The University of Hong Kong
Department of Electrical and Electronic Engineering

ELEC6026 Digital Signal Processing

Assignment 2: Markowitz mean-variance portfolio
(5 marks)

1. Objectives:

The purpose of this assignment is to study the Markowitz mean-variance portfolio, which is an application of Digital Signal Processing to Finance. The details will not be examined in the final exam.

2. Equipment Required

MATLAB: MATLAB is an interactive environment for high level simulation of general numerical applications such as numerical linear algebra, signal processing, etc. Refer to http://www.mathworks.com/help/matlab/ for more details.

If you do not have the MATLAB software, you may use the PC workstations located at the CYC103 Laboratory.

3. Submission Format and Deadline

A brief written report addressing the problems listed in **section 4** should be submitted either in Microsoft Word or PDF format and **uploaded to the Moodle System by Nov 30, 2015 23:55 (Mon) (GMT +8:00).**

4. Procedure and Analysis (10 marks)

The assignment package 'ass2.zip' contains the following files:

- `Ass2.m`: The script will generate 200 time-instants of 10 random assets.

- `ARMAX_GARCH_K_Toolbox (folder)`: A time series toolbox written by Mr. Alexandros Gabrielsen. The function "garch" can estimate and perform 1-step-ahead mean and variance prediction based on the ARMA-GARCH model.

The assignment has two parts.

In part 1, the theory of Markowitz mean-variance portfolio is studied and you are asked to derive the required expressions for solving the mean-variance portfolio.

In part 2, you are given the MATLAB® codes. Run "Ass2.m" to generate the mean-variance portfolio. You will be asked questions related to the mean variance portfolio generated by "Ass2.m".

Part 1. Theory of Markowitz mean-variance portfolio

a) A portfolio is a combination of different assets. In a portfolio optimization problem, one seeks to optimize the proportion of wealth allocated to different assets in order to reach certain specified target return. Since the return of the assets are often subjected to the randomness due to different kind of uncertainties, such as market risk, credit risk and etc., investors attempt to make more efficient choices. If there are two assets with same return, the one with lower variance is considered a better choice. Based on this concept, minimizing the variance of the portfolio is one of the investment strategies.

More precisely, consider J risky assets, whose rates of return are given by the random variable $\boldsymbol{x} = [x_1, x_2, ..., x_J]^T$. Suppose we are given N time-instants of the assets, denoted by $\boldsymbol{x}(n)$, $n = 1,2,...N$. In the Markowitz mean-variance portfolio, one seeks to construct a portfolio

$$y(n) = \sum_{j=0}^{J} w_j x_j(n) = \boldsymbol{w}^T \boldsymbol{x}(n),$$

and find the weight of wealth invested in asset j, denoted by w_j, $j = 1,2,...J$, $\sum_{j=1}^{J} w_j = 1$, which minimize the variance of the portfolio, i.e. $\text{var}(y(n))$, and reaching a target expected return $E(y(n)) = t$ at the same time.

i) Show that the expected return $E(y(n))$ of the portfolio can be written as

$$E(y(n)) = \boldsymbol{w}^T \boldsymbol{\mu}_x,$$

where $\boldsymbol{\mu}_x = E[\boldsymbol{x}(n)]$.

(1 marks)

ii) Show that the variance of the portfolio $\text{var}(y(n))$ can be written as

$$\text{var}(y(n)) = \boldsymbol{w}^T \boldsymbol{C}_{xx} \boldsymbol{w},$$

where $\boldsymbol{C}_{xx} = E[\bar{\boldsymbol{x}}(n)\bar{\boldsymbol{x}}(n)^T] = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \cdots & \sigma_{J1} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{J2} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{1J} & \sigma_{2J} & \cdots & \sigma_{JJ} \end{bmatrix}$ is the covariance of $\boldsymbol{x}(n)$.

$\bar{\boldsymbol{x}}(n) = \boldsymbol{x}(n) - \boldsymbol{\mu}_x$ is the centered return.

(0.5 marks)

iii) The mean-variance portfolio optimization problem can be written as:

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \tfrac{1}{2}\boldsymbol{w}^T \boldsymbol{C}_{xx} \boldsymbol{w} \\ \text{subject to} \quad & \boldsymbol{w}^T \boldsymbol{\mu}_x = \text{t} \quad , \\ & \boldsymbol{w}^T \boldsymbol{1}_J = 1 \end{aligned}$$

where $\boldsymbol{1}_J$ is a $J \times 1$ vector of ones. By differentiating the Lagrangian $L(\boldsymbol{w}, \lambda_1, \lambda_2) = \frac{1}{2}\boldsymbol{w}^T \boldsymbol{C}_{xx} \boldsymbol{w} - \lambda_1(\boldsymbol{w}^T \boldsymbol{1}_J - 1) - \lambda_2(\boldsymbol{w}^T \boldsymbol{\mu}_x - t)$ w.r.t. w , the Lagrange multipliers λ_1, λ_2 and set the gradient to zero. Show that the optimum weight vector $\boldsymbol{w}^*$ that minimizes the variance and reaching the target return is

$$\boldsymbol{w}^* = \boldsymbol{C}_{xx}^{-1}(\lambda_1 \boldsymbol{1}_J + \lambda_2 \boldsymbol{\mu}_x).$$

(1 marks)

iii) (*) By substituting $\boldsymbol{w}^* = \boldsymbol{C}_{xx}^{-1}(\lambda_1 \boldsymbol{1}_J + \lambda_2 \boldsymbol{\mu}_x)$ into the constraints $\boldsymbol{w}^T \boldsymbol{\mu}_x = \mathrm{t}$ and $\boldsymbol{w}^T \boldsymbol{1}_J = 1$, show that the optimal Lagrange multipliers are

$$\lambda_1^* = \frac{c - bt}{ac - b^2} \text{ and } \lambda_2^* = \frac{at - b}{ac - b^2},$$

where $a = \boldsymbol{1}_J^T \boldsymbol{C}_{xx}^{-1} \boldsymbol{1}_J$, $b = \boldsymbol{1}_J^T \boldsymbol{C}_{xx}^{-1} \boldsymbol{\mu}_J$ and $c = \boldsymbol{\mu}_J^T \boldsymbol{C}_{xx}^{-1} \boldsymbol{\mu}_J$.

(0.5 marks)

Part 2. MATLAB Implementation of Markowitz mean-variance portfolio

b) Run the MATLAB® code "Ass2.m". It contains three parts:

1. 200 time-instants of the return of 10 assets are randomly generated. They are assumed to be independent of each other and are normally distributed.

2. The autoregressive moving average and general autoregressive conditional heteroscedasticity (ARMA(P,Q)-GARCH(R,S)) is a time series model in which the mean and variance of the assets are modelled by the ARMA and the GARCH processes respectively. The mean and variance of the assets in future can be predicted using the ARMA-GARCH. The parameters P,Q, R and S are the model orders. In this assignment, ARMA(1,1)-GARCH(1,1) is used to predict the 1-step ahead mean and covariance return of the assets, denoted as "MFpred" and "VFpred" respectively. (The assets are assumed independent of each other). The code is adopted from the ARMAX-GARCH-K Toolbox provided by Mr. Alexandros Gabrielsen.

3. The optimal $\boldsymbol{w}^*$ is computed for expected returns ranging from $t = -0.1$ to $t = 0.1$ using "MFpred" and "VFpred".

i.) The code "Ass2.m" will plot a graph that shows the expected return against the standard deviation of the portfolio using "Ass2.m". From the graph, find the best expected return μ_1 when the standard deviation of the portfolio is $\sigma_1 = 10^{-3}$. Similarly, find the best expected return μ_2 when $\sigma_2 = 1.6 \times 10^{-3}$.

(1 marks)

ii.) By comparing the slopes of the curve at (μ_1, σ_1) and (μ_2, σ_2), suggest whether (μ_1, σ_1) or (μ_2, σ_2) gives a better portfolio. (You may draw a tangent to the curve by hand as illustrated in Fig. 1 to find the slopes at the two locations, exact value of the slopes are not required)

(1 marks)

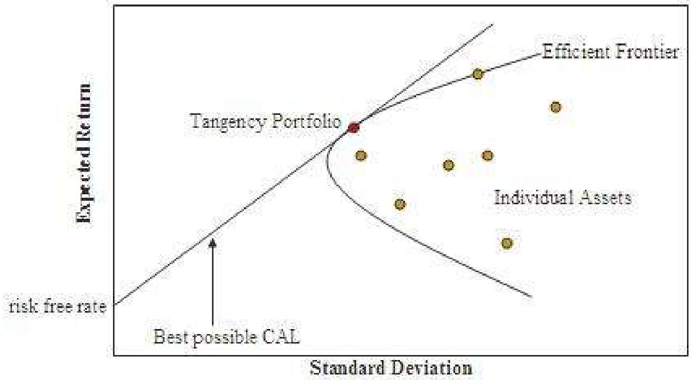

Fig. 1. An illustration of Tangency Portfolio (extracted from https://en.wikipedia.org/wiki/Efficient_frontier).

*** END ***