MULTIDIMENSIONAL DISCRETE-TIME SIGNALS AND SYSTEMS

CONTENTS

REFERENCES:

D. E. DUDGEON AND R. M. MERSEREAU,

MULTIDIMENSIONAL DIGITAL SIGNAL PROCESSING.

PHI., 1984.

1. TWO DIMENSIONAL DISCRETE SIGNALS

A two-dimensional (2-D) discrete signal (sequence) is a sequence defined over the set of ordered pairs of integers:

$$x = \{x(n_1, n_2), -\infty < n_1, n_2 < \infty\}. \quad \textbf{(1-1)}$$

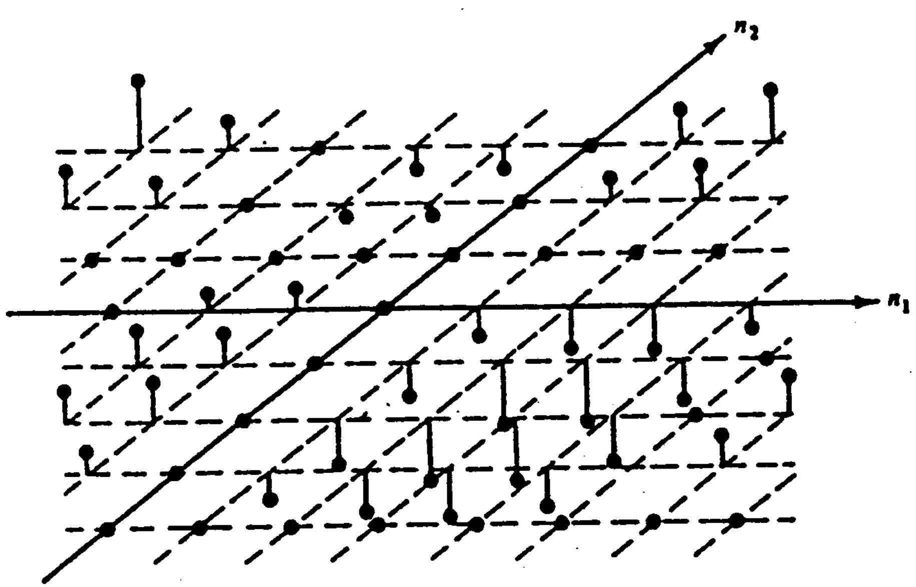
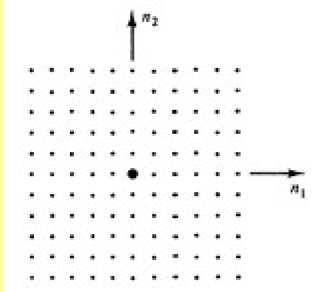


Figure 1.1 Graphical representation of a two-dimensional sequence.

1.1 Some special sequences

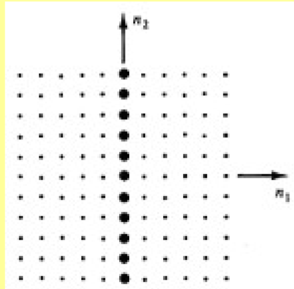
- **2D unit impulse,** $\delta(n_1, n_2)$

$$\delta(n_1,n_2) = \begin{cases} 1, n_1 = n_2 = 0 \\ 0 otherwise \end{cases}. \tag{1-2}$$

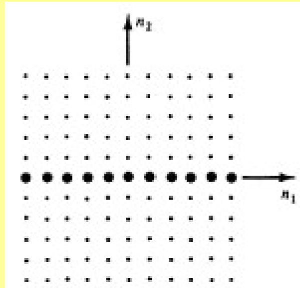


- **2D line impulse**

$$x(n_1,n_2) = \delta(n_1). \tag{1-3a}$$

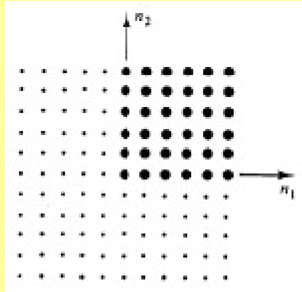


$$y(n_1,n_2) = \delta(n_2). \tag{1-3b}$$



■ 2D unit step sequence

$$u(n_1,n_2)=\begin{cases}1 & n_1\geq 0, n_2\geq 0\\ 0 & otherwise\end{cases}. \qquad (1\text{-}4)$$



$$u(n_1,n_2)=u(n_1)u(n_2).$$

■ Exponential Sequences:

$$x(n_1,n_2)=a^{n_1}b^{n_2};a,b:complex. \qquad (1\text{-}5)$$

1.2 Separable sequences

Any sequence that can be expressed as the product of 1-D sequences as follows:

$$x(n_1,n_2)=x_1(n_1)x_2(n_2). \qquad (1\text{-}6)$$

is called separable.

1.3 Finite-extent sequences

Finite-extent 2D sequences are zero outside a region of finite extent in the (n_1, n_2)-plane. This region is called the region of support.

1.4 Periodic sequences

$$\tilde{x}(n_1 + N_{11}, n_2 + N_{21}) = \tilde{x}(n_1, n_2). \quad \text{(1-7a)}$$

$$\tilde{x}(n_1 + N_{12}, n_2 + N_{22}) = \tilde{x}(n_1, n_2). \quad \text{(1-7b)}$$

$$D = \left|\det \boldsymbol{N}\right| = N_{11}N_{22} - N_{12}N_{21} \neq 0. \quad \text{(1-7c)}$$

One period of such a sequence is contained in the parallelogram-shaped region whose adjacent sides are formed by $\boldsymbol{N}_1 = (N_{11}, N_{21})^T$ and $\boldsymbol{N}_2 = (N_{12}, N_{22})^T$. The number of samples in this region is $|D| = |\det \boldsymbol{N}|$.

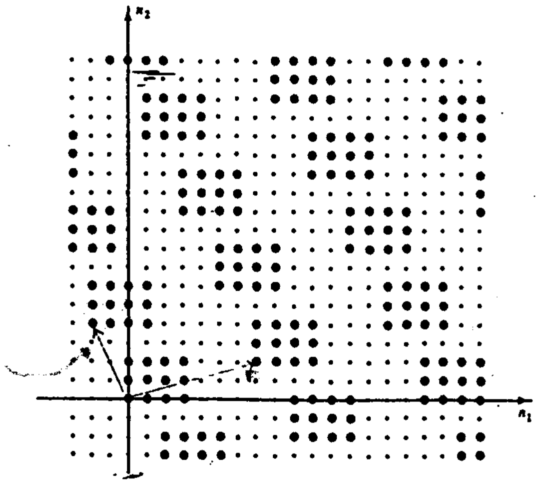


Figure 1.8 Periodic sequence with periodicity vectors (7, 2)′ and (−2, 4)′.

$\tilde{x}(\boldsymbol{n})$ is an *M*-dimensional periodic sequence if there exist *M* linearly independent *M*-dimensional integer vectors, ***N**i*, such that

$$\tilde{x}(\boldsymbol{n} + \boldsymbol{N}_i) = \tilde{x}(\boldsymbol{n}), \ i = 1, \ldots, M. \tag{1-8}$$

$\boldsymbol{N} = [\boldsymbol{N}_1, \boldsymbol{N}_2, \cdots, \boldsymbol{N}_M]$ is called the periodicity matrix and:

$$\tilde{x}(\boldsymbol{n} + \boldsymbol{N}\boldsymbol{r}) = \tilde{x}(\boldsymbol{n}). \qquad \boldsymbol{n} = [n_1, n_2]^T. \tag{1-9}$$

2. MULTIDIMENSIONAL SYSTEMS

A system is an operator that maps one signal (the input $x(n_1,n_2)$) into another (the output $y(n_1,n_2)$).

$$y = T[x] \qquad\qquad x(n_1,n_2) \longrightarrow \boxed{T[\cdot]} \longrightarrow y(n_1,n_2)$$

2.1 Fundamental operations on multidimensional signals

Addition:

$$y(n_1,n_2) = x(n_1,n_2) + w(n_1,n_2) \tag{2-1}$$

Multiplication by a constant:

$$y(n_1,n_2) = cx(n_1,n_2) \tag{2-2}$$

Shifting:

$$y(n_1,n_2) = x(n_1 - m_1, n_2 - m_2) \tag{2-3}$$

Multiplication of sequences:

$$y(n_1,n_2) = c(n_1,n_2)x(n_1,n_2) \tag{2-4}$$

Any 2-D sequence can be written as a sum of weighted and shifted 2-D unit impulses:

$$x(n_1,n_2) = \sum_{k_1=\infty}^{\infty}\sum_{k_2=\infty}^{\infty} x(k_1,k_2)\delta(n_1 - k_1, n_2 - k_2). \tag{2-5}$$

2.1.1 Linear systems

A system is linear if and only if it satisfies:

If $y_1 = L[x_1]; y_2 = L[x_2]$, (2-6a)

then $ay_1 + by_2 = L[ax_1 + bx_2]$. (2-6b)

If the system is linear, we have:

$$y(n_1, n_2) = L\left[\sum_{k_1=\infty}^{\infty}\sum_{k_2=\infty}^{\infty} x(k_1, k_2)\delta(n_1 - k_1, n_2 - k_2)\right].$$

$$y(n_1, n_2) = \sum_{k_1=\infty}^{\infty}\sum_{k_2=\infty}^{\infty} x(k_1, k_2)L[\delta(n_1 - k_1, n_2 - k_2)] = \sum_{k_1=\infty}^{\infty}\sum_{k_2=\infty}^{\infty} x(k_1, k_2)h_{k_1k_2}(n_1, n_2). \quad (2\text{-}7)$$

where $h_{k_1k_2}$ is the response of the system to a unit impulse located at (k_1, k_2).

2.1.2 Shift-Invariant systems

A **shift-invariant system** is one for which a shift in the input sequence implies a corresponding shift in the output sequence.

If $y(n_1,n_2) = T[x(n_1,n_2)]$, the system is shift invariant iff

$$T[x(n_1-m_1,n_2-m_2)] = y(n_1-m_1,n_2-m_2). \tag{2-8}$$

for all sequences *x* and all integer shifts (m_1,m_2).

2.1.3 Linear Shift-Invariant systems

The principle of shift invariance implies:

$$h_{k_1 k_2}(n_1, n_2) = h_{00}(n_1 - k_1, n_2 - k_2). \quad \textbf{(2-9)}$$

Defining $h(n_1, n_2) = h_{00}(n_1, n_2)$, **(2-7) is simplified to:**

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) h(n_1 - k_1, n_2 - k_2). \quad \textbf{(2-10)}$$

This is a 2-D convolution sum. Using vector notation, the output of an M-D LTI system is:

$$y(\boldsymbol{n}) = \sum_{\boldsymbol{k}} x(\boldsymbol{k}) h(\boldsymbol{n} - \boldsymbol{k}). \quad \boldsymbol{n} = [n_1, n_2]^T, \ \boldsymbol{k} = [k_1, k_2]^T. \quad \textbf{(2-11)}$$

2.1.4 Separable systems

A separate system is an LTI system whose impulse response is a separable sequence.

$$h(n_1,n_2) = h_1(n_1)h_2(n_2). \tag{2-12}$$

The output of the system is:

$$y(n_1,n_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} x(n_1-k_1,n_2-k_2)h_1(k_1)h_2(k_2)$$

$$= \sum_{k_1=-\infty}^{\infty} h_1(k_1)\left[\sum_{k_2=-\infty}^{\infty} x(n_1-k_1,n_2-k_2)h_2(k_2)\right] = \sum_{k_1=-\infty}^{\infty} h_1(k_1)y(n_1-k_1,n_2)$$

The 2-D convolution is then reduced to a number of 1-D convolutions.

2.1.5 Stable systems

Bound input, bounded output stable (BIBO): When $|x(n_1,n_2)| \le B,$, there must exist a *B*' such that $|y(n_1,n_2)| \le B'$, for all (n_1,n_2). A necessary and sufficient condition for an LTI system to be BIBO stable is that its impulse response be absolutely summable:

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} |h(n_1,n_2)| = S_1 < \infty. \tag{2-13}$$

A weaker form of stability is mean-square stability. An LTI system is mean-square stable if:

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} |h(n_1,n_2)|^2 = S_1 < \infty. \tag{2-14}$$

Multidimensional stability is far more difficult both to understand and to test than 1-D. Commonly used filters are of finite impulse duration (FIR).

3. FREQUENCY-DOMAIN CHARACTERIZATION OF SIGNALS AND SYSTEMS

3.1 Frequency response of a 2-D LTI system

If the input is a complex sinusoidal of the form

$$x(n_1,n_2) = \exp(j\omega_1 n_1 + j\omega_2 n_2). \quad \text{(3-1)}$$

The output of an LTI system is:

$$y(n_1,n_2) = \exp(j\omega_1 n_1 + j\omega_2 n_2)H(\omega_1,\omega_2), \quad \text{(3-2)}$$

where $$H(\omega_1,\omega_2) = \sum_{n_1}\sum_{n_2} h(n_1,n_2)\exp(-j\omega_1 n_1 - j\omega_2 n_2), \quad \text{(3-3)}$$

or $$H(\boldsymbol{\omega}) = \sum_{\boldsymbol{n}} h(\boldsymbol{n})\exp(-j\boldsymbol{\omega}^T\boldsymbol{n}),\ \boldsymbol{\omega} = [\omega_1,\omega_2]^T,\ \boldsymbol{n} = [n_1,n_2]^T \quad \text{(3-4)}$$

is called the systems's frequency response. This is also the Fourier transform of the impulse sequence. It is periodic in the horizontal and vertical frequency variables with a periodic 2π.

3.1.2 Frequency response and inverse Fourier transform

The inverse relationship can be derived by multiplying both sizes of equation (3-3) by a complex sinusoid and integrating over a square in the frequency plane.

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} H(\omega_1,\omega_2)\exp(j\omega_1 k_1 + j\omega_2 k_2)d\omega_1 d\omega_2$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{n_1}\sum_{n_2} h(n_1,n_2)\exp(-j\omega_1(n_1-k_1) - j\omega_2(n_2-k_2))d\omega_1 d\omega_2 \qquad \text{(3-5)}$$

$$= \sum_{n_1}\sum_{n_2} h(n_1,n_2)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-j\omega_1(n_1-k_1)d\omega_1\right]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-j\omega_2(n_2-k_2)d\omega_2\right]$$

The term inside the bracket can be shown to be the unit impulse at $n_1 - k_1(n_2 - k_2)$. Finally we have the inverse Fourier transform of $H(\omega_1,\omega_2)$ as:

$$h(n_1,n_2) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} H(\omega_1,\omega_2)\exp(j\omega_1 n_1 + j\omega_2 n_2)d\omega_1 d\omega_2 . \qquad \text{(3-6)}$$

3.1.3 Multidimensional Fourier transform

The Fourier transform pair are:

$$X(\omega_1,\omega_2)=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}x(n_1,n_2)\exp(-j\omega_1 n_1-j\omega_2 n_2), \qquad \text{(3-7)}$$

$$x(n_1,n_2)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}X(\omega_1,\omega_2)\exp(j\omega_1 n_1+j\omega_2 n_2)d\omega_1 d\omega_2. \qquad \text{(3-8)}$$

Suppose that we have a 2-D LTI system L[.] with impulse response, $h(n_1,n_2)$ and a frequency response $H(\omega_1,\omega_2)$. Using the linearity property and the Fourier representation of $x(n_1,n_2)$, we have:

$$\begin{aligned} y(n_1,n_2)=L[x(n_1,n_2)]&=\frac{1}{4\pi^2}L\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}X(\omega_1,\omega_2)\exp(j\omega_1 n_1+j\omega_2 n_2)d\omega_1 d\omega_2\right] \\ &=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}X(\omega_1,\omega_2)L[\exp(j\omega_1 n_1+j\omega_2 n_2)]d\omega_1 d\omega_2 \end{aligned} \qquad \text{(3-9)}$$

Using (3-2), we get:

$$y(n_1,n_2)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}H(\omega_1,\omega_2)X(\omega_1,\omega_2)\exp(j\omega_1 n_1+j\omega_2 n_2)d\omega_1 d\omega_2 . \qquad \textbf{(3-10)}$$

Comparing with the Fourier transform representation of $y(n_1,n_2)$, we have

$$Y(\omega_1,\omega_2)=H(\omega_1,\omega_2)X(\omega_1,\omega_2). \qquad \textbf{(3-11)}$$

This result is the convolution theorem: the Fourier transform of the convolution of two 2-D sequences is the product of their Fourier transforms.

The Fourier tranform can be shown to exist whenever the sequence is absolutely summable. When the sequence is only squared summable, their Fourier transform is well defined except at points of discontinuity.

4. SAMPLING CONTINUOUS SIGNAL

We shall only consider periodic sampling with rectangular geometry. Other sampling such as the hexagonal sampling is possible, however, it is rather involved and will not be discussed here.

In rectangular sampling, the discrete signal, $x(n_1, n_2)$, is obtained from the analog signal, $x_a(t_1, t_2)$

$$x(n_1, n_2) = x_a(n_1T_1, n_2T_2), \tag{4-1}$$

where T_1 and T_2 are sampling intervals or periods.

Define the 2-D Fourier Transform relations for continusous signals:

$$X_a(\Omega_1, \Omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_a(t_1, t_2)\exp(-j\Omega_1 t_1 - j\Omega_2 t_2)dt_1 dt_2, \tag{4-2}$$

Using (3-7) and (3-8), we can write

$$x(n_1,n_2)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X_a(\Omega_1,\Omega_2)\exp(j\Omega_1 n_1 T_1+j\Omega_2 n_2 T_2)d\Omega_1 d\Omega_2, \qquad \text{(4-3)}$$

Making the substituting $\omega_1=\Omega_1 T_1;\omega_2=\Omega_2 T_2$ and let $SQ(k_1,k_2)$ be the square

$$-\pi+2\pi k_1\le\omega_1\le\pi+2\pi k_1, \text{n} -\pi+2\pi k_2\le\omega_2\le\pi+2\pi k_2$$

(4-3) can be written as:

$$x(n_1,n_2)=\frac{1}{4\pi^2}\sum_{k_1}\sum_{k_2}\iint_{SQ(k_1,k_2)}\frac{1}{T_1T_2}X_a(\frac{\omega_1}{T_1},\frac{\omega_1}{T_2})\exp(j\omega_1 n_1+j\omega_2 n_2)d\omega_1 d\omega_2$$

$$=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T_1T_2}\sum_{k_1}\sum_{k_2}X_a(\frac{\omega_1-2\pi k_1}{T_1},\frac{\omega_2-2\pi k_2}{T_2})\right] \qquad \text{(4-4)}$$

$$\times\exp(j\omega_1 n_1+j\omega_2 n_2)\exp(-j2\pi k_1 n_1+j2\pi k_2 n_2)d\omega_1 d\omega_2$$

Using the inverse discrete time Fourier transform relation, we have:

$$X(\omega_1,\omega_2)=\frac{1}{T_1T_2}\sum_{k_1}\sum_{k_2}X_a\left(\frac{\omega_1-2\pi k_1}{T_1},\frac{\omega_2-2\pi k_2}{T_2}\right), \qquad (4\text{-}5)$$

This is the relationship between the discrete-time and continuous- time Fourier transform of the discrete-time and continuous-time signals, respectively.

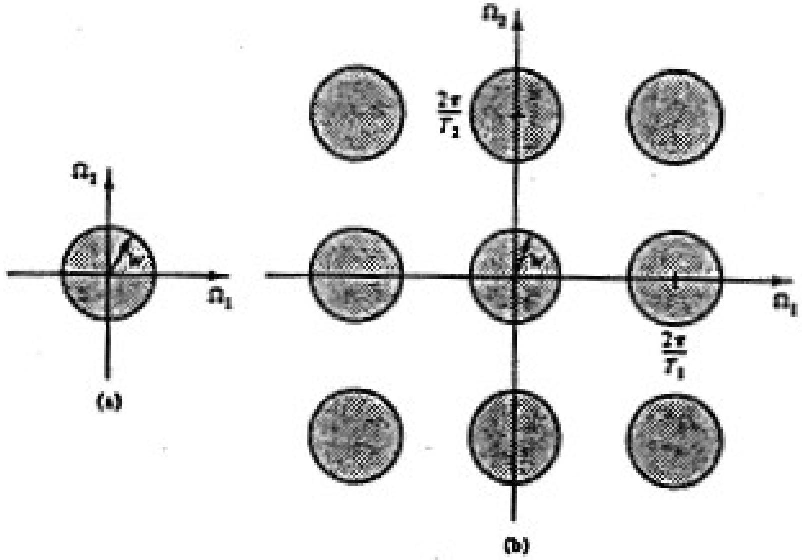


Figure 1.24 (a) Fourier transform of a continuous bandlimited signal. (b) Periodic extension of the transform.

It can be seen that if $X_a(\Omega_1,\Omega_2)$ is bandlimited to

$$X_a(\Omega_1,\Omega_2) = 0 \text{ for } |\Omega_1| \geq \frac{\pi}{T_1}, |\Omega_2| \geq \frac{\pi}{T_2}, \qquad \text{(4-6)}$$

then $X(\Omega_1,\Omega_2) = \dfrac{1}{T_1 T_2} X_a(\Omega_1,\Omega_2)$, **for** $|\Omega_1| \leq \dfrac{\pi}{T_1}, |\Omega_2| \leq \dfrac{\pi}{T_2}$.

As long as $X_a(\Omega_1,\Omega_2)$ satisfies (4-6), it can be recovered from $X(\Omega_1 T_1,\Omega_2 T_1)$ by ideal lowpass filtering.