FILTER DESIGN TECHNIQUES

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FOURIER TRANSFORM (Appendix A)

The Fourier transform is the extension of Fourier Series to *nonperiodic signals*

$$\Im[x(t)] = X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \qquad \text{(Fourier Transform)} \tag{A.1-a}$$

The inverse Fourier transform of *X*(*f*) is

$$\Im^{-1}[X(f)] = x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \tag{A.1-b}$$

The Fourier transform of a signal is called the *spectrum of the signal* and it is in general a complex function of the frequency *f*.

Properties

1. If $x(t)$ is a **real-valued signal**, then $X(f)$ satisfies the **Hermitian symmetry**:

$$X(-f) = X^*(f) \qquad \text{(A.2)}$$

2. **Duality:**

$$\Im[X(t)] = x(-f) \qquad \text{(A.3)}$$

3. **Modulation:** Multiplication by an exponential in the time domain corresponds to a frequency shift in the frequency domain

$$\Im[e^{j2\pi f_0 t} x(t)] = X(f - f_0) \qquad \text{(A.4)}$$

$$\Im[x(t)\cos(2\pi f_0 t)] = \frac{1}{2}[X(f - f_0) + X(f + f_0)]$$

4. **Convolution: Convolution in the time domain is equivalent to multiplication in the frequency domain, and vice versa.**

If $\Im[x(t)] = X(f)$ **and** $\Im[y(t)] = Y(f)$**, then** **(A.5)**

$$\Im[x(t) * y(t)] = X(f)Y(f)$$
$$\Im[x(t)y(t)] = X(f) * Y(f).$$

5. **Parseval's relation:**

If $\Im[x(t)] = X(f)$ **and** $\Im[y(t)] = Y(f)$**, then** **(A.6)**

$$\int_{-\infty}^{+\infty} x(t)y^*(t)dt = \int_{-\infty}^{+\infty} X(f)y^*(f)df$$
$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df.$$

Table 1.1 Table of Fourier transform pairs

$x(t)$	$X(f)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f t_0}$	$\delta(f-f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\delta(f-f_0)-\frac{1}{2j}\delta(f+f_0)$
$\Pi(t)$	$\text{sinc}(f)$
$\text{sinc}(t)$	$\Pi(f)$
$\Lambda(t)$	$\text{sinc}^2(f)$
$\text{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t}u_{-1}(t), \quad \alpha>0$	$\frac{1}{\alpha+j2\pi f}$
$te^{-\alpha t}u_{-1}(t), \quad \alpha>0$	$\frac{1}{(\alpha+j2\pi f)^2}$
$e^{-\alpha\|t\|}, \quad \alpha>0$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\text{sgn}(t)$	$\frac{1}{j\pi f}$
$u_{-1}(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)$

Sampling Theorem (Appendix B)

B.1 Periodic Sampling

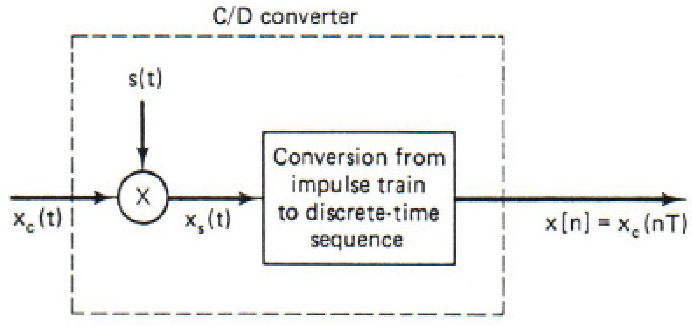
Typical method of obtaining a discrete-time representation of a continuous-time signal, $x_c(t)$, is through periodic sampling:

$$x[n] = x_c(nT), \; -\infty < n < \infty \qquad \textbf{(B.1)}$$

where

T **is the sampling period,**

$f_s = \frac{1}{T}$ **is the sampling frequency, in samples per second.**



(a)

Mathematical Model of Sampling

The sampled data waveform $x_s(t)$ is an impulse train, which zero except at integer multiples of T where the amplitude of the impulse (unit area) is equal to the sampled data.

The sampled value of $x_c(t)$ is represented in $x[n]$.

It is indexed by the integer variable n which contains no info. on the sampling rate and lead to time normalization.

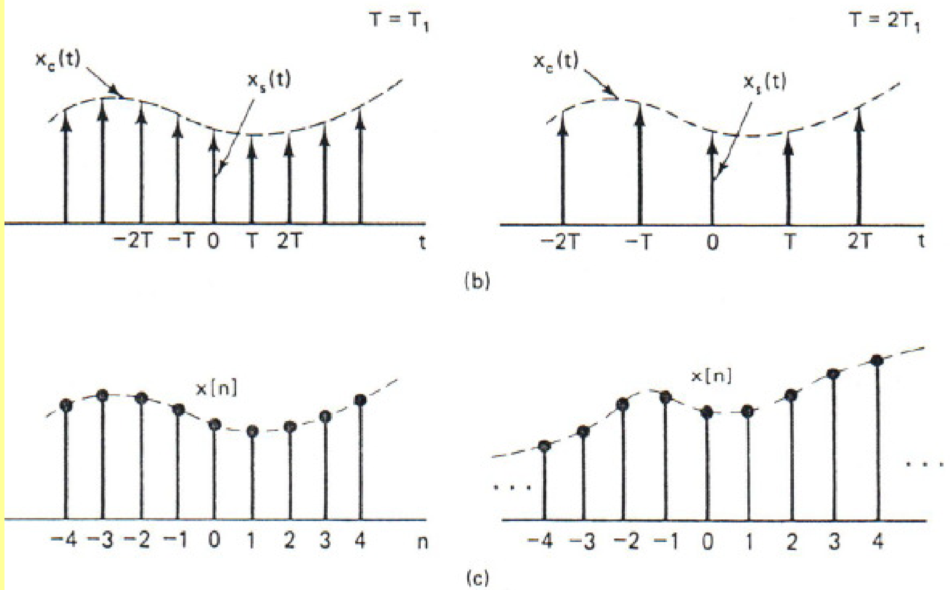


Figure 3.2 Sampling with a periodic impulse train followed by conversion to a discrete-time sequence. (a) Overall system. (b) $x_s(t)$ for two sampling rates. The dashed envelope represents $x_c(t)$. (c) The output sequence for the two different sampling rates.

B.2 Frequency-Domain Representation of Sampling

The modulating signal $s(t)$ is a periodic impulse train

$$s(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT) \qquad \text{(B.2)}$$

where $\delta(t)$ is the Dirac delta function.

Consequently,

$$x_s(t)=x_c(t)s(t)=x_c(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT)=\sum_{n=-\infty}^{+\infty}x(nT)\delta(t-nT). \qquad \text{(B.3)}$$

The Fourier transform of $x_s(t)$ $(X_s(j\Omega))$ is the convolution of the Fourier transform of $x_c(t)$ $(X_c(j\Omega))$ and $s(t)$ $(S(j\Omega))$.

$$X_s(j\Omega)=\frac{1}{2\pi}X_c(j\Omega)*S(j\Omega) \qquad \text{(B.4)}$$

where * denotes the operation of frequency domnain convolution.

Since

$$S(j\Omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_s)$$

where $\Omega_s = 2\pi/T$ is the sampling frequency in radian/s, it follows that

$$X_s(j\Omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j\Omega-kj\Omega_s). \tag{B.5}$$

Hence, the Fourier transform of $x_s(t)$ consists of the sum of periodically shifted copies of $X_c(j\Omega)$ by integer multiples of the sampling frequency.

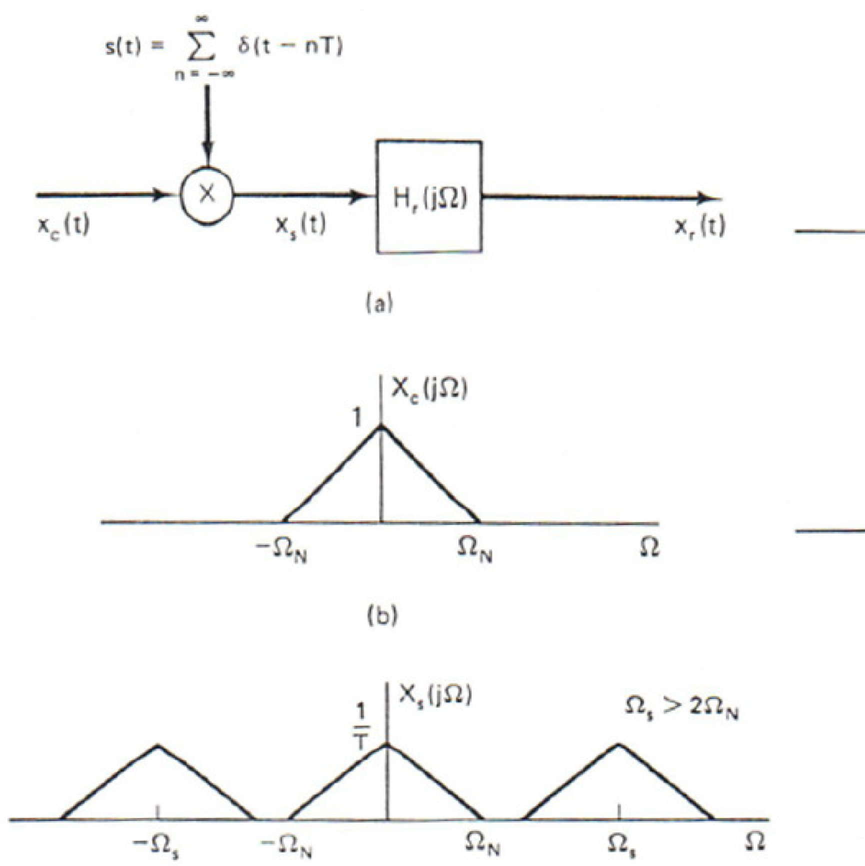


Figure 3.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.

Suppose that $x_c(t)$ is bandlimited with highest nonzero frequency component Ω_N, i.e.

$$X_c(j\Omega) = 0, \ |\Omega| > \Omega_N \quad \textbf{(B.6)}$$

When

$$|\Omega_s| > 2\Omega_N \tag{B.7}$$

$x_c(t)$ **can be recovered from** $x_s(t)$ **by passing it through an ideal lowpass filter (LPF) with frequency response:**

$$H_r(j\Omega) = \begin{cases} T & |\Omega| < \Omega_s \\ 0 & otherwise \end{cases} \tag{B.8}$$

and

$$X_c(j\Omega) = H_r(j\Omega) X_s(j\Omega) \tag{B.9}$$

If the inequality (B.7) does not hold, i.e., if $|\Omega_s| \le 2\Omega_N$**, the copies of** $X_c(j\Omega)$ **overlap so that when they are added together,** $X_c(j\Omega)$ **cannot be recovered by lowpass filtering and is referred to as aliasing.**

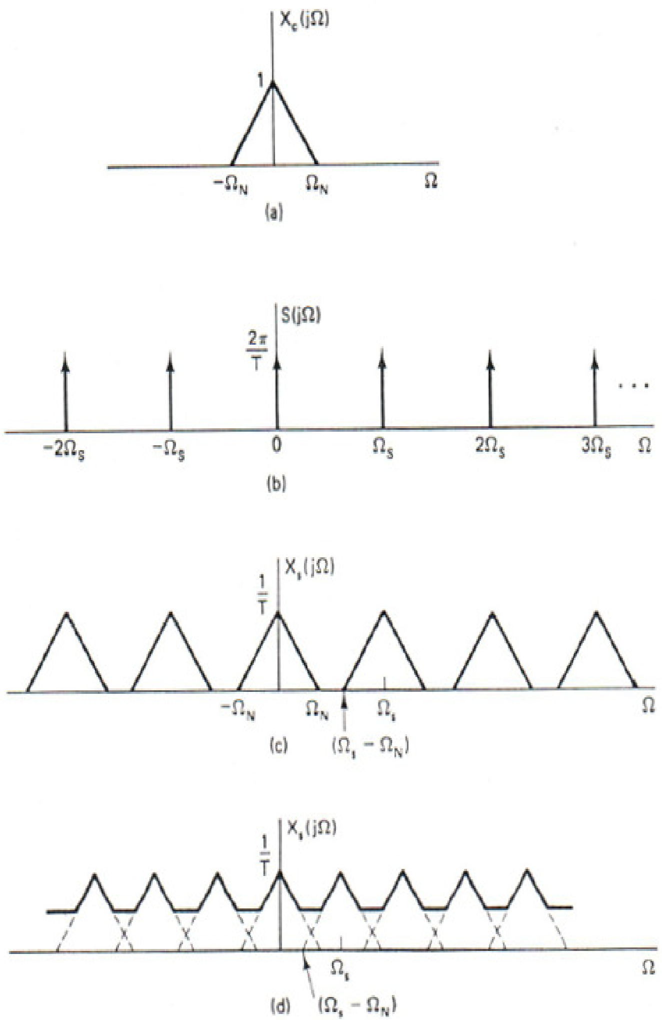


Figure 3.3 Effect in the frequency domain of sampling in the time domain. (a) Spectrum of the original signal. (b) Spectrum of sampling function. (c) Spectrum of sampled signal with $\Omega_S > 2\Omega_N$. (d) Spectrum of sampled signal with $\Omega_S < 2\Omega_N$.

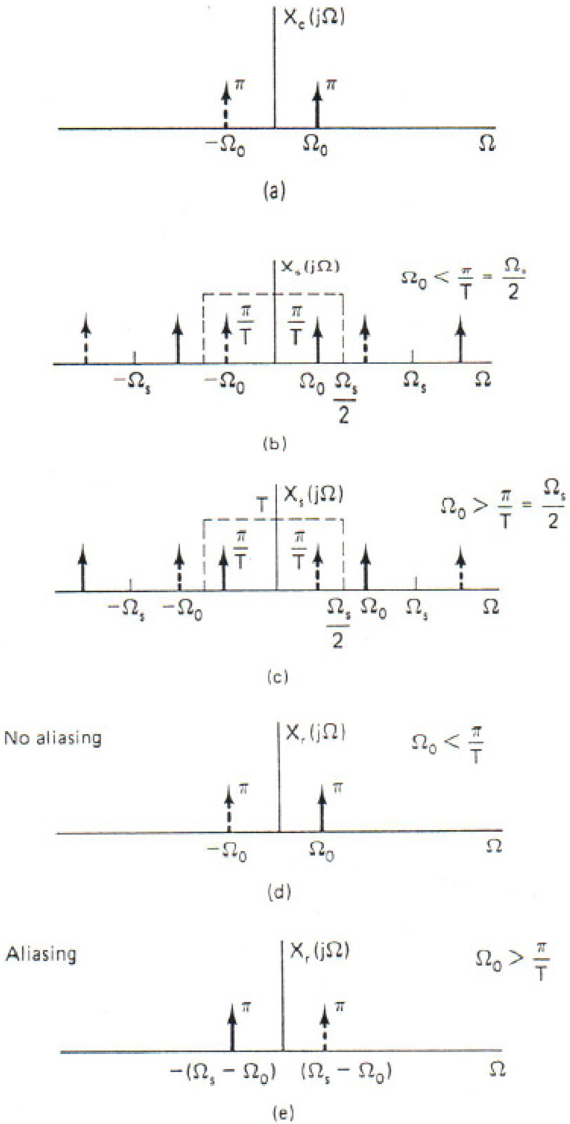


Figure 3.5 The effect of aliasing in the sampling of a cosine signal.

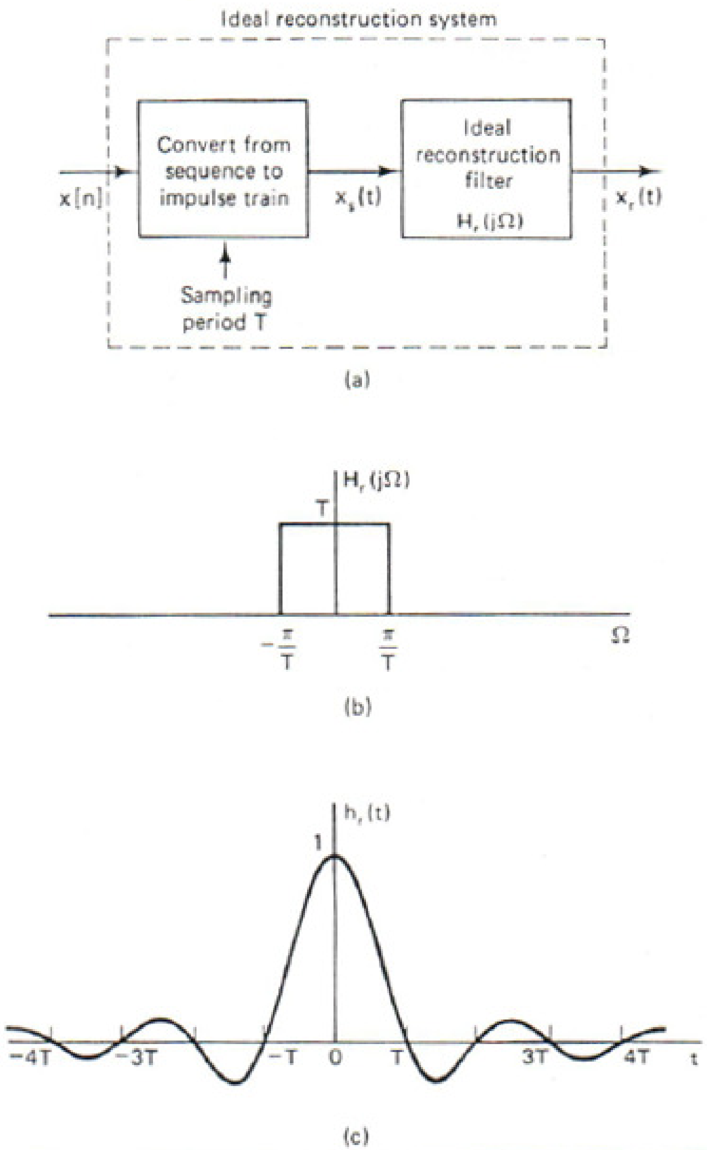


Figure 3.6 (a) Block diagram of an ideal bandlimited signal reconstruction system. (b) Frequency response of an ideal reconstruction filter. (c) Impulse response of an ideal reconstruction filter.

Nyquist Sampling Theorem

Let $x_c(t)$ be a bandlimited signal with

$$X_c(j\Omega) = 0 \quad \text{for } |\Omega| > \Omega_N \tag{B.10}$$

Then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT)$, $n = 0, \pm 1, \pm 2, \ldots$ if

$$\Omega_S = 2\pi / T > 2\Omega_N. \tag{B.11}$$

Ω_N is called the Nyquist frequency, and the frequency $2\Omega_N$ that must be exceeded by the sampling frequency is called the Nyquist rate.

Next we established the relationship between the discrete-time and continuous-time Fourier transform.

Applying the Fourier transform to $x_s(t)$ in (B.3) gives:

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\exp(-j\Omega nT). \qquad \textbf{(B.12)}$$

Since

$$x[n] = x_c(nT) \qquad \textbf{(B.13)}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\exp(-j\omega n), \text{ (DT-FT)} \qquad \textbf{(B.14)}$$

it follows that

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T}). \qquad \textbf{(B.15)}$$

From (B.5) and (B.15), we have

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(-j\Omega - jk\Omega_s) \qquad \textbf{(B.16)}$$

$$\Leftrightarrow \quad X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(-j\frac{\omega}{T} - j2\pi\frac{k}{T}). \qquad \textbf{(B.17)}$$

$X(e^{j\omega})$ **is simply a frequency-scaled version of** $X_s(j\Omega)$ **with** $\omega = \Omega T$**.**

B.3 Reconstruction of a Bandlimited Signal from its Samples

If the conditions of the sampling theorem are met, the original signal can be reconstructed by passing the modulated impulse train through an appropriate lowpass filter (LPF).

If the continuous-time filter has frequency response, $H_r(j\Omega)$, and impulse response, $h_r(t)$, then the output of the filter (CT-convolution) will be

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT). \qquad \textbf{(B.18)}$$

The ideal LPF has a gain *T* and cutoff frequency Ω_c with impulse response

$$h_r(t) = \frac{\sin \pi t / T}{\pi t / T} \qquad \textbf{(B.19)}$$

It then follows that

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T}. \qquad \textbf{(B.20)}$$

The LPF interpolates between the impulses of $x_s(t)$ to construct continuous-time (CT) signal, $x_c(t)$.

In terms of the discrete-time Fourier transform, the interpolation process reads:

$$X_r(j\Omega) = H_r(j\Omega)X(e^{j\Omega T}). \tag{B.21}$$

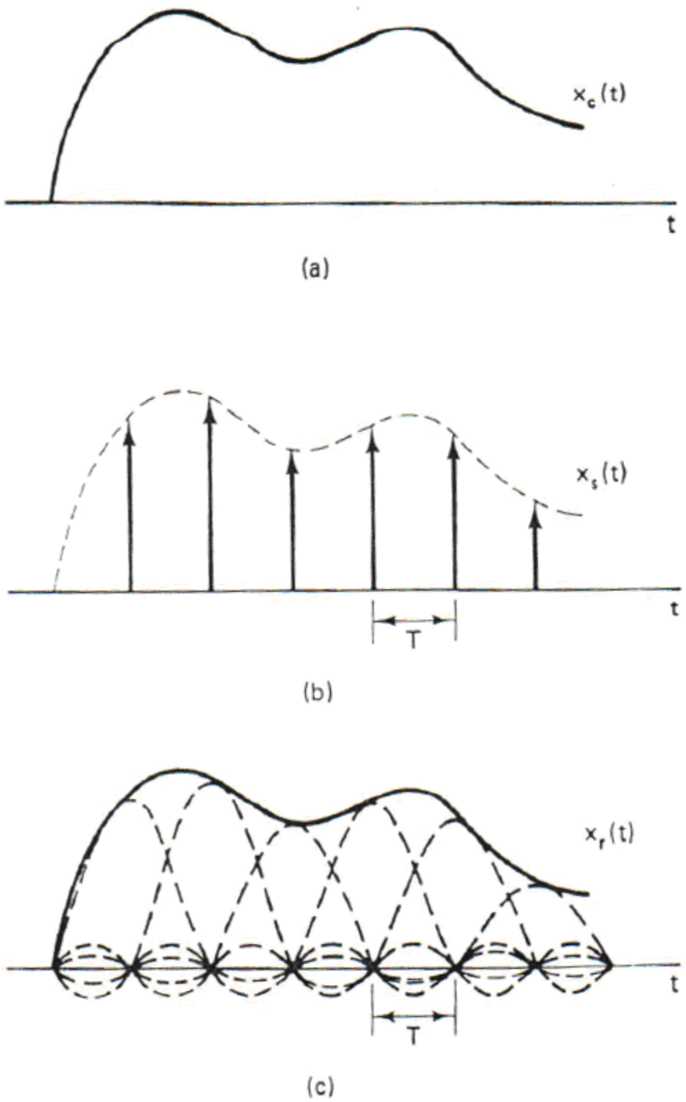


Figure 3.7 Ideal bandlimited interpolation.

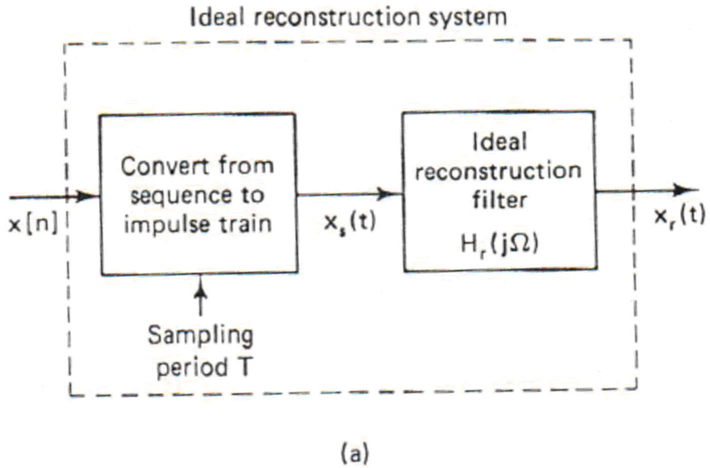
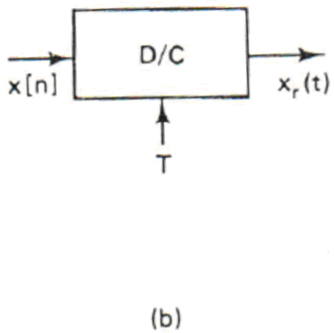


Figure 3.8 (a) Ideal bandlimited signal reconstruction.



(b) Equivalent representation as an ideal D/C converter.

B.4 Discrete-Time Processing of Continuous-Time Signals

A major application of discrete-time systems is in the processing of continuous-time signals.

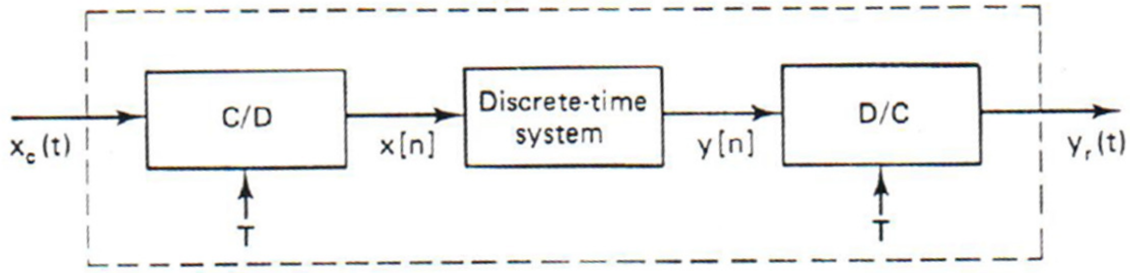


Figure 3.9 Discrete-time processing of continuous-time signals.

The C/D converter produces a discrete-time signal

$$x[n] = x_c(nT), \tag{B.22}$$

and from (B.17),

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right). \tag{B.23}$$

The D/C converter creates a CT output signal of the form:

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin\pi(t-nT)/T}{\pi(t-nT)/T}. \quad \text{(B.24)}$$

The continuous-time and discrete-time Fourier transform of $y_r(t)$ and $y[n]$ are related by (using (B.21))

$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T}). \quad \text{(B.25)}$$

If $y[n] = x[n]$, we have the identity system in which $x_c(t) = y_r(t)$.

If the discrete-time system is LTI with frequency response, $H(e^{j\omega})$, we then have

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}). \quad \text{(B.26)}$$

10. Structure of a digital signal processing system

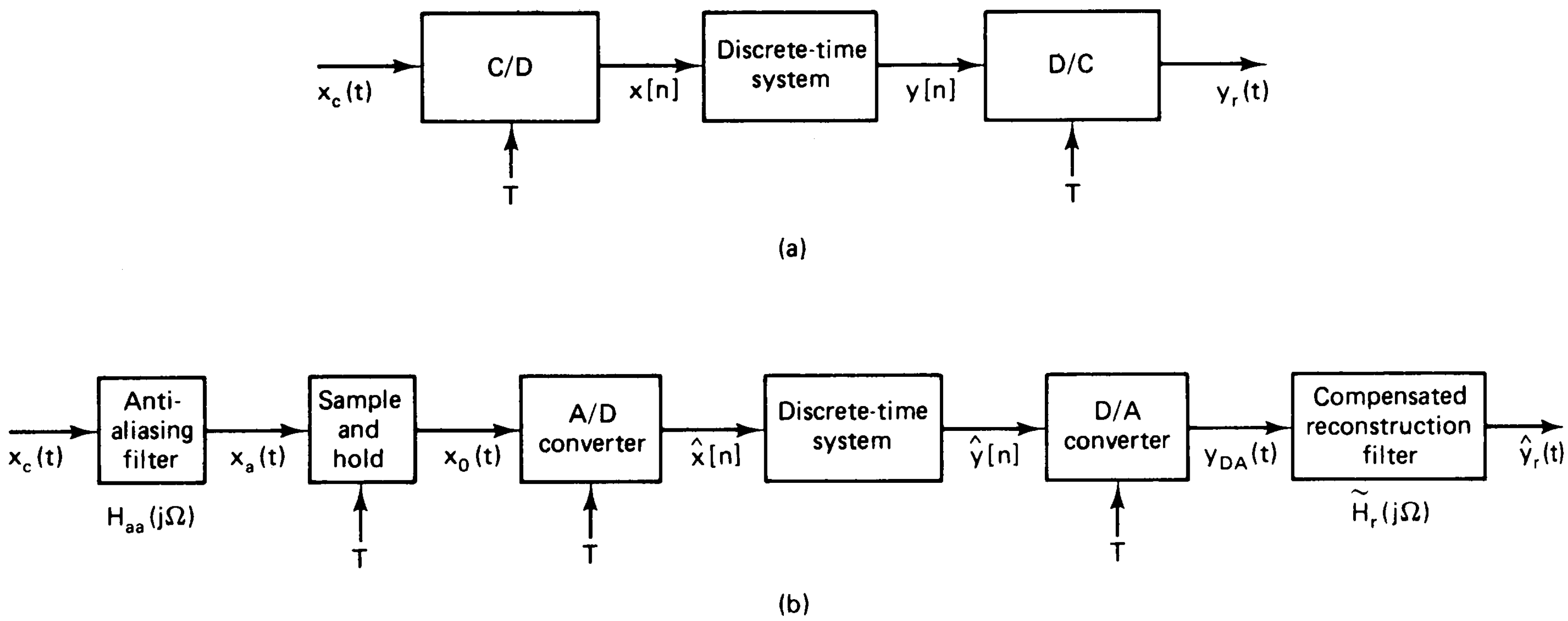


Fig. 10.1 (Fig. 3.26 in Oppenheim's book)

Relationship between continuous-time Fourier transform and DTFT:

$$x[n] = x_c(nT).$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \qquad \textbf{(10.1)}$$

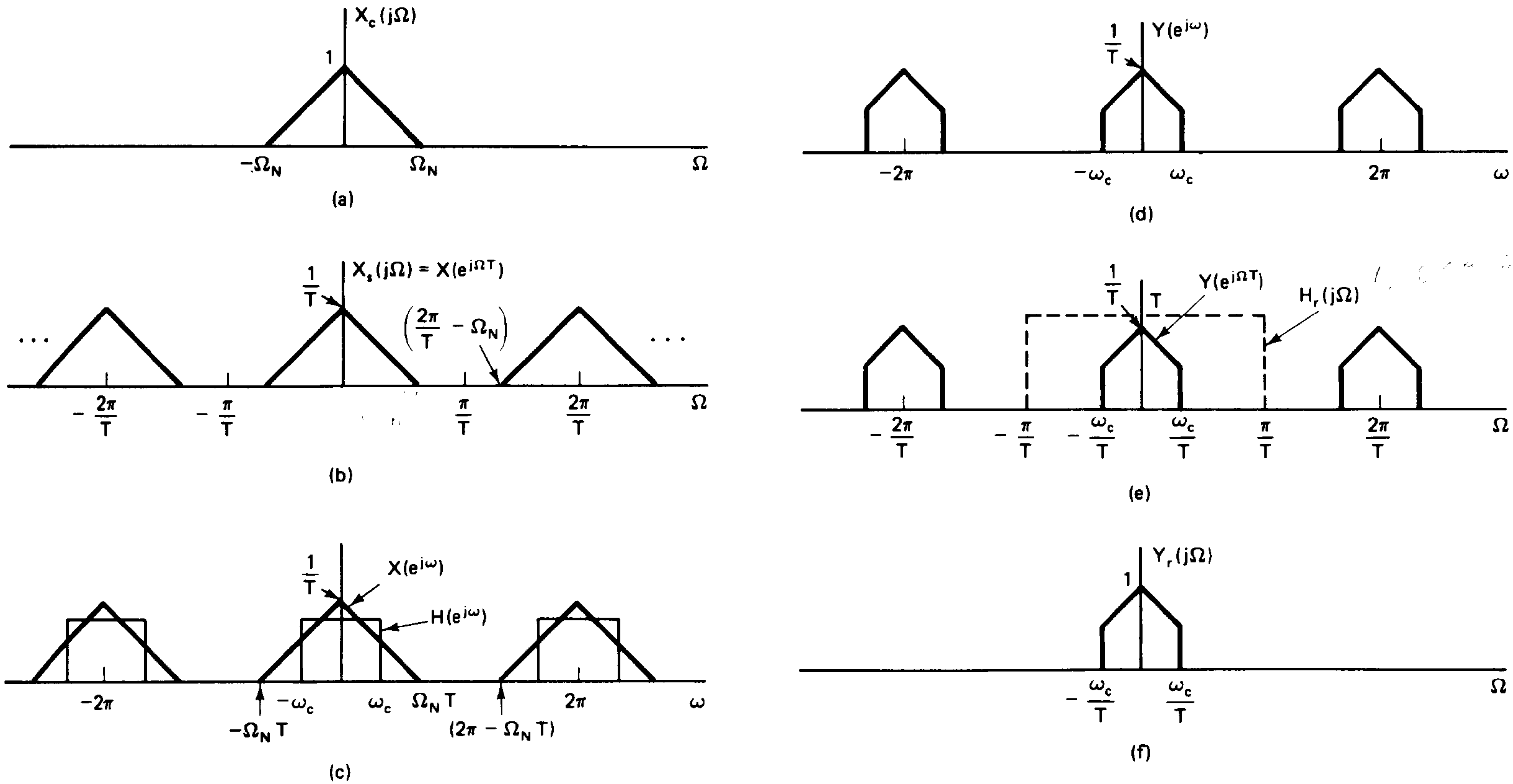


Figure 3.11 (a) Fourier transform of a bandlimited input signal. (b) Fourier transform of sampled input plotted as a function of continuous-time frequency Ω. (c) Fourier transform $X(e^{j\omega})$ of sequence of samples and frequency response $H(e^{j\omega})$ of discrete-time system plotted vs. ω. (d) Fourier transform of output of discrete-time system. (e) Fourier transform of output of discrete-time system and frequency response of ideal reconstruction filter plotted vs. Ω. (f) Fourier transform of output.

Signal reconstruction:

$$y_r[n] = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}.$$

$$Y_r(j\Omega) = H_r(j\Omega) Y(e^{i\Omega T}). \qquad (10.2)$$

After passing through a LTI filter with $H(e^{j\omega})$, the DT-FT of output y[n] is

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}). \qquad \textbf{(10.3)}$$

From (10.2) and (10.3), the continuous-time Fourier transform of output *y*(*t*) is

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}) \qquad \textbf{(10.4)}$$

If sampling theorem is satisfied (i.e. $X_c(j\Omega) = 0, \text{for} \, |\Omega| \geq \pi/T$), then

$$H_r(j\Omega)X(e^{j\Omega T}) = X_c(j\Omega)$$

and (10.4) becomes

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega) & |\Omega| < \pi/T, \\ 0 & |\Omega| \geq \pi/T. \end{cases}.$$

Thus, the equivalent analog filter of $H(e^{j\omega})$ is

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \pi/T, \\ 0 & |\Omega| \geq \pi/T. \end{cases}. \qquad \textbf{(10.5)}$$

Filter specifications in continuous and discrete-time domains

(Example 7.1 in Oppenhiem's book)

Consider a discrete-time system that is to be used lowpass filter a continuous-time signal using the basic configuration in Fig. 10.1 (a). Suppose that the sampling rate is 10^4 samples/sec (T=10^{-4} sec).

(What is the cutoff frequency of the ideal anti-aliasing filter? What is the maximum operating frequency without aliasing?)

The specifications are:

1. The gain $|H_{eff}(j\Omega)|$ should be within ± 0.01 $(0.086\,\mathrm{dB})$ of unity (zero dB) in the frequency band $0 \le \Omega \le 2\pi(2000)$.
2. The gain should be no greater than ± 0.001 $(\text{-}60\,\mathrm{dB})$ in the frequency band $2\pi(3000) \le \Omega$.

This is illustrated in the following figure. The parameters are

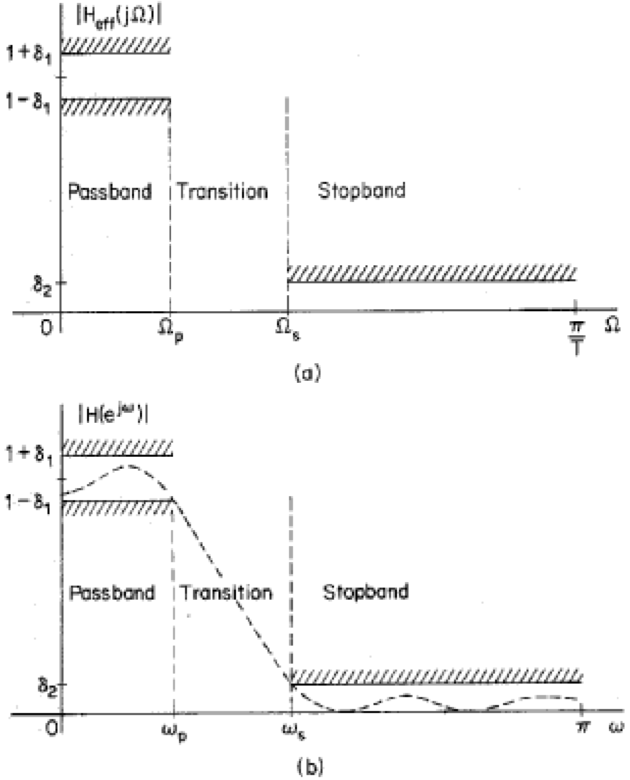


Figure 7.2 (a) Specifications for effective frequency response of overall system in Fig. 7.1 for the case of lowpass filter. (b) Corresponding specifications for the discrete-time system in Fig. 7.1.

$\delta_1 = 0.01(20\log_{10}(1+\delta_1) = 0.086\ \ dB)$ **(passband ripple);**

$\delta_2 = 0.001(20\log_{10}\delta_2 = -60\ \ dB)$ **(stopband ripple);**

$\Omega_p = 2\pi(2000)$ **(passband cutoff frequency);**

$\Omega_s = 2\pi(3000)$ **(stopband cutoff frequency)**

Because of (10.5), the equivalent specifications in the digital domain are:

1. $(1-\delta_1) \le |H(e^{j\omega})| \le (1+\delta_1) \quad |\omega| \le \omega_p$,
2. $|H(e^{j\omega})| \le \delta_2 \quad \omega_s \le |\omega| \le \pi$,

Since the sampling period is $T = 10^{-4}$ **sec., we have**

$$\omega_p = \Omega_p \cdot T = 2\pi(2000)\cdot 10^{-4} = 0.4\pi \text{ radians,}$$

and $\omega_s = \Omega_s \cdot T = 2\pi(3000)\cdot 10^{-4} = 0.6\pi$ **radians.**

The transition bandwidth $\Delta\omega = \omega_s - \omega_p = 0.6\pi - 0.4\pi = 0.2\pi$.

FIR Vs IIR Filters

FIR Filters	IIR Filters
Contains only zeros (always stable)	Contain zeros and poles
Can have exact linear phase response -> No phase distortion	Phase response is usually non-linear, especially at band edges
Quantization effects are less severe	After coefficient quantization, a pole can move to outside of the unit circle (cause instability)
Sharp cutoff and low passband delay requires long filter length.	Potentially lower filter order for low passband delay and sharp cutoff.
Optimal design using Parks-McClellan algorithm or SOCP (linear and passband linear phase).	Require nonlinear optimization. Perform model order reduction (MOR) of a long optimal FIR filter.

11. Ideal frequency-selective filters

The frequency response of an **ideal lowpass filter** is

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}, \qquad (11.1)$$

where ω_c is the cutoff frequency. Frequencies components below ω_c pass through the filter without any distortion, while those above are suppressed. In practice, we can only approximate (11.1).

- From the inverse DT-FT, the corresponding impulse response is

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{lp}(e^{j\omega})e^{jn\omega}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{jn\omega}d\omega$$

$$= \frac{1}{2\pi jn}[e^{jn\omega_c} - e^{-jn\omega_c}] = \frac{\sin(n\omega_c)}{\pi n}. \quad (11.2)$$

Its impulse response extends from $-\infty$ to $+\infty$ and the system is **not computationally realizable**. The phase response is zero.

Linear Phase filters

Shifting theorem:

$$\Im[x(t-\alpha)] = \int_{-\infty}^{\infty} x(t-\alpha)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi(\tau+\alpha)}d\tau$$

$$= e^{-j2\pi f\alpha}\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi\tau}d\tau = e^{-j2\pi f\alpha} \times X_c(f).$$

The Fourier transform of a signal with a time shifting of α is equal to the multiplication of its Fourier transform by $e^{-j2\pi f\alpha}$.

The factor $e^{-j2\pi f\alpha}$ has a unit magnitude and its phase is $-2\pi f\alpha = -\Omega\alpha$, which is a linear function of Ω.

Recall the relationship between the DT-FT of the sample $x(nT_s) = x[n]$ and the FT of $x(t)$. $\left(X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)\right)$

Assuming the sampling theorem is satisfied, the DT-FT of $x_\alpha(t) = x(t-\alpha)$ is

$$X_\alpha(e^{j\omega}) = e^{-j\omega\alpha/T} X_c(j\frac{\omega}{T}) = e^{-j\omega\alpha} X(j\omega) \quad , -\pi < \omega \le \pi,$$

where $\alpha' = \alpha / T$ is the normalized shift in the discrete-time domain.

Since the ideal lowpass filter in (11-1) is non-causal, we can shift the ideal impulse response to the right so that it becomes causal. The frequency response is then given by

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}, \ |\omega| < \pi$$

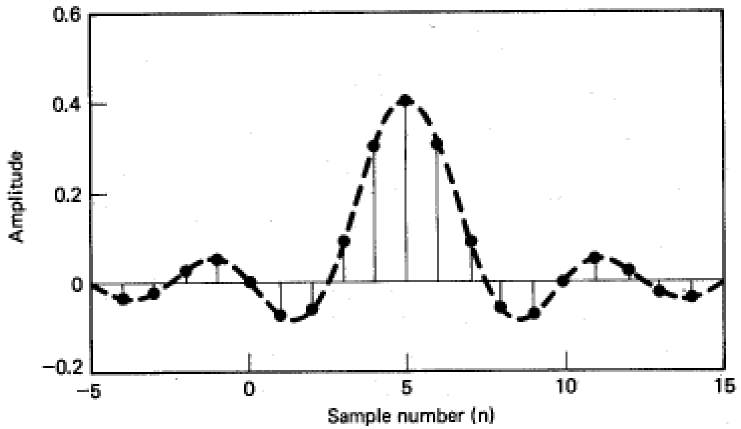
Example: The ideal lowpass filter $H_{lp}(e^{j\omega})$ has frequency response

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

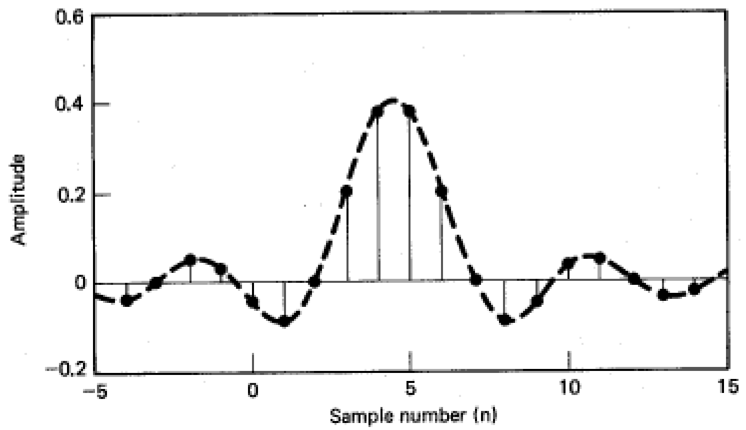
Its impulse response is $h_{lp}[n] = \dfrac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)}$. (From inverse DT-FT)

Note the time shift of the impulse response.

Symmetric and Anti-symmetric impulse responses



(a)



(b)

- Note that when α is an integer, the impulse response is symmetric about $n = n_d$.

$$h_{lp}[2n_d - n] = h_{lp}[n]$$

- If α is an integer plus one-half then

$$h_{lp}[2n_d - n] = h_{lp}[n].$$

- The point of symmetry is α which is a half integer.

- For $\alpha = 4.3$, there is no symmetry in the filter coefficients.

In general, a linear-phase system has frequency response

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}, |\omega| < \pi.$$

- For FIR filter, we can impose symmetry on the impulse response coefficient about α when

  2α is an integer (it is called half-sample symmetry).

  α is an integer (it is called full-sample symmetry).

- This is a sufficient condition for the system to have linear phase but not necessary (for $\alpha = 4.3$ above, if the coefficients are exact, it can be linear phase. Will it be true if the coefficients are quantized?)

Linear Phase FIR Filters

There are four types of FIR generalized linear-phase systems.

Type I FIR Linear Phase System

M is an even integer, symmetric impulse response

$$h[n] = h[M-n], \quad 0 \le n \le M,$$

and

$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos(\omega k)\right) \quad \textbf{[Delay } \alpha = M/2 \textbf{]}$$

where $a[0] = h\left[\frac{M}{2}\right]$ **and** $a[k] = 2h\left[\frac{M}{2} - k\right]$, $k = 1, \ldots, \frac{M}{2}$.

The proof for the rest are left as exercise.

Type II FIR Linear Phase System

M is an odd integer, symmetric impulse response

$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{(M+1)/2} b[k]\cos\left(\omega(k-\tfrac{1}{2})\right)\right)$$

where $b[k] = 2h\left[\frac{M+1}{2} - k\right]$, $k = 1,\ldots,\frac{M+1}{2}$.

Type III FIR Linear Phase System

M is an even integer, antisymmetric impulse response

$$h[n] = -h[M-n],\ \ 0 \le n \le M\ ,$$

and $H(e^{j\omega}) = je^{-j\omega M/2}\left(\sum_{k=1}^{M/2} c[k]\sin(\omega k)\right)$

where $c[k] = 2h\left[\frac{M}{2} - k\right]$, $k = 1,\ldots,\frac{M}{2}$.

Type IV FIR Linear Phase System

M is an odd integer, antisymmetric impulse response

$$H(e^{j\omega}) = je^{-j\omega M/2}\left(\sum_{k=1}^{(M+1)/2} d[k]\sin(\omega(k-\tfrac{1}{2}))\right)$$

where $d[k] = 2h\left[\dfrac{M+1}{2} - k\right]$, $k = 1,\ldots,\dfrac{M+1}{2}$.

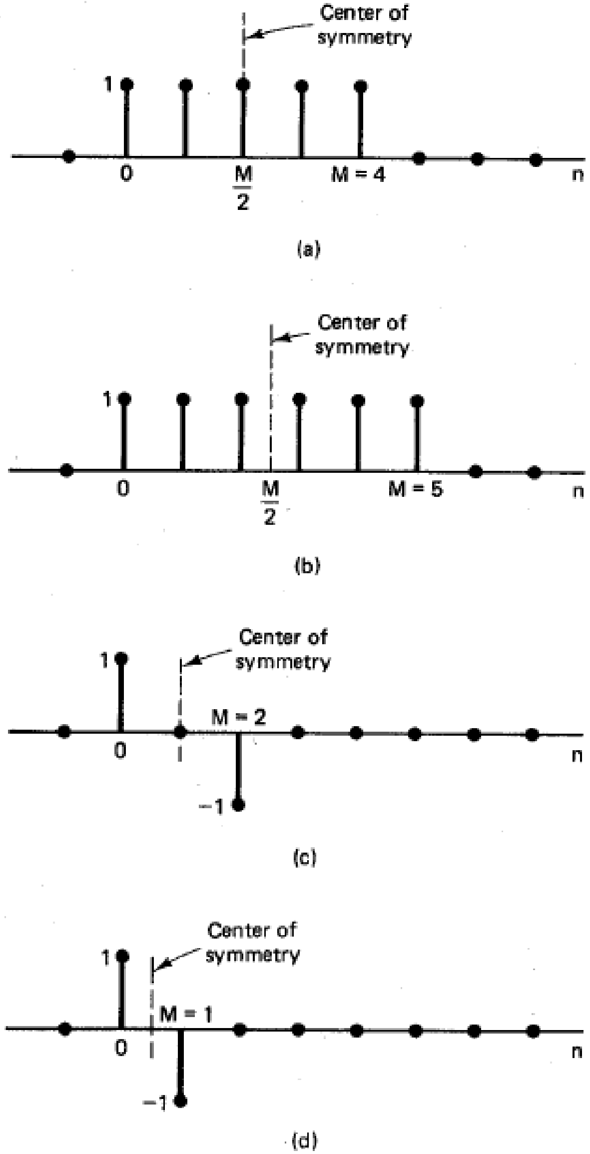


Figure 5.33 Examples of FIR linear phase systems. (a) Type I, M even, $h[n] = h[M - n]$. (b) Type II, M odd, $h[n] = h[M - n]$. (c) Type III, M even, $h[n] = -h[M - n]$. (d) Type IV, M odd, $h[n] = -h[M - n]$.

Zero Locations for FIR Linear Phase System (left as exercise)

- **For type I and II, $H(z)$ can be expressed as**

$$H(z)=\sum_{n=0}^{M}h[M-n]z^{-n}=\sum_{k=M}^{0}h[k]z^{k}z^{-M}=z^{-M}H(z^{-1})$$

If $z_0=re^{j\theta}$ is a zero of $H(z)$,then

$$H(z_0)=z_0^{-M}H(z_0^{-1})=0$$

and $z_0^{-1}=r^{-1}e^{-j\theta}$ is also a zero of $H(z)$. When $h[n]$ is real, then $z_0^*=re^{-j\theta}$ will also be a zero of $H(z)$, so will $(z_0^*)^{-1}=r^{-1}e^{j\theta}$.

When $h[n]$ is real, each complex zero not on the unit circle will be part of a set of four conjugate reciprocal zeros of the form

$$(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})(1-r^{-1}e^{j\theta}z^{-1})(1-r^{-1}e^{-j\theta}z^{-1})$$

Zeros on the unit circle come in pairs of the form

$$(1-e^{j\theta}z^{-1})(1-e^{-j\theta}z^{-1})$$

Zero at $z=\pm 1$ **can appear by itself and** $H(z)$ **may have factors**

$$(1\pm z^{-1})$$

Since

$$H(-1)=(-1)^{M}H(-1)$$

If M is odd, $z=-1$ **must be zero.**

- **For Type III and IV, we have**

$$H(z)=-z^{-M}H(z^{-1})$$

$H(z)$ **have a zero at** $z=1$ **for both M even and M odd and** $z=-1$ **is a zero of** $H(z)$ **if M is even.**

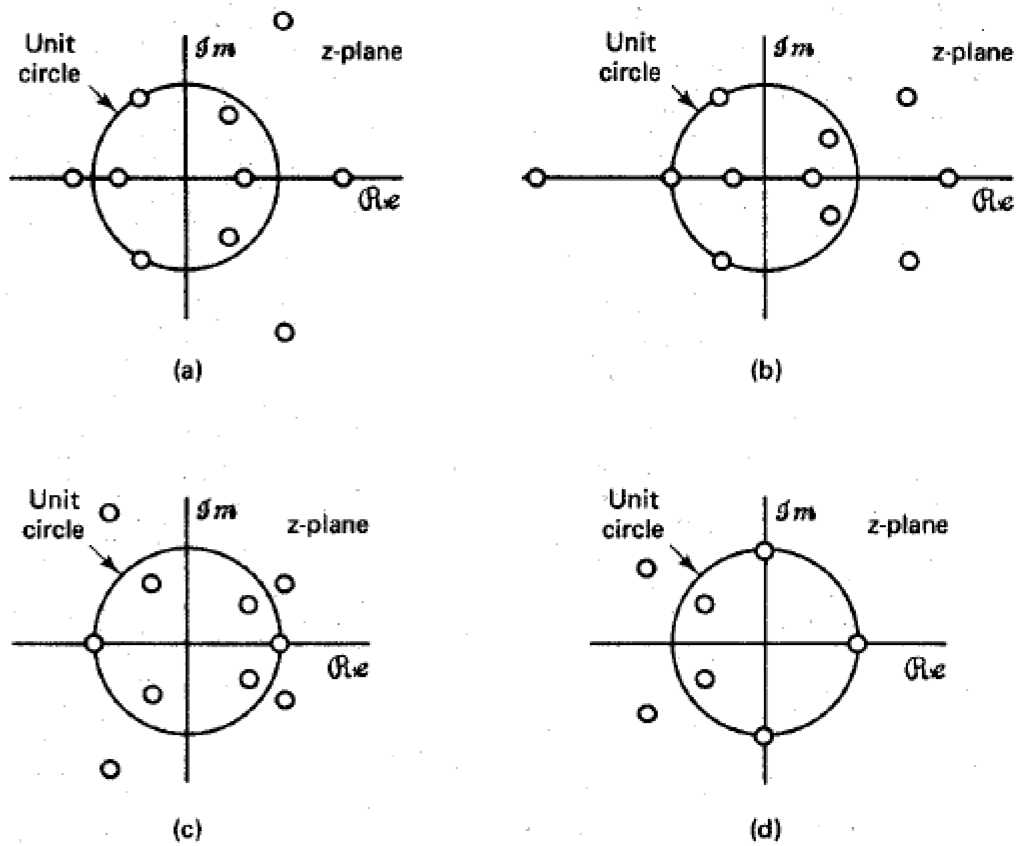
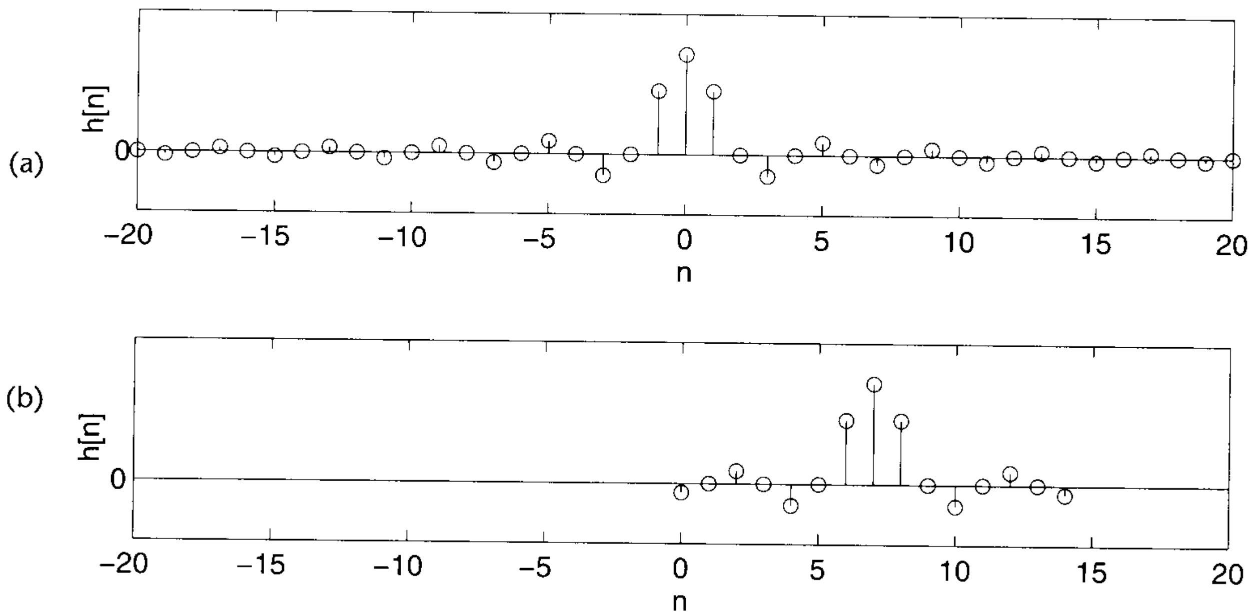


Figure 5.38 Typical zero plots for linear phase systems. (a) Type I. (b) Type II. (c) Type III. (d) Type IV.

12. Windowing method

Fig. 15.26
Impulse response of a non-recursive filter:
(a) non-causal with an infinite number of coefficients;
(b) causal with 15 coefficients



- **Note, the ideal impulse response is symmetric around n=0. In general, filters with symmetric and anti-symmetric impulse response have perfect linear-phase (i.e. no phase distortion). This is not possible for IIR filters.**
- **In windowing method, the impulse response is truncated by multiplying the ideal response by a window and shifted it to the right to make it causal (h[n]=0, n<0).**

12.1 Designing Linear phase FIR filters by windowing

The impulse response is

$$h_{lp}[n] = \frac{\sin[\omega_c(n-(M/2))]}{\pi(n-(M/2))} w[n], \quad \text{n=0,…,M,}$$

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & otherwise \end{cases}. \quad \text{(window function)} \qquad (12.1)$$

Note, the shift (M/2), or system delay, is a half-integer if M is odd and a integer if M is even.

- **A commonly used window is the Kaiser window**

$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M \\ 0, & otherwise \end{cases}. \qquad (12.2)$$

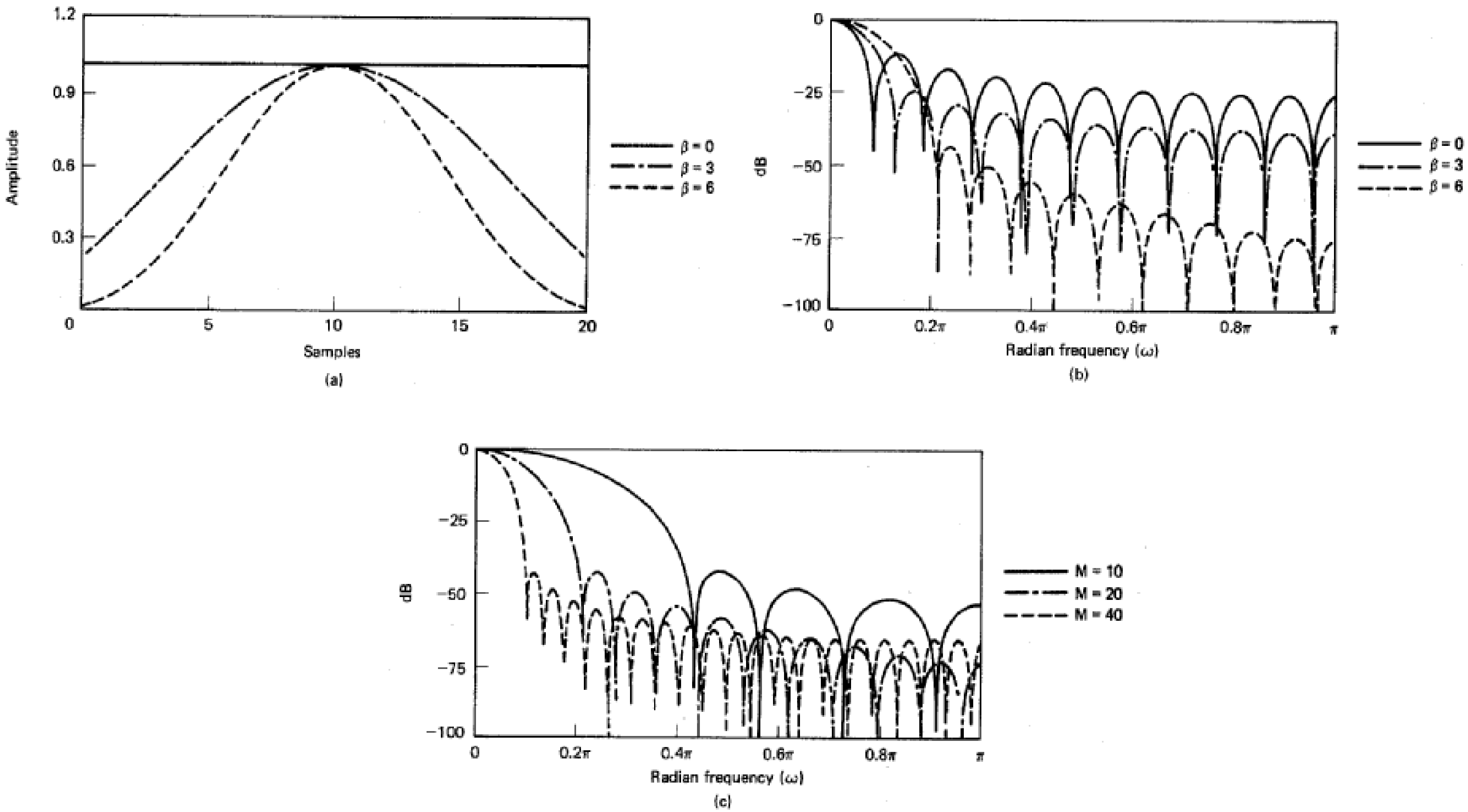


Figure 7.32 (a) Kaiser windows for $\beta = 0$, 3, and 6 and $M = 20$. (b) Fourier transforms corresponding to windows in (a). (c) Fourier transforms of Kaiser windows with $\beta = 6$ and $M = 10$, 20, and 40.

From the modulation theorem

$$h[n] = h_d[n]w[n] \quad \leftrightarrow \quad H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta,$$

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & otherwise \end{cases}. \quad w[n] \leftrightarrow W(e^{j\omega}). \tag{12.3}$$

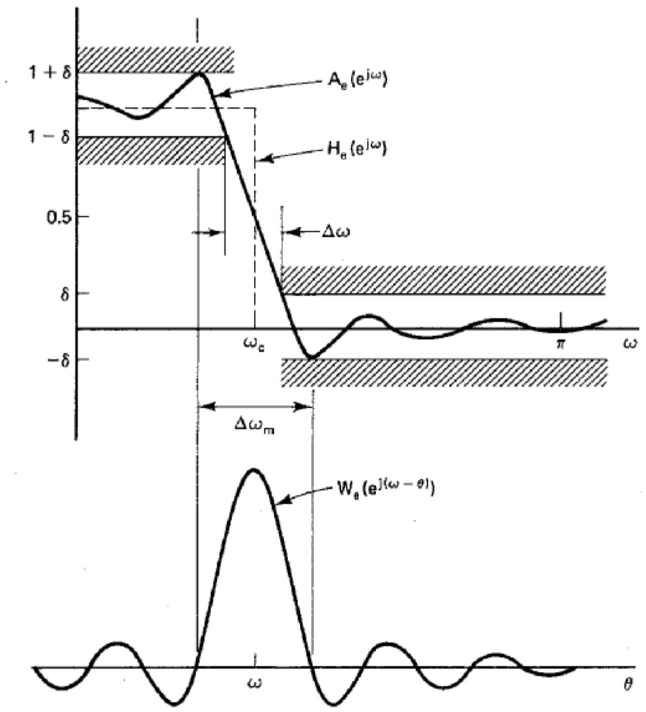


Figure 7.31 Illustration of type of approximation obtained at a discontinuity of the ideal frequency response.

ω_p : passband cutoff frequency.

ω_s : stopband cutoff frequency.

$\Delta\omega = \omega_s - \omega_p$:transition bandwidth.

δ : passband/stopband ripples

$A = -20\log_{10}\delta$ (dB) : stopband attenuation.

- **The passband and stopband ripples (stopband attenuation) are nearly identical.**
- **The transition bandwidth $\Delta\omega$ is inversely proportional to filter length.**

The parameter $\Delta\omega$ and filter length (*M+1*) can be determined empirically:

$$A = -20\log_{10}\delta \quad \text{(dB)}$$

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50, \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50, \\ 0.0 & A < 21 \end{cases} \qquad (12.4)$$

$$M = \frac{A-8}{2.285 \cdot \Delta\omega}.$$

Examples:

The specifications are

$$\omega_p = 0.4\pi, \omega_s = 0.6\pi, \delta_1 = 0.01 \text{ and } \delta_2 = 0.001.$$

Since window method inherently has $\delta_1 = \delta_2$, we must set

$$\delta = \min(\delta_1, \delta_2) = 0.001.$$

The cutoff frequency is $\omega_c = \dfrac{\omega_p + \omega_s}{2} = 0.5\pi$

The values of *M* and β are obtained from (12.4) as

$$\beta = 5.653, \quad M = 37$$

The impulse response of the filter is then given by

$$h[n] = \begin{cases} \dfrac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)} \cdot \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)} & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

(Since *M* is odd, the filter is of type II.) The peak approximation error is slightly greater than $\delta = 0.001$. Increasing M to 38 results in a type I filter for which $\delta = 0.0008$.

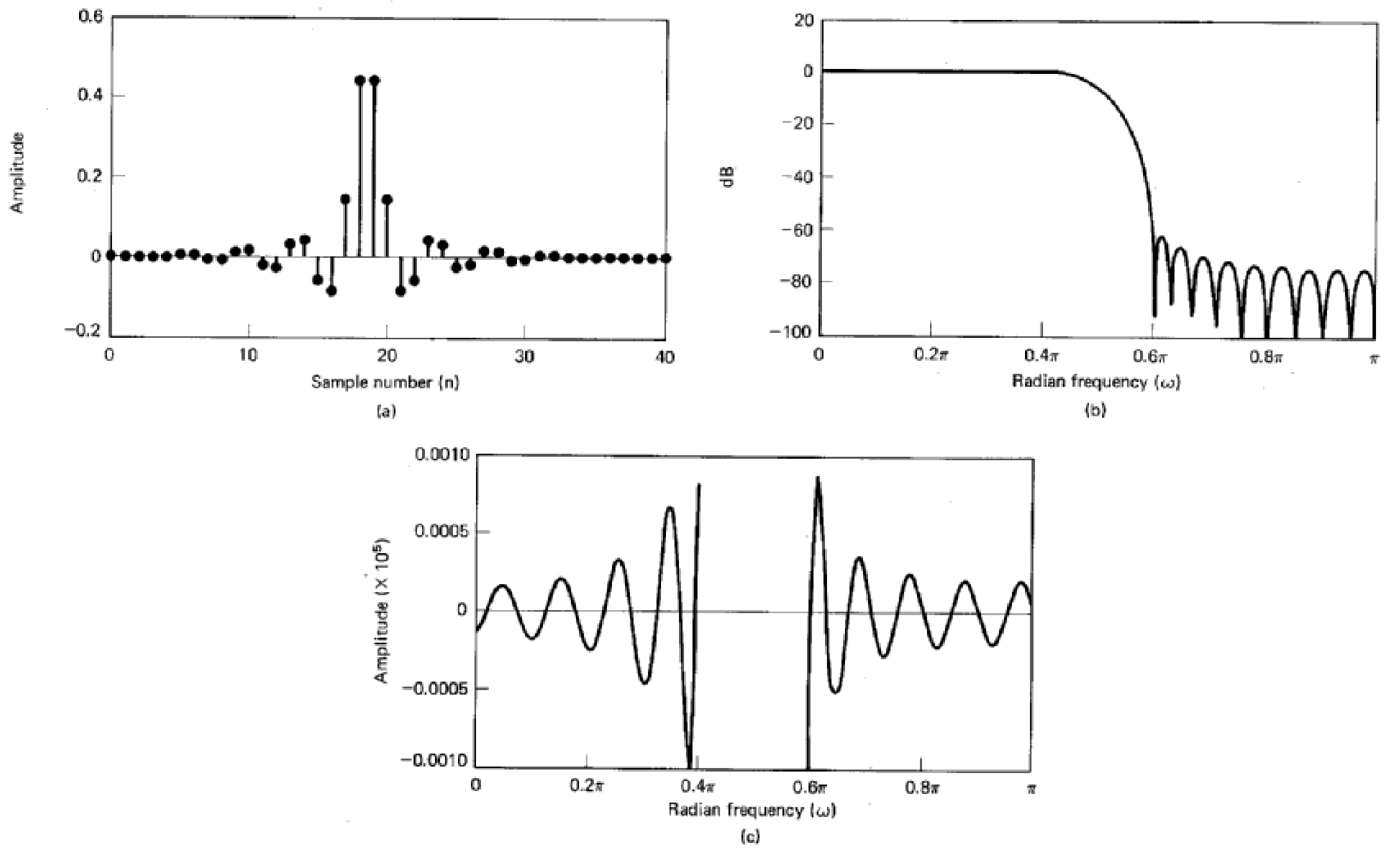


Figure 7.33 Response functions for Example 7.11. (a) Impulse response ($M = 37$). (b) Log magnitude. (c) Approximation error.

Exercise:

1. Show that the DT-FT of $h_{lp}[n] = \dfrac{\sin[\omega_c(n-(M/2))]}{\pi(n-(M/2))}$ **is given by**

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

What is the phase response of the digital filter? Is it a linear function of ω **(i.e. linear phase)?**

13. Highpass, bandpass, and bandstop filters

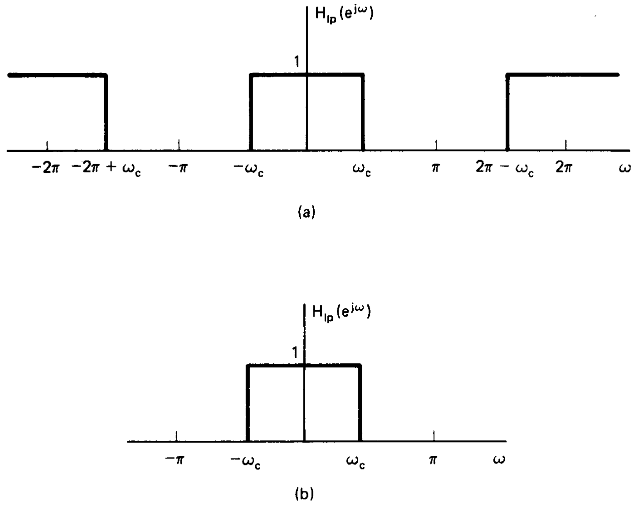


Figure 2.17 Ideal lowpass filter showing (a) periodicity of the frequency response, (b) one period of the periodic frequency response.

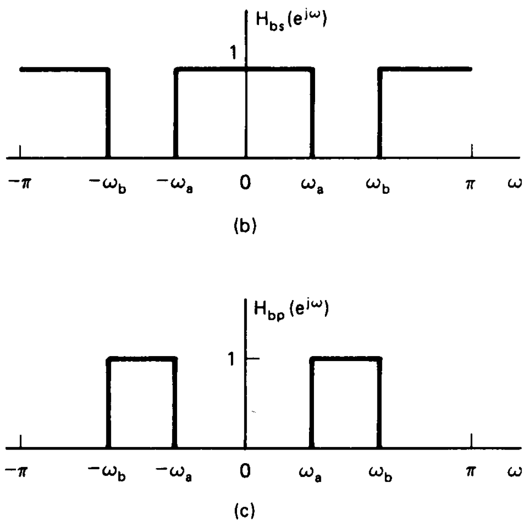


Figure 2.18 Ideal frequency-selective filters. (a) Highpass filter. (b) Bandstop filter. (c) Bandpass filter. In each case, the frequency response is periodic with period 2π. Only one period is shown.

(Lowpass filters) (Bandstop and bandpass filters)

- **Windowing method is also applicable to the design of these filters, δ in (12.4) should be the minimum ripple value in the various bands. $\Delta\omega$ in (12.4) should be the minimum transition bandwidth in the various bands.**

Highpass filter design

An ideal highpass filter with generalized linear-phase has frequency response

$$H_{hp}(e^{j\omega}) = \begin{cases} 0, & 0 \le |\omega| \le \omega_c \\ e^{-j\omega M/2}, & \omega_c \le |\omega| \le \pi \end{cases}. \qquad \textbf{(13.1)}$$

and impulse response (taking the inverse DT-FT of (5-1))

$$h_{hp}[n] = \frac{\sin \pi(n - M/2)}{\pi(n - M/2)} - \frac{\sin \omega_c (n - M/2)}{\pi(n - M/2)} \quad , -\infty < n < \infty \qquad \textbf{(13.2)}$$

Suppose that $\omega_s = 0.35\pi, \omega_p = 0.5\pi$, **and** $\delta_1 = \delta_2 = \delta = 0.021$.

Applying Kaiser's formula yields the required values of $\beta = 2.6$ **and** ***M*** **= 24.**

The filter is type I with a delay of *M*/2 = 12 samples.

The actual peak approximation error is $\delta = 0.0213$ rather than 0.021 as specified. Since type II FIR linear-phase systems are generally not appropriate for either highpass or bandstop filter, because of the zero at $\omega = \pi$, we increase *M* to 26.

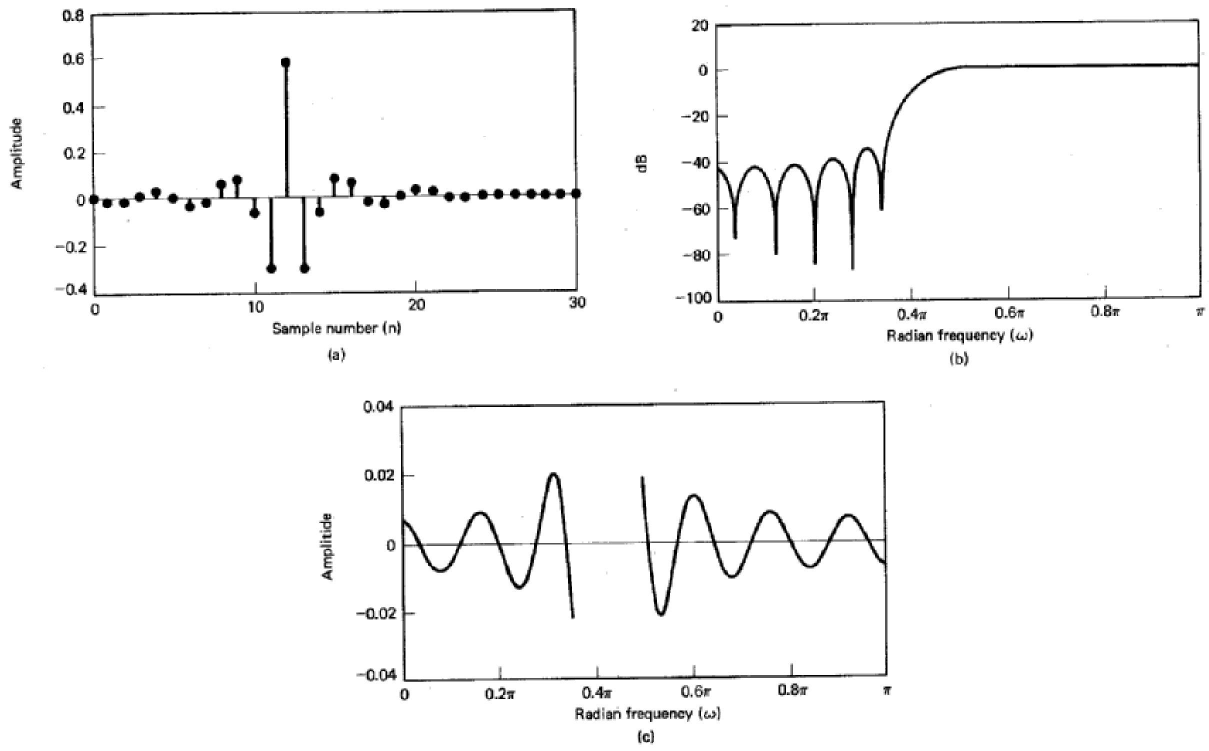


Figure 7.34 Response functions for type I FIR highpass filter. (a) Impulse response ($M = 24$). (b) Log magnitude. (c) Approximation error.

14. Optimal approximation of FIR filters

- **The windowing method does not permit individual control over the approximation errors in different bands. For many applications, better filters result from the minimization of the maximum error or a frequency-weighted error criterion.**
- **The Parks-McClellan algorithm reformulates the filter design problem as a polynomial approximation problem.**

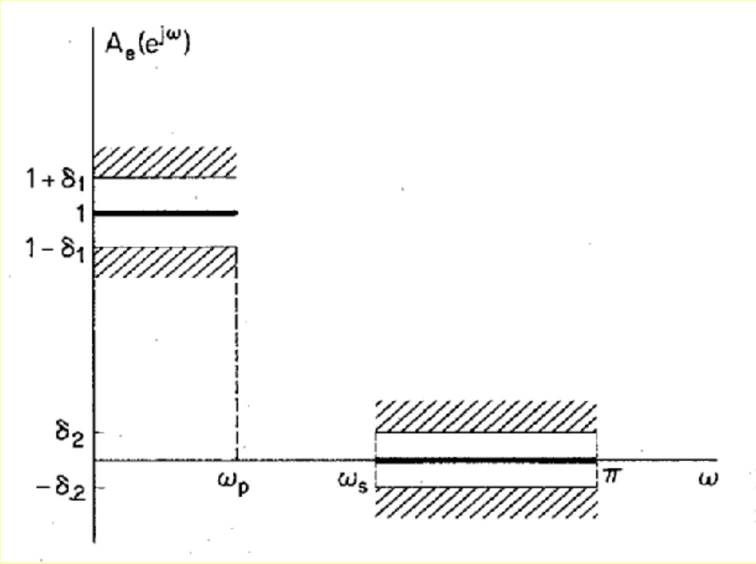
Rewrite $A_e(e^{j\omega})$ of a zero-phase filter as an *L*th-order polynomial in $\cos\omega$:

$$A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k (\cos\omega)^k, \qquad (14.1)$$

where $P(x) = \sum_{k=0}^{L} a_k x^k$.

Define the approximation error function to be

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - A_e(e^{j\omega})]. \qquad \textbf{(14.2)}$$



$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_p \\ 0, & \omega_s \le |\omega| \le \pi \end{cases} \qquad \textbf{(14.3)}$$

- **The error function $E(\omega)$, the weighting function $W(\omega)$, and the desired frequency response $H_d(e^{j\omega})$ are defined only over closed subintervals of $0 \le \omega \le \pi$.**
- **The approximating function $A_e(e^{j\omega})$ is not constrained in the transition region.**

The approximation errors are weighted differently in different approximation intervals using the weighting function $W(\omega)$.

For the present problem:

$$W(\omega) = \begin{cases} 1/K, & 0 \le |\omega| \le \omega_p \\ 1 & \omega_s \le |\omega| \le \pi \end{cases} \tag{14.3}$$

where $K = \delta_1 / \delta_2$. Using a minimax criterion, the best approximation is

$$\min_{\{h_e[n]:0 \le n \le L\}} \max_{\omega \in F} |E(\omega)| \tag{14.4}$$

where *F* is the closed subset $0 \le \omega \le \pi$ such that $0 \le \omega \le \omega_p$ or $\omega_s \le \omega \le \pi$.

14.1 Alternation theorem

Let F_P denote the closed subset consisting of the disjoint union of closed subsets of the real axis *x*. $P(x)$ denotes an *r*th-order polynomial.

Also $D_p(x)$ denotes a given desired function of *x* that is continuous on F_P; $W_p(x)$ is a positive function, continuous on F_P, and $E_p(x)$ denotes the weighted error

$$E_p(x) = W_p(x)[D_p(x) - p(x)].$$

The maximum error $\| E \|_\infty$ is defined as

$$\| E \|_\infty = \max_{x \in F_p} | E_p(x) |.$$

- **A necessary and sufficient condition that $P(x)$ is the unique r^{th}-order polynomial that minimizes $\| E \|_\infty$ is that $E_p(x)$ exhibits at least (r+2) alternations, i.e., there must exist at least (r+2) values x_i in F_P such that $x_1 < x_2 < \ldots < x_{r+2}$ and such that $E_p(x_i) = -E_p(x_{i+1}) = \pm \| E \|$ for $i = 1,2,\ldots,(r+1)$.**

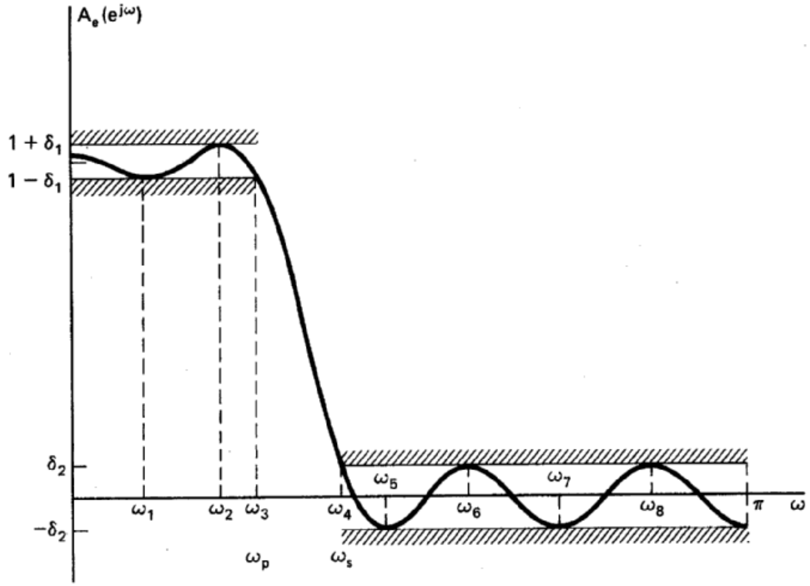


Figure 7.42 Typical example of a lowpass filter approximation that is optimal according to the alternation theorem for $L = 7$.

14.2 Optimal Type-I Lowpass filters

For type-I filters

$$P(\cos\omega) = \sum_{k=0}^{L} a_k (\cos\omega)^k \quad \textbf{(14.5)}$$

$$D_p(\omega) = \begin{cases} 1, & \cos\omega_p \le |\cos\omega| \le 1 \\ 0, & -1 \le |\cos\omega| \le \cos\omega_s \end{cases} \quad \textbf{(14.6)}$$

$$W_p(\omega) = \begin{cases} 1/K, & \cos\omega_p \le |\cos\omega| \le 1 \\ 0, & -1 \le |\cos\omega| \le \cos\omega_s \end{cases} \quad \textbf{(14.7)}$$

$$E_p(\cos\omega) = W_p(\cos\omega)[D_p(\cos\omega) - P(\cos\omega)] \quad \textbf{(14.8)}$$

The alternation theorem then states that a set of coefficients a_k in (14.5) will correspond to the filter representing the unique best approximation to the ideal lowpass filter with the ratio δ_1/δ_2 fixed at K and with passband and

stopband edges ω_p and ω_s if and only if $E_p(\cos\omega)$ exhibits at least (*L*+2) alternations on F_P . Such approximations are called equiripple approximations.

For type-I lowpass filter, the maximum possible number of alternations of the error is (*L*+3) .

- **Alternations will always occur at ω_p and ω_s.**
- **All points with zero slope inside the passband and the stopband will correspond to alternations. Including the possible alternations at $\omega = 0$ and π, the maximum number of alternations including the two at the band edges ω_p, and ω_s is (*L*+3).**
- **If either of the alternations at ω_p or ω_s is removed, the maximum number of alternations reduces to (*L*+1) violating the alternation theorem. Similar argument suggests the filter will be equiripple except possibly at $\omega = 0/\pi$**

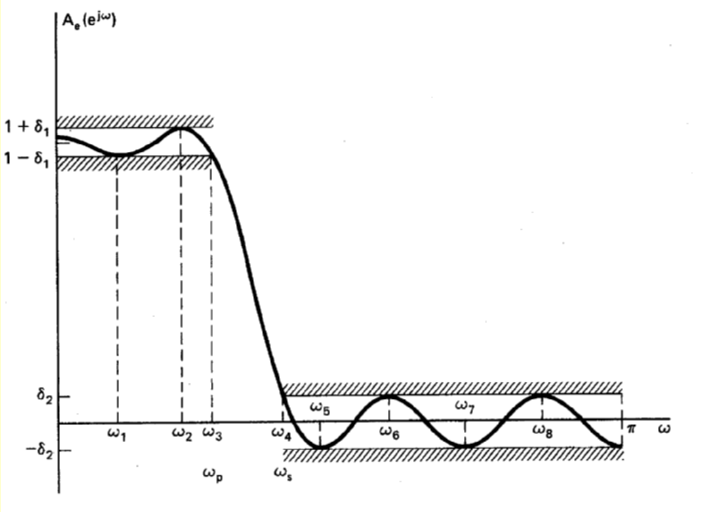


Figure 7.42 Typical example of a lowpass filter approximation that is optimal according to the alternation theorem for $L = 7$.

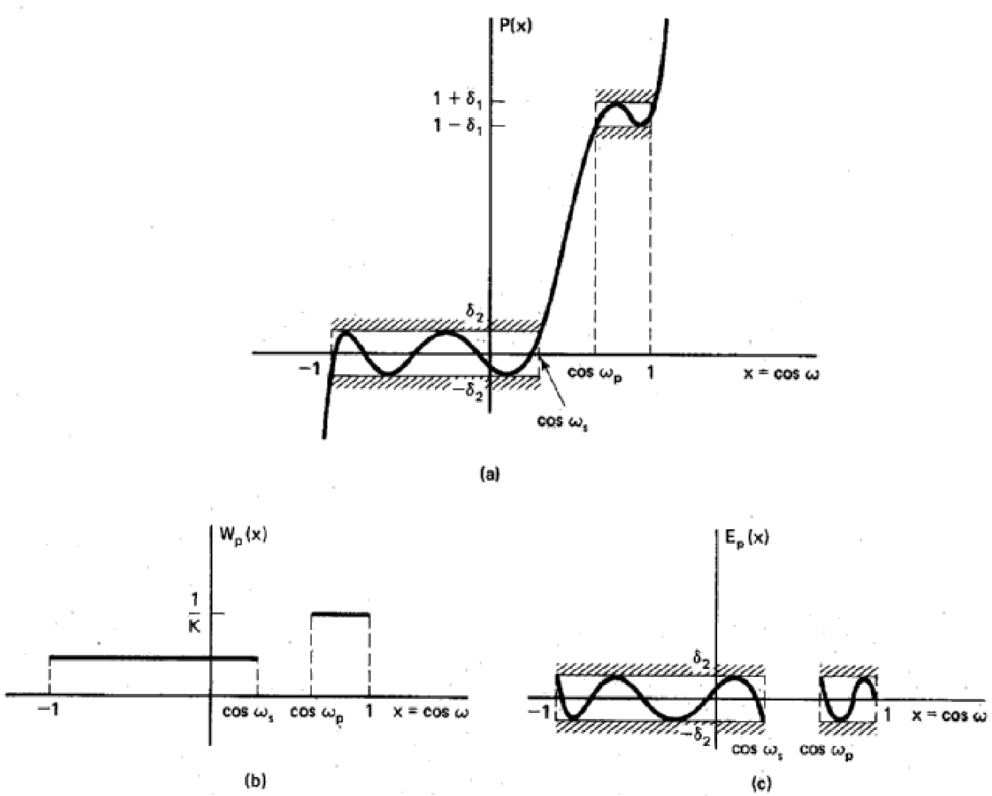


Figure 7.43 Equivalent polynomial approximation functions as a function of $x = \cos\omega$. (a) Approximating polynomial. (b) Weighting function. (c) Approximation error.

Similar procedure can be applied to type –II, -III, and –IV FIR filters.

14.4 The Parks-McCllelan Algorithm (Details not examined)

From the alternation theorem, the optimum filter $A_e(e^{j\omega})$ will satisfy:

$$W(\omega_i)\cdot[H_d(e^{j\omega_i})-A_e(e^{j\omega_i})]=(-1)^{i+1}\delta,\ \text{for } i=1,2,...,(L+2) \qquad \textbf{(14.14)}$$

The procedure begins by guessing a set of alternation frequencies ω_i, $i=1,2,...,(L+2)$. The set of equations (14.14) can be solved for a_k and δ. A more efficient alternative is to use polynomial interpolation. The polynomial so obtained can be used to evaluate $A_e(e^{j\omega})$ and also $E(\omega)$ on a dense set of frequencies in the passband and stopband. If $|E(\omega)|<\delta$ for all ω in the passband and stopband, then the optimum approximation has been found. Otherwise, the Remez exchange method is used to obtain a completely new set of extremal frequencies defined by the (L+2) largest peaks of the error curve.

If there is a maximum of the error function at both 0 and π, then the frequencies at which the greatest errors occur is taken as the new estimate of alternation frequencies.

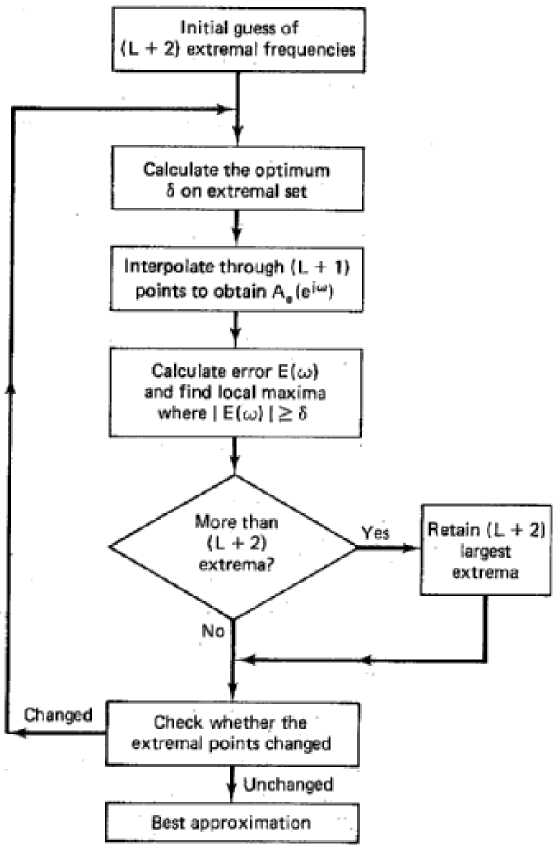


Figure 7.48 Flowchart of Parks–McClellan algorithm.

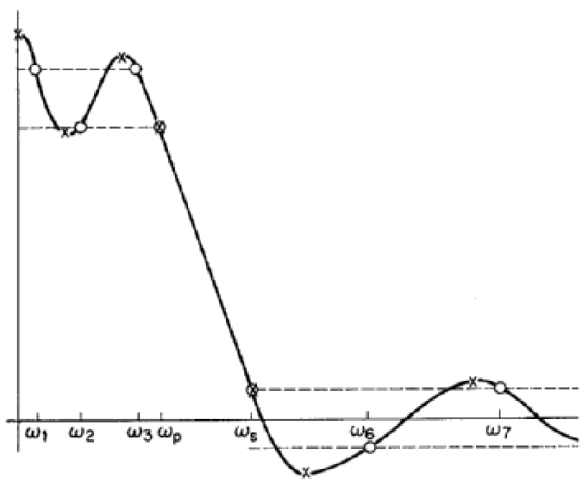


Figure 7.47 Illustration of the Parks–McClellan algorithm for equiripple approximation.

- **Kaiser obtained the following simplified formula for determining *M* given the transition width and pass- and stopband ripples:**

$$M = \frac{-10\log_{10}(\delta_1\delta_2) - 13}{2.324\Delta\omega}, \qquad \textbf{(14.15)}$$

where $\Delta\omega = \omega_s - \omega_p$.

This can be used to estimate the required filter order.

Examples

LOWPASS FILTER :

The specifications are:

$\omega_p = 0.4\pi, \omega_s = 0.6\pi, \delta_1 = 0.01$ **and** $\delta_2 = 0.001$.

- **Substituting into (14.15) gives *M* = 26. This filter fails to meet the original specifications and we must increase *M* to 27.**
- **For the same specifications, the Kaiser window method requires a value of *M* = 38 to meet or exceed the specifications.**

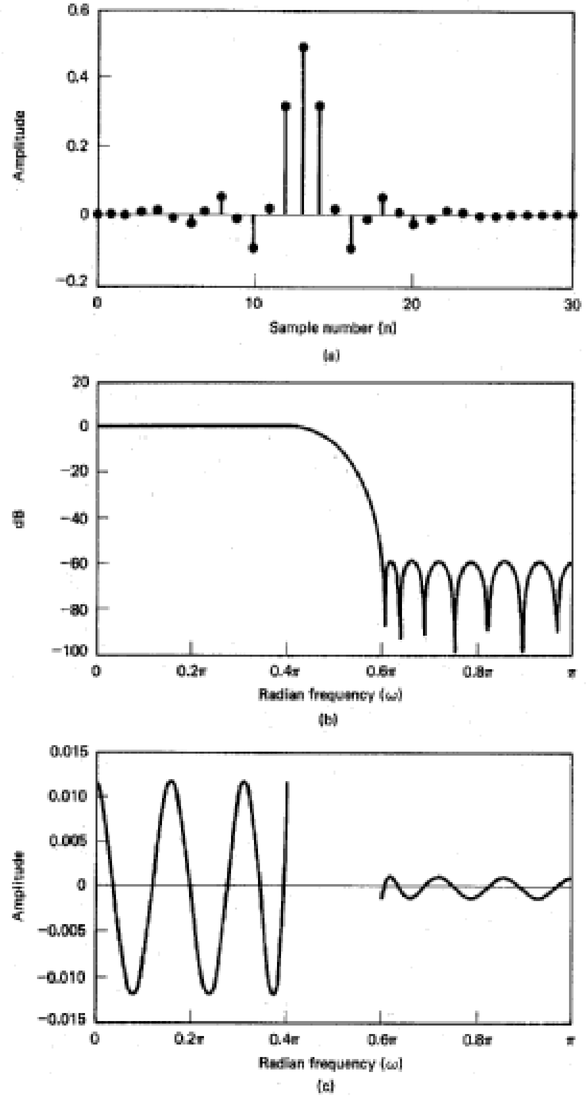


Figure 7.50 Optimum type I FIR lowpass filter for $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $K = 10$, and $M = 26$. (a) Impulse response. (b) Log magnitude. (c) Approximation error (unweighted).

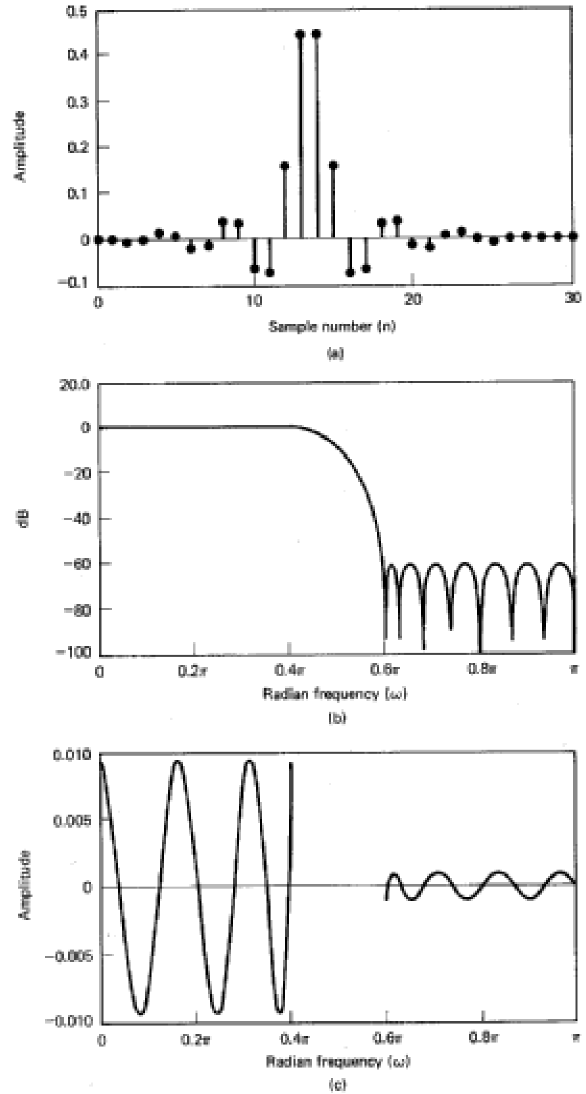
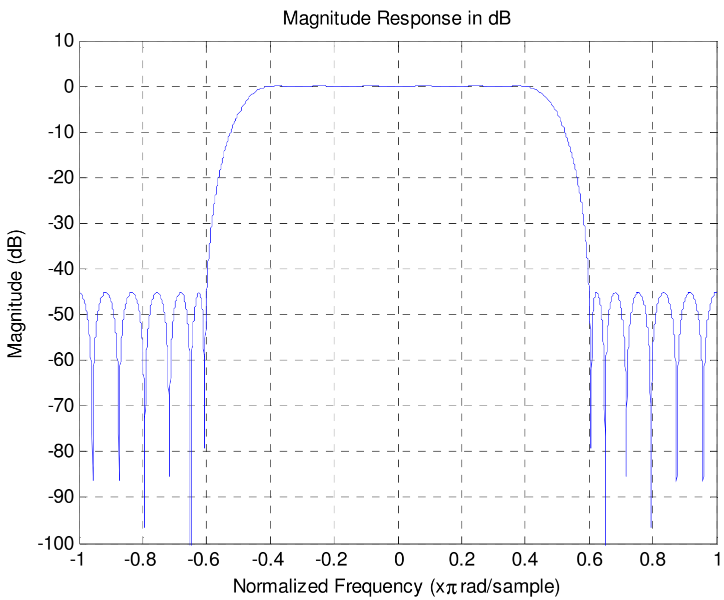


Figure 7.51 Optimum type II FIR lowpass filter for $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $K = 10$, and $M = 27$. (a) Impulse response. (b) Log magnitude. (c) Approximation error (unweighted).

MATLAB command



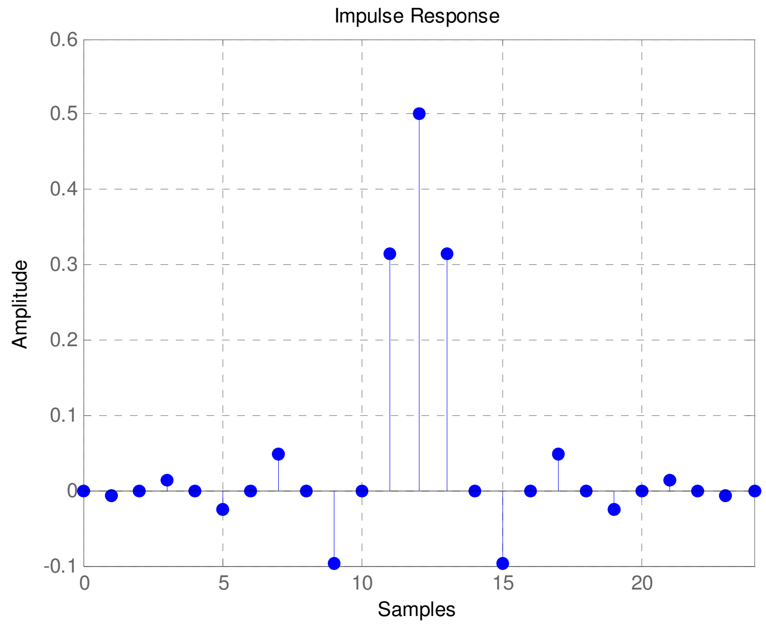
Park-McClellan algorithm. MATLAB COMMAND: b=remez(N,f,m).

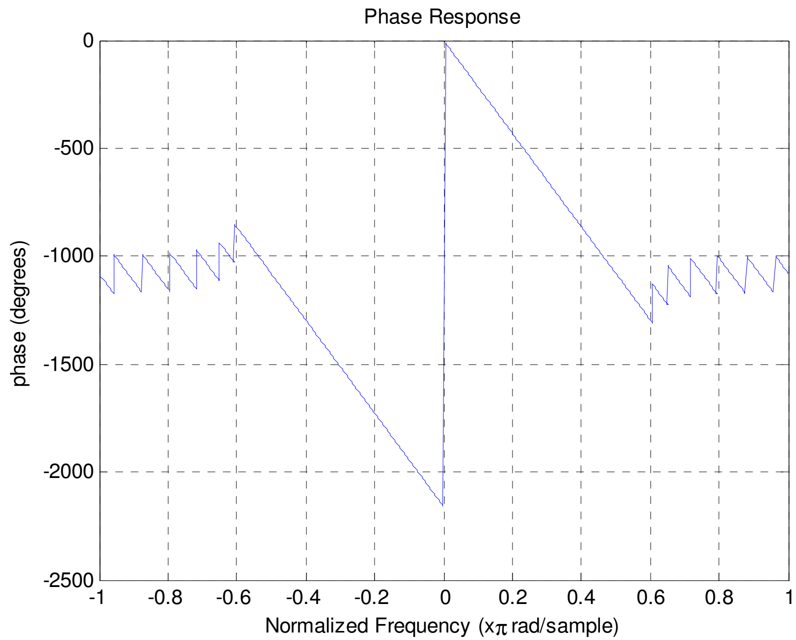
b = filter coefficient vector,

N = filter length,

m = weighting in different frequency band,

f = frequency band).Filter Specification: N=25, $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$.


Impulse Response
Amplitude
Samples
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
5
10
15
20


Phase Response
phase (degrees)
Normalized Frequency (xπ rad/sample)
0
-500
-1000
-1500
-2000
-2500
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

15. Design of Digital IIR Filters From Continuous-time Filters

- The traditional approach to design DT-IIR filters involves the transformation of a continuous-time filter to a discrete-time filter meeting prescribed specifications.

- This is a reasonable approach because:
  - The art of continuous-time IIR filter design is highly advanced. Many useful continuous-time IIR design method have relatively simple closed-form design formulas.
  - The approximation methods for continuous-time IIR filters do not lead to simple closed-form design formulas when applied directly to the DT.
  - If approximate linear phase is required in passband to avoid phase distortion, direct design method using nonlinear optimization or MOR is required.

In such transformations we generally require that the essential properties of the continuous-time frequency response be preserved in the frequency response of the resulting discrete-time filter:

- **The imaginary axis of the s-plane to map onto the unit circle of the z-plane.**
- **Stable continuous-time filter should be transformed to a stable discrete-time filter.**

15.1. Bilinear Transformation

Let $H_c(s)$ and $H(z)$ be the frequency response of the continuous-time and the discrete-time systems, respectively. The bilinear transformation replaces *s* by

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad \textbf{(15.1)}$$

$$H(z) = H_c(s)\Big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \tag{15.2}$$

- **Causal stable continuous time filters are mapped by bilinear transform to causal stable discrete-time filters.**

Solving for z in (15.2), we obtain

$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s}. \tag{15.3}$$

Substituting $s = \sigma + j\Omega$ into gives

$$z = \frac{1+\sigma(T_d/2)+j\Omega(T_d/2)}{1-\sigma(T_d/2)-j\Omega(T_d/2)}. \tag{15.4}$$

If $\sigma < 0$, then from (15.4), $|z| < 1$ for any value of Ω. Similarly, $\sigma > 0$, then $|z| > 1$ for all Ω. Pole of $H_c(s)$ is in the left-half s-plane, its image in the z-plane will be inside the unit circle.

- The $j\Omega-$axis of the s-plane maps onto the unit circle.

Substituting $s = j\Omega$ (15.3), we obtain

$$z = \frac{1 + j\Omega(T_d / 2)}{1 - j\Omega(T_d / 2)} \tag{15.5}$$

It is clear that $|z| = 1$ for all values of s on the $j\Omega-$axis. Solving for Ω gives

$$\Omega = \frac{2}{T_d}\tan(\omega/2) \tag{15.6}$$

$$\Leftrightarrow \qquad \omega = 2\arctan(\Omega T_d / 2). \tag{15.7}$$

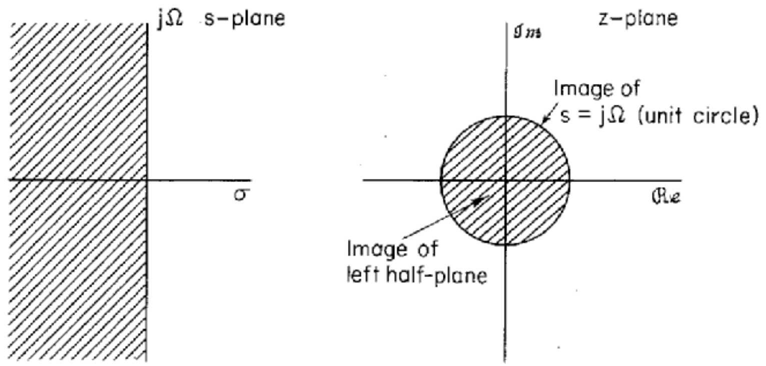


Figure 7.7 Mapping of the *s*-plane onto the *z*-plane using the bilinear transformation.

The ranges of frequencies $0 \le \Omega \le \infty$ maps to $0 \le \omega \le \pi$ while the range $-\infty \le \Omega \le 0$ maps to $-\pi \le \omega \le 0$.

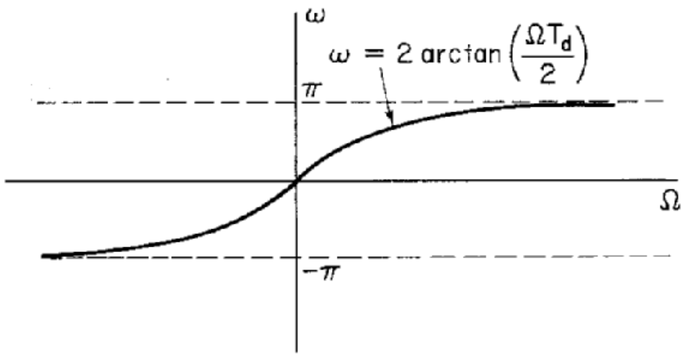


Figure 7.8 Mapping of the continuous-time frequency axis onto the unit circle by bilinear transformation.

- **The bilinear transformation avoids the problem of aliasing because it maps the entire imaginary axis of the s-plane onto the unit circle in the z-plane. However, there will be a nonlinear compression of the frequency axis, leading to considerably phase distortion, esp. around band edges.**

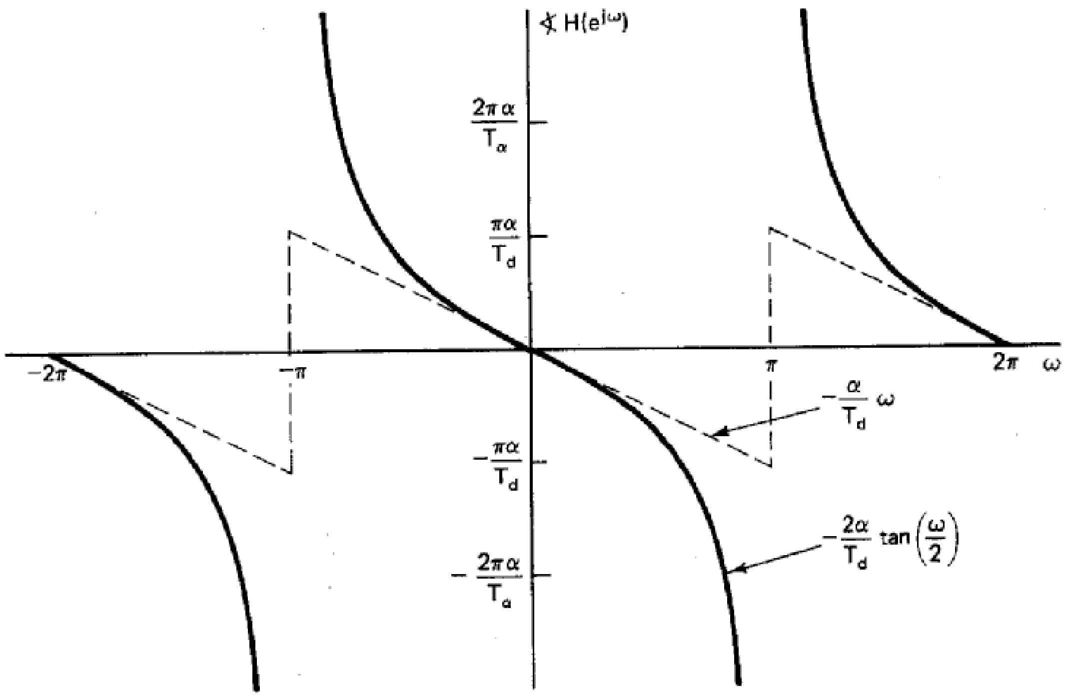


Figure 7.11 Illustration of the effect of the bilinear transformation on a linear phase characteristic. (Dashed line is linear phase and solid line is phase resulting from bilinear transformation.)

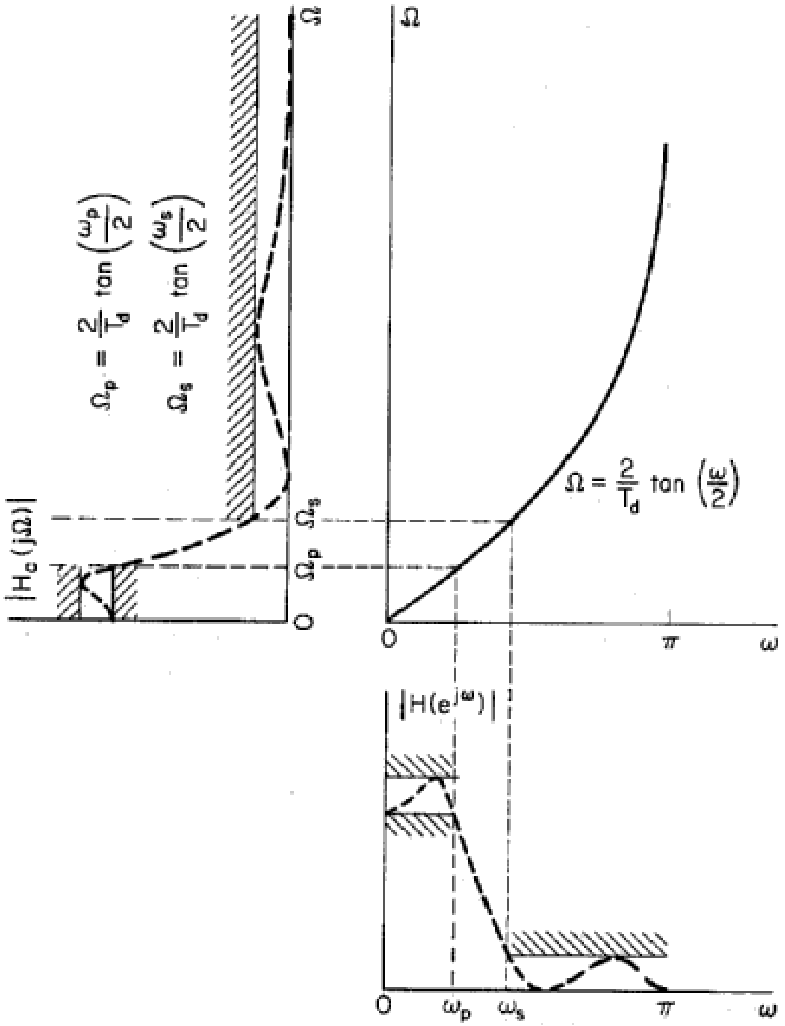


Figure 7.10 Frequency warping inherent in the bilinear transformation of a continuous-time lowpass filter into a discrete-time lowpass filter. To achieve the desired discrete-time cutoff frequencies, the continuous-time cutoff frequencies must be prewarped as indicated.

- **The bilinear transformation is useful only when this compression can be tolerated or compensated for, as in the case of filters that approximate ideal piecewise-constant magnitude response characteristics.**

Fig. 7.10 depicts the mapping of a continuous-time frequency response and tolerance scheme to a corresponding discrete-time frequency response and tolerance scheme.

If the critical frequencies (such as the passband and stopband edge frequencies) of the continuous-time filter are prewarped according to

$$\Omega = \frac{2}{T_d}\tan(\omega/2), \tag{15.8}$$

then when the continuous-time filter is transformed to the discrete-time filter using the bilinear transform, the discrete-time filter will meet the desired specifications.

- **Typical frequency-selective continuous-time approximations are Butterworth, Chebyshev, and elliptic filters.**

- **Butterworth continuous-time filter is monotonic in the passband and in the stopband.**

- **A type I Chebyshev filter has an equiripple characteristic in the passband and it varies monotonically in the stopband.**

- **A type II Chebyshev filter is monotonic in the passband and equiripple in the stopband.**

- **An elliptic filter is equiripple in both the passband and the stopband**

- **These properties will be preserved when the filter is mapped to a digital filter with the bilinear transformations.**

- **Though the bilinear transformation can be used effectively in mapping a piecewise-constant magnitude response characteristic from the s-plane to the z-plane, the distortion in the frequency axis also manifests itself as a warping of the phase response of the filter.**

Substituting the bilinear transformation into the ideal linear phase factor $e^{-s\alpha}$, the phase angle is $-\frac{2\alpha}{T_d}\tan(\omega/2)$.

- **Therefore, the bilinear transformation could not obtain a linear phase digital filter from a linear phase continuous-time filter.**

Example 7.1

Consider a discrete-time filter that is to be used to lowpass filter a continuous-time signal using the basic configuration of Fig. 7.1. Specifically, we want the overall system of Fig. 7.1 to have the following properties when the sampling rate is 10^4 samples/s ($T = 10^{-4}$ s):

1. The gain $|H_{eff}(j\Omega)|$ should be within ± 0.01 (0.086 dB) of unity (zero dB) in the frequency band $0 \le \Omega \le 2\pi(2000)$.
2. The gain should be no greater than 0.001 (-60 dB) in the frequency band $2\pi(3000) \le \Omega$.

Such a set of lowpass specifications on $|H_{eff}(j\Omega)|$ can be depicted as in Fig. 7.2(a), where the limits of tolerable approximation error are indicated by the shaded horizontal lines. For this specific example, the parameters of Fig. 7.2(a) would be

$$\delta_1 = 0.01 \; (20 \log_{10} (1 + \delta_1) = 0.086 \text{ dB}),$$

$$\delta_2 = 0.001 \; (20 \log_{10} \delta_2 = -60 \text{ dB}),$$

$$\Omega_p = 2\pi(2000),$$

$$\Omega_s = 2\pi(3000).$$

Since the sampling rate is 10^4 samples/s, the gain of the ideal system is identically zero above $\Omega = 2\pi(5000)$ due to the ideal discrete-to-continuous (D/C) converter.

The tolerance scheme for the digital filter is shown in Fig. 7.2(b). It is the same as Fig. 7.2(a) except that it is plotted as a function of normalized frequency ($\omega = \Omega T$) and it need only be plotted in the range $0 \le \omega \le \pi$, since the remainder can be inferred from symmetry properties (assuming $h[n]$ is real) and periodicity of $H(e^{j\omega})$. From Eq. (7.1b) it follows that the *passband*, within which the magnitude of the frequency response must approximate unity with an error of $\pm\delta_1$, is

$$(1 - \delta_1) \le |H(e^{j\omega})| \le (1 + \delta_1), \qquad |\omega| \le \omega_p, \tag{7.2}$$

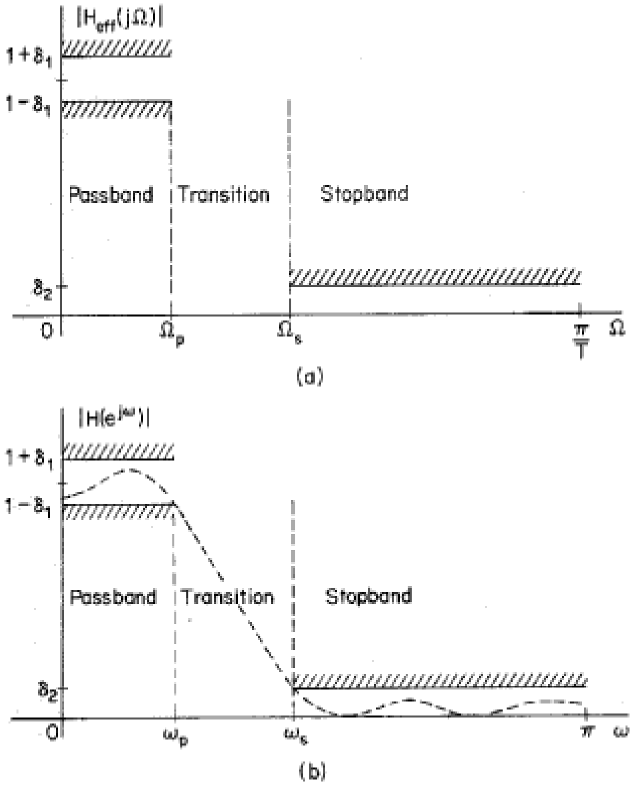


Figure 7.2 (a) Specifications for effective frequency response of overall system in Fig. 7.1 for the case of lowpass filter. (b) Corresponding specifications for the discrete-time system in Fig. 7.1.

where in this example $\delta_1 = 0.01$ and $\omega_p = 2\pi(2000)\cdot 10^{-4} = 0.4\pi$ radians. The other approximation band is the *stopband*, in which the magnitude response must approximate zero with an error less than δ_2; i.e.,

$$|H(e^{j\omega})| \le \delta_2, \qquad \omega_s \le |\omega| \le \pi. \tag{7.3}$$

In this example, $\delta_2 = 0.001$ and $\omega_s = 2\pi(3000)\cdot 10^{-4} = 0.6\pi$ radians. The passband cutoff frequency ω_p and the stopband cutoff frequency ω_s are given in terms of z-plane angle. To approximate the ideal lowpass filter in this way with a realizable system, we must provide a transition band of nonzero width $(\omega_s - \omega_p)$ in which the magnitude response changes smoothly from passband to stopband. The dashed curve in Fig. 7.2(b) is the magnitude response of a system that meets the prescribed specification.

16 COMPUTER AIDED DESIGN OF DESCRETE-TIME IIR FILTERS

Analytical formulas do not exist for the design of either continuous-time and discrete-time filters to match arbitrary frequency-response specifications. In these cases, optimization techniques such as Decky's method are often used.