SECONG ORDER CONE PROGRAMMING

CONTENTS

REFERENCES:

S. BOYD AND L. VANDENBERGHE, CONVEX OPTIMIZATION. CAMBRIDGE UNIVERSITY PRESS, 2004. (Available from http://www.stanford.edu/~boyd/cvxbook/)

1. Second-order Cone

1.1 Euclidean Norm

$$\|\boldsymbol{x}\|_2 = \sqrt{\boldsymbol{x}^T \boldsymbol{x}} = \sqrt{x_1^2 + \cdots + x_N^2}\,, \tag{1}$$

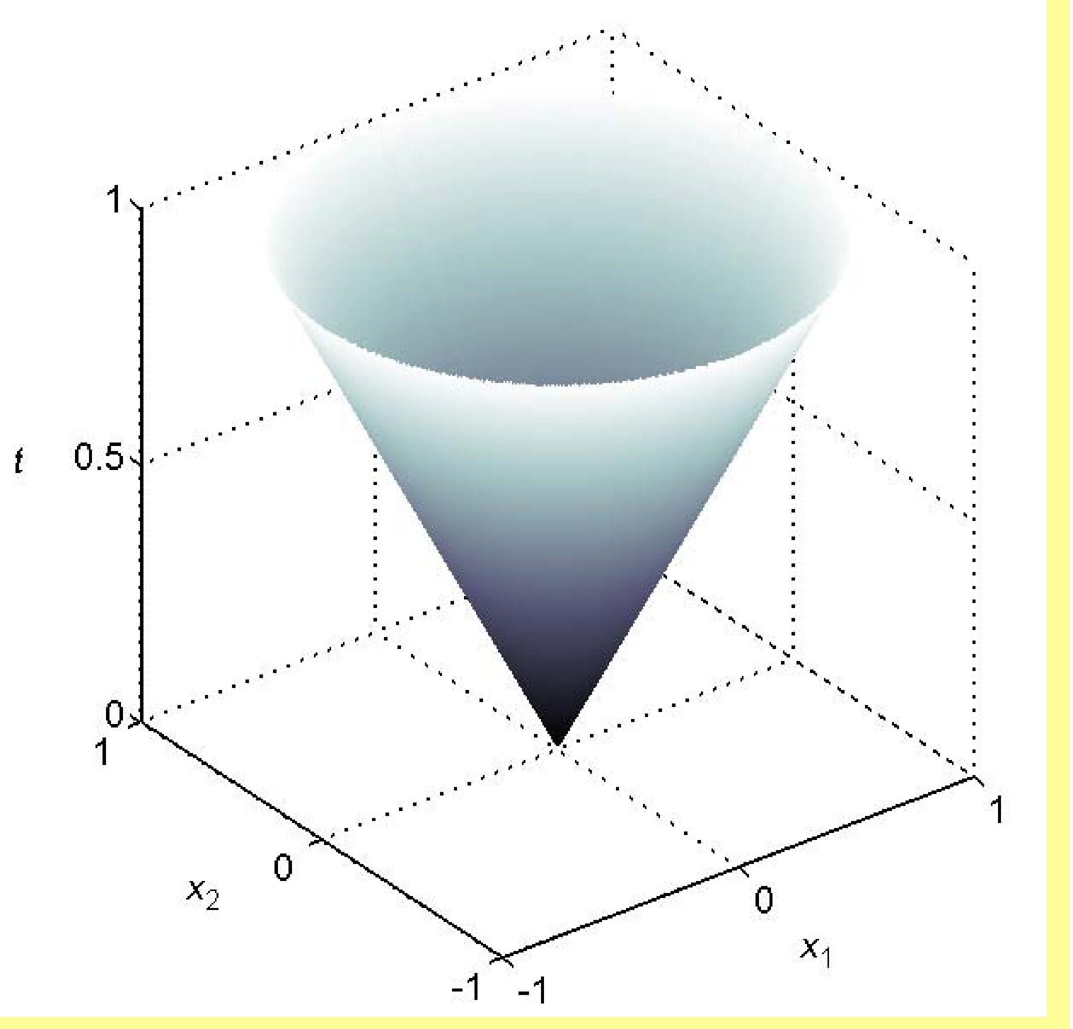
1.2 Second-order Cone

$$\begin{aligned} C &= \{(\boldsymbol{x}, t) \in \Re^{N+1} \mid \ \|\boldsymbol{x}\|_2 \le t\} \\ &= \left\{ \begin{bmatrix} \boldsymbol{x} \\ t \end{bmatrix} \ \middle| \ \begin{bmatrix} \boldsymbol{x} \\ t \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ t \end{bmatrix} \le 0,\ t \ge 0 \right\} \end{aligned} \tag{2}$$

Example:

Boundary of second-order cone in $\Re^3$.

$$\{(x_1, x_2, t) \mid \sqrt{x_1^2 + x_2^2} \le t\}$$



2. Second-order Cone Programming

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & \left\| \boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{b}_i \right\|_2 \le \boldsymbol{c}_i^T \boldsymbol{x} + \boldsymbol{d}_i, \quad i = 1, \ldots, M, \\ & \boldsymbol{F}\boldsymbol{x} = \boldsymbol{g}, \end{aligned} \tag{3}$$

where $\boldsymbol{x} \in \Re^N$ is the optimization variable, $\boldsymbol{A}_i \in \Re^{N_i \times N}$, $\boldsymbol{F} \in \Re^{P \times N}$, and $\left\| \boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{b}_i \right\| \le \boldsymbol{c}_i^T \boldsymbol{x} + \boldsymbol{d}_i$ for $n = 1, \ldots, N$ are second-order cone constraints.

- When $\boldsymbol{c}_i = 0$ for $i = 1, \ldots, M$, the SOCP reduces to quadratically constrained quadratic programming (QCQP).
- When $\boldsymbol{A}_i = 0$ for $i = 1, \ldots, M$, the SOCP reduces to linear programming (LP).

3. Design Example: FIR Filter Design

- **Frequency response of a general digital FIR filter:**

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n} = \boldsymbol{h}^T(\boldsymbol{c}(\omega) - j\boldsymbol{s}(\omega)),$$

where $\boldsymbol{h} = [h(0),\ldots,h(N-1)]^T$, $\boldsymbol{c}(\omega) = [1,\ldots,\cos((N-1)\omega)]^T$ **and**

$\boldsymbol{s}(\omega) = [0,\ldots,\sin((N-1)\omega)]^T$.

- **Minimax Design Criterion:**

$$\min_{\boldsymbol{h}} \max_{\omega\in\Omega} W(\omega)\,|H(e^{j\omega}) - H_d(\omega)|,$$

where Ω is the frequency interval of interest; $W(\omega)$ is a positive weighting function; $H_d(\omega)$ is the desired frequency response.

- **Discretizing $\omega \in \Omega$ into M equally spaced samples, gives**

$$\begin{aligned} &\min_{\boldsymbol{h}} && \delta \\ &\text{subject to} && \delta - \{\alpha_R^2(\omega_i) + \alpha_I^2(\omega_i)\}^{1/2} \geq 0, \quad i = 1, \ldots, M, \end{aligned}$$

where $\alpha_R(\omega) = W(\omega) \cdot \{\boldsymbol{h}^T \boldsymbol{c}(\omega) - \mathrm{Re}[H_d(\omega)]\}$,

$\alpha_I(\omega) = W(\omega) \cdot \{\boldsymbol{h}^T \boldsymbol{s}(\omega) + \mathrm{Im}[H_d(\omega)]\}$.

- **Equivalent SOCP problem (with variable $\boldsymbol{x} = [\delta \quad \boldsymbol{h}^T]^T$):**

$$\begin{aligned} &\min_{\boldsymbol{x}} && \boldsymbol{c}^T \boldsymbol{x} \\ &\text{subject to} && \boldsymbol{c}^T \boldsymbol{x} \geq \left\| \boldsymbol{A}_i \boldsymbol{x} - \boldsymbol{b}_i \right\|_2, \quad i = 1, \ldots, M, \end{aligned}$$

where $\boldsymbol{c} = [1 \quad \boldsymbol{O}_N^T]^T$, $\boldsymbol{A}_i = W(\omega_i)\begin{bmatrix} 0 & \boldsymbol{c}(\omega_i)^T \\ 0 & \boldsymbol{s}(\omega_i)^T \end{bmatrix}$, $\boldsymbol{b}_i = W(\omega_i)\begin{bmatrix} -\mathrm{Re}[H_d(\omega_i)] \\ \mathrm{Im}[H_d(\omega_i)] \end{bmatrix}$

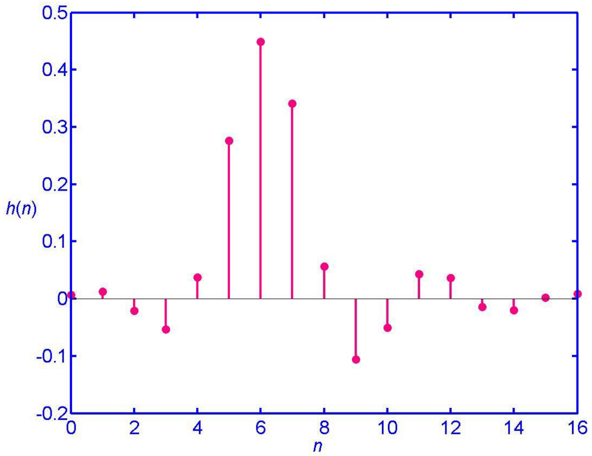
$\boldsymbol{O}_N$ **is an N row zero vector.**

Example (low-delay lowpass FIR filter)

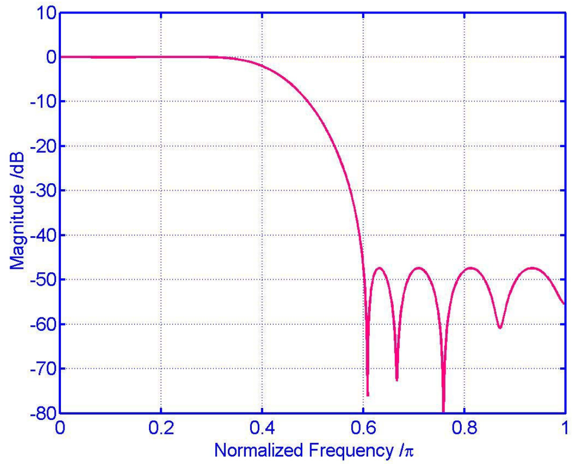
- **Desired response:** $H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau}, & 0 < \omega < \omega_p \\ 0, & \omega_s < \omega < \pi \end{cases}$

  where group delay $\tau = (N-1)/2 - D$; D is delay reduction parameter.

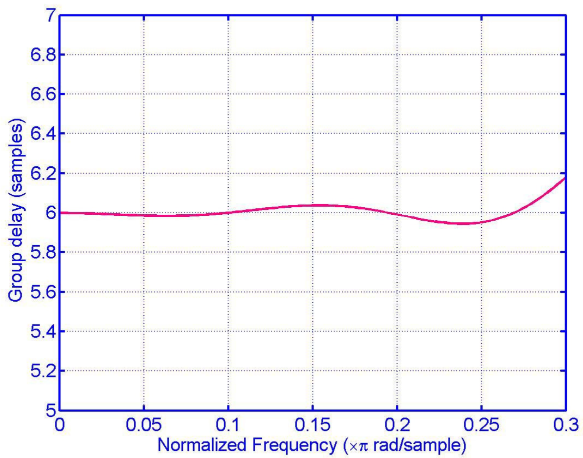
- **Specifications:** $N = 17$, $D = 2$, $\omega_p = 0.3\pi$, $\omega_s = 0.6\pi$, $M = 200$ and $W(\omega) = 1, \forall\omega$.



Impulse Response



Frequency Response



Group Delay Response