# TWO DIMENSIONAL FIR FILTERS

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## REFERENCES:

D.E. Dudgeon and R.M. Mersereau, MULTIDIMENSIONAL DIGITAL SIGNAL PROCESSING. ENGLEWOOD CLIFFS, NJ: PRENTICE-HALL, INC., 1984.

## 1. FIR FILTERS

- An FIR (finite impulse response) or nonrecursive filter is one whose impulse response is of finite duration. The impulse response is always absolutely summable and the FIR filters are always stable.
- IIR filters have infinite impulse response and are usually implemented as a multidimensional difference equation of finite order. They may or may not be stable but may be less complex to realize than equivalent FIR filters. Design is quite different from 1-D IIR filters.

A zero-phase FIR filter satisfies:

$$H(\omega_1,\omega_2) = H^*(\omega_1,\omega_2) \Leftrightarrow h(n_1,n_2) = h^*(n_1,n_2). \qquad (1)$$

i.e the frequency and impulse responses are real. Linear-phase filters are readily obtained from a zero-phase FIR filter by inserting appropiate number of delays.

# 2. IMPLEMENTATION OF FIR FILTERS

## 2.1 Direct Convolution

**Since the impulse response of a FIR filter is of finite extent, its output can be computed with the convolution sum:**

$$y(n_1,n_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} h(k_1,k_2)x(n_1-k_1,n_2-n_2). \quad \textbf{(2-1)}$$

**Due to the symmetry of the impulse response in the zero-phase filter, the number of multiplications and additions can be approximately halved.**

## 2.2 Discrete Fourier Transform Implementation of FIR Filters

**Let** $w(n_1,n_2)$ **be the linear convolution of** $h(n_1,n_2)$ **and** $x(n_1,n_2)$**:**

$$w(n_1,n_2) = x(n_1,n_2) ** h(n_1,n_2). \quad \textbf{(2-1)}$$

**Taking Fourier transforms of both sides:**

$$W(\omega_1,\omega_2) = H(\omega_1,\omega_2)X(\omega_1,\omega_2), \qquad \textbf{(2-2)}$$

**and sampling** $\omega_1$ **and** $\omega_2$ **respectively at** $2\pi k_1/N_1$ **and** $2\pi k_2/N_2$ **gives:**

$$W(k_1,k_2) = H(k_1,k_2)X(k_1,k_2). \qquad \textbf{(2-3)}$$

$y(n_1,n_2)$**, the inverse DFT of *W*, will then be the circular convolution of** $h(n_1,n_2)$ **and** $x(n_1,n_2)$**. If** $N_1$ **and** $N_2$ **are chosen to be sufficiently large,** $y(n_1,n_2) = w(n_1,n_2)$**.**

**Discrete Fourier transform implementation of FIR, using the fast Fourier transform (FFT), is very efficient for high order filters but it requires lot of storage. Convolution sum has high arithmetic complexity but moderate storage. A compromise is to use *block convolution* such as the *overlap save and overlap add methods*.**

# 3. Design of FIR Filters

## 3.1 Windowing Methods

Let $i(n_1, n_2)$ and $I(\omega_1, \omega_2)$ be the impulse and frequency responses for the ideal filter, respectively. Similarly, let $h(n_1, n_2)$ and $H(\omega_1, \omega_2)$ be the impulse and frequency responses of the filter to be designed.

- In the windowing methods, $h(n_1, n_2)$ is obtained by multiplying $i(n_1, n_2)$ by a finite length window function $w(n_1, n_2)$ of finite support ***R***.

$$h(n_1, n_2) = i(n_1, n_2) w(n_1, n_2). \qquad \text{(3-1-1)}$$

$H(\omega_1, \omega_2)$ is obtained by a frequency-domain convolution of $I(\omega_1, \omega_2)$ with $W(\omega_1, \omega_2)$:

$$H(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} I(\Omega_1, \Omega_2) W(\omega_1 - \Omega_1, \Omega_2 - \omega_2) d\Omega_1 d\Omega_2 . \qquad \text{(3-1-2)}$$

**In many filter design problems, the desired filter behavior is specified through $I(\omega_1,\omega_2)$ rather than $i(n_1,n_2)$. we must either compute $i(n_1,n_2)$ analytically or approximate it by sampling $I(\omega_1,\omega_2)$ densely over $(\omega_1,\omega_2)$ and then performing an IDFT. To minimize aliasing error, the length of the IDFT should be several times larger than the extent of R.**

**For zero-phase filter, the window should also be zero-phase. The 2-D window is usually generated from 1-D windows as follows**

$$w_R(n_1,n_2)=w_1(n_1)w_2(n_2). \tag{3-1-3a}$$

$$w_C(n_1,n_2)=w(\sqrt{n_1^2+n_2^2}). \tag{3-1-3b}$$

**(3-1-3a) has a rectangular region of support. (3-1-3b) has nearly circular region of support.**

**The Fourier transform of the rectangular window is:**

$$W_R(\omega_1,\omega_2) = W_1(\omega_1)W_2(\omega_2). \qquad \textbf{(3-1-4)}$$

**The Fourier transform of $w_C(n_1,n_2)$ resembles, but differs in detail from, a circularly rotated version of the 1-D Fourier transform of w(t).**

- **Common windows are rectangular, Hanning, and Kaiser windows.**

## Design Example

**Design an (11x11)-point FIR with ideal filter response:**

$$I(\omega_1,\omega_2) = \begin{cases} 1, \omega_1^2 + \omega_2^2 \le (0.4\pi)^2 \\ 0, otherwise, -\pi \le \omega_1,\omega_2 \le \pi \end{cases}. \qquad \textbf{(3-1-5)}$$

**For zero-phase filter, the region of support is:**

$$R = \{(n_1,n_2): -5 \le n_1,n_2 \le 5\}. \qquad \textbf{(3-1-6)}$$

**The ideal impulse response $i(n_1,n_2)$, is computed by taking the inverse Fourier transform of $I(\omega_1,\omega_2)$:**

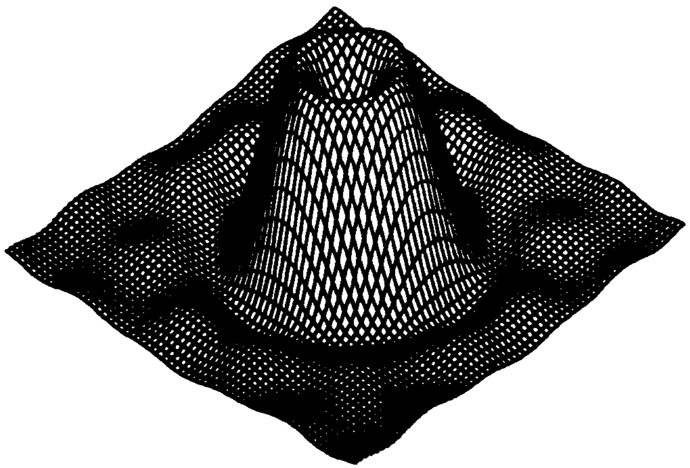
$$i(n_1,n_2)=\frac{0.2J_1(0.4\pi\sqrt{n_1^2+n_2^2})}{\sqrt{n_1^2+n_2^2}}. \tag{3-1-7}$$

**where $J_1(x)$ is the Bessel function of the first kind of order 1.**

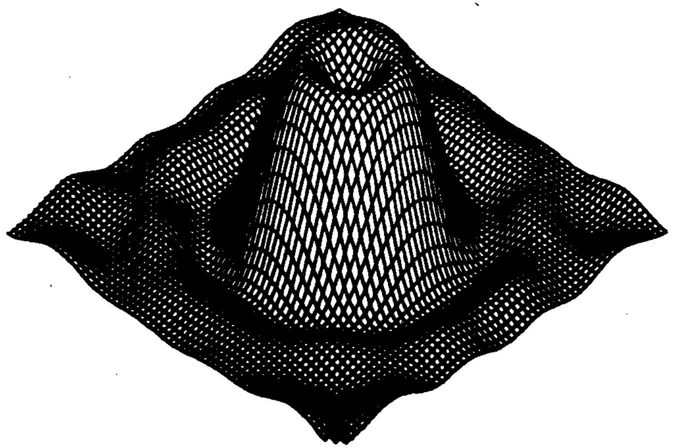
**Using Kaiser window, the two windows are:**

$$w_R(n_1,n_2)=\begin{cases}\dfrac{I_0[\alpha\sqrt{1-(n_1/5)^2}]I_0[\alpha\sqrt{1-(n_2/5)^2}]}{I_0^2[\alpha]}, & |n_1|\le 5,|n_2|\le 5\\ 0 & otherwise\end{cases} \tag{3-1-8}$$

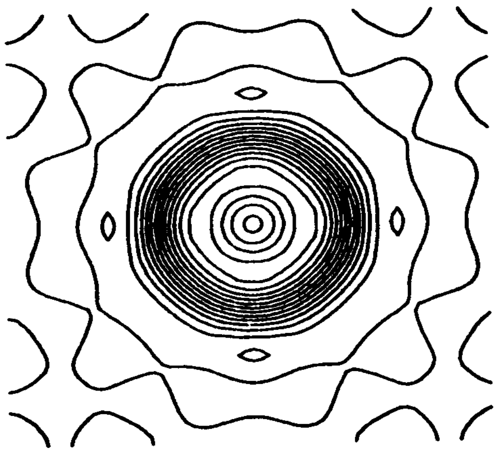
$$w_C(n_1,n_2)=\begin{cases}\dfrac{I_0[\alpha\sqrt{1-((n_1^2+n_2^2)/25)^2}]}{I_0^2[\alpha]}, & n_1^2+n_2^2\le 25.\\ 0 & otherwise\end{cases} \tag{3-1-9}$$



(a)

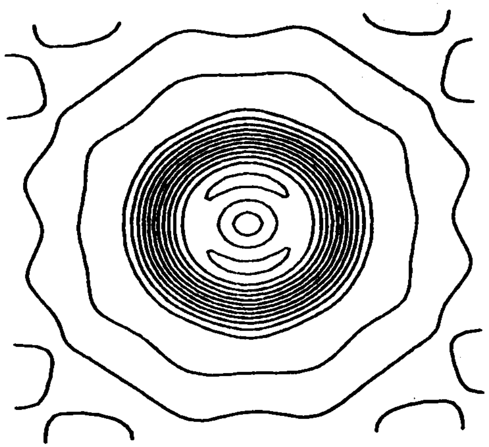


(c)



(b)

**Figure 3.4** Frequency responses of two 11 × 11 FIR lowpass filters designed using windows. (a) Perspective plot of a design based on an outer product window. (b) Contour plot. (c) Perspective plot of a design based on a rotated window. (d) Contour plot. The contours are shown for values of *H* between −0.1 and 1.1 in increments of 0.1. (Courtesy of Russell M. Mersereau, from *Two-Dimensional Digital Signal Processing I*, Thomas S. Huang, ed., *Topics in Applied Physics*, Vol. 42, © 1981 Springer-Verlag.)



(d)

**Figure 3.4** (*Continued*)

**Following the approach of Kaiser in 1-D case, a formula for the filter order *N* in terms of its specification $\delta_p, \delta_s$ and $\Delta\omega = \omega_s - \omega_p$ can be experimentally developed:**

$$N_R \approx \frac{-20\log_{10}\sqrt{\delta_p \delta_s} - 8}{2.10 \cdot \Delta\omega}. \qquad \textbf{(3-1-10)}$$

$$N_C \approx \frac{-20\log_{10}\sqrt{\delta_p \delta_s} - 7}{2.18 \cdot \Delta\omega}. \qquad \textbf{(3-1-11)}$$

**The filter order is approximately the same.**

**The window parameter $\alpha$ can be determined from the desired attenuation of the filter as follows**

$$ATT = -20\log_{10}\sqrt{\delta_p \delta_s}. \qquad \textbf{(3-1-12)}$$

**For the outer product windows, $\alpha$ has been experimentally determined to be**

$$\alpha \approx \begin{cases} 0.42(ATT-19.3)^{0.4}+0.089(ATT-19.3), & 20 < ATT < 60 \\ 0, & ATT < 20 \end{cases} \qquad \textbf{(3-1-13)}$$

**For the rotated window, $\alpha$ is approximately given by:**

$$\alpha \approx \begin{cases} 0.56(ATT-20.2)^{0.4}+0.083(ATT-20.2), & 20 < ATT < 60 \\ 0, & ATT < 20 \end{cases} \qquad \textbf{(3-1-14)}$$

## 3.2 Optimal least squares FIR Filter Design

**Let** $$E(\omega_1,\omega_2) = H(\omega_1,\omega_2) - I(\omega_1,\omega_2) \quad \textbf{(3-2-1)}$$

**be the error function in the filter approximation. One approach to design the filter is to choose the coefficients of the filter to mininimize some function of this error, such as its $L_P$ norm:**

$$E_p = \sqrt[p]{\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|E(\omega_1,\omega_2)\right|^p d\omega_1 d\omega_2} \,. \quad \textbf{(3-2-2)}$$

**Filters designed using different error criteria can be quite different.**

**The frequency response of an FIR filter with support R is given by:**

$$H(\omega_1,\omega_2) = \sum_{(n_1,n_2)\in R} h(n_1,n_2)\exp(-jn_1\omega_1 - jn_2\omega_2)$$

**(3-2-3)**

$$= \sum_{i=1}^{F} a(i)\phi_i(\omega_1,\omega_2) \,.$$

The functions $\{\phi_i(\omega_1,\omega_2)\}$ are called the basis functions of the approximation. For the zero-phase filter,

$$\phi_i(\omega_1,\omega_2)=\begin{cases}2\cos(n_1\omega_1+n_2\omega_2) & (n_1,n_2)\neq(0,0)\\ 1 & (n_1,n_2)=(0,0)\end{cases} \qquad \text{(3-2-4)}$$

## 3.2.1 Least-Squares Design

The function to be minimized is:

$$E_2=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|E(\omega_1,\omega_2)\right|^2 d\omega_1 d\omega_2 \qquad \text{(3-2-5)}$$

Substituting (3-2-3) into (3-2-5), one gets

$$E_2=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\sum_{i=1}^{F}a(i)\phi_i(\omega_1,\omega_2)-I(\omega_1,\omega_2)\right|^2 d\omega_1 d\omega_2 \qquad \text{(3-2-6)}$$

**Differentiating (3-20) wrt to a(i) and setting the derivatives to 0 gives the set of F linear equations:**

$$\sum_{i=1}^{F} a(i)\phi_{ik} = I_k, k = 1,2,...,F. \quad \textbf{(3-2-7)}$$

**where** $$\phi_{ik} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\phi_i(\omega_1,\omega_2)\phi_k(\omega_1,\omega_2)d\omega_1 d\omega_2 \text{ and} \quad \textbf{(3-2-8)}$$

$$I_k = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} I(\omega_1,\omega_2)\phi_k(\omega_1,\omega_2)d\omega_1 d\omega_2 . \quad \textbf{(3-2-9)}$$

**If $\{\phi_i(\omega_1,\omega_2)\}$'s are orthogonal, the solution is simply: $a(i) = I_i / \phi_{ii}$. Filter designed using least squares may not be satisfactory due to large passband and stopband ripples. The integrals (3-2-8) and (3-2-9) may also be difficult to evaluate and it can be approximated by the following summation:**

$$E_2 = \sum_m W_m [H(\omega_{1m}, \omega_{2m}) - I(\omega_{1m}, \omega_{2m})] \quad \textbf{(3-2-10)}$$

$\omega_{1m}, \omega_{2m}$ **are called the constraint frequencies.**

- **Finding the coefficients a(i) that minimize $E_2$ again involves the solution of a systems of linear equations in F unknowns.**

**Although the design of filters to minimize an $L_P$ error norm other than p=2 can be approached similarly, the equations that result are not linear and their solutions are not simple. General optimization techniques will be required.**

## 3.2.2 Design of Zero-Phase Equiripple FIR Filters

In the 1-D case, the equiripple solution can be proved to be unique and there are no numerical difficulties to limit the filter order, and the algorithm (the Park-McClellan algorithm) converges rapidly.

- For higher-dimensional cases, due primarily to the absence of a factorization theorem, the algorithms are slower to converge, more difficult to understand, and somewhat limited in their capabilities.

- An approach is to design the filters by minimizing the $L_\infty$ norm. Very good approximation of equripple filters with high order can be designed using nonlinear optimization, weighted least squares (WLS), and more recently semidefinite programming (SDP).

## 3.3 FIR Filter Design Using McClellan Transformation

The idea is to convert a 1-D zero-phase FIR filter into a multidimensional one through a substitution of variables. It is attractive for a number of reasons:

1. The designs are easy to perform;
2. 1-D FIR filter design methods are well understood;
3. using an optimal 1-D filters, it might be possible to design optimal MD filters; and
4. efficient implementation are avaibable.

For a zero phase filter, we can write:

$$H(\omega) = \sum_{n=0}^{N} a(n)\cos(n\omega) = \sum_{n=0}^{N} a(n)T_n[\cos(\omega)], \tag{3-3-1}$$

where $T_n$ is the n-th Chebyshev polynomial.

**If we make the substitution:**

$$F(\omega_1, \omega_2) \to \cos\omega, \qquad \textbf{(3-3-2)}$$

**we obtain the following 2-D frequency response:**

$$H(\omega_1, \omega_2) = \sum_{n=0}^{N} a(n) T_n[F(\omega_1, \omega_2)]. \qquad \textbf{(3-3-3)}$$

**An M-dimensional frequency response results if an M-dimensional transformation function is used.**

**The simplest choice is the frequency response of a zero-phase (3x3 filter):**

$$F(\omega_1, \omega_2) = A + B\cos\omega_1 + C\cos\omega_2$$
$$+ D\cos(\omega_1 - \omega_2) + E\cos(\omega_1 + \omega_2). \qquad \textbf{(3-3-4)}$$

**where constants A, B, C, and D are free parameters.**

## Design Example: Circular Symmetric 2D Filters

**Consider the transformation with A = -0.5, B= C = 0.5, D = E = 0.25. This is the original transformation proposed by McCllelan.**

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**The isopotentials close to the center are nearly circular and those towards the outside look like squares. The transformation function is:**

$$F(\omega_1, \omega_2) = \tfrac{1}{2}(-1 + \cos\omega_1 + \cos\omega_2 + \cos\omega_1 \cos\omega_2). \qquad \textbf{(3-3-5)}$$

**For $\omega_2 = 0$, we have**

$$F(\omega_1, 0) = \cos\omega_1. \qquad \textbf{(3-3-6)}$$

**Thus,**

$$H(\omega_1, 0) = H(\omega_1). \qquad \textbf{(3-3-7)}$$

**For this transformation function, the prototype frequency response becomes a cross-sectional slice of the 2-D frequency response. A 1-D lowpass filter thus produces an 2-D lowpass filter, and a 1-D bandpass filter produces an 2-D ring bandpass filter.**

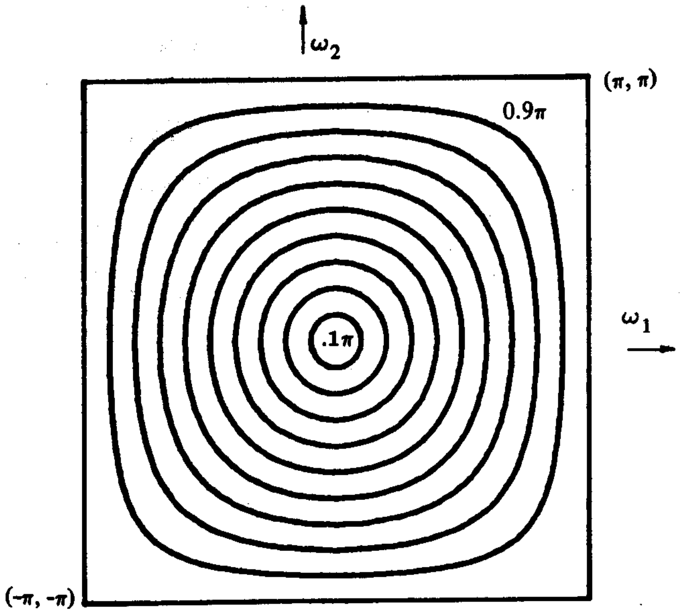


**Figure 3.11** Contours of constant value for the transformation of first order with $A = -\frac{1}{2}$, $B = C = \frac{1}{2}$, $D = E = \frac{1}{4}$. The contours are shown for values of $\omega$ in increments of $0.1\pi$. (Courtesy of Russell M. Mersereau, from *Two-Dimensional Digital Signal Processing I*, Thomas S. Huang, ed., *Topics in Applied Physics*, Vol. 42, © 1981 Springer-Verlag.)

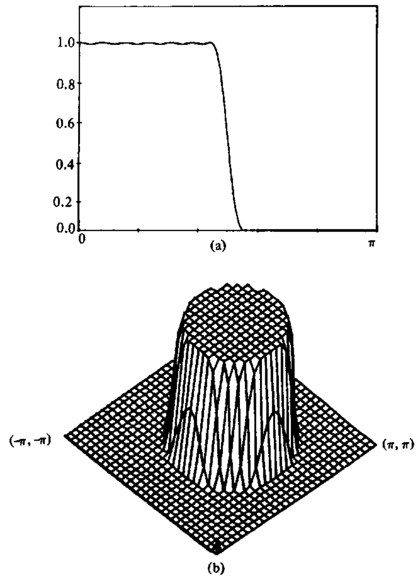


**Figure 3.12** Parts (b) and (d) show the 2-D filters that result from the transformation function of Figure 3.11 used with the lowpass and bandpass prototypes shown in parts (a) and (c). (Courtesy of Russell M. Mersereau, from *Two-Dimensional Digital Signal Processing I*, Thomas S. Huang, ed., *Topics in Applied Physics*, Vol. 42, © 1981 Springer-Verlag.)

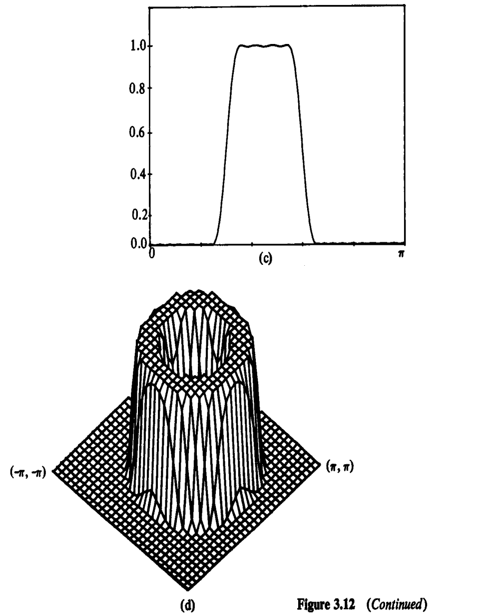


**Figure 3.12** (*Continued*)

Furthermore, the height of the passband and stopband ripples in the prototype become the ripple heights in the final design.