Communications Signal Processing (S. C. Chan, scchan@eee.hku.hk)

CONTENTS

Background

Digital Transmission through bandlimited channels

Detection theory and baseband transmission

Digital Transmission via Carrier Modulation

Source and Channel Coding*

Multiple access systems and Interference suppression*

Textbook : J. Proakis and M. Salehi: Contemporary communication systems using MATLAB, 1st Edition. Brooks/Cole, Thomson Learning. 2000. (2nd Edition available on 2004).

Signal and Linear Systems

CONTENTS

Textbook : J. Proakis and M. Salehi: Contemporary communication systems using MATLAB, 1st Edition. Brooks/Cole, Thomson Learning. 2000. (2nd Edition available on 2004).

1. Fourier Series

- **The input x(t) and the output y(t) of a linear time-invariant system with impulse response h(t) is given by the convolution integral:**

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = h(t) * x(t) \qquad (1.1)$$

Example 1: **Input x(t) is a complex exponential $x(t) = Ae^{j2\pi f_0 t}$.**

$$y(t) = \int_{-\infty}^{\infty} h(\tau)Ae^{j2\pi f_0 (t-\tau)}d\tau = Ae^{j2\pi f_0 t}H(f_0).$$

where $H(f_0) = \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f_0 \tau}d\tau$ is the frequency response of the system.

- **The frequency response of a LTI system is equal to the Fourier transform of its impulse response.**

- **The output is also a complex sinusoidal with the same frequency. Its amplitude and phase is modified by the frequency response of the system $H(f_0)$ at f_0**

Example 2: **Response of a LTI system to a periodic input x(t) with period T_0.**

Expand x(t) as Fourier series:

$$x(t)=\sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0} \quad \text{where } x_n=\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi nt/T_0}dt \qquad (1.2)$$

where $f_0=1/T_0$ is the fundamental frequency and nf_0 is the nth harmonics.

Using (1.1), we have

$$y(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau=\int_{-\infty}^{\infty} h(\tau)\sum_{n=-\infty}^{\infty} x_n e^{j2\pi n(t-\tau)/T_0}d\tau$$

$$=\sum_{n=-\infty}^{\infty} x_n\left(\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi n\tau/T_0}d\tau\right)e^{j2\pi nt/T_0}=\sum_{n=-\infty}^{\infty} x_n H\left(\frac{n}{T_0}\right)e^{j2\pi nt/T_0}.$$

The output is also periodic with the same period. The Fourier series coefficient of y(t) is $y_n = H(\frac{n}{T_0})x_n$.

Exercise (illustrative problem 1.4 in textbook):

A triangular pulse train x(t) with period $T_0 = 2$ is defined in one period as

$$\Lambda(t) = \begin{cases} t+1 & -1 \le t \le 0 \\ -t+1 & 0 \le t \le 1 \\ 0 & otherwise \end{cases}$$

1. Determine the Fourier series coefficients of x(t).
2. Plot the discrete spectrum of x(t).
3. Assume that this signal passes through an LTI system whose impulse response is given by

$$h(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & otherwise \end{cases},$$

plot the discrete spectrum and the output y(t). Plots of x(t) and y(t) are as follows:

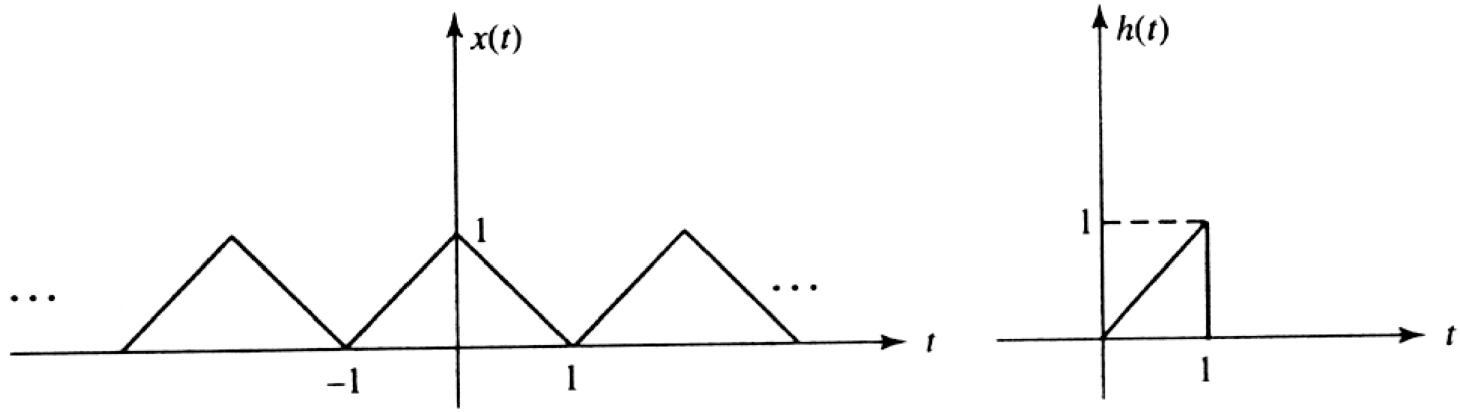


Figure 1.10 The input signal and the system impulse response

Verify the matlab code from the textbook.

Also see the link:

http://www.brookscole.com/cgi-brookscole/course_products_bc.pl?discipline_number=38&subject_code=EE28&fid=M20b&product_isbn_issn=0534371736

2. Fourier Transform (FT)

The Fourier transform is the extension of Fourier Series to nonperiodic signals:

$$\Im[x(t)] = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt \qquad \textbf{(Fourier Transform)} \qquad \textbf{(2-1a)}$$

The inverse Fourier transform of X(f) is

$$\Im^{-1}[X(f)] = x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df \qquad \textbf{(Inverse FT)} \qquad \textbf{(2-1b)}$$

The Fourier transform of a signal is called the spectrum of the signal and it is in general a complex function of *f*.

2.1 Properties

1. If x(t) is a real-valued signal, then X(f) satisfies the Hermitian symmetry:

$$X(-f) = X^*(f) \tag{2-2}$$

2. Duality:

$$\Im[X(t)] = x(-f) \tag{2-3}$$

3. Modulation: Multiplication by an exponential in the time domain corresponds to a frequency shift in the frequency domain

$$\begin{aligned} &\Im[e^{j2\pi f_0 t} x(t)] = X(f - f_0) \\ &\Im[x(t)\cos(2\pi f_0 t)] = \frac{1}{2}[X(f - f_0) + X(f + f_0)] \end{aligned} \tag{2-4}$$

4. Convolution : Convolution in the time domain is equivalent to multiplication in the frequency domain, and vice versa.

If $\Im[x(t)] = X(f)$ and $\Im[y(t)] = Y(f)$, then

$$\Im[x(t) * y(t)] = X(f)Y(f)$$

$$\Im[x(t)y(t)] = X(f) * Y(f).$$

(2-5)

5. Parseval's relation :

If $\Im[x(t)] = X(f)$ and $\Im[y(t)] = Y(f)$, then

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

(2-6)

Exercise:

Go through illustrative problems 1.5, 1.6, 1.7.

Table 1.1 Table of Fourier transform pairs

$x(t)$	$X(f)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f t_0}$	$\delta(f-f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\delta(f-f_0)-\frac{1}{2j}\delta(f+f_0)$
$\Pi(t)$	$\operatorname{sinc}(f)$
$\operatorname{sinc}(t)$	$\Pi(f)$
$\Lambda(t)$	$\operatorname{sinc}^2(f)$
$\operatorname{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t}u_{-1}(t), \quad \alpha>0$	$\frac{1}{\alpha+j2\pi f}$
$te^{-\alpha t}u_{-1}(t), \quad \alpha>0$	$\frac{1}{(\alpha+j2\pi f)^2}$
$e^{-\alpha\|t\|}, \quad \alpha>0$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\operatorname{sgn}(t)$	$\frac{1}{j\pi f}$
$u_{-1}(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)$

3. Sampling Theorem

The sampling theorem says that a bandlimited signal – that is, a signal whose Fourier transform vanishes for $|f| > W$ for some W – can be completely described in terms of its sample values taken at intervals $T_s \le 1/(2W)$. $f_s = (2W)$ is called the **Nyquist rate.**

- The signal x(t) can be reconstructed from the samples $x[n] = x(nT_s),\ n = -\infty, \ldots, \infty$, as

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\sin c(2W(t - nT_s)) \quad \text{(3-1)}$$

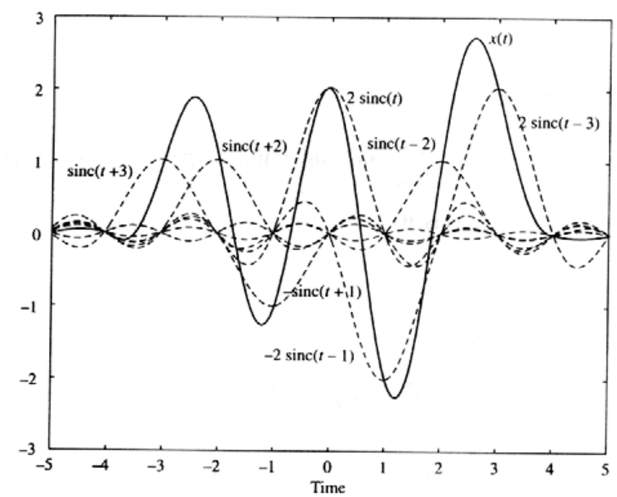


Figure 1.17 Representation of the sampling theorem

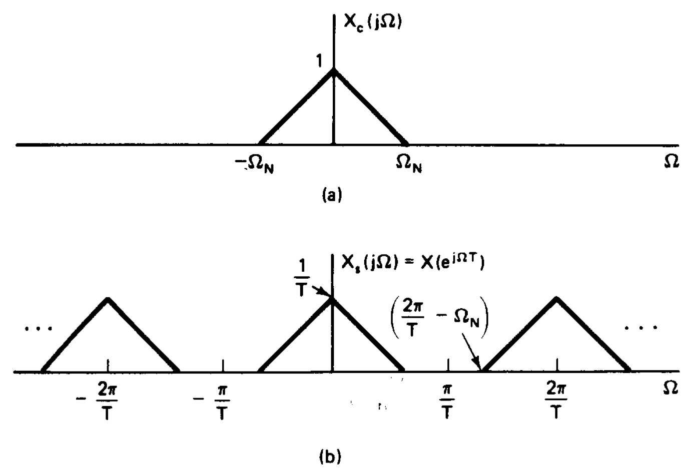
Proof:

The sampled waveform $x_\delta(t)$ can be written as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

has a Fourier transform given by

$$X_\delta(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s}) \text{ .}$$



Passing $x_\delta(t)$ through a lowpass filter with a bandwidth of W and a gain of T_s in the passband will reproduce the original signal.

4. Discrete-time Signals and Systems

(A. V. Oppenheim: Discrete-time signal processing. PHI, 1989)

The discrete-time Fourier transform pair is defined as follows:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{(DT-FT)} \qquad (4\text{-}1)$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{(Inverse DT-FT)} \qquad (4\text{-}2)$$

- **The DT-FT represents the frequency components of *x*[*n*] at digital radian frequency ω. It is obtained by substituting $z = e^{j\omega}$ in the z-transform**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

- **The inverse DT-FT synthesizes the sequence $x[n]$ from infinitesimally small complex sinusoids of form $\frac{1}{2\pi}X(e^{j\omega})e^{jn\omega}d\omega$.**

4.1 Discrete Fourier transform (DFT)

Consider the DT-FT of a finite length sequence: x[n], n=0,…,N-1.

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$ **(DT-FT)**

Sampling ω regularly at $\omega_k = 2\pi k / N$ (spacing $2\pi / N$), k=0,1,…,N-1, we obtain the discrete Fourier transform (DFT).

$$X[k] = X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi nk/N)} \quad (4\text{-}3)$$ **(DFT)**

Examples: Compute the DFT of the following sequence

$$x[n] = \begin{cases} 1 & \mathrm{n} = 0,\ldots,4 \\ 0 & \mathrm{n} = 5,\ldots,\mathrm{N}-1 \end{cases}, \quad X[k] = \sum_{n=0}^{4} e^{-j(2\pi nk/N)} = \frac{1 - e^{-j5(2\pi k/N)}}{1 - e^{-j(2\pi k/N)}}$$

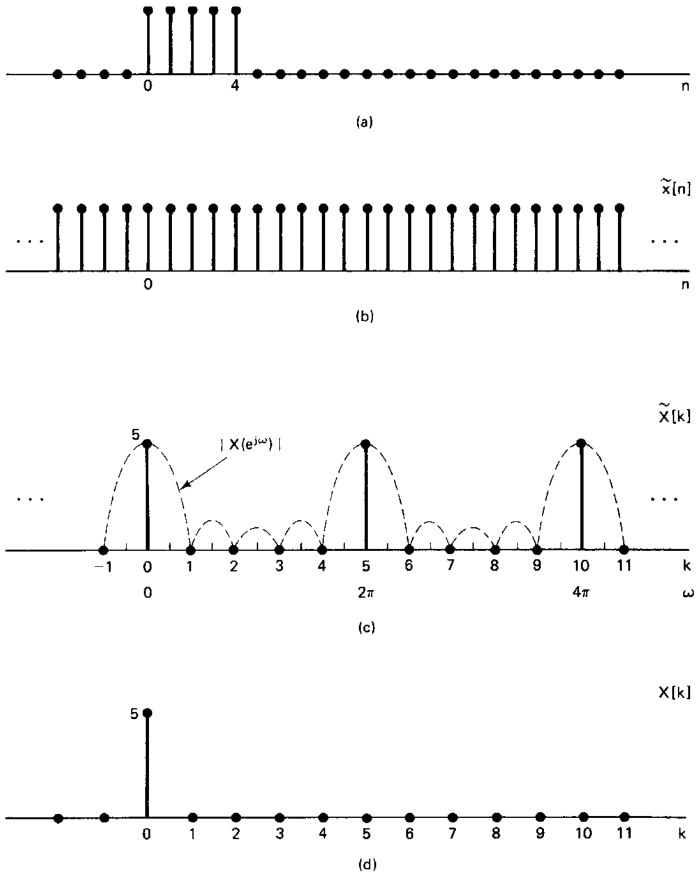


Figure 8.10 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 5$. (c) Fourier series coefficients $\tilde{X}[k]$ for $\tilde{x}[n]$. To emphasize that the Fourier series coefficients are samples of the Fourier transform, $|X(e^{j\omega})|$ is also shown. (d) DFT of $x[n]$.

N=5

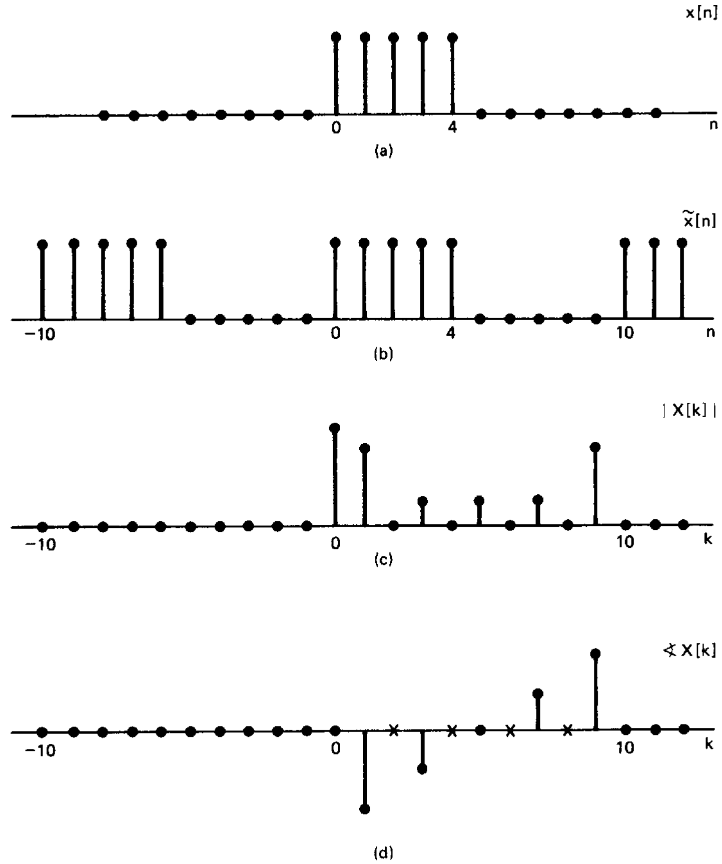


Figure 8.11 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 10$. (c) DFT magnitude. (d) DFT phase. (x's indicate indeterminate value.)

N=11

- **DFT as uniform samples of the DT-FT in the frequency domain** (spacing $2\pi / N$).

4.2 Inverse Discrete Fourier transform (IDFT)

DFT is an orthogonal transformation and it has a simple inversion formula (the inverse DFT):

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j(2\pi nk/N)} \qquad \textbf{(4-4)}$$

4.3 z-transform and LTI system

For a LTI system, the output y[n] is given by the discrete-time convolution of the input x[n] and its impulse response h[n].

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]. \qquad (4\text{-}5)$$

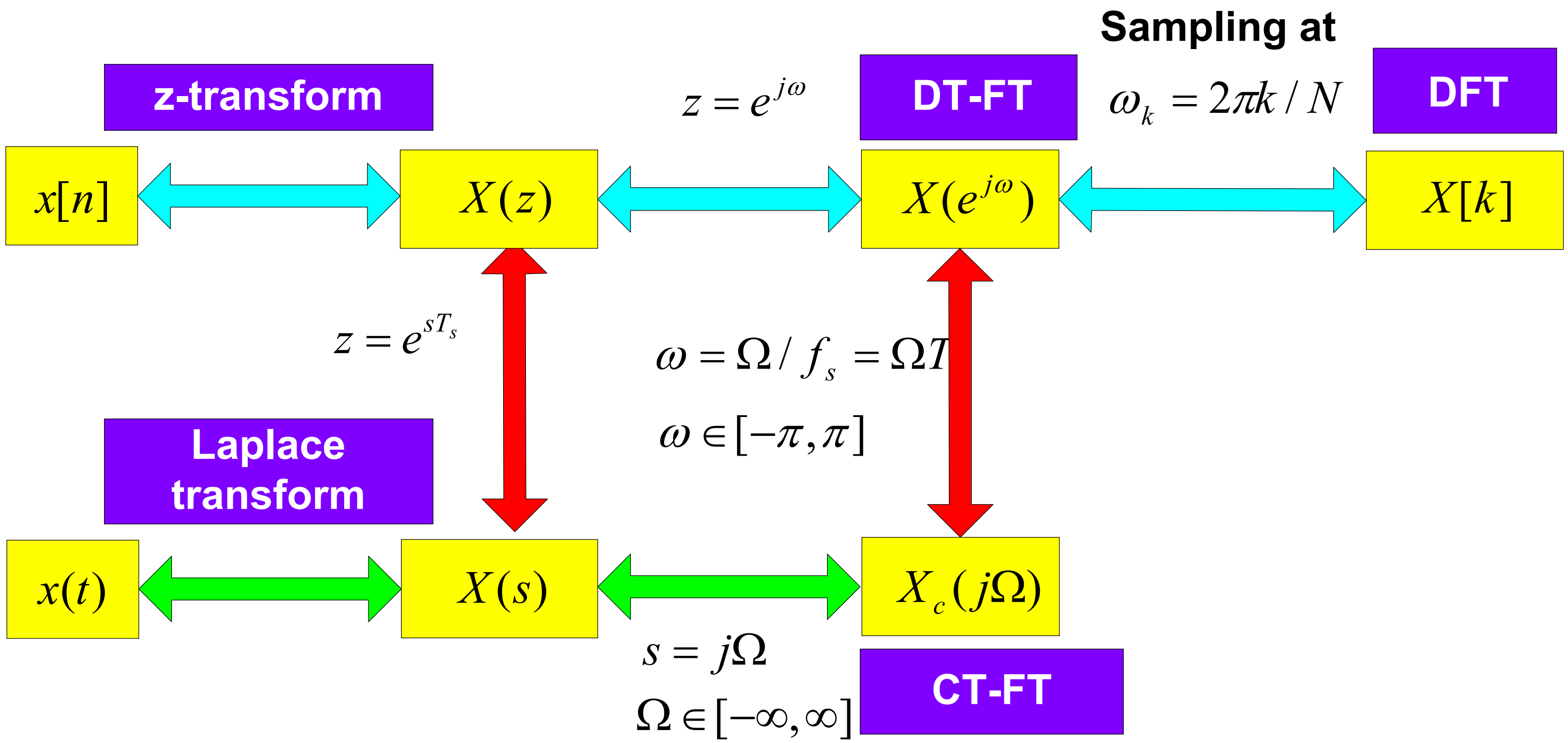
Using the convolution theorem, we have

$$y[n] = h[n] * x[n] \xleftrightarrow{Z} H(z)X(z) = Y(z). \qquad (4\text{-}6)$$

where $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$ is the z-transform of h[n]. Since the discrete-time Fourier transforms (if they exist) of *h*[*n*] and *x*[*n*] are the z-transform evaluated on the unit circle, we have

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}). \qquad (4\text{-}7)$$

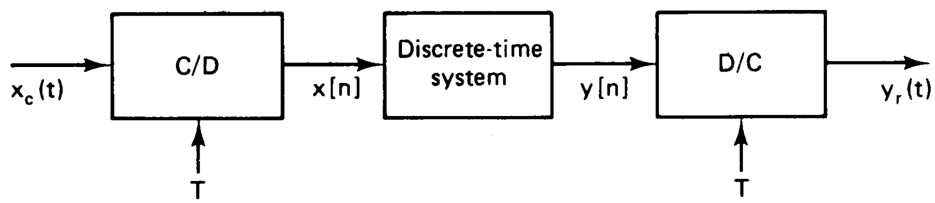
RELATIONSHIP BETWEEN THE TRANSFROMATIONS



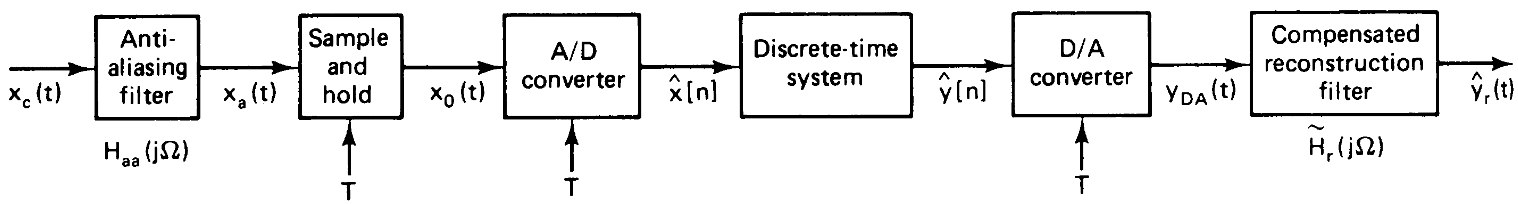
If x(t) is bandlimited and the sampling rate is greater than the Nyquist rate, then x(t) can be recovered from x[n]. Anti-aliasing filter with bandwidth fmax has to be applied to x(t) to avoid aliasing.

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \qquad \textbf{(10.1)}$$

5. Structure of a digital signal processing system



(a)



(b)

(Fig. 3.26 in Oppenheim's book)

Relationship between continuous-time Fourier transform and DTFT:

$$x[n] = x_c(nT).$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \qquad \textbf{(5.1)}$$

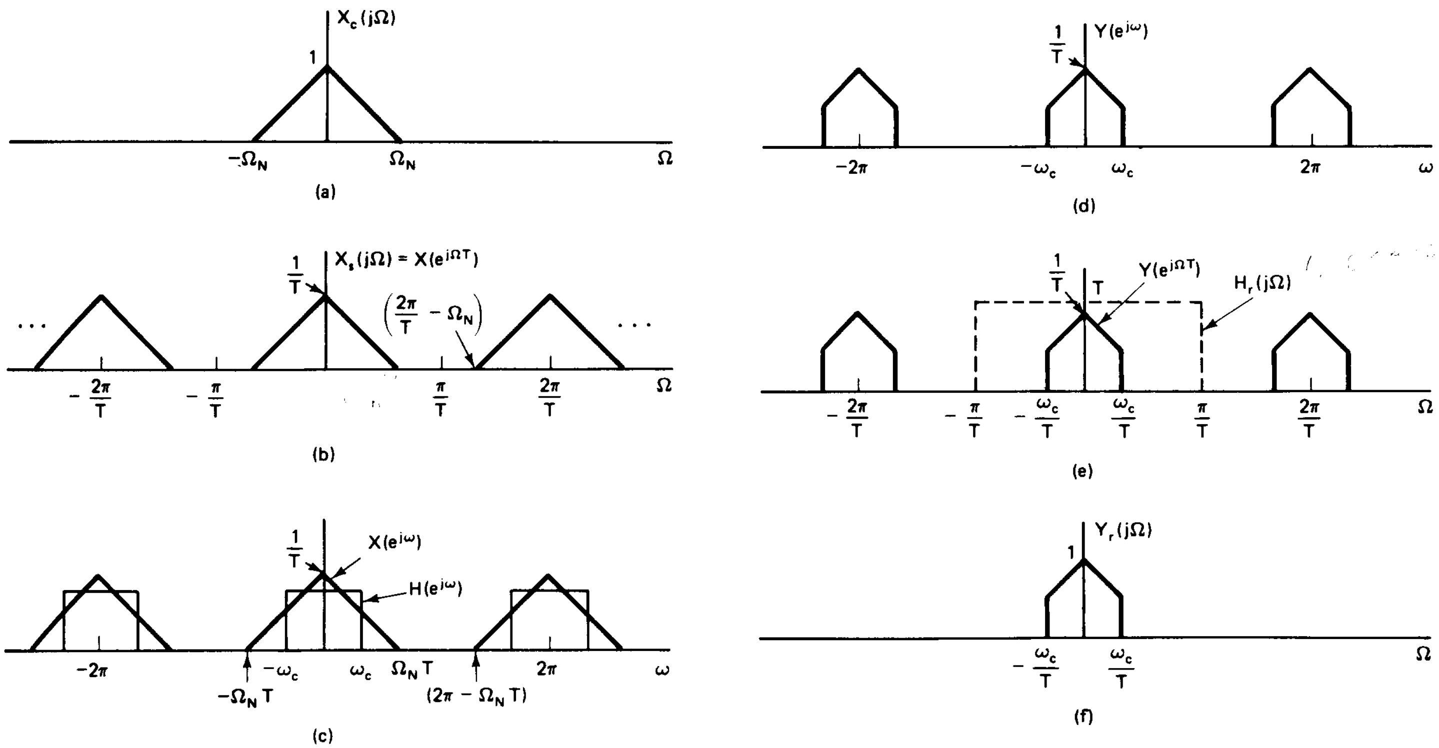


Figure 3.11 (a) Fourier transform of a bandlimited input signal. (b) Fourier transform of sampled input plotted as a function of continuous-time frequency Ω. (c) Fourier transform $X(e^{j\omega})$ of sequence of samples and frequency response $H(e^{j\omega})$ of discrete-time system plotted vs. ω. (d) Fourier transform of output of discrete-time system. (e) Fourier transform of output of discrete-time system and frequency response of ideal reconstruction filter plotted vs. Ω. (f) Fourier transform of output.

Signal reconstruction:

$$y_r[n] = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}.$$

$$Y_r(j\Omega) = H_r(j\Omega) Y(e^{i\Omega T}). \qquad \textbf{(5.2)}$$

After passing through a LTI filter with $H(e^{j\omega})$, the DT-FT of output y[n] is

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}). \qquad (5.3)$$

From (10.2) and (10.3), the continuous-time Fourier transform of output y(t) is

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}) \qquad (5.4)$$

If sampling theorem is satisfied (i.e $X_c(j\Omega) = 0, \text{ for } |\Omega| \geq \pi/\mathrm{T}$), then

$$H_r(j\Omega)X(e^{j\Omega T}) = X_c(j\Omega)$$

and (10.4) becomes

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(e^{j\Omega T}) & |\Omega| < \pi/T, \\ 0 & |\Omega| \geq \pi/T. \end{cases}.$$

Thus, the equivalent analog filter of $H(e^{j\omega})$ is

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \pi/T, \\ 0 & |\Omega| \geq \pi/T. \end{cases}. \qquad (5.5)$$

6. Useful Terminology

Signals with finite energy are called energy-type signals:

$$E_X = \int_{-\infty}^{\infty} x^2(t)dt \quad (< \infty) \qquad \text{(Energy)} \tag{6-1}$$

- The Energy spectral density of an energy-type signal gives the distribution of energy at various frequencies of the signal:

$$G_X(f) = |X(f)|^2 \tag{6-2}$$

Therefore $$E_X = \int_{-\infty}^{\infty} G_X(f)df = \int_{-\infty}^{\infty} |X(f)|^2 df . \tag{6-3}$$

- The autocorrelation function of x(t) is

$$R_X(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)d\tau = x(\tau) * x(-\tau) . \tag{6-4}$$

Using the convolution theorem, we have

$$G_X(f) = \Im[R_X(\tau)]. \tag{6-5}$$

- The Fourier transform of the autocorrelation function of x(t) is equal to its energy spectral density.

Signals with positive and finite power are called power-type signals:

$$P_X = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt \quad (< \infty) \tag{6-6}$$

- All periodic signals are power signals.
- The time-average autocorrelation function of a power signal x(t) is

$$R_X(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)d\tau \tag{6-7}$$

- The power-spectral density is given by

$$S_X(f) = \Im[R_X(\tau)] \tag{6-8}$$

The total power is $P_X = \int_{-\infty}^{\infty} S_X(f)df$. (6-9)

Random Processes

CONTENTS

Probability density and distribution functions

Power spectrum of random processes

Lowpass and bandpass processes

Textbook : J. Proakis and M. Salehi: Contemporary communication systems using MATLAB. Brooks/Cole, Thomson Learning. 2000.

1 Probability density and distribution functions

- **Random Variables**

  Let X be a random variable (r.v.), the probability distribution function

  $$P_X(x) = P\{X \le x\}$$

  is the probability of the random variable *X* less than or equal to a value *x*.

- If $P_X(x)$ is continuous, then the probability density function (pdf) is

  $$p_X(x) = dP_X(x)/dx$$

  $P_X(x)$ and $p_X(x)$ satisfy the properties:

  $$p_X(x) \ge 0, \forall x; \qquad \int_{-\infty}^{\infty} p_X(x)dx = 1 \tag{1.1a}$$

  $$P\{a < X \le b\} = P_X(b) - P_X(a) = \int_a^b p_X(x)dx; \tag{1.1b}$$

  $$P_X(-\infty) = 0;\ P_X(\infty) = 1. \tag{1.1c}$$

Expectation of Random Variables

If g(x) is a function of the r.v., the expectation of the new r.v. g(X) is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x) dx \tag{1-2}$$

Mean value: $$\mu_X = E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{1-3}$$

Mean Square value: $$\chi_X^2 = E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{1-4}$$

Variance: $$\sigma_X^2 = E[(X-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 p_X(x) dx \tag{1-5}$$

n-th moment: $$E[X^n] \tag{1-6}$$

n-th central moment: $$E[(X-\mu_X)^n] \tag{1-7}$$

The variance measures the spread of the distribution about its mean value. σ_X is called the standard deviation or root-mean-square value.

Bivariate Distribution

Let X and Y be two r.v.'s whose mean values and variance are denoted by $\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2$, respectively.

The joint probability distribution $P_{XY}(x, y)$ and joint pdf are defined by:

$$P_{XY}(x, y) = P\{X \le x, Y \le y\} \qquad \text{(1-8)}$$

$$p_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} P_{XY}(x, y); \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{XY}(x, y)dxdy = 1 \qquad \text{(1-9)}$$

The probability that the r.v.'s X and Y are jointly in ranges (a,b) and (c.d) is:

$$P\{a < X \le b, c < Y \le d\} = \int_a^b \int_c^d p_{XY}(x, y)dxdy \qquad \text{(1-10)}$$

The r.v.'s X and Y are said to be statistically independent if

$$p_{XY}(x, y) = p_X(x)p_Y(y) \qquad \text{(1-11)}$$

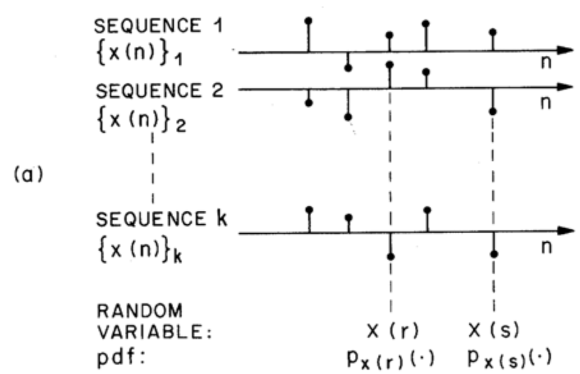
The expectation of a function g(X,Y) of X and Y is:

$$E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)p_{XY}(x, y)dxdy \qquad \text{(1-12)}$$

Random Processes

- **Communications waveforms are not deterministic - the signal generating rule is either not known or is so complex as to make precise signal description impossible. They are usually modeled as random or stochastic signals.**

Figure 2.11 Random processes in (a) one and (b) two dimensions [Rosenfield and Kak, 1976. Reprinted with permission].



Each sequence from a random process is considered to be a member of an ensemble of signals. Each realization of the random process is chosen randomly from the ensemble by nature.

Stationary Processes

- **At each time instant n, x(n) of the sequence {x(n)} is a particular value of a random variable X(n) which can be described by a set of probability distributions function** $P_{x(n)}(x) = P\{X(n) \le x\}$**.**

- **In case of non-stationary processes, all expectations will be functions of time index n. In wide-sense stationary process, the first two moments of the distribution are independent of time:**

$$\mu_{x(n)} = \mu_{x(n+k)} = \mu_x; \qquad \sigma^2_{x(n)} = \sigma^2_{x(n+k)} = \sigma^2_x \qquad \text{(1-13)}$$

Hence the first and second-order statistics are independent of time shift. In strict-sense stationary process, none of the statistics is affected by a time shift.

Correlation functions

- Correlation functions are used to describe the behavior of r.v.'s X(n) and X(m) of a random process.

Autocovariance function:

$$C_{xx}(m,n) = E[X(m) - \mu_{x(m)})(X(n) - \mu_{x(n)})] \quad \text{(1-14)}$$

Autocorrelation function (acf):

$$R_{xx}(m,n) = E[X(m)X(n)] = C_{xx}(m,n) + \mu_{x(m)}\mu_{x(n)} \quad \text{(1-15)}$$

For zero mean process, the two functions are identical. Both the acf and autocovariance functions depend on the time variables m and n (or m and time lag k = n-m). For wide-sense stationary signals, the joint pdf $p_{x(m)x(n)}(x,y)$ depends only on time shift implying that the acf depends only on n-m=k. We may then write: $R_{xx}(k) = E[X(n)X(n+k)]$ (1-16)

2 Power spectrum of random processes

A stationary random process X(t) is characterized in the frequency domain by its power spectrum $S_x(f)$, which is the Fourier transform of the autocorrelation function $R_x(\tau)$ of the random process.

$$S_X(f) = \Im[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau \qquad \text{(2-1)}$$

$$R_X(\tau) = \Im^{-1}[S_X(f)] = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df \qquad \text{(2-2)}$$

White Noise

is frequently used to model thermal noise (and interference – central limit theorem) in communications systems.

A random process *X*(*t*) is called a white process if it has a flat power spectrum – that is, if $S_X(f)$ is a constant of all f.

- The spectrum of thermal noise $S_n(f)$ achieves its maximum, $kT/2$, at f=0 and goes to 0 slowly as f goes to infinity, where k is the Boltzmann's constant and T is the temperature in kelvins.
- At room temperature, $S_n(f)$ drops to 90% of its maximum at about f = $f \approx 2\times10^{12}$ Hz.

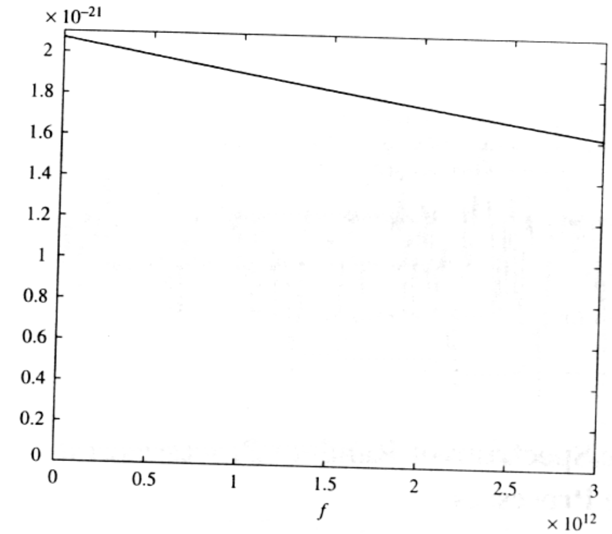


Figure 2.9 Plot of $S_n(f)$ in (2.4.4)

- Therefore, thermal noise can be considered white with power spectrum $kT/2 = N_0$ in practical communications systems.

- The two-sided spectrum density $S_n(f)$ is $N_0/2$. The autocorrelation function $R_x(\tau)=\int_{-\infty}^{\infty}x(t)x(t+\tau)dt$ is

$$R_x(\tau)=\int_{-\infty}^{\infty}S_x(f)e^{j2\pi f\tau}df=\frac{N_0}{2}\int_{-\infty}^{\infty}e^{j2\pi f\tau}df=\frac{N_0}{2}\delta(\tau),$$

where $\delta(\tau)$ is the unit impulse.

- For discrete time signals, the autocorrelation sequence is defined as

$$R_x(m)=\frac{1}{N-m}\sum_{n=1}^{N-m}x[n]x[n+m],\ m=0,1,\ldots,M$$

$$=\frac{1}{N-|m|}\sum_{n=|m|}^{N}x[n]x[n+m],\ m=-1,\ldots,-M.$$

- To avoid the discontinuity at the two ends, a window w[n] tapering at the two ends (Hanning, Hamming, etc) is usually multiplied to x[n]x[n+m].

- **The power spectrum can be computed from the DT-FT of $R_x[m]$:**

$$S_x(f) = \sum_{m=-M}^{M} R_x[m] e^{-j2\pi fm}, \quad ,$$

If *f* is evaluated uniformly over the unit circle, it can be computed efficiently by the DFT.

Exercise (go through illustrative problems 2.4 and 2.5).

3 Power spectrum of random processes

- **Filtering is frequently used to limit out-of-band interference and noise in communications systems. It is desirable to study how it affects our signals.**
- **Consider a stationary random process X(t) passing through a linear time-invariant filter with impulse response h(t). The output of the filter is**

$$Y(t) = \int_{-\infty}^{\infty} X(t)h(t-\tau)d\tau \, .$$

The mean value of Y(t) is

$$m_y = E[Y(t)] = \int_{-\infty}^{\infty} E[X(t)h(t-\tau)]d\tau = m_x \int_{-\infty}^{\infty} h(t-\tau)d\tau = m_x H(0), \quad \text{(3-1)}$$

where *H*(0) is the frequency response *H*(*f*) of the filter evaluated at *f*=0.

The autocorrelation function of *Y*(t) is

$$R_y(\tau) = E[Y(\tau)Y(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(\tau)X(\alpha)]h(t-\tau)h(t+\tau-\alpha)d\tau d\alpha$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(\tau-\alpha)h(t-\tau)h(t+\tau-\alpha)\,d\tau d\alpha \text{.}$$

Taking the Fourier transform, we obtain the power spectrum of Y(t) as

$$S_y(f) = S_x(f)\,|H(f)|^2, \qquad \textbf{(3-2)}$$

- **For discrete-time signals, we have**

$$m_y = E[Y[n]] = \sum_{k=0}^{\infty} h[k]E[X[n-k]] = m_x \sum_{k=0}^{\infty} h[k] = m_x H(0), \qquad \textbf{(3-3)}$$

And $$R_y(m) = E[Y(n)Y(n+m)] = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} h[k]h[l]E[X(n-k)X(n+m-l)]$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} h[k]h[l]R[m-l+k] \text{.}$$

Taking the DT-FT, we obtain the power spectrum of Y[n] as

$$S_y(f) = S_x(f)\,|H(f)|^2, \qquad \textbf{(3-4)}$$

where $S_x(f) = \sum_{m=-\infty}^{\infty} R_x[m]e^{-j2\pi fm}$, $S_y(f) = \sum_{m=-\infty}^{\infty} R_y[m]e^{-j2\pi fm}$.

Exercise: Go through the description and m-files in illustrative problem 2.8.

4. Lowpass and Bandpass processes

- A random process is called **lowpass** if its power spectrum is large in the vicinity of f=0 and small (approaching 0) at high frequencies. In other words, a lowpass random process has most of its power concentrated at low frequencies.
- A lowpass random process X(t) is **bandlimited** if the power spectrum $S_x(f)=0$ for |f|>*B*. The parameter *B* is called the bandwidth of the random process.
- A random process is called **bandpass** if its power spectrum is large in a band of frequencies centered in the neighborhood of a central frequencies $\pm f_0$ and relatively small outside of this band of frequencies. A random process is called **narrowband** if its bandwidth *B*<< f_0.

- **A bandpass random process X(*t*) can be represented as**

$$X(t) = X_c(t)\cos(2\pi f_0 t) - X_s(t)\sin(2\pi f_0 t)$$

where $X_c(t)$ and $X_s(t)$ are called the in-phase and quadrature components of X(t).

- **If X(t) is a zero-mean, stationary random process, the processes $X_c(t)$ and $X_s(t)$ are also zero-mean, jointly stationary processes [covered in communications systems].**

The autocorrelation functions of $X_c(t)$ and $X_s(t)$ are identical and are given by

$$R_c(\tau) = R_s(\tau) = R_x(\tau)\cos(2\pi f_0 t) + \hat{R}_x(\tau)\sin(2\pi f_0 t)$$

where $\hat{R}_x(\tau)$ is the Hilbert transform of the autocorrelation function $R_x(\tau)$: $\hat{R}_x(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(t)}{\tau - t} dt$.

Exercise : Go through the description and m-files in illustrative problems 2.9 and 2.10.