ADVANCED DIGITAL SIGNAL PROCESSING

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- **DISCRETE-TIME SIGNALS AND SYSTEMS**
- **MULTI-DIMENSIONAL SIGNALS AND SYSTEMS**
- **RANDOM PROCESSES AND APPLICATIONS**
- **ADAPTIVE SIGNAL PROCESSING**

DISCRETE-TIME SIGNALS AND SYSTEMS

- DISCRETE-TIME SIGNALS AND SYSTEMS
- Z-TRANSFORM
- DISCRETE-TIME AND DISCRETE FOURIER TRANSFORMS
- FILTER DESIGN

REFERENCES:

A.V. OPPENHEIM AND R.W. SCHAFER,

DISCRETE-TIME SIGNAL PROCESSING.

ENGLEWOOD CLIFFS, NJ: PRENTICE-HALL, INC., 1989.

DISCRETE-TIME SIGNALS AND SYSTEMS

DISCRETE-TIME SIGNALS

DISCRETE-TIME SYSTEMS

LINEAR AND TIME-INVARIANT SYSTEMS

IMPULSE RESPONSE

DEFINITION

Continuous-time signals: **signals defined along a continuum of times $x(t)$.**

Discrete-time signals: **signals defined at discrete times $x[n]$**

Digital signals: **both time and amplitude are discrete**

($x[n]$ amplitude quantized to say 16-bits)

Advantages of digital/discrete-time systems:

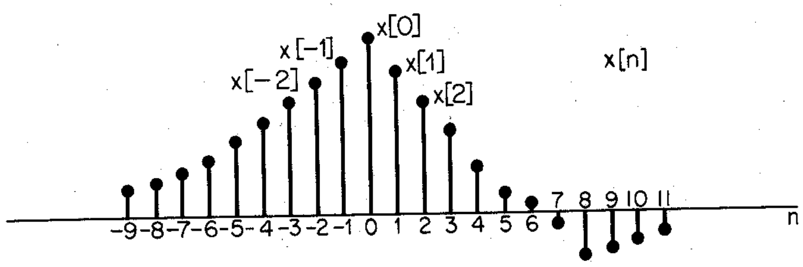
1) Signals are represented as strings of “0” and ‘1”. High noise immunity. Transmission and storage (copy!!) are almost free from errors, if it is done properly.
2) Sophisticated, flexible, and accurate processing (imagine computing with resistors, capacitors, and operational amplifiers!).
3) Efficient realization using digital signal processors (microprocessors doing real-time processing), very large scale integration (VLSI) circuits, etc. …..

1. DISCRETE-TIME SIGNALS

Discrete-time signals are represented mathematically as sequences of numbers:

$$\{x[n]\}, \quad -\infty < n < \infty \qquad \textbf{(1.1)}$$

where *n* is an integer.



Such sequences may arise from periodic sampling of an analog signal $x_a(t)$.

$$x[n] = x_a(nT), \quad -\infty < n < \infty \qquad \textbf{(1.2)}$$

***T* is the period in seconds.**

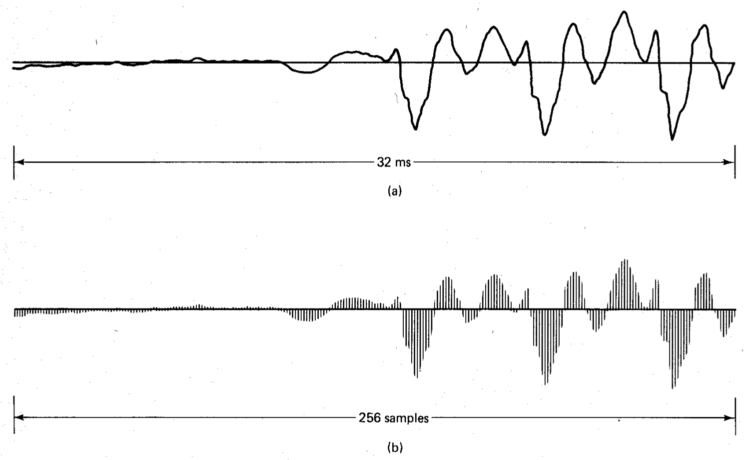


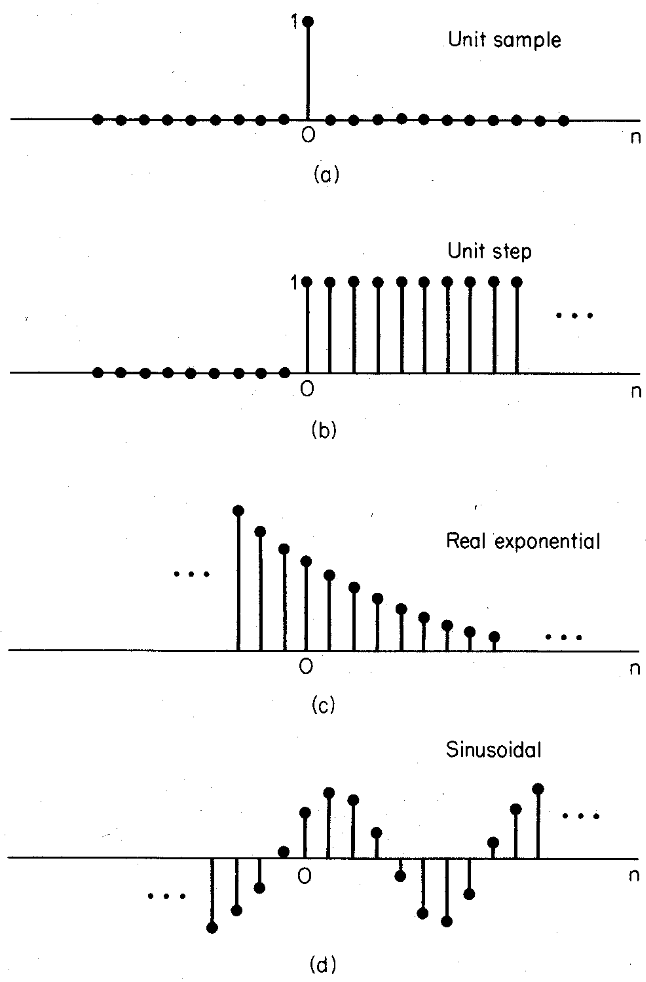
Figure 2.2 (a) Segment of a continuous-time speech signal. (b) Sequence of samples obtained from part (a) with $T = 125\ \mu s$.

1.1. BASIC SEQUENCES AND OPERATORS

1) $y[n]$ is a delayed sequence of $x[n]$ if

$$y[n] = x[n - n_0] \qquad (1.3)$$

where n_0 is an integer.



2) Unit sample sequence (or impulse)

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \qquad (1.4)$$

3) Unit Step sequence

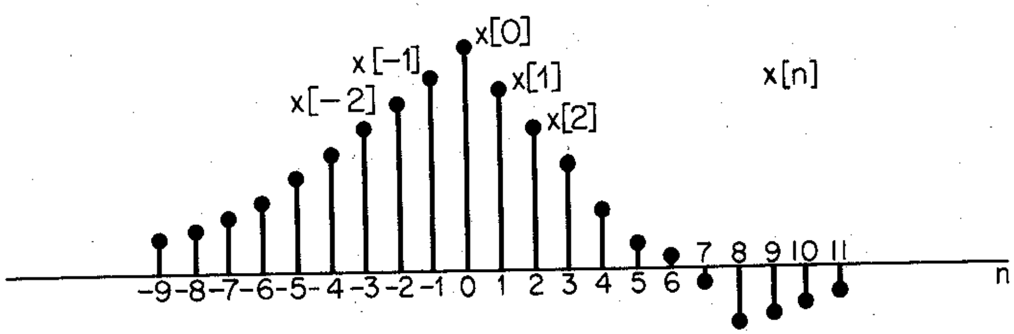
$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad (1.5)$$

4) Exponential and sinusoidal sequences

$$x[n] = A\alpha^n \qquad (1.6)$$

- **Any sequence, $x[n]$, can be expressed as a sum of scaled and delayed impulses**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad \textbf{(1.7)}$$



Example:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{k=-\infty}^{n} \delta[k].$$

- **For complex α and *A*,**

$$\alpha = |\alpha| \cdot e^{j\omega_0} \quad \textbf{and} \quad A = |A| \cdot e^{j\phi} \qquad \textbf{(1.8)}$$

$$x[n] = A\alpha^n = |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi) \qquad \textbf{(1.9)}$$

The sequence oscillates with an exponentially growing envelop if $|\alpha| > 1$ or with an exponentially decaying envelop if $|\alpha| < 1$.

- **When $|\alpha| = 1$, we obtain the complex exponential sequence**

$$x[n] = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi), \qquad \textbf{(1.10)}$$

where ω_0 and ϕ are the frequency and phase of the complex exponential or sinusoidal, respectively.

2. DISCRETE-TIME SYSTEMS

A discrete-time system is defined mathematically as a **transformation or operator** that maps an input sequence with values $x[n]$ into an output sequence with values $y[n]$.

$$y[n] = T\{x[n]\} \qquad (2.1)$$

$$x[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n]$$

Example 2.1 The Ideal Delay System

$$y[n] = x[n - n_d], \quad -\infty < n < \infty \qquad (2.2)$$

n_d is a fixed integer called the delay of the system. The input is shifted right by n_d samples.

Example 2.2 Moving Average

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k] \qquad (2.3)$$

The system averages the $(M_1 + M_2 + 1)$ samples from $n + M_1$ to $n - M_2$.

2.1 MEMORYLESS SYSTEMS

A system is referred to as memoryless if the output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n.

Example 2.3

$$y[n] = (x[n])^2 \qquad (2.4)$$

2.2 LINEAR SYSTEMS

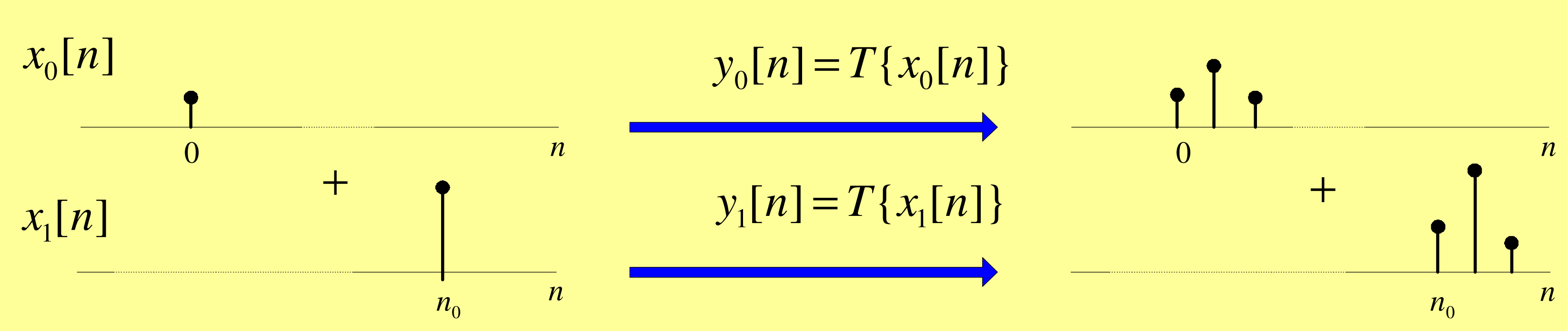
Linear systems satisfy the principle of superposition:

Let $y_1[n]$ **and** $y_2[n]$ **be the responses of a system when the inputs are respectively** $x_1[n]$ **and** $x_2[n]$**. The system is linear if and only if**

$$T\{x_1[n]+x_2[n]\} = T\{x_1[n]\}+T\{x_2[n]\} \qquad \textbf{(2.5)} \qquad \textbf{(additive property)}$$

and $T\{ax[n]\} = aT\{x[n]\} = ay[n]$ **(scaling property)**

where *a* is an arbitrary constant.



or equivalently

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\} \qquad (2.6)$$

- Principle of superposition allows us to add the contributions from individual inputs together, and there are no "coupling" effects between them.

EXERCISE: Show that systems of Examples 2.1 and 2.2 are linear systems while that in Example 2.3 is nonlinear. (Show that they either satisfy or do not satisfy (2.6)).

Example 2.4 Accumulator

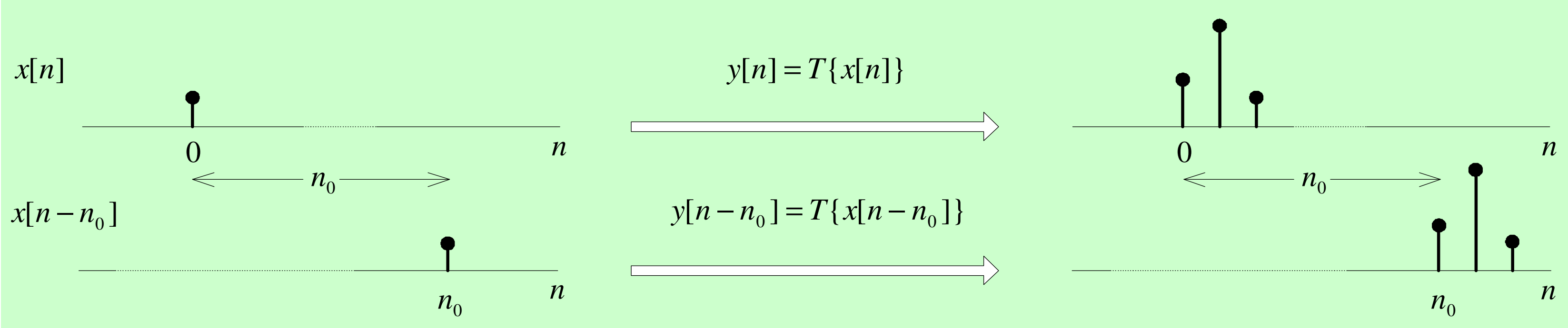
$$y[n] = \sum_{k=-\infty}^{n} x[k].$$

The output $y[n]$ is the sum of all previous inputs up to the current time instant *n*.

2.3 TIME-INVARIANT SYSTEMS

A time-invariant system is one for which a time shift of the input sequence causes a corresponding shift in the output sequence, i.e.

If $y[n] = T\{x[n]\}$, **then** $y[n-n_0] = T\{x[n-n_0]\}$ **for all integer** n_0. **(2.7)**



Example 2.5 Downsampler (Left as an exercise)

$$y[n] = x[Mn], \quad -\infty < n < \infty \qquad (2.8)$$

M is a positive integer.

It discards $(M-1)$ samples every M samples. It is not time-invariant unless $M=1$.

2.2.4 CAUSALITY

A system is **causal** if for every choice of n_0, the output sequence value at index $n = n_0$ depends only on the input sequence values for $n \le n_0$.

This implies that: If $x_1[n] = x_2[n]$ for $n \le n_0$, then $y_1[n] = y_2[n]$ for $n \le n_0$.

Example 2.6 Forward difference system

$$y[n] = x[n+1] - x[n] \qquad (2.9)$$

The system is non-causal (finitely non-causal) because it involves a future input sample $x[n+1]$.

- Non-causal systems are more difficult to implement (e.g. $y[n] = 0.9 \cdot y[n+1] + x[n]$- solving a difference equation with a proper initial condition!).

2.5 STABILITY

- **A system is stable in the bounded-input bounded-output (BIBO) sense if and only if "every" bounded input sequence produces a bounded output sequence.**

The input $x[n]$ is bounded if there exists a fixed positive finite value B_x such that

$$|x[n]| \le B_x < \infty \quad \text{for all } n. \tag{2.10}$$

Stability requires that for every bounded input there exists a fixed positive finite value B_y such

that : $$|y[n]| \le B_y < \infty \quad \text{for all } n. \tag{2.11}$$

- **Examples 2.1, 2.2, 2.3, and 2.5 are stable systems.**

- **The accumulator of Example 2.4 is unstable because for $x[n] = u[n]$ (the unit step input),**

$$y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0, & n < 0 \\ (n+1), & n \geq 0 \end{cases} \qquad (2.12)$$

There is no fixed finite value B_y such that $(n+1) \leq B_y < \infty$ for all *n*.

3. LINEAR TIME-INVARIANT SYSTEMS

A linear system is completely characterized by its impulse response.

Let $h_k[n] = T\{\delta[n-k]\}$ be the response of the system to $\delta(n-k)$, an impulse occurring at $n = k$. In general, $h_k(n)$ depends on both *n* and *k*.

Consider the output of a system T{.} to *x*[n]:

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} \qquad (3.1)$$

Use (1.7) and (2.1)

From the principle of superposition, we have

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n] \qquad (3.2)$$

With the additional constraint of time-invariant ($y[n-n_0] = T\{x[n-n_0]\}$), we have $h_k[n] = h(n-k)$ and

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n] \quad (3.3)$$

* denotes discrete-time convolution.

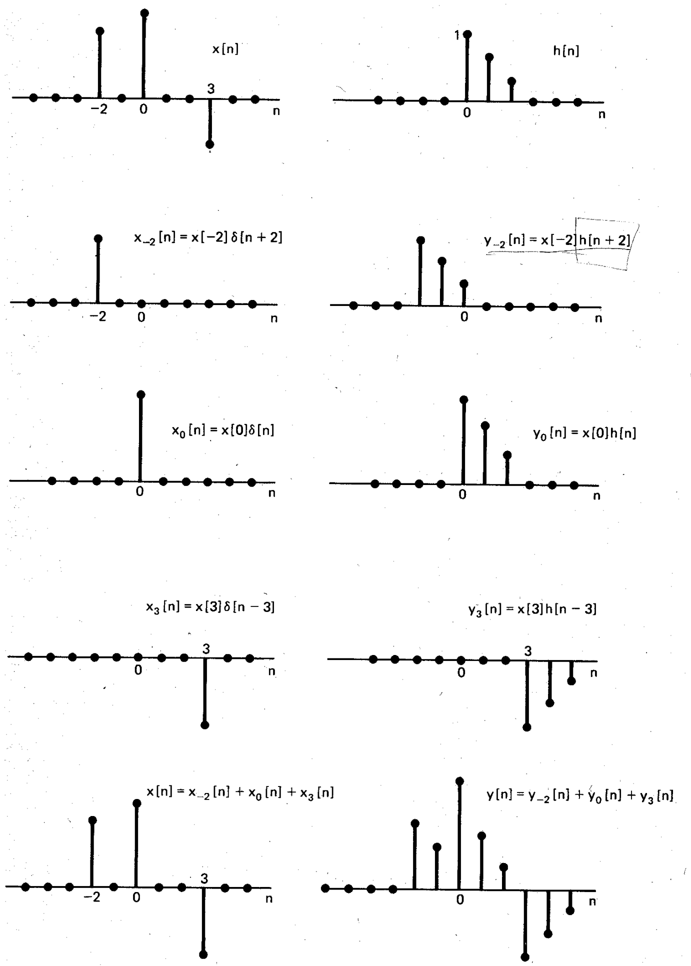


Figure 2.8 Representation of the output of a linear time-invariant system as the superposition of responses to individual samples of the input.

Direct Computation of discrete-time convolution

- It can also be computed efficiently using discrete Fourier transform (DFT).
- How about if the impulse response is of infinite length?

FINITE-DURATION IMPULSE RESPONSE (FIR) SYSTEMS

If the impulse response $h[n]$ of a LTI system is of finite duration, i.e.

$$h[n] \neq 0 \qquad -\infty < N_1 \leq n \leq N_2 < \infty,$$

it is called a finite-duration impulse response (FIR) filter or system.

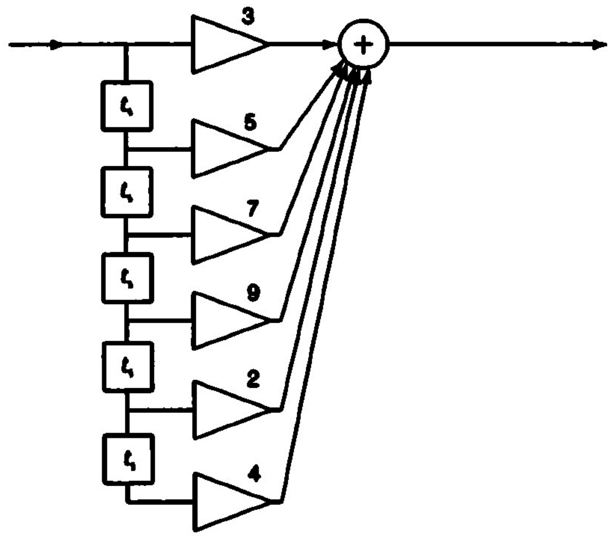
$$y[n] = \sum_{k=0}^{M} h[k] \cdot x[n-k] \qquad (3.4)$$

- **The impulse response is**

  $h[n]$=0, n<0 (causal),

  $h[0]$=3, $h[1]$=5, $h[2]$=7, $h[3]$=9,
  $h[4]$=2, $h[5]$=4.
  $h[n]$=0, n>5.

- **M is called the order/degree of the system. M+1 is the filter length.**



Structure (signal flow graph) of a non-recursive filter.

- **For causal system, $h[n] = 0$ for $n < 0$. Other commonly used names of FIR filters are non-recursive filters and moving average (MA) filters.**

INFINITE-DURATION IMPULSE RESPONSE (IIR) SYSTEMS

If the impulse response $h[n]$ of a LTI system is of infinite duration, it is called infinite-duration impulse response (IIR) systems.

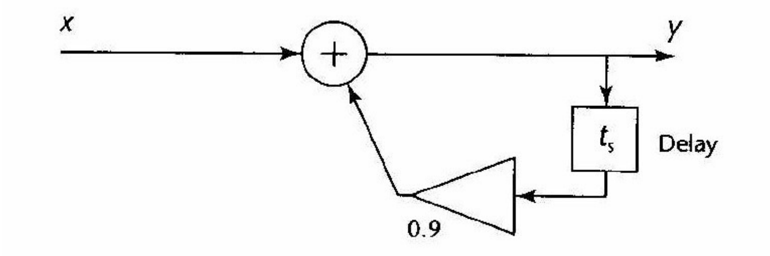
Example 1: The impulse response of the accumulator is infinite in duration, which belongs to the class of IIR systems.

Example 2: $y[n] = a \cdot y[n-1] + x[n]$. The filter output is obtained through a recurrence relation, instead of from the discrete-time convolution.

- The impulse response is

  $h[n]=0$, $n<0$ (causal),

  $h[0]=1$, $h[1]=a$, $h[2]=a^2$, ….



Structure (signal flow graph) of a simple recursive filter.

- The impulse response is equal to $h[n] = a^n u[n]$. Is it stable? Under what condition is it stable?

4. LINEAR CONSTANT COEFFICIENT DIFFERENCE EQUATIONS

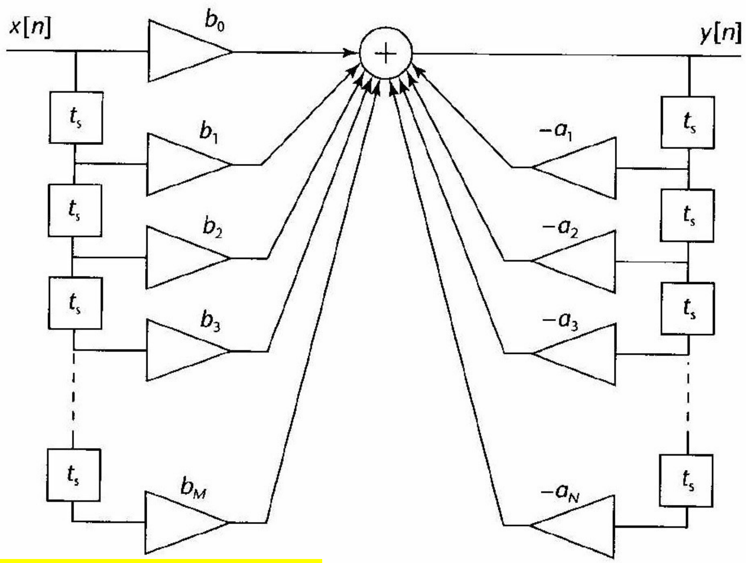
An important subclass of LTI systems are those with input $x[n]$ and output $y[n]$ satisfy an *N*th-order linear constant-coefficient difference equation of form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \quad a_1 = 1 \quad \textbf{(4.1)}$$

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \quad \textbf{(4.2)}$$

Feedback **Feedfoward**

- **Although the impulse response is of infinite duration and its output is still given by the discrete-time convolution of *x*[*n*] and *h*[*n*], it is not employed to compute the system output.**



Feedforward (non-recursive) part

Feedback (recursive) part

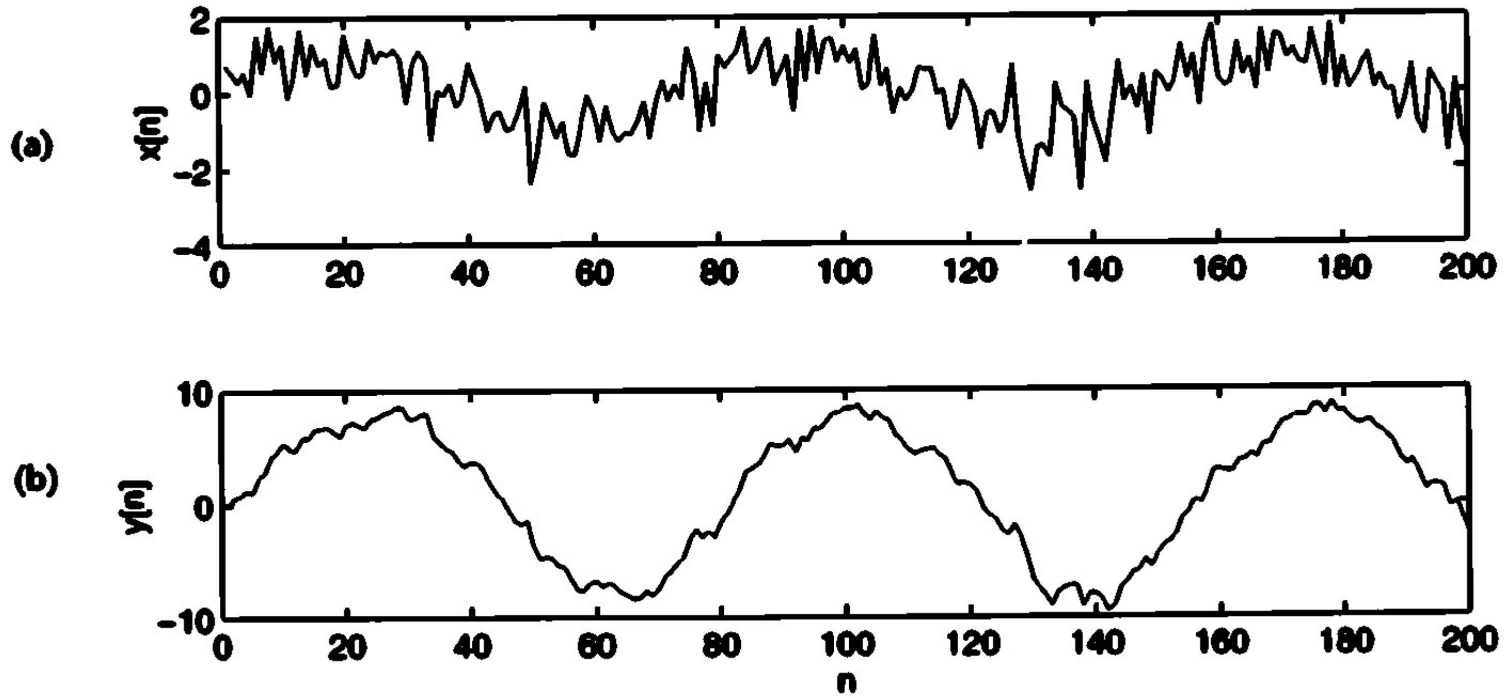
Structure of a recursive (IIR) digital network

MATLAB simulation

•The MATLAB file contains data for n=1:200

```
        x(n)=sin(n/12)+.6*randn;
        % this generates the artificial data set of a random
        % Gaussian distributed random variable added to a sinusoid
    end
    save C14data.mat x
    load C14data.mat
    % this calls up the data file of this name and
    % hence places the variable x in the MATLAB workplace
        y=zeros(size(x));
        % this initializes the output values to be zero
    for n=2:200
        % we start at n=2 so that the index of the first
        % element of the output array to be addressed is one
        y(n)=x(n)+.9*y(n-1);
    end
    subplot(2,1,1); plot(x);
    ylabel('x[n]')
    subplot(2,1,2); plot(y);
    ylabel('y[n]'),xlabel('n')
```

MATLAB simulation .....



Effect of simple recursive filtering (a) input signal (b) output signal.

- The noise is reduced and the sinusoid is enhanced. A filter can do a lot more…

5. PROPERTIES OF LTI SYSTEMS

1) Commutative:

$$x[n] * h[n] = h[n] * x[n] \qquad \textbf{(5.1)}$$

Letting $m = n - k$ in (3.3) leads to the desire results.

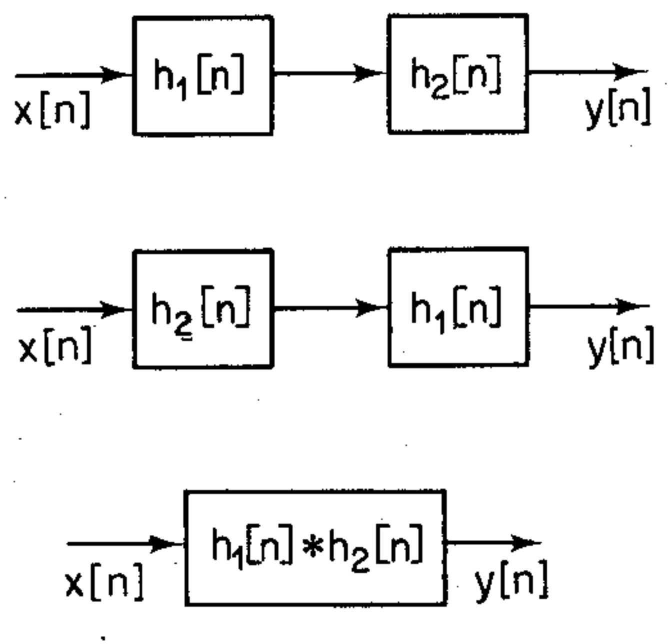
2) Distributive:

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n] \qquad \textbf{(5.2)}$$

3) Systems in Cascade:

$$h[n] = h_1[n] * h_2[n] \qquad \textbf{(5.3)}$$

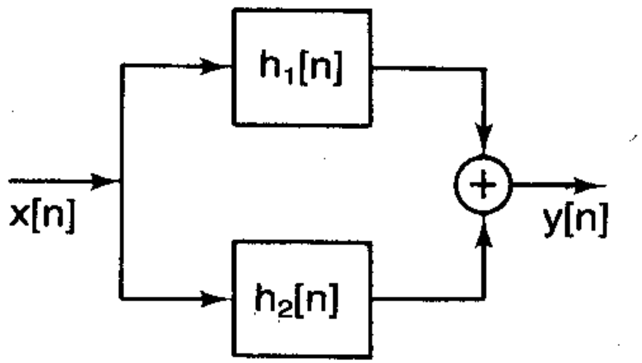
Consider the response of the system to an impulse



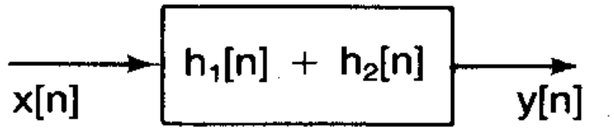
System in cascade.

4) Systems in Parallel

$$h[n] = h_1[n] + h_2[n] \qquad \textbf{(5.4)}$$



(a)



(b)

Consider the response of the system to a single impulse.

6. STABILITY OF LTI SYSTEMS (the proof can be omitted for 1st reading)

Linear time-invariant systems are stable if and only if the impulse response is absolutely summable, i.e. if

$$S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty \qquad (6.1)$$

Proof:

Sufficient: From (3.3),

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| < \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \qquad (6.2)$$

If $x[n]$ is bounded so that $|x[n]| \leq B_x$, then

$$|y[n]| \leq B_x \sum_{k=-\infty}^{\infty} |h[k]| \qquad (6.3)$$

Therefore, $S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$ **implies the system is stable.**

Necessary: Since a unstable system does not necessary give a unbounded output for every input. We must show that if $S=\infty$, then a bounded input can be found that will cause an unbound output. The sequence is

$$x[n]=\begin{cases}\dfrac{h^*[-n]}{|h[-n]|}, & h[n]\neq 0\\ 0, & h[n]=0\end{cases}, \quad \text{* complex conjugate.} \qquad (6.4)$$

is bounded by unity. However,

$$y[0]=\sum_{k=-\infty}^{\infty}x[-k]h[k]=\sum_{k=-\infty}^{\infty}\frac{|h[k]|^2}{|h[k]|}=S$$

Thus, if $S=\infty$, the system is unstable. For the system to be stable $S<\infty$.

- For the ideal delay, moving average, forward difference, and backward difference examples, it is clear that $S<\infty$ since their impulse responses are only of finite duration.

- **FIR systems are always be stable as long as each of the impulse response is finite in magnitude.**
- **For IIR systems, it is easier to infer the stability from the poles of their z-transform.**