qquad \textbf{(10.5)}$$

10.1 Filter specifications in continuous and discrete-time domains

(Example 7.1 in Oppenhiem's book)

Consider a discrete-time system that is to be used lowpass filter a continuous-time signal using the basic configuration in Fig. 10.1 (a). Suppose that the sampling rate is 10^4 samples/sec (T=10^{-4} sec).

(What is the cutoff frequency of the ideal anti-aliasing filter? What is the maximum operating frequency without aliasing?)

The specifications are:

1. The gain $|H_{eff}(j\Omega)|$ should be within ± 0.01 $(0.086\ \mathrm{dB})$ of unity (zero dB) in the frequency band $0 \le \Omega \le 2\pi(2000)$.
2. The gain should be no greater than ± 0.001 $(\text{-}60\ \mathrm{dB})$ in the frequency band $2\pi(2000) \le \Omega$.

This is illustrated in the following figure. The parameters are

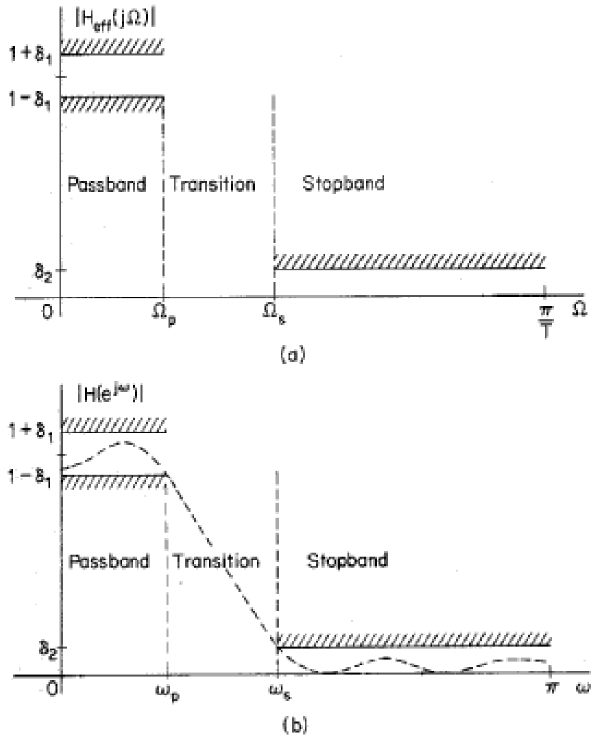


Figure 7.2 (a) Specifications for effective frequency response of overall system in Fig. 7.1 for the case of lowpass filter. (b) Corresponding specifications for the discrete-time system in Fig. 7.1.

$\delta_1 = 0.01(20\log_{10}(1+\delta_1) = 0.086 \;\; dB)$ **(passband ripple);**

$\delta_2 = 0.01(20\log_{10}\delta_2 = -60 \;\; dB)$ **(stopband ripple);**

$\Omega_p = 2\pi(2000)$ **(passband cutoff frequency);**

$\Omega_s = 2\pi(3000)$ **(stopband cutoff frequency)**

Because of (10.5), the equivalent specifications in the digital domain are:

1. $(1-\delta_1) \leq |H(e^{j\omega})| \leq (1+\delta_1) \quad |\omega| \leq \omega_p$,
2. $|H(e^{j\omega})| \leq \delta_2 \quad \omega_s \leq |\omega| \leq \pi$,

Since the sampling period is *T*=10^{-4} sec., we have

$$\omega_p = \Omega_p \cdot T = 2\pi(2000) \cdot 10^{-4} = 0.4\pi \text{ radians,}$$

and $\omega_s = \Omega_s \cdot T = 2\pi(3000) \cdot 10^{-4} = 0.6\pi$ **radians.**

The transition bandwidth $\Delta\omega = \omega_s - \omega_p = 0.6\pi - 0.4\pi = 0.2\pi$.

11. Ideal frequency-selective filters

The frequency response of the ideal lowpass filter is

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}, \qquad \textbf{(11.1)}$$

where ω_c is the cutoff frequency. Frequencies components below ω_c pass through the filter without any distortion, while those above are suppressed. In practice, we can only approximate (11.1).

- **From the inverse DT-FT, the corresponding impulse response is**

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{lp}(e^{j\omega})e^{jn\omega}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{jn\omega}\,d\omega$$

$$= \frac{1}{2\pi jn}[e^{jn\omega_c} - e^{-jn\omega_c}] = \frac{\sin(n\omega_c)}{\pi n}. \qquad \textbf{(11.2)}$$

Its impulse response extends from $-\infty$ to $+\infty$ and the system is not computationally realizable. The phase response is zero.

Linear-phase filters

Shifting theorem:

$$\Im[x(t-\alpha)] = \int_{-\infty}^{\infty} x(t-\alpha)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f(\tau+\alpha)}d\tau$$

$$= e^{j2\pi f\alpha} \cdot \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}d\tau = e^{-j2\pi f\alpha} \cdot X_c(f)$$

- **The Fourier transform of a signal with a time shift of α is equal to the multiplication of its Fourier transform by $e^{-j2\pi f\alpha}$.**
- **The factor $e^{-j2\pi f\alpha}$ has a unit magnitude and its phase is $-2\pi f \cdot \alpha = -\Omega\alpha$, which is a linear function of Ω.**

Because of the relationship between the DT-FT of the sample $x(nT_s) = x[n]$ and the FT of $x(t)$. $X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\frac{\omega}{T} - j\frac{2\pi k}{T})$

Assuming the sampling theorem is satisfied, the DT-FT of $x_\alpha(t) = x(t-\alpha)$ is

$$X_\alpha(e^{j\omega}) = e^{-j\omega\alpha/T} X_c(j\frac{\omega}{T}) = e^{-j\omega\alpha'} X(e^{j\omega}), \quad -\pi < \omega \le \pi,$$

where $\alpha' = \alpha / T$ is the normalized shift in the discrete-time domain.

- **Since the ideal lowpass filter in (11-1) is non-causal, we can shift the ideal impulse response to the right so that it becomes causal. The frequency response is then given by**

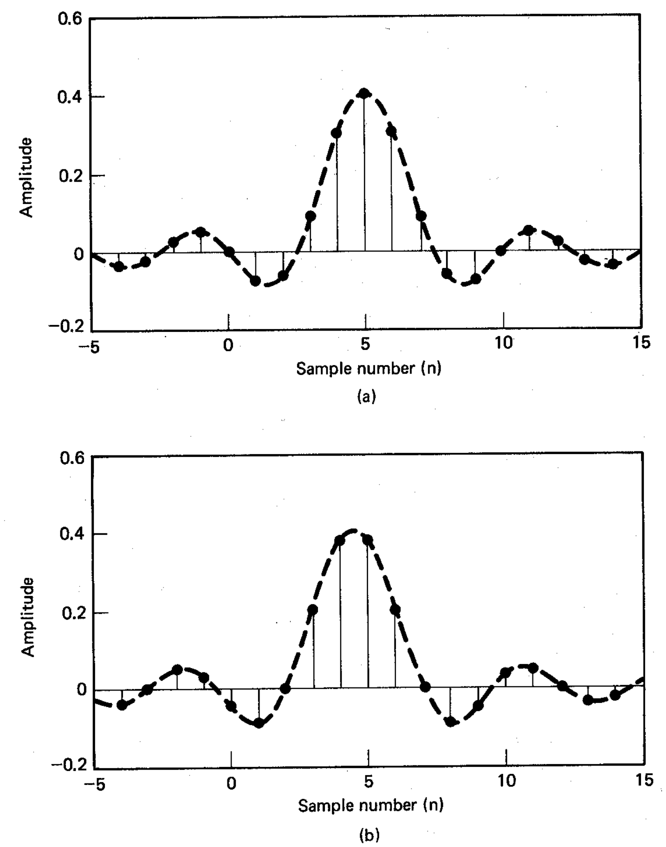
$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{-j\omega\alpha}, \qquad |\omega| < \pi$$

Example: **The ideal lowpass filter $H_{lp}(e^{j\omega})$ has frequency response**

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

and impulse response is $h_{lp}[n] = \dfrac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)}$. (from inverse DT-FT).

Symmetric and Anti-symmetric impulse responses



- **Note that when α is an integer, the impulse response is symmetric about $n = n_d$.**

$$h_{lp}[2n_d - n] = h_{lp}[n].$$

- **If α is an integer plus one-half then**

$$h_{lp}[2n_d - n] = h_{lp}[n].$$

The point of symmetry is α, which is not an integer.

- For $\alpha = 4.3$, there is no symmetry at all.

In general, a linear-phase system has frequency response

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{-j\omega\alpha}, \qquad |\omega| < \pi .$$

- For FIR filters, we can impose symmetry on the impulse response about α when

  2α is an integer (it is called half-sample symmetry).
  α is an integer (it is called full-sample symmetry).

  This is a sufficient condition for the system to have linear phase but not necessary.

Linear phase FIR filters

There are four types of FIR generalized linear-phase systems.

Type I FIR linear phase systems:

M **an even integer, symmetric impulse response**

$$h[n] = h[M-n], \qquad 0 \le n \le M$$

and

$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos(\omega k)\right) \quad \text{[Delay } \alpha = M/2\text{]}$$

where $a[0] = h\left[\frac{M}{2}\right]$ **and** $a[k] = 2h\left[\frac{M}{2} - k\right]$, $k = 1, \ldots, \frac{M}{2}$.

The proof for the rest are left as exercise.

Type II FIR linear phase systems:

***M* an odd integer, symmetric impulse response**

$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{(M+1)/2} b[k]\cos\left(\omega\left(k-\tfrac{1}{2}\right)\right)\right)$$

where $b[k] = 2h\left[\frac{M+1}{2} - k\right]$**,** $k = 1, \ldots, \frac{M+1}{2}$**.**

Type III FIR linear phase systems:

***M* an even integer, antisymmetric impulse response**

$$h[n] = -h[M-n], \qquad 0 \le n \le M$$

and $$H(e^{j\omega}) = je^{-j\omega M/2}\left(\sum_{k=1}^{M/2} c[k]\sin(\omega k)\right)$$

where $c[k] = 2h\left[\frac{M}{2} - k\right]$**,** $k = 1, \ldots, \frac{M}{2}$**.**

Type IV FIR linear phase systems:

M an odd integer, antisymmetric impulse response

$$H(e^{j\omega}) = je^{-j\omega M/2}\left(\sum_{k=1}^{(M+1)/2} d[k]\sin\left(\omega\left(k-\tfrac{1}{2}\right)\right)\right)$$

where $d[k] = 2h\left[\frac{M+1}{2} - k\right]$**,** $k = 1,\ldots,\frac{M+1}{2}$

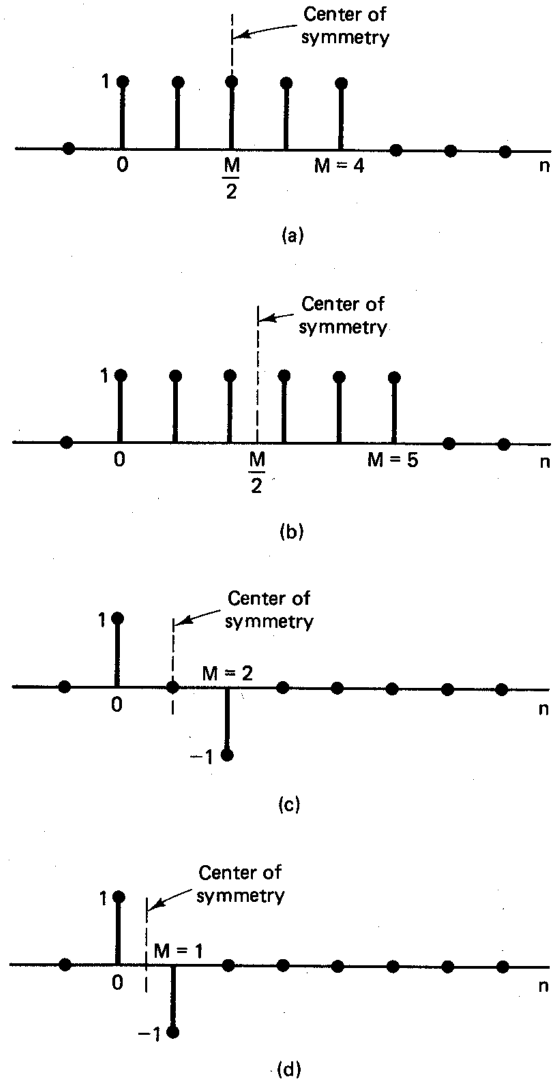


Figure 5.33 Examples of FIR linear phase systems. (a) Type I, M even, $h[n] = h[M - n]$. (b) Type II, M odd, $h[n] = h[M - n]$. (c) Type III, M even, $h[n] = -h[M - n]$. (d) Type IV, M odd, $h[n] = -h[M - n]$.

Zero locations for FIR linear phase systems (left as exercise)

- **For type-I and -II, $H(z)$ can be expressed as**

$$H(z)=\sum_{n=0}^{M}h[M-n]z^{-n}=\sum_{k=M}^{0}h[k]z^{k}z^{-M}=z^{-M}H(z^{-1})$$

If $z_0=re^{j\theta}$ is a zero of $H(z)$, then

$$H(z_0)=z_0^{-M}H(z_0^{-1})=0$$

and $z_0^{-1}=r^{-1}e^{-j\theta}$ is also a zero of $H(z)$. When $h[n]$ is real, then $z_0^{*}=re^{-j\theta}$ will also be a zero of $H(z)$, so will $\left(z_0^{*}\right)^{-1}=r^{-1}e^{j\theta}$.

When $h[n]$ is real, each complex zero not on the unit circle will be part of a set of four conjugate reciprocal zeros of the form

$$(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})(1-r^{-1}e^{j\theta}z^{-1})(1-r^{-1}e^{-j\theta}z^{-1})$$

Zeros on the unit circle come in pairs of the form

$$(1-e^{j\theta}z^{-1})(1-e^{-j\theta}z^{-1})$$

Zero at $z=\pm 1$ can appear by itself and $H(z)$ may have factors

$$(1\pm z^{-1})$$

Since

$$H(-1)=(-1)^{M}H(-1).$$

If *M* is odd, $z=-1$ must be zero.

- **For type III and IV, we have**

$$H(z)=-z^{-M}H(z^{-1})$$

$H(z)$ have a zero at $z=1$ for both *M* even and *M* odd and $z=-1$ is a zero of $H(z)$ if *M* is even

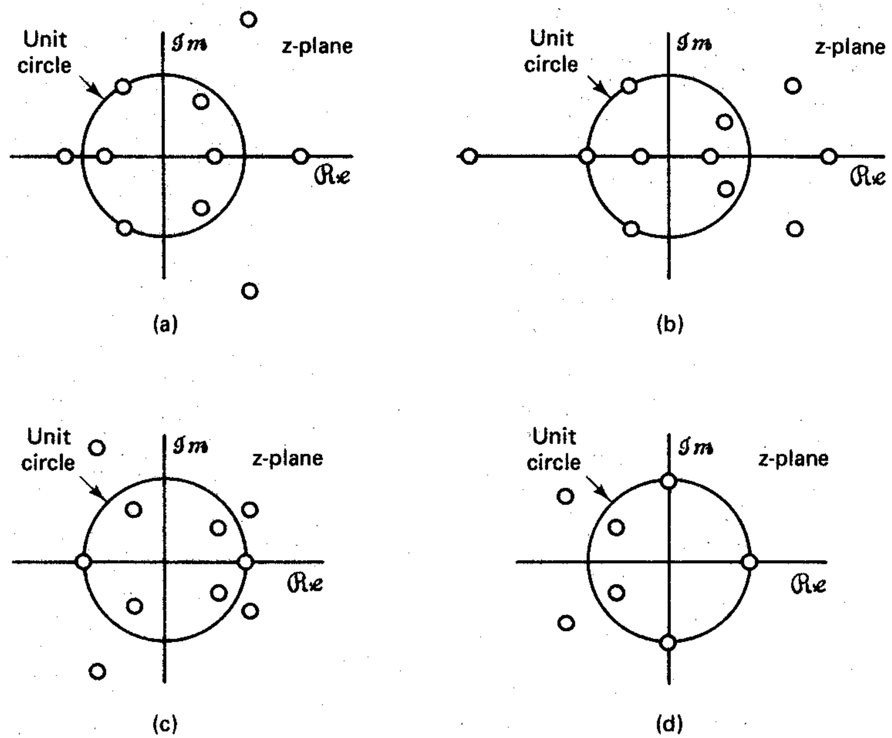
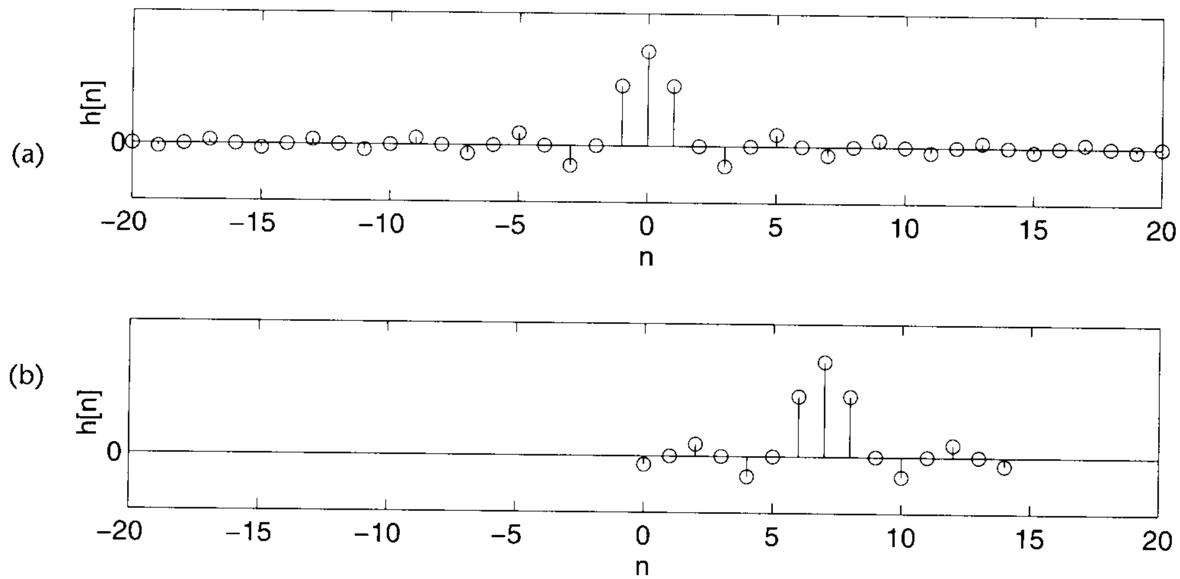


Figure 5.38 Typical zero plots for linear phase systems. (a) Type I. (b) Type II. (c) Type III. (d) Type IV.

12. Windowing method

Fig. 15.26
Impulse response of a non-recursive filter:
(a) non-causal with an infinite number of coefficients;
(b) causal with 15 coefficients



- **Note, the ideal impulse response is symmetric around n=0. In general, filters with symmetric and anti-symmetric impulse response have perfect linear-phase (i.e. no phase distortion). This is not possible for IIR filters.**
- **In windowing method, the impulse response is truncated by multiplying the ideal response by a window and shifted it to the right to make it causal (h[n]=0, n<0).**

12.1 Designing Linear phase FIR filters by windowing

The impulse response is

$$h_{lp}[n] = \frac{\sin[\omega_c(n-(M/2))]}{\pi(n-(M/2))} w[n], \quad \textbf{n=0,…,M,}$$

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & otherwise \end{cases}. \quad \textbf{(window function)} \qquad \textbf{(12.1)}$$

Note, the shift (M/2), or system delay, is an integer if M is odd and a half-integer if M is even.

- **A commonly used window is the Kaiser window**

$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M \\ 0, & otherwise \end{cases}. \qquad \textbf{(12.2)}$$

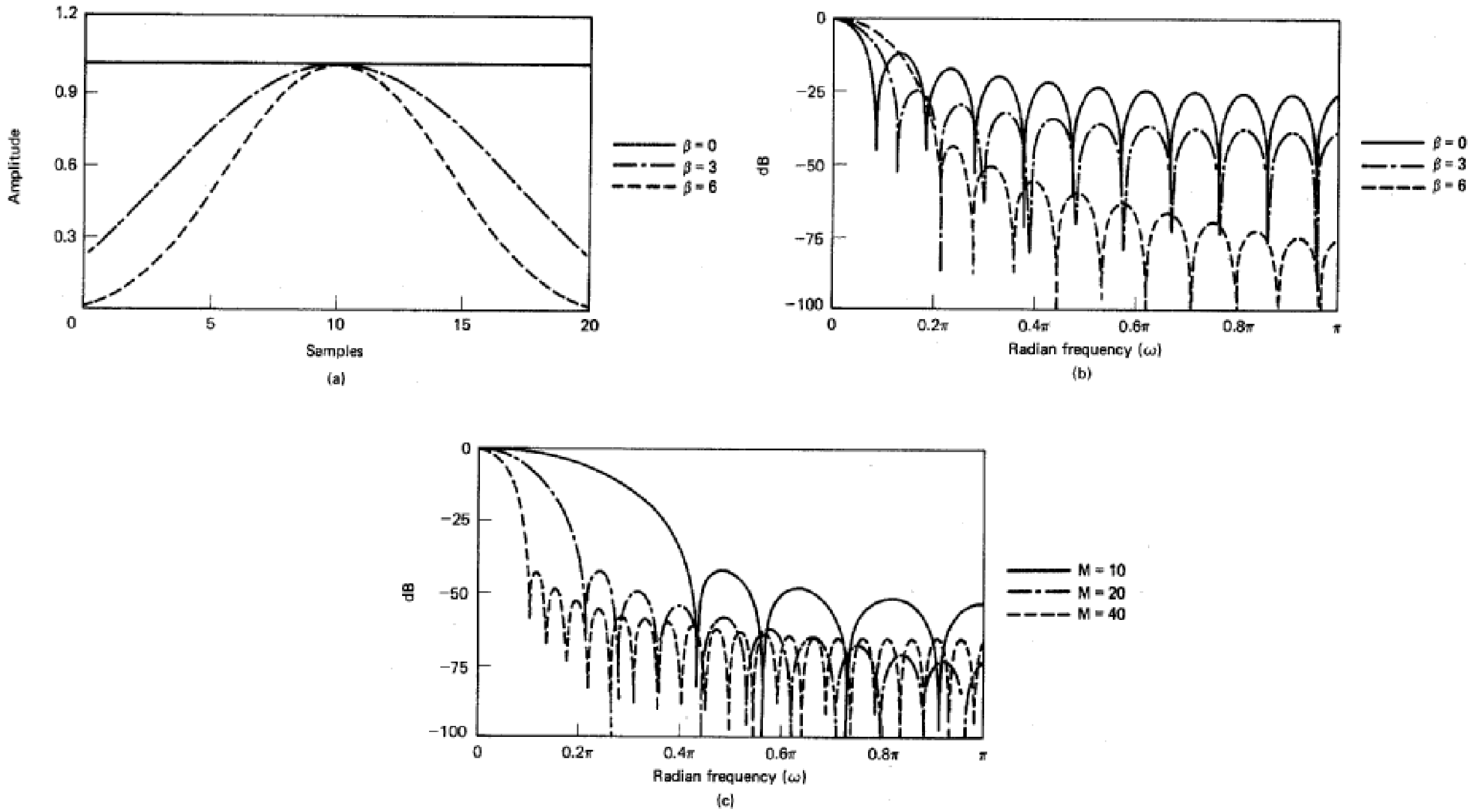


Figure 7.32 (a) Kaiser windows for $\beta = 0$, 3, and 6 and $M = 20$. (b) Fourier transforms corresponding to windows in (a). (c) Fourier transforms of Kaiser windows with $\beta = 6$ and $M = 10$, 20, and 40.

From the modulation theorem

$$h[n] = h_d[n]w[n] \;\leftrightarrow\; H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta,$$

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & otherwise \end{cases}. \quad w[n] \leftrightarrow W(e^{j\omega}). \tag{12.3}$$

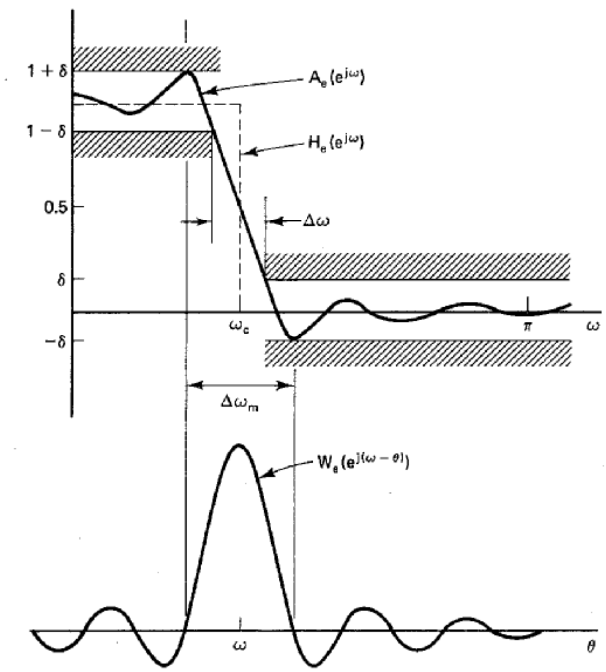


Figure 7.31 Illustration of type of approximation obtained at a discontinuity of the ideal frequency response.

ω_p : passband cutoff frequency.

ω_s : stopband cutoff frequency.

$\Delta\omega = \omega_s - \omega_p$:transition bandwidth.

δ : passband/stopband ripples

$A = -20\log_{10}\delta$ (dB) : stopband attenuation.

- **The passband and stopband ripples (stopband attenuation) are nearly identical.**
- **The transition bandwidth $\Delta\omega$ is inversely proportional to filter length.**

The parameter $\Delta\omega$ and filter length (*M+1*) can be determined empirically:

$$A = -20\log_{10}\delta \quad \text{(dB)}$$

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50, \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50, \\ 0.0 & A < 21 \end{cases} \qquad \text{(12.4)}$$

$$M = \frac{A-8}{2.285\cdot\Delta\omega}.$$

Examples:

The specifications are

$$\omega_p = 0.4\pi, \omega_s = 0.6\pi, \delta_1 = 0.01 \text{ and } \delta_2 = 0.001.$$

Since window method inherently has $\delta_1 = \delta_2$, we must set

$$\delta = \min(\delta_1, \delta_2) = 0.001.$$

The cutoff frequency is $\omega_c = \dfrac{\omega_p + \omega_s}{2} = 0.5\pi$

The values of *M* and β are obtained from (12.4) as

$$\beta = 5.653, \quad M = 37$$

The impulse response of the filter is then given by

$$h[n] = \begin{cases} \dfrac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)} \cdot \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)} & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

(Since *M* is odd, the filter is of type II.) The peak approximation error is slightly greater than $\delta = 0.001$. Increasing M to 38 results in a type I filter for which $\delta = 0.0008$.

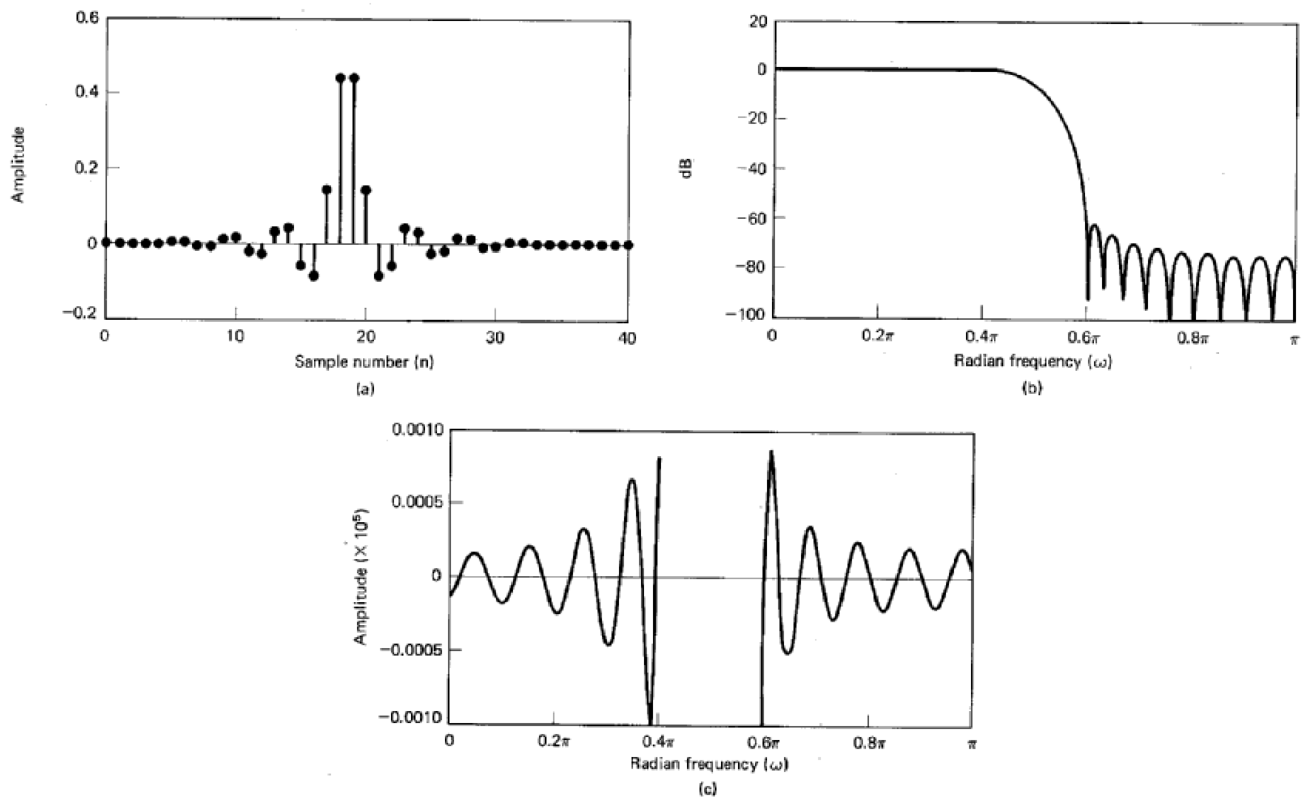


Figure 7.33 Response functions for Example 7.11. (a) Impulse response ($M = 37$). (b) Log magnitude. (c) Approximation error.

Exercise:

1. Show that the DT-FT of $h_{lp}[n] = \dfrac{\sin[\omega_c(n-(M/2))]}{\pi(n-(M/2))}$ **is given by**

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

What is the phase response of the digital filter? Is it a linear function of ω **(i.e. linear phase)?**

2. see tutorial sheets.

13. Highpass, bandpass, and bandstop filters

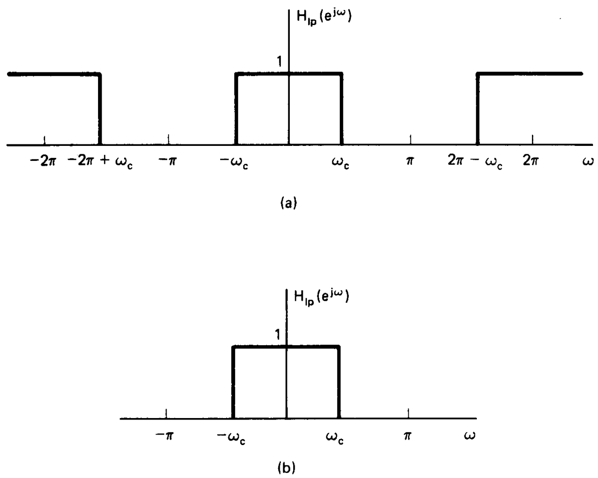


Figure 2.17 Ideal lowpass filter showing (a) periodicity of the frequency response, (b) one period of the periodic frequency response.

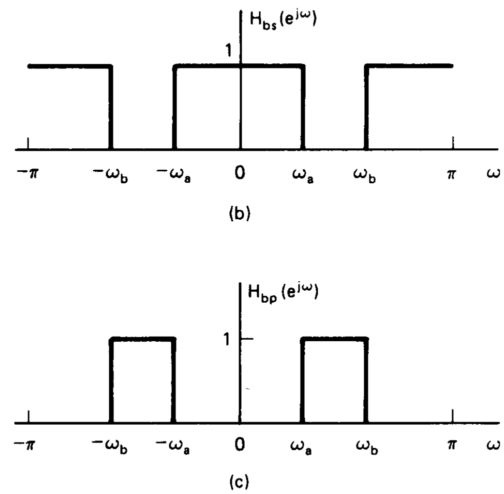


Figure 2.18 Ideal frequency-selective filters. (a) Highpass filter. (b) Bandstop filter. (c) Bandpass filter. In each case, the frequency response is periodic with period 2π. Only one period is shown.

(Lowpass filters) (Bandstop and bandpass filters)

- **Windowing method is also applicable to the design of these filters, δ in (12.4) should be the minimum ripple value in the various bands. $\Delta\omega$ in (12.4) should be the minimum transition bandwidth in the various bands.**

Highpass filter design

An ideal highpass filter with generalized linear-phase has frequency response

$$H_{hp}(e^{j\omega}) = \begin{cases} 0, & 0 \le |\omega| \le \omega_c \\ e^{-j\omega M/2}, & \omega_c \le |\omega| \le \pi \end{cases}. \qquad \textbf{(13.1)}$$

and impulse response (taking the inverse DT-FT of (5-1))

$$h_{hp}[n] = \frac{\sin \pi(n - M/2)}{\pi(n - M/2)} - \frac{\sin \omega_c (n - M/2)}{\pi(n - M/2)} \quad , -\infty < n < \infty \qquad \textbf{(13.2)}$$

Suppose that $\omega_s = 0.35\pi, \omega_p = 0.5\pi,$ **and** $\delta_1 = \delta_2 = \delta = 0.021$**.**

Applying Kaiser's formula yields the required values of $\beta = 2.6$ **and** ***M*** **= 24.**

The filter is type I with a delay of ***M*****/2 = 12 samples.**

The actual peak approximation error is $\delta = 0.0213$ rather than 0.021 as specified. Since type II FIR linear-phase systems are generally not appropriate for either highpass or bandstop filter, because of the zero at $\omega = \pi$, we increase *M* to 26.

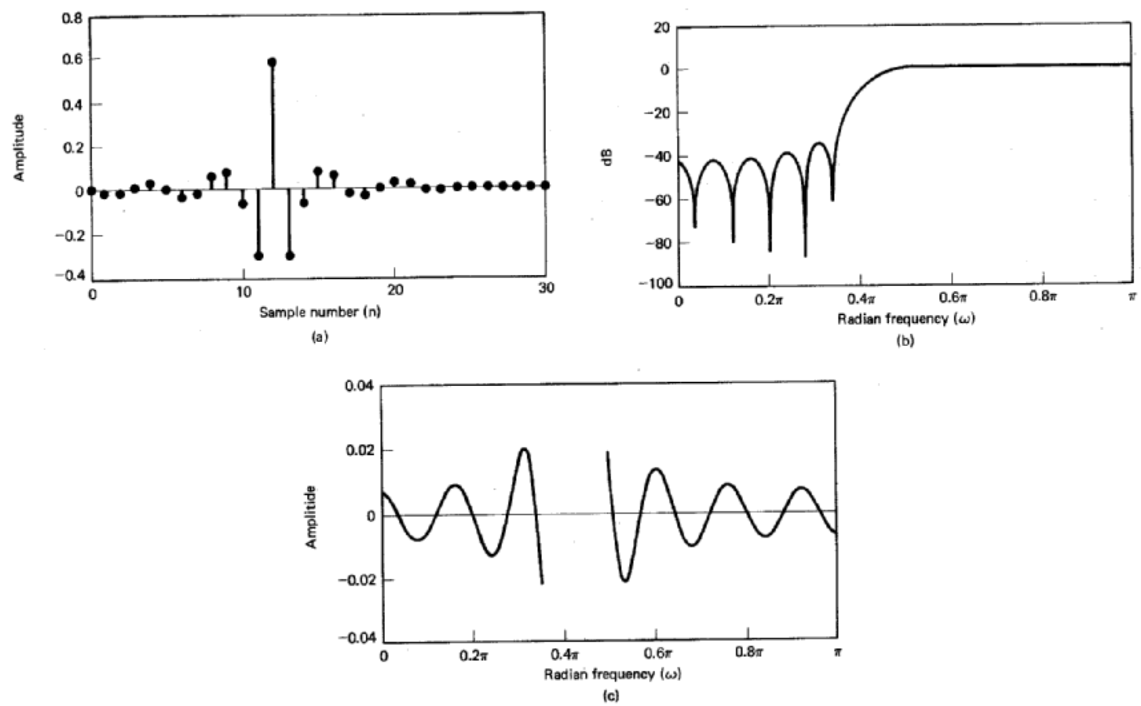


Figure 7.34 Response functions for type I FIR highpass filter. (a) Impulse response ($M = 24$). (b) Log magnitude. (c) Approximation error.

14. Optimal approximation of FIR filters

- **The windowing method does not permit individual control over the approximation errors in different bands. For many applications, better filters result from the minimization of the maximum error or a frequency-weighted error criterion.**
- **The Parks-McClellan algorithm reformulates the filter design problem as a polynomial approximation problem.**

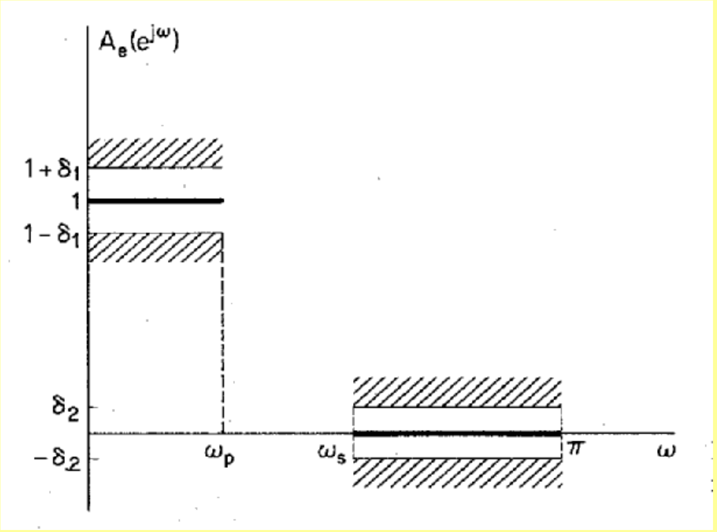
Rewrite $A_e(e^{j\omega})$ **of a zero-phase filter as an** ***L*****th-order polynomial in** $\cos\omega$**:**

$$A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k (\cos\omega)^k = P(\cos\omega) \qquad \textbf{(14.1)}$$

where $P(x) = \sum_{k=0}^{L} a_k x^k$ **.**

Define the approximation error function to be

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - A_e(e^{j\omega})]. \qquad \textbf{(14.2)}$$



$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_p \\ 0, & \omega_s \le |\omega| \le \pi \end{cases} \qquad \textbf{(14.3)}$$

- **The error function $E(\omega)$, the weighting function $W(\omega)$, and the desired frequency response $H_d(e^{j\omega})$ are defined only over closed subintervals of $0 \le \omega \le \pi$.**
- **The approximating function $A_e(e^{j\omega})$ is not constrained in the transition region.**

The approximation errors are weighted differently in different approximation intervals using the weighting function $W(\omega)$.

For the present problem:

$$W(\omega) = \begin{cases} 1/K, & 0 \le |\omega| \le \omega_p \\ 1 & \omega_s \le |\omega| \le \pi \end{cases} \qquad \textbf{(14.3)}$$

where $K = \delta_1 / \delta_2$. Using a minimax criterion, the best approximation is

$$\min_{\{h_e[n]:0 \le n \le L\}} \max_{\omega \in F} |E(\omega)| \qquad \textbf{(14.4)}$$

where *F* is the closed subset $0 \le \omega \le \pi$ such that $0 \le \omega \le \omega_p$ or $\omega_s \le \omega \le \pi$.

Alternation theorem

Let F_P denote the closed subset consisting of the disjoint union of closed subsets of the real axis *x*. $P(x)$ denotes an *r*th-order polynomial.

Also $D_p(x)$ denotes a given desired function of *x* that is continuous on F_P; $W_p(x)$ is a positive function, continuous on F_P, and $E_p(x)$ denotes the weighted error

$$E_p(x) = W_p(x)[D_p(x) - p(x)].$$

The maximum error $\| E \|_\infty$ is defined as

$$\| E \|_\infty = \max_{x \in F_p} | E_p(x) |.$$

- **A necessary and sufficient condition that $P(x)$ is the unique r^{th}-order polynomial that minimizes $\|E\|_\infty$ is that $E_p(x)$ exhibits at least (r+2) alternations, i.e., there must exist at least (r+2) values x_i in F_P such that $x_1 < x_2 < \ldots < x_{r+2}$ and such that $E_p(x_i) = -E_p(x_{i+1}) = \pm\|E\|$ for $i = 1,2,\ldots,(r+1)$.**

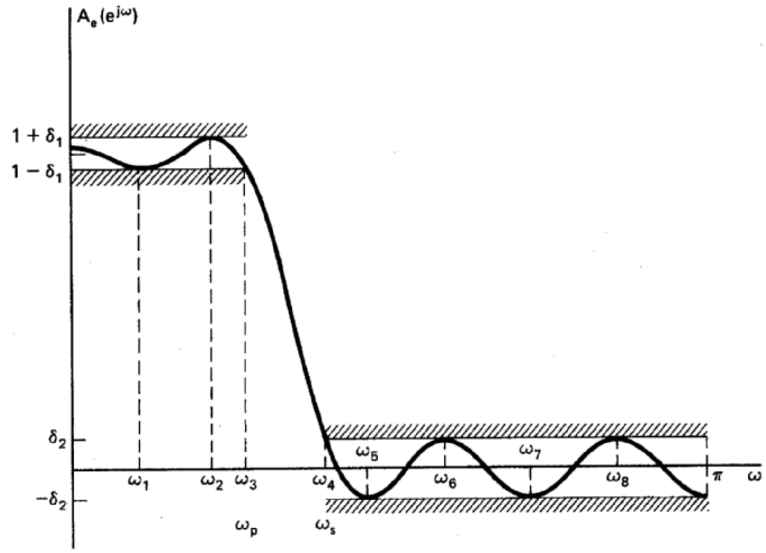


Figure 7.42 Typical example of a lowpass filter approximation that is optimal according to the alternation theorem for $L = 7$.

Optimal Type-I Lowpass filters

For type-I filters

$$P(\cos\omega) = \sum_{k=0}^{L} a_k (\cos\omega)^k \qquad \textbf{(14.5)}$$

$$D_p(\omega) = \begin{cases} 1, & \cos\omega_p \le |\cos\omega| \le 1 \\ 0, & -1 \le |\cos\omega| \le \cos\omega_s \end{cases} \qquad \textbf{(14.6)}$$

$$W_p(\omega) = \begin{cases} 1/K, & \cos\omega_p \le |\cos\omega| \le 1 \\ 0, & -1 \le |\cos\omega| \le \cos\omega_s \end{cases} \qquad \textbf{(14.7)}$$

$$E_p(\cos\omega) = W_p(\cos\omega)[D_p(\cos\omega) - P(\cos\omega)] \qquad \textbf{(14.8)}$$

The alternation theorem then states that a set of coefficients a_k in (14.5) will correspond to the filter representing the unique best approximation to the ideal lowpass filter with the ratio δ_1/δ_2 fixed at *K* and with passband and

stopband edges ω_p and ω_s if and only if $E_p(\cos\omega)$ exhibits at least (L+2) alternations on F_P. Such approximations are called equiripple approximations.

For type-I lowpass filter, the maximum possible number of alternations of the error is (L+3) .

- **Alternations will always occur at ω_p and ω_s.**
- **All points with zero slope inside the passband and the stopband will correspond to alternations.**

Taking the derivative of $P(\cos\omega)$, we have

$$\frac{dP(\cos(\omega))}{d\omega} = -\sin(\omega)\cdot\left(\sum_{k=0}^{L-1}(k+1)a_{k+1}(\cos(\omega))^k\right) \qquad \textbf{(14.9)}$$

which is always zero at the (L-1) roots of the (L-1)st order polynomial in (6.10).

Including the possible alternations at $\omega = 0$ and π, the maximum number of alternations including the two at the band edges ω_p, and ω_s is (L+3).

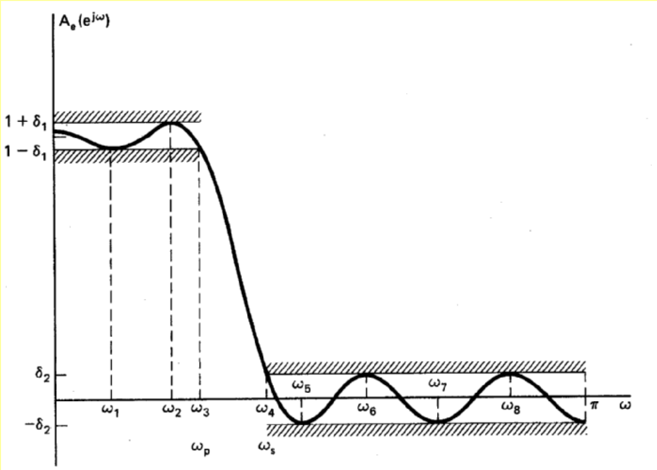


Figure 7.42 Typical example of a lowpass filter approximation that is optimal according to the alternation theorem for $L = 7$.

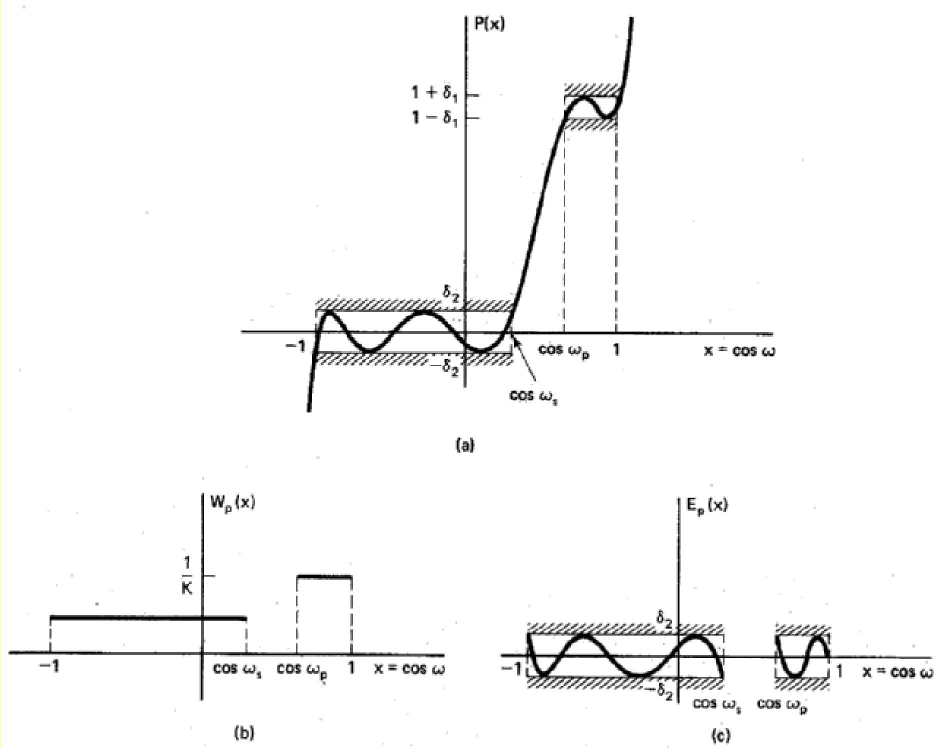


Figure 7.43 Equivalent polynomial approximation functions as a function of $x = \cos\omega$. (a) Approximating polynomial. (b) Weighting function. (c) Approximation error.

- **If either of the alternations at ω_p or ω_s is removed, the maximum number of alternations reduces to (*L*+1) violating the alternation theorem. Similar argument shows that the filter will be equiripple except possibly at $\omega = 0$ or π.**

Optimal Type-II Lowpass filters (left for self study)

For type-II filter

$$H(e^{j\omega}) = \cos(\omega/2)\left(\sum_{n=0}^{(M-1)/2} \tilde{b}[n]\cdot\cos(\omega n)\right) \qquad \textbf{(14.10)}$$

or equivalently

$$H(e^{j\omega}) = e^{-j\omega M/2}\cos(\omega/2)P(\cos\omega) \qquad \textbf{(14.11)}$$

where $P(\cos\omega) = \sum_{k=0}^{L} a_k(\cos\omega)^k$ **. The desired function to be approximated is**

$$H_d(e^{j\omega}) = D_P(\cos\omega) = \begin{cases} 1/\cos(\omega/2) & 0 \le |\omega| \le \omega_p \\ 1 & \omega_s \le |\omega| \le \pi \end{cases} \qquad \textbf{(14.12)}$$

and the weighting function is

$$W(\omega) = W_p(\cos\omega) = \begin{cases} \cos(\omega/2)/K & 0 \le |\omega| \le \omega_p \\ \cos(\omega/2) & \omega_s \le |\omega| \le \pi \end{cases} \qquad \textbf{(14.13)}$$

A similar set of issues arises in the design of type-III and type- IV linear-phase filters.

The Parks-McCllelan Algorithm

From the alternation theorem, the optimum filter $A_e(e^{j\omega})$ will satisfy:

$$W(\omega_i)\cdot[H_d(e^{j\omega_i})-A_e(e^{j\omega_i})]=(-1)^{i+1}\delta, \text{ for } i=1,2,\ldots,(L+2) \qquad \textbf{(14.14)}$$

The procedure begins by guessing a set of alternation frequencies ω_i, $i=1,2,\ldots,(L+2)$. The set of equations (14.14) can be solved for a_k and δ. A more efficient alternative is to use polynomial interpolation. The polynomial so obtained can be used to evaluate $A_e(e^{j\omega})$ and also $E(\omega)$ on a dense set of frequencies in the passband and stopband. If $|E(\omega)|<\delta$ for all ω in the passband and stopband, then the optimum approximation has been found. Otherwise, the Remez exchange method is used to obtain a completely new set of extremal frequencies defined by the (L+2) largest peaks of the error curve.

If there is a maximum of the error function at both 0 and π, then the frequencies at which the greatest errors occur is taken as the new estimate of alternation frequencies.

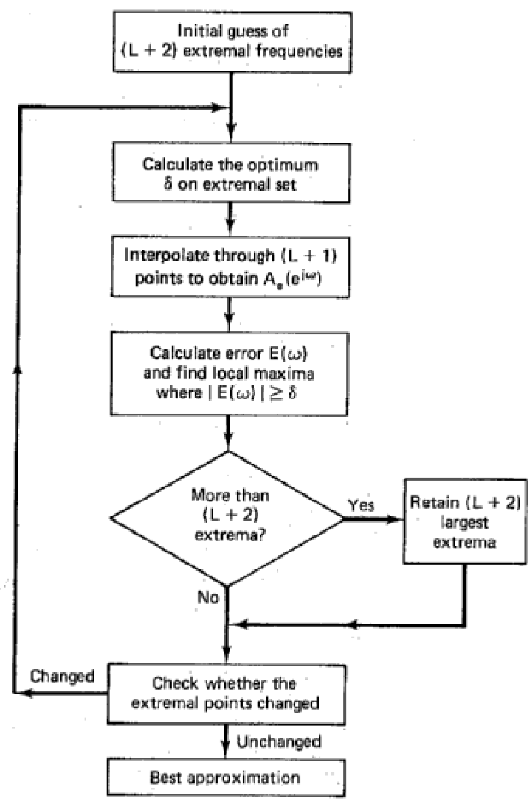


Figure 7.48 Flowchart of Parks–McClellan algorithm.

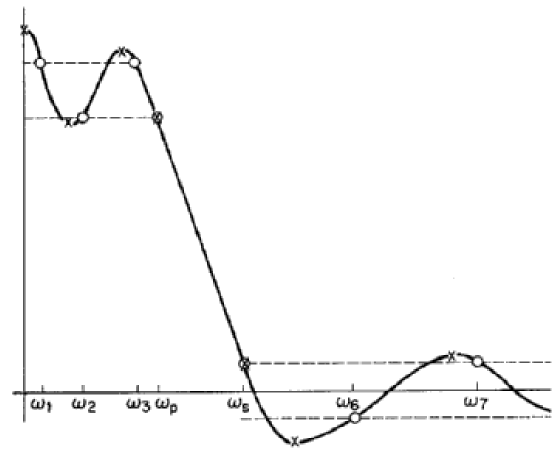


Figure 7.47 Illustration of the Parks–McClellan algorithm for equiripple approximation.

If given values of δ_1 and δ_2 are desired, the algorithm just described can be employed to determine a filter with prescribed values of δ_1 and δ_2 by fixing ω_p and *M* and varying ω_s until the desired δ_1 and δ_2 are obtained.

- **Kaiser obtained the following simplified formula for determining *M* given the transition width and pass- and stopband ripples:**

$$M = \frac{-10\log_{10}(\delta_1\delta_2) - 13}{2.324\Delta\omega}, \qquad \textbf{(14.15)}$$

where $\Delta\omega = \omega_s - \omega_p$

Examples

LOWPASS FILTER :

The specifications are:

$$\omega_p = 0.4\pi, \omega_s = 0.6\pi, \delta_1 = 0.01 \text{ and } \delta_2 = 0.01.$$

- **Substituting into (14.15) gives *M* = 26. This filter fails to meet the original specifications and we must increase *M* to 27.**

- **For the same specifications, the Kaiser window method requires a value of *M* = 38 to meet or exceed the specifications.**

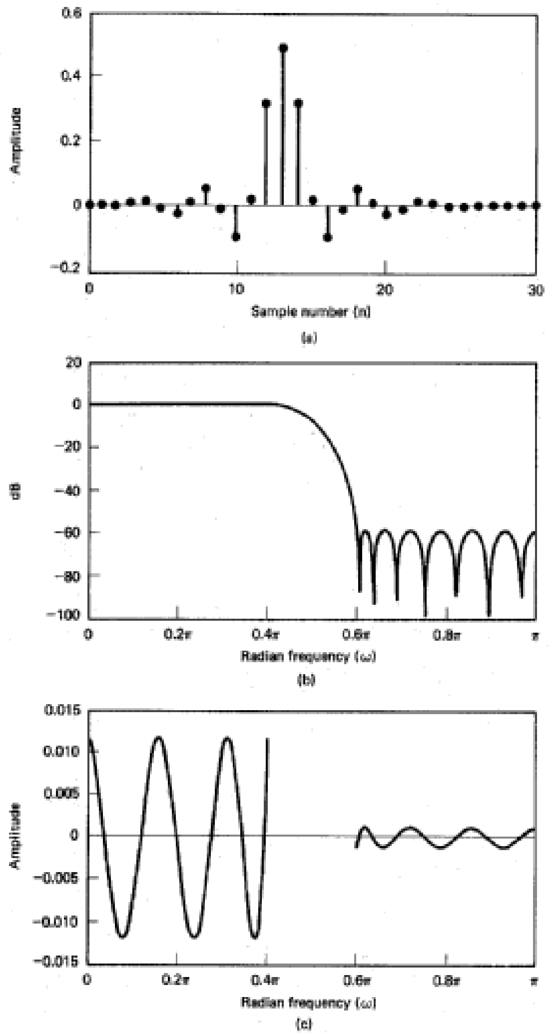


Figure 7.50 Optimum type I FIR lowpass filter for $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $K = 10$, and $M = 26$. (a) Impulse response. (b) Log magnitude. (c) Approximation error (unweighted).

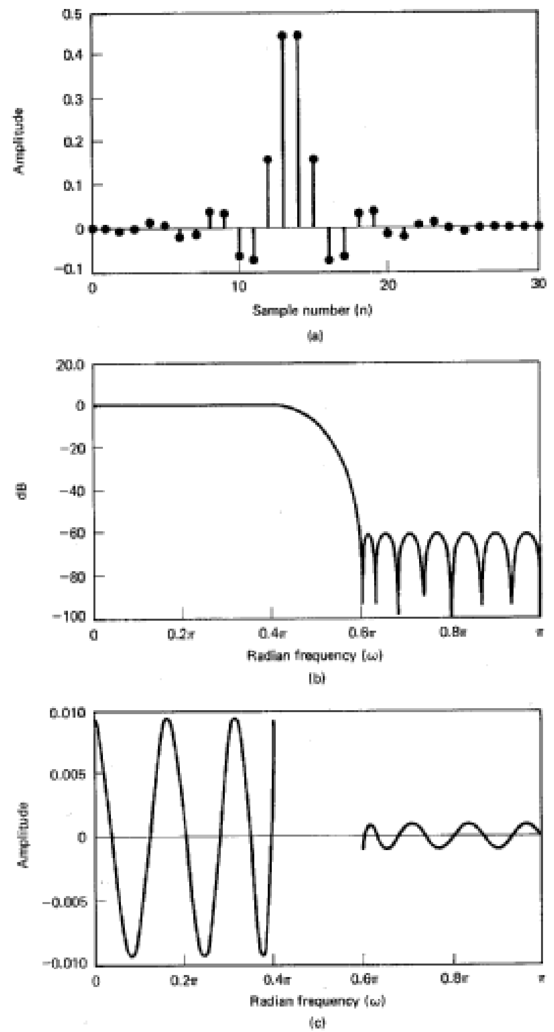
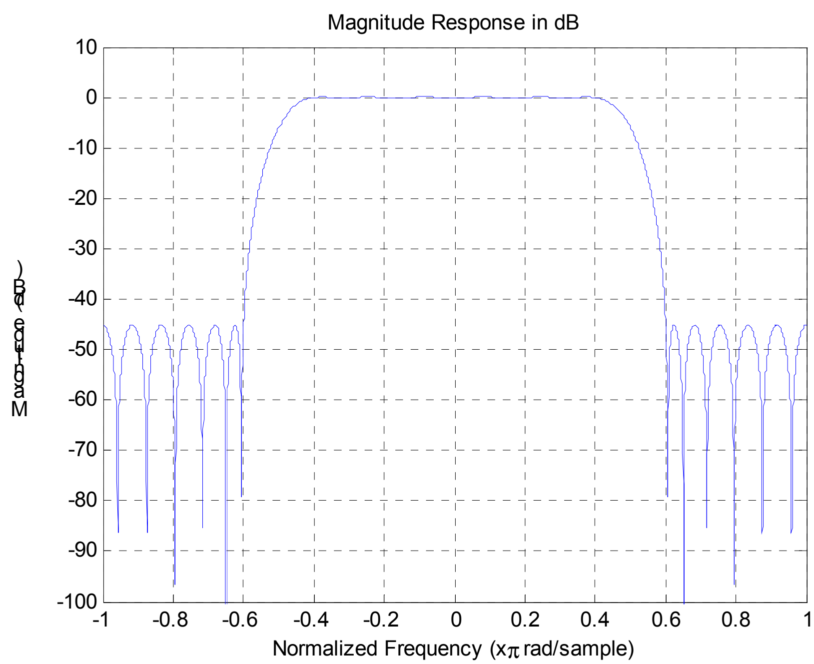


Figure 7.51 Optimum type II FIR lowpass filter for $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $K = 10$, and $M = 27$. (a) Impulse response. (b) Log magnitude. (c) Approximation error (unweighted).

MATLAB command



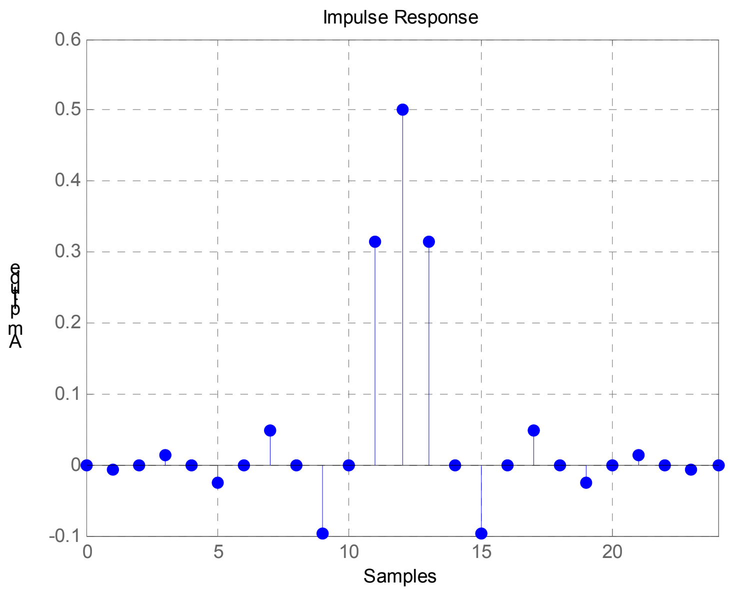
Park-McClellan algorithm. MATLAB COMMAND: b=remez(N,f,m).

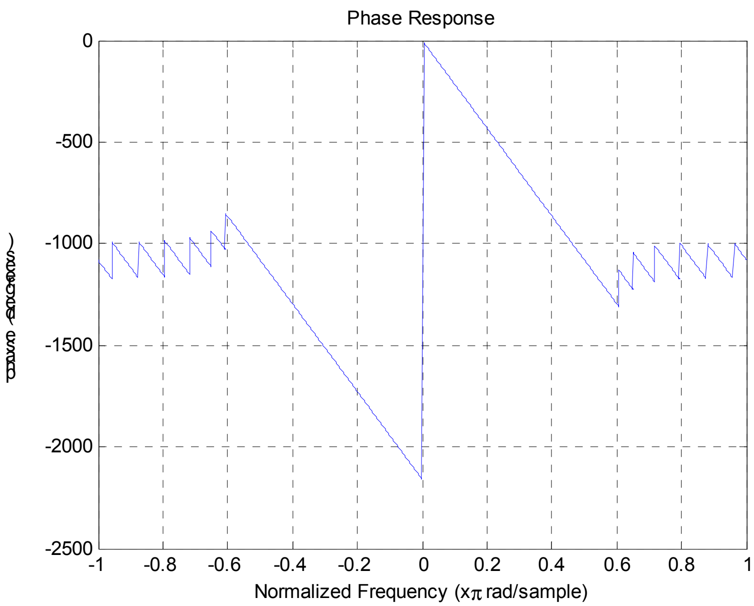
b = filter coefficient vector,

N = filter length,

m = weighting in different frequency band,

f = frequency band).Filter Specification: N=25, $\omega_p = 0.4\pi$**,** $\omega_s = 0.6\pi$**.**


Impulse Response
Amplitude
Samples
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
5
10
15
20


Phase Response
Phase (degrees)
Normalized Frequency (×π rad/sample)
0
-500
-1000
-1500
-2000
-2500
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1