QCB 508 – Week 4

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Spring 2017

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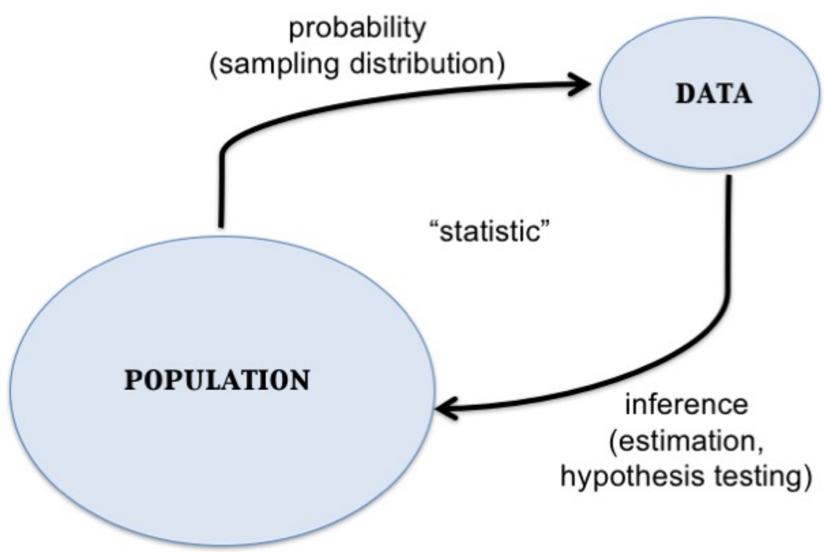
Probability and Statistics

Roles In Data Science

Probabilistic modeling and/or statistical inference are required in data science when the goals include:

1. Characterizing randomness or "noise" in the data
2. Quantifying uncertainty in models we build or decisions we make from the data
3. Predicting future observations or decisions in the face of uncertainty

Central Dogma of Inference



Data Analysis Without Probability

It is possible to do data analysis without probability and formal statistical inference:

- Descriptive statistics can be reported without utilizing probability and statistical inference
- Exploratory data analysis and visualization tend to not involve probability or formal statistical inference
- Important problems in machine learning do not involve probability or statistical inference.

Probability

Sample Space

- The **sample space** Ω is the set of all **outcomes**
- We are interested in calculating probabilities on relevant subsets of this space, called **events**: $A \subseteq \Omega$
- Examples —
  - Two coin flips: $\Omega = \{\text{HH, HT, TH, TT}\}$
  - Netflix movie rating: $\Omega = \{1, 2, 3, 4, 5\}$
  - Number of lightning strikes on campus: $\Omega = \{0, 1, 2, 3, \ldots\}$
  - Height of adult humans in meters: $\Omega = [0, \infty)$

Measure Theoretic Probabilty

$$(\Omega, \mathcal{F}, \Pr)$$

- Ω is the sample space
- $\mathcal{F}$ is the σ-algebra of events where probability can be measured
- $\Pr$ is the probability measure

Mathematical Probability

A proper mathematical formulation of a probability measure should include the following properties:

1. The probability of any even A is such that $0 \leq \Pr(A) \leq 1$
2. If Ω is the sample space then $\Pr(\Omega) = 1$
3. Let A^c be all outcomes from Ω that are not in A (called the *complement*); then $\Pr(A) + \Pr(A^c) = 1$
4. For any n events such that $A_i \cap A_j = \varnothing$ for all $i \neq j$, then $\Pr\left(\cup_{i=1}^n A_i\right) = \sum_{i=1}^n \Pr(A_i)$, where $\varnothing$ is the empty set

Union of Two Events

The probability of two events are calculated by the following general relationship:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

where we note that $\Pr(A \cap B)$ gets counted twice in $\Pr(A) + \Pr(B)$.

Conditional Probability

An important calclation in probability and statistics is the conditional probability. We can consider the probability of an event A, conditional on the fact that we are restricted to be within event B. This is defined as:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Independence

Two events A and B by definition independent when:

- $\Pr(A|B) = \Pr(A)$
- $\Pr(B|A) = \Pr(B)$
- $\Pr(A \cap B) = \Pr(A) \Pr(B)$

All three of these are equivalent.

Bayes Theorem

A common approach in statistics is to obtain a conditional probability of two events through the opposite conditional probability and their marginal probability. This is called Bayes Theorem:

$$\Pr(B|A) = \frac{\Pr(A|B) \Pr(B)}{\Pr(A)}$$

This forms the basis of *Bayesian Inference* but has more general use in carrying out probability calculations.

Law of Total Probability

For events $A_1, \ldots, A_n$ such that $A_i \cap A_j = \varnothing$ for all $i \neq j$ and $\cup_{i=1}^n A_i = \Omega$, it follows that for any event B:

$$\Pr(B) = \sum_{i=1}^{n} \Pr(B|A_i) \Pr(A_i).$$

Random Variables

Definition

A random variable X is a function from Ω to the real numbers:

$$X : \Omega \to \mathbb{R}$$

For any outcome in Ω, the function $X(\omega)$ produces a real value.

We will write the range of X as

$$\mathcal{R} = \{X(\omega) : \omega \in \Omega\}$$

where $\mathcal{R} \subseteq \mathbb{R}$.

Distributon of RV

We define the probability distribution of a random variable through its **probability mass function** (pmf) for discrete rv's or its **probability density function** (pdf) for continuous rv's.

We can also define the distribution through its **cumulative distribution function** (cdf). The pmf/pdf determines the cdf, and vice versa.

Discrete Random Variables

A discrete rv X takes on a discrete set of values such as $\{1, 2, \ldots, n\}$ or $\{0, 1, 2, 3, \ldots\}$.

Its distribution is characterized by its pmf

$$f(x) = \Pr(X = x)$$

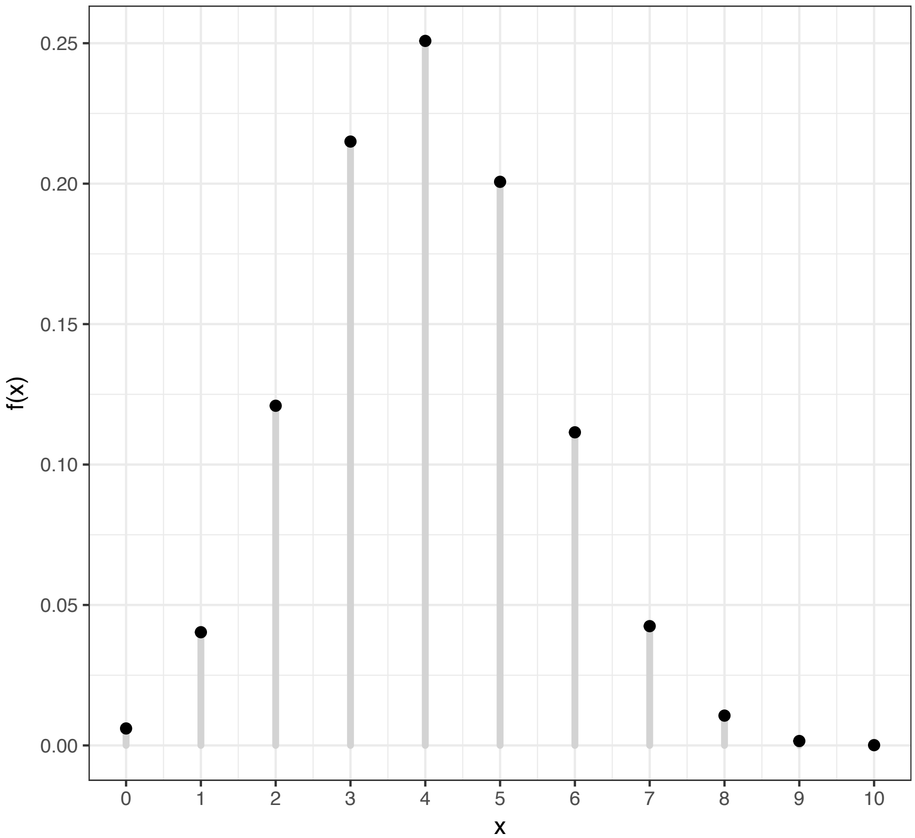
for $x \in \{X(\omega) : \omega \in \Omega\}$ and $f(x) = 0$ otherwise.

Its cdf is

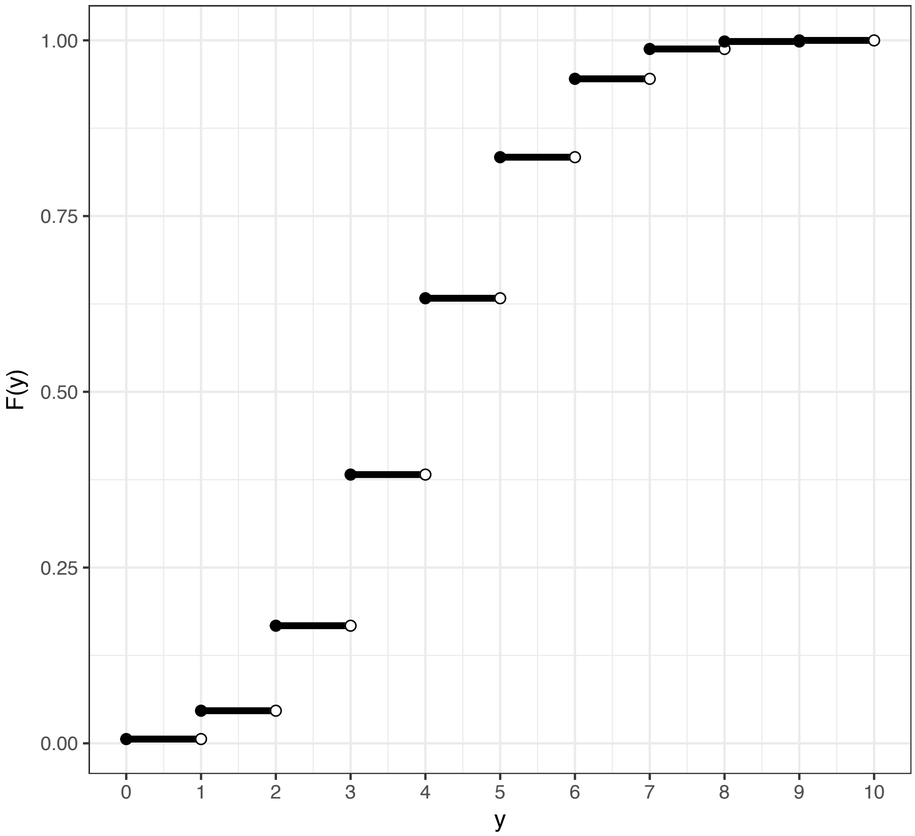
$$F(y) = \Pr(X \leq y) = \sum_{x \leq y} \Pr(X = x)$$

for $y \in \mathbb{R}$.

Example: Discrete PMF



Example: Discrete CDF



Probabilities of Events Via Discrete CDF

Examples:

Probability	CDF	PMF
$\Pr(X \leq b)$	$F(b)$	$\sum_{x \leq b} f(x)$
$\Pr(X \geq a)$	$1 - F(a-1)$	$\sum_{x \geq a} f(x)$
$\Pr(X > a)$	$1 - F(a)$	$\sum_{x > a} f(x)$
$\Pr(a \leq X \leq b)$	$F(b) - F(a-1)$	$\sum_{a \leq x \leq b} f(x)$
$\Pr(a < X \leq b)$	$F(b) - F(a)$	$\sum_{a < x \leq b} f(x)$

Continuous Random Variables

A continuous rv X takes on a continuous set of values such as $[0, \infty)$ or $\mathbb{R} = (-\infty, \infty)$.

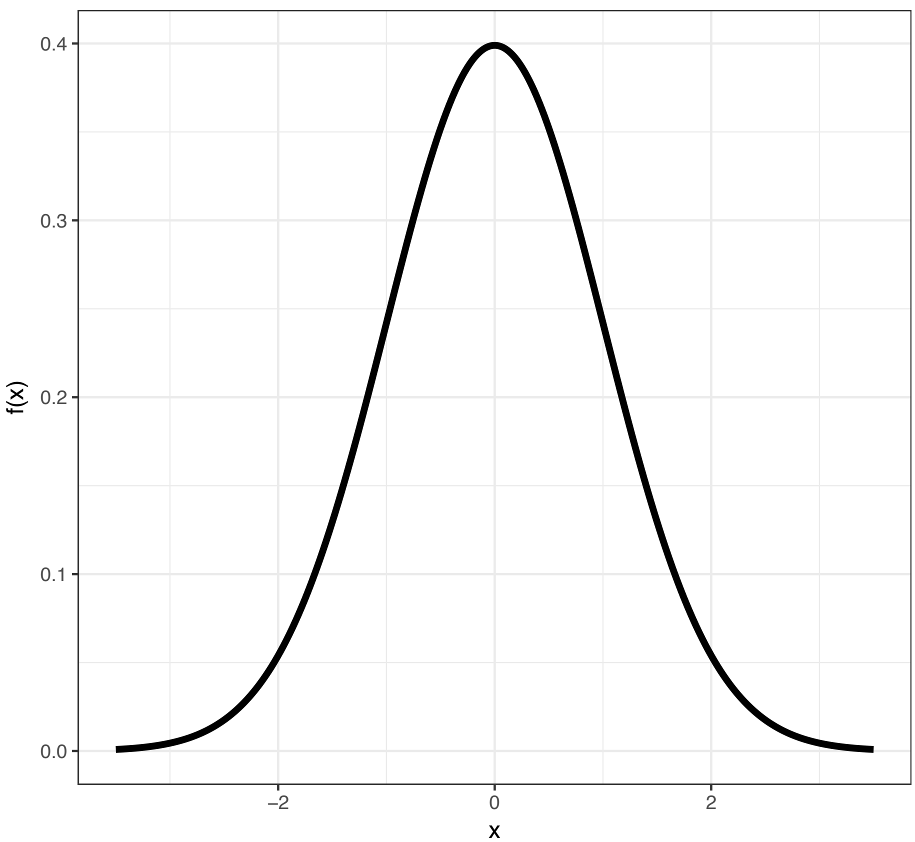
The probability that X takes on any specific value is 0; but the probability it lies within an interval can be non-zero. Its pdf $f(x)$ therefore gives an infinitesimal, local, relative probability.

Its cdf is

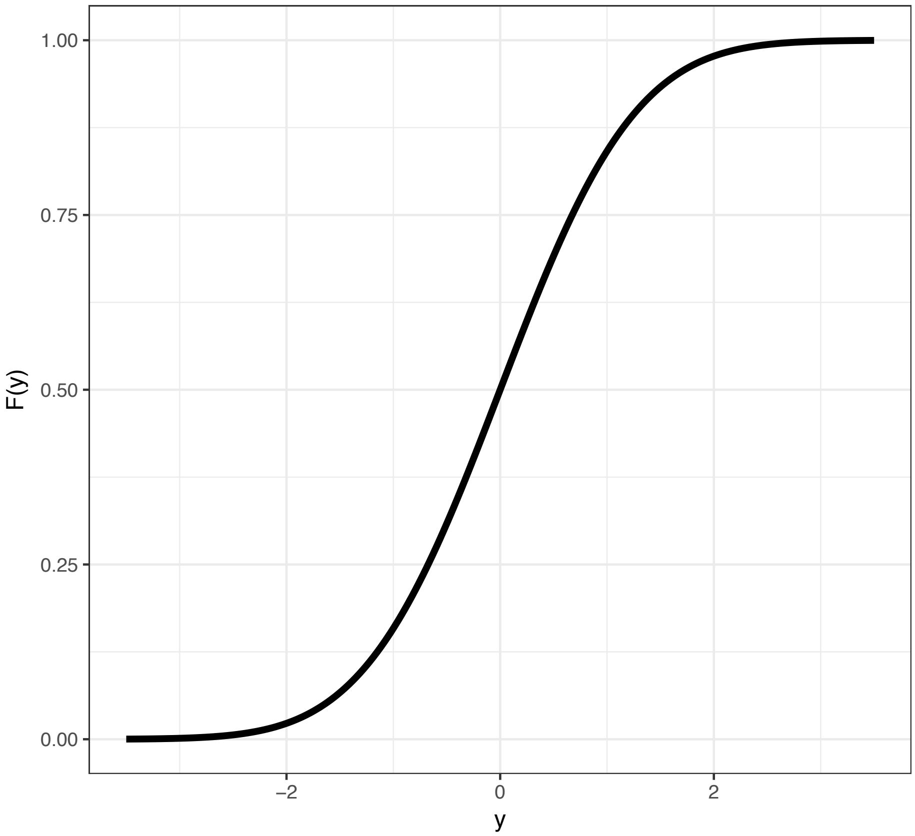
$$F(y) = \Pr(X \leq y) = \int_{-\infty}^{y} f(x)dx$$

for $y \in \mathbb{R}$.

Example: Continuous PDF



Example: Continuous CDF

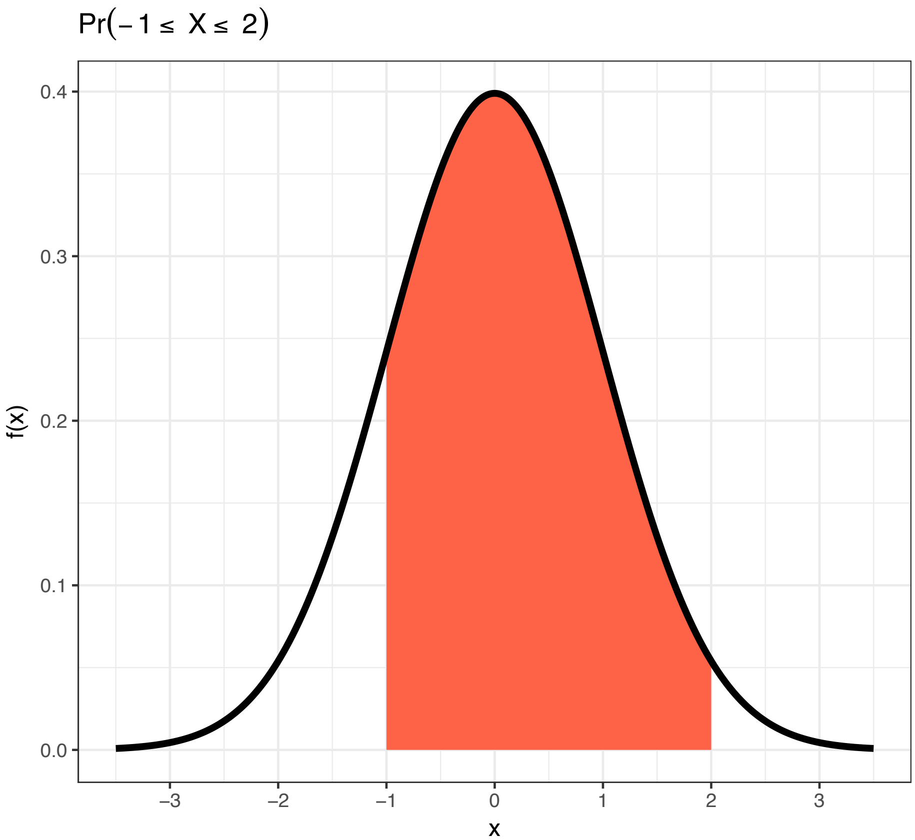


Probabilities of Events Via Continuous CDF

Examples:

Probability	CDF	PDF
$\Pr(X \leq b)$	$F(b)$	$\int_{-\infty}^{b} f(x)dx$
$\Pr(X \geq a)$	$1 - F(a)$	$\int_{a}^{\infty} f(x)dx$
$\Pr(X > a)$	$1 - F(a)$	$\int_{a}^{\infty} f(x)dx$
$\Pr(a \leq X \leq b)$	$F(b) - F(a)$	$\int_{a}^{b} f(x)dx$
$\Pr(a < X \leq b)$	$F(b) - F(a)$	$\int_{a}^{b} f(x)dx$

Example: Continuous RV Event



Note on PMFs and PDFs

PMFs and PDFs are defined as $f(x) = 0$ outside of the range of X, $\mathcal{R} = \{X(\omega) : \omega \in \Omega\}$. That is:

Also, they sum or integrate to 1:

$$\sum_{x \in \mathcal{R}} f(x) = 1$$

$$\int_{x \in \mathcal{R}} f(x) dx = 1$$

Using measure theory, we can consider both types of rv's in one framework, and we would write:

$$\int_{-\infty}^{\infty} dF(x) = 1$$

Note on CDFs

Properties of all cdf's, regardless of continuous or discrete underlying rv:

- They are right continuous with left limits
- $\lim_{x \to \infty} F(x) = 1$
- $\lim_{x \to -\infty} F(x) = 0$
- The right derivative of $F(x)$ equals $f(x)$

Sample Vs Population Statistics

We earlier discussed measures of center and spread for a set of data, such as the mean and the variance.

Analogous measures exist for probability distributions.

These are distinguished by calling those on data "sample" measures (e.g., sample mean) and those on probability distributions "population" measures (e.g., population mean).

Expected Value

The **expected value**, also called the "population mean", is a measure of center for a rv. It is calculated in a fashion analogous to the sample mean:

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{x \in \mathcal{R}} x \; f(x) && \text{(discrete)} \\
\mathrm{E}[X] &= \int_{-\infty}^{\infty} x \; f(x) \; dx && \text{(continuous)} \\
\mathrm{E}[X] &= \int_{-\infty}^{\infty} x \; dF(x) && \text{(general)}
\end{aligned}$$

Variance

The **variance**, also called the "population variance", is a measure of spread for a rv. It is calculated in a fashion analogous to the sample variance:

$$\mathrm{Var}(X) = \mathrm{E}\left[(X - \mathrm{E}[X])^2 \right]; \qquad \mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$$

$$\text{Var}(X) = \sum_{x \in \mathcal{R}} (x - \text{E}[X])^2 \; f(x) \qquad \text{(discrete)}$$

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \text{E}[X])^2 \; f(x) \; dx \qquad \text{(continuous)}$$

Covariance

The **covariance**, also called the "population covariance", measures how two rv's covary. It is calculated in a fashion analogous to the sample covariance:

$$\text{Cov}(X, Y) = \text{E}\left[(X - \text{E}[X])(Y - \text{E}[Y])\right]$$

Note that $\text{Cov}(X, X) = \text{Var}(X)$.

Correlation

The population **correlation** is calculated analogously to the sample correlation:

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\text{SD}(X)\,\text{SD}(Y)}$$

Moment Generating Functions

The **moment generating function** (mgf) of a rv is defined to be

$$m(t) = \text{E}\left[e^{tX}\right]$$

whenever this expecation exists.

Under certain conditions, the **moments** of a rv can then be obtained by:

$$\text{E}\left[X^k\right] = \frac{d^k}{dt^k} m(0).$$

Random Variables in R

The pmf/pdf, cdf, quantile function, and random number generator for many important random variables are built into R. They all follow the form, where `<name>` is replaced with the name used in R for each specific distribution:

- `d<name>`: pmf or pdf

- `p<name>`: cdf
- `q<name>`: quantile function or inverse cdf
- `r<name>`: random number generator

To see a list of random variables, type `?Distributions` in R.

Discrete RVs

Uniform (Discrete)

This simple rv distribution assigns equal probabilities to a finite set of values:

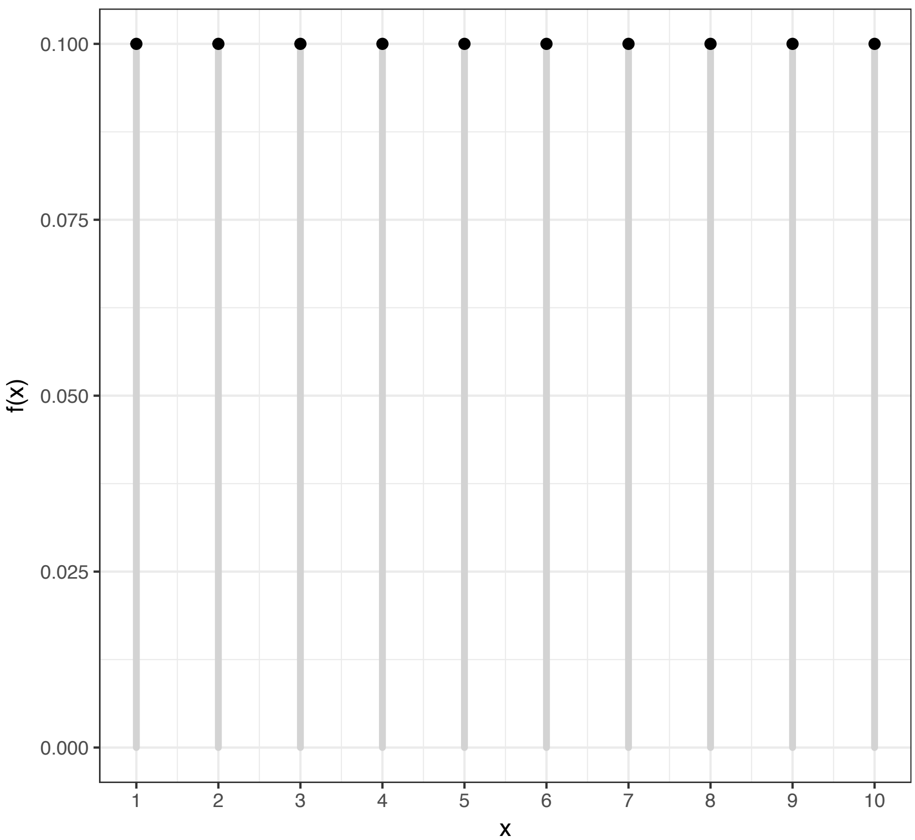
$$X \sim \text{Uniform}\{1, 2, \ldots, n\}$$

$$\mathcal{R} = \{1, 2, \ldots, n\}$$

$$f(x; n) = 1/n \text{ for } x \in \mathcal{R}$$

$$\text{E}[X] = \frac{n+1}{2}, \ \text{Var}(X) = \frac{n^2-1}{12}$$

Uniform (Discrete) PMF



Uniform (Discrete) in R

There is no family of functions built into R for this distribution since it is so simple. However, it is possible to generate random values via the `sample` function:

```
> n <- 20L
> sample(x=1:n, size=10, replace=TRUE)
 [1] 15 17 18  6  4  6  2  7  2 19
>
> x <- sample(x=1:n, size=1e6, replace=TRUE)
> mean(x) - (n+1)/2
[1] 0.002405
> var(x) - (n^2-1)/12
[1] -0.002146536
```

Bernoulli

A single success/failure event, such as heads/tails when flipping a coin or survival/death.

$$X \sim \text{Bernoulli}(p)$$

$$\mathcal{R} = \{0, 1\}$$

$$f(x; p) = p^x(1-p)^{1-x} \text{ for } x \in \mathcal{R}$$

$$\text{E}[X] = p, \ \text{Var}(X) = p(1-p)$$

Binomial

An extension of the Bernoulli distribution to simultaneously considering n independent success/failure trials and counting the number of successes.

$$X \sim \text{Binomial}(n, p)$$

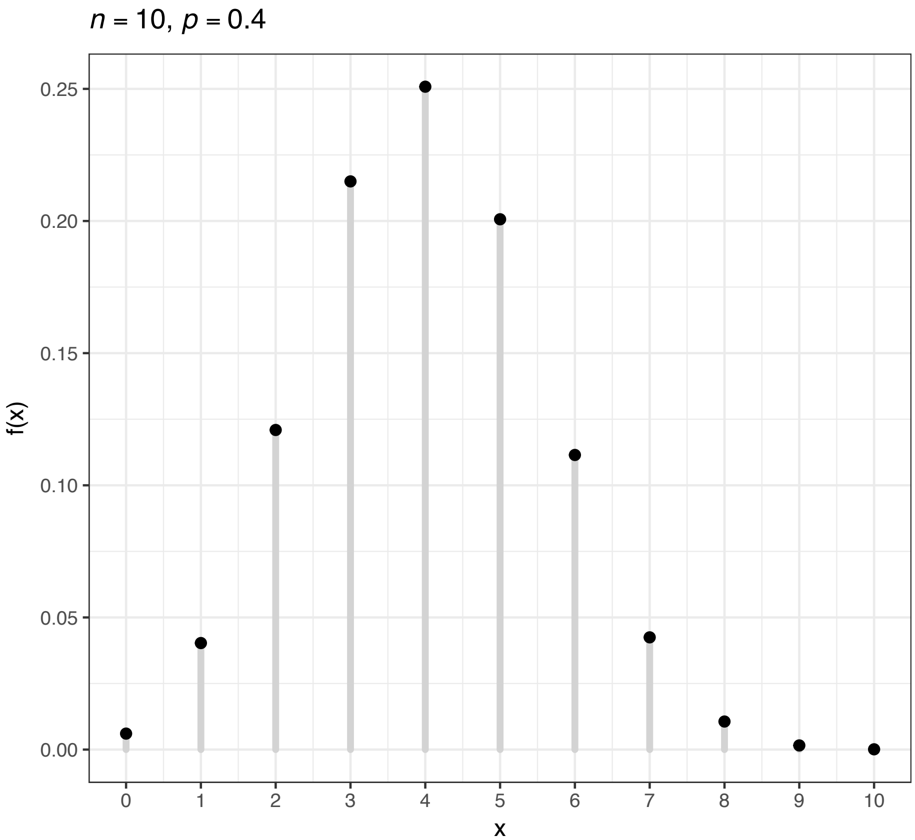
$$\mathcal{R} = \{0, 1, 2, \ldots, n\}$$

$$f(x; p) = \binom{n}{x} p^x(1-p)^{n-x} \text{ for } x \in \mathcal{R}$$

$$\text{E}[X] = np, \ \text{Var}(X) = np(1-p)$$

Note that $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ is the number of unique ways to choose x items from n without respect to order.

Binomial PMF



Binomial in R

```
> str(dbinom)
function (x, size, prob, log = FALSE)
```

```
> str(pbinom)
function (q, size, prob, lower.tail = TRUE, log.p = FALSE)
```

```
> str(qbinom)
function (p, size, prob, lower.tail = TRUE, log.p = FALSE)
```

```
> str(rbinom)
function (n, size, prob)
```

Poisson

Models the number of occurrences of something within a defined time/space period, where the occurrences are independent. Examples: the number of lightning strikes on campus in a given year; the number of emails received on a given day.

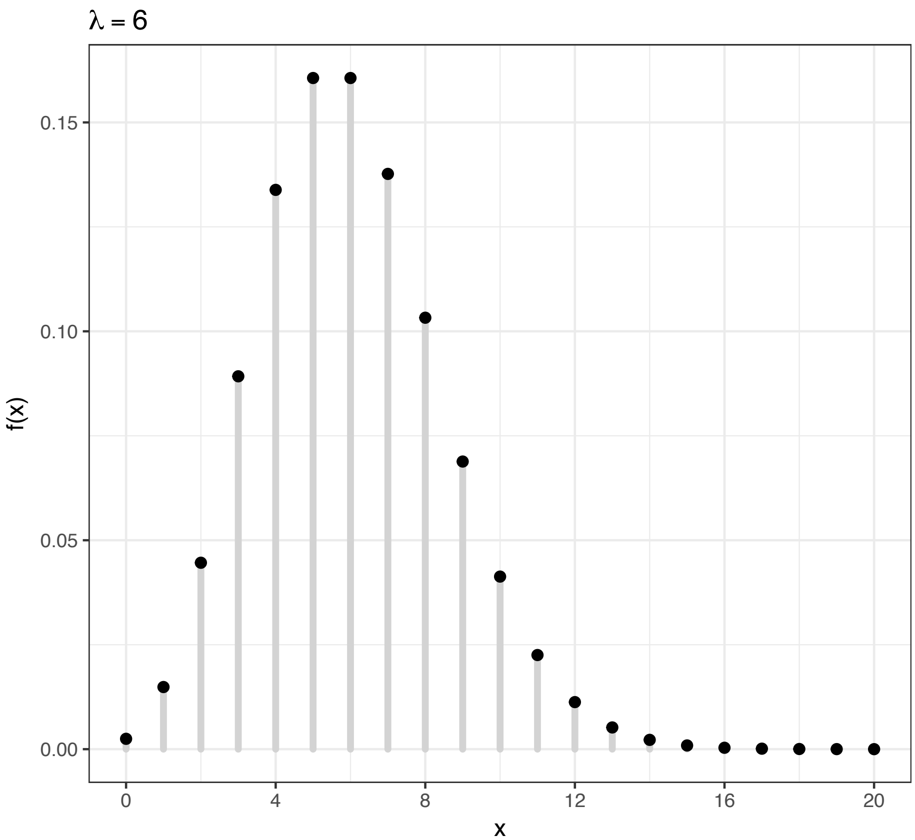
$$X \sim \text{Poisson}(\lambda)$$

$$\mathcal{R} = \{0, 1, 2, 3, \ldots\}$$

$$f(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \text{ for } x \in \mathcal{R}$$

$$\text{E}[X] = \lambda, \ \text{Var}(X) = \lambda$$

Poisson PMF



Poisson in R

```
> str(dpois)
function (x, lambda, log = FALSE)
```

```
> str(ppois)
function (q, lambda, lower.tail = TRUE, log.p = FALSE)
```

```
> str(qpois)
function (p, lambda, lower.tail = TRUE, log.p = FALSE)
```

```
> str(rpois)
function (n, lambda)
```

Continuous RVs

Uniform (Continuous)

Models the scenario where all values in the unit interval [0,1] are equally likely.

$$X \sim \text{Uniform}(0, 1)$$

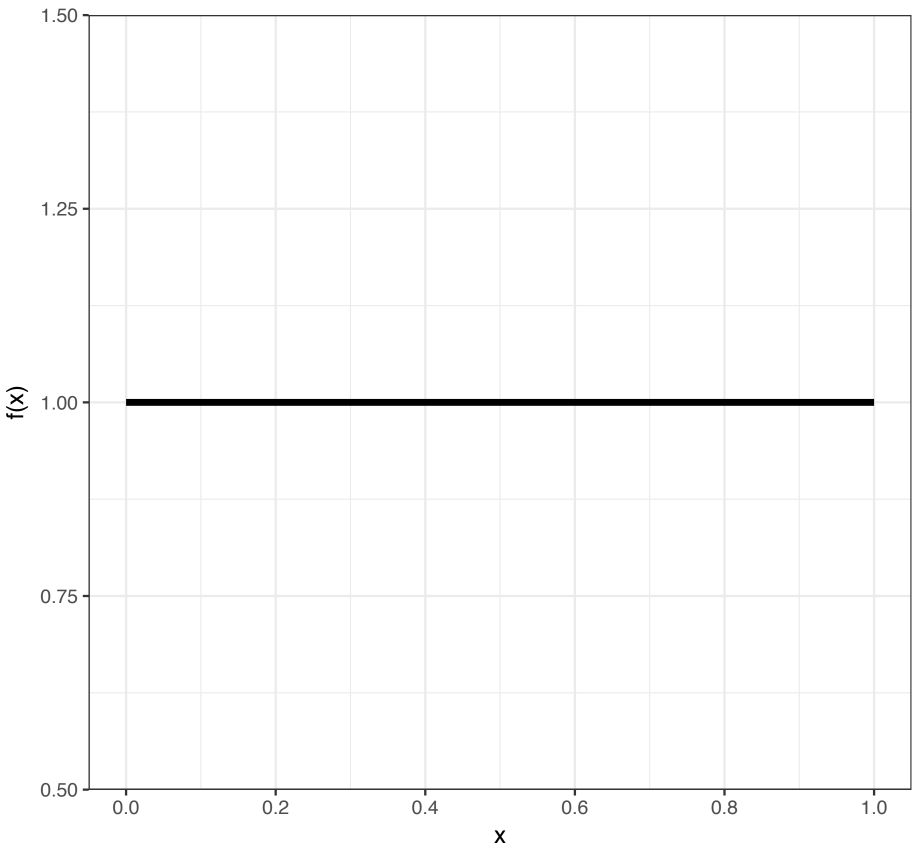
$$\mathcal{R} = [0, 1]$$

$$f(x) = 1 \text{ for } x \in \mathcal{R}$$

$$F(y) = y \text{ for } y \in \mathcal{R}$$

$$\text{E}[X] = 1/2, \ \text{Var}(X) = 1/12$$

Uniform (Continuous) PDF



Uniform (Continuous) in R

```
> str(dunif)
function (x, min = 0, max = 1, log = FALSE)
```

```
> str(punif)
function (q, min = 0, max = 1, lower.tail = TRUE, log.p = FALSE)
```

```
> str(qunif)
function (p, min = 0, max = 1, lower.tail = TRUE, log.p = FALSE)
```

```
> str(runif)
function (n, min = 0, max = 1)
```

Exponential

Models a time to failure and has a "memoryless property".

$$X \sim \text{Exponential}(\lambda)$$

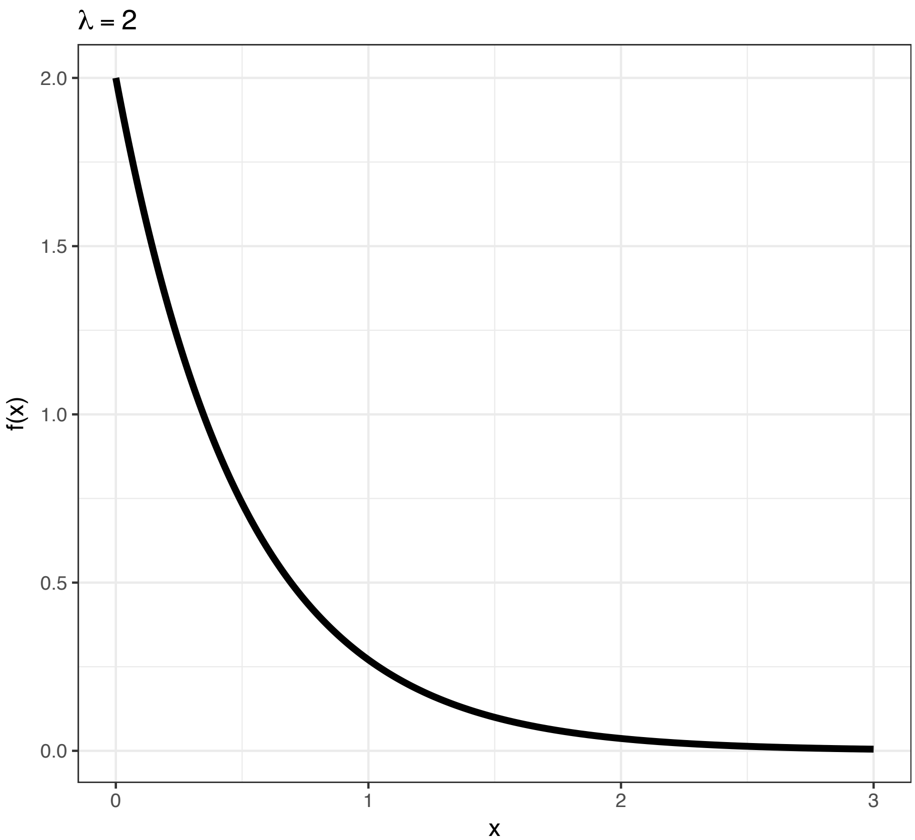
$$\mathcal{R} = [0, \infty)$$

$$f(x; \lambda) = \lambda e^{-\lambda x} \text{ for } x \in \mathcal{R}$$

$$F(y; \lambda) = 1 - e^{-\lambda y} \text{ for } y \in \mathcal{R}$$

$$\text{E}[X] = \frac{1}{\lambda},\ \text{Var}(X) = \frac{1}{\lambda^2}$$

Exponential PDF



Exponential in R

```
> str(dexp)
function (x, rate = 1, log = FALSE)
```

```
> str(pexp)
function (q, rate = 1, lower.tail = TRUE, log.p = FALSE)
```

```
> str(qexp)
function (p, rate = 1, lower.tail = TRUE, log.p = FALSE)
```

```
> str(rexp)
function (n, rate = 1)
```

Beta

Yields values in $(0, 1)$, so often used to generate random probabilities, such as the p in Bernoulli(p).

$$X \sim \text{Beta}(\alpha, \beta) \text{ where } \alpha, \beta > 0$$

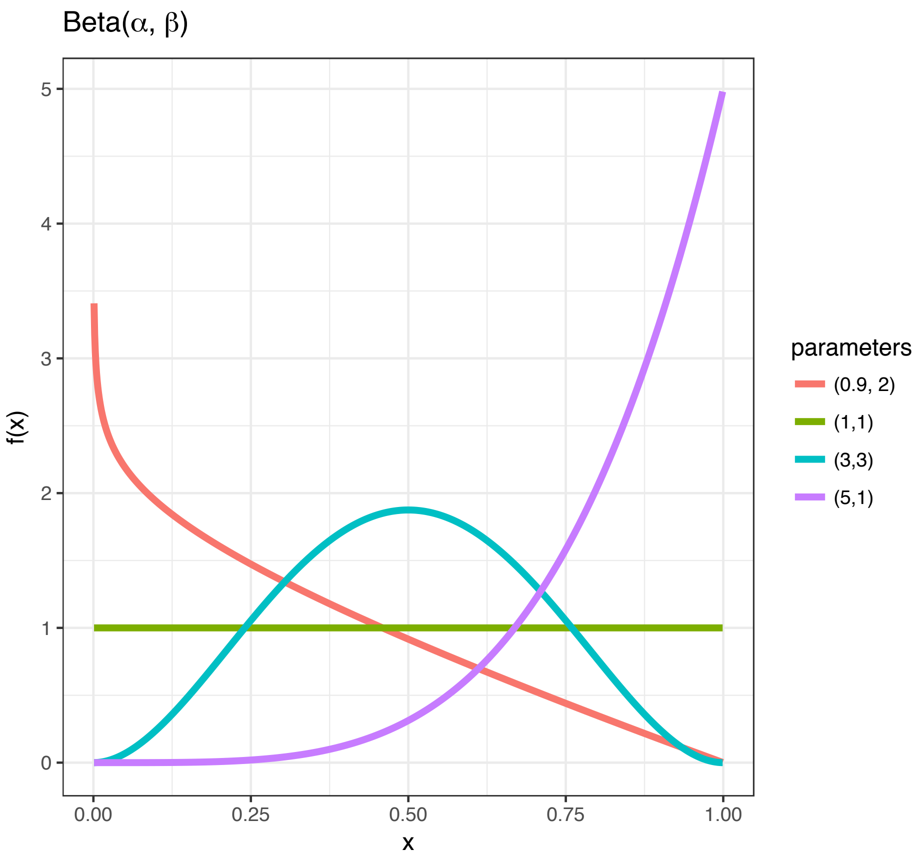
$$\mathcal{R} = (0, 1)$$

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \text{ for } x \in \mathcal{R}$$

where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$.

$$\text{E}[X] = \frac{\alpha}{\alpha + \beta},\ \text{Var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

Beta PDF



Beta in R

```
> str(dbeta) #shape1=alpha, shape2=beta
function (x, shape1, shape2, ncp = 0, log = FALSE)
```

```
> str(pbeta)
function (q, shape1, shape2, ncp = 0, lower.tail = TRUE,
    log.p = FALSE)
```

```
> str(qbeta)
function (p, shape1, shape2, ncp = 0, lower.tail = TRUE,
    log.p = FALSE)
```

```
> str(rbeta)
function (n, shape1, shape2, ncp = 0)
```

Normal

Due to the Central Limit Theorem (covered later), this "bell curve" distribution is often observed in properly normalized real data.

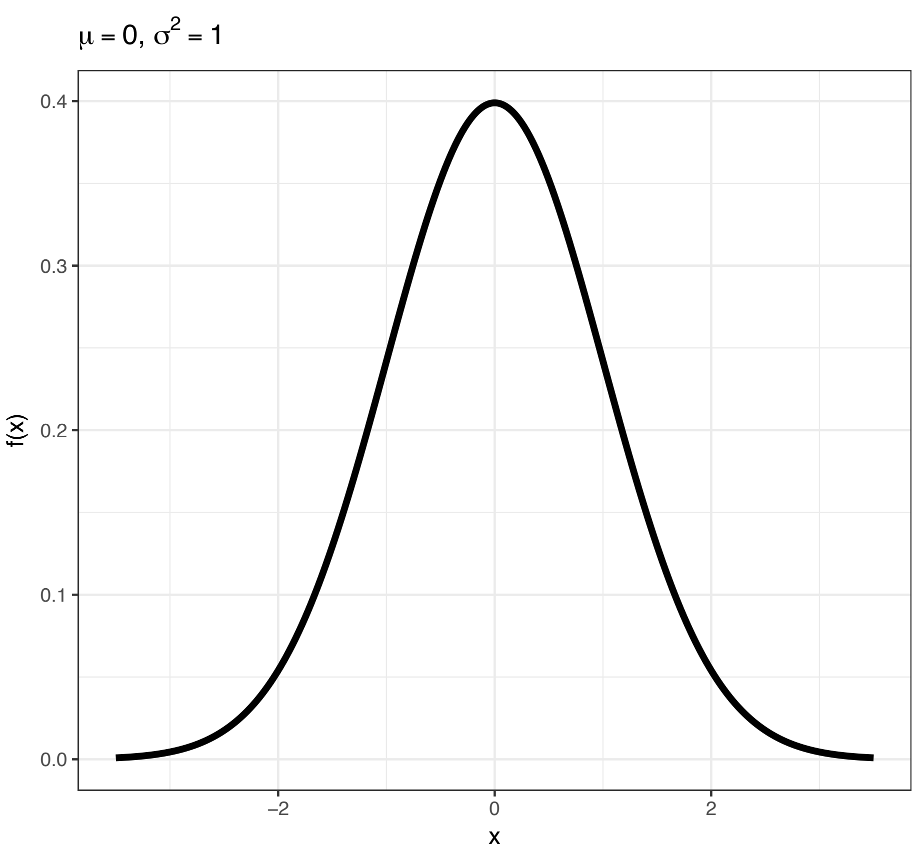
$$X \sim \text{Normal}(\mu, \sigma^2)$$

$$\mathcal{R} = (-\infty, \infty)$$

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{ for } x \in \mathcal{R}$$

$$\text{E}[X] = \mu, \ \text{Var}(X) = \sigma^2$$

Normal PDF



Normal in R

```
> str(dnorm) #notice it requires the STANDARD DEVIATION, not the variance
function (x, mean = 0, sd = 1, log = FALSE)
```

```
> str(pnorm)
function (q, mean = 0, sd = 1, lower.tail = TRUE, log.p = FALSE)
```

```
> str(qnorm)
function (p, mean = 0, sd = 1, lower.tail = TRUE, log.p = FALSE)
```

```
> str(rnorm)
function (n, mean = 0, sd = 1)
```

Sums of Random Variables

Linear Transformation of a RV

Suppose that X is a random variable and that a and b are constants. Then:

$$\mathrm{E}\left[a + bX\right] = a + b\,\mathrm{E}[X]$$

$$\mathrm{Var}\left(a + bX\right) = b^2\,\mathrm{Var}(X)$$

Sums of Independent RVs

If $X_1, X_2, \ldots, X_n$ are independent random variables, then:

$$\mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}[X_i]$$

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

Sums of Dependent RVs

If $X_1, X_2, \ldots, X_n$ are independent random variables, then:

$$\mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}[X_i]$$

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2\sum_{i \neq j} \mathrm{Cov}(X_i, X_j)$$

Means of Random Variables

Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed (iid) random variables. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be their sample mean. Then:

$$\mathrm{E}\left[\overline{X}_n\right] = \mathrm{E}[X_i]$$

$$\mathrm{Var}\left(\overline{X}_n\right) = \frac{1}{n}\mathrm{Var}(X_i)$$

Convergence of Random Variables

Sequence of RVs

Let $Z_1, Z_2, \ldots$ be an infinite sequence of rv's.

An important example is

$$Z_n = \overline{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}.$$

It is useful to be able to determine a limiting value or distribution of $\{Z_i\}$.

Convergence in Distribution

$\{Z_i\}$ converges in distribution to Z, written

$$Z_n \xrightarrow{D} Z$$

if

$$F_{Z_n}(y) = \Pr(Z_n \leq y) \rightarrow \Pr(Z \leq y) = F_Z(y)$$

as $n \rightarrow \infty$ for all $y \in \mathbb{R}$.

Convergence in Probability

$\{Z_i\}$ converges in probability to Z, written

$$Z_n \xrightarrow{P} Z$$

if

$$\Pr(|Z_n - Z| \leq \epsilon) \to 1$$

as $n \to \infty$ for all $\epsilon > 0$.

Note that it may also be the case that $Z_n \xrightarrow{P} \theta$ for a fixed, nonrandom value θ.

Almost Sure Convergence

$\{Z_i\}$ converges almost surely (or "with probability 1") to Z, written

$$Z_n \xrightarrow{a.s.} Z$$

if

$$\Pr\left(\{\omega : |Z_n(\omega) - Z(\omega)| \overset{n\to\infty}{\longrightarrow} 0\}\right) = 1.$$

Note that it may also be the case that $Z_n \xrightarrow{a.s.} \theta$ for a fixed, nonrandom value θ.

Strong Law of Large Numbers

Suppose $X_1, X_2, \ldots, X_n$ are iid rv's with population mean $\mathrm{E}[X_i] = \mu$ where $\mathrm{E}[|X_i|] < \infty$. Then

$$\overline{X}_n \xrightarrow{a.s.} \mu.$$

Central Limit Theorem

Suppose $X_1, X_2, \ldots, X_n$ are iid rv's with population mean $\mathrm{E}[X_i] = \mu$ and variance $\mathrm{Var}(X_i) = \sigma^2$. Then as $n \to \infty$,

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{D} \mathrm{Normal}(0, \sigma^2)$$

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} \text{Normal}(0, 1)$$

Example: Calculations

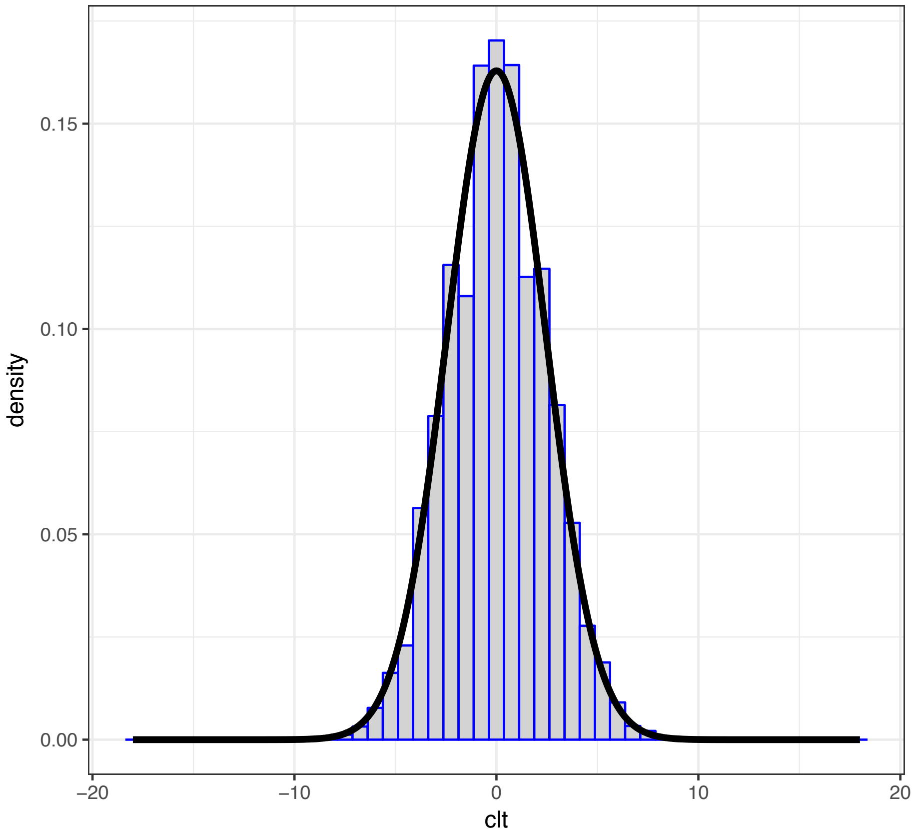
Let $X_1, X_2, \ldots, X_{40}$ be iid Poisson(λ) with $\lambda = 6$.

We will form $\sqrt{40}(\overline{X} - 6)$ over 10,000 realizations and compare their distribution to a Normal(0, 6) distribution.

```
> x <- replicate(n=1e4, expr=rpois(n=40, lambda=6),
+                  simplify="matrix")
> x_bar <- apply(x, 2, mean)
> clt <- sqrt(40)*(x_bar - 6)
>
> df <- data.frame(clt=clt, x = seq(-18,18,length.out=1e4),
+                   y = dnorm(seq(-18,18,length.out=1e4),
+                              sd=sqrt(6)))
```

Example: Plot

```
> ggplot(data=df) +
+   geom_histogram(aes(x=clt, y=..density..), color="blue",
+                   fill="lightgray", binwidth=0.75) +
+   geom_line(aes(x=x, y=y), size=1.5)
```



Joint Distributions

Bivariate Random Variables

For a pair of rv's X and Y defined on the same probability space, we can define their joint pmf or pdf. For the discrete case,

$$\begin{aligned} f(x, y) &= \Pr(\{\omega : X(\omega) = x\} \cap \{\omega : Y(\omega) = y\}) \\ &= \Pr(X = x, Y = y). \end{aligned}$$

The joint pdf is defined analogously for continuous rv's.

Events for Bivariate RVs

Let $A_x \times A_y \subseteq \mathbb{R} \times \mathbb{R}$ be an event. Then $\Pr(X \in A_x, Y \in A_y)$ is calculated by:

$$\sum_{x \in A_x} \sum_{y \in A_y} f(x,y) \qquad \text{(discrete)}$$

$$\int_{x \in A_x} \int_{y \in A_y} f(x,y) dy dx \qquad \text{(continuous)}$$

$$\int_{x \in A_x} \int_{y \in A_y} f(x,y) dF_Y(y) dF_X(x) \qquad \text{(general)}$$

Marginal Distributions

We can calculate the marginal distribution of X (or Y) from their joint distribution:

$$f(x) = \int_{-\infty}^{\infty} f(x,y) dF_Y(y)$$

Independent Random Variables

Two rv's are independent when their joint pmf or pdf factor:

$$f(x,y) = f(x)f(y)$$

This means, for example in the continuous case,

$$\begin{aligned}
\Pr(X \in A_x, Y \in A_y) &= \int_{x \in A_x} \int_{y \in A_y} f(x,y) dy dx \\
&= \int_{x \in A_x} \int_{y \in A_y} f(x) f(y) dy dx \\
&= \Pr(X \in A_x) \Pr(Y \in A_y)
\end{aligned}$$

Conditional Distributions

We can define the conditional distribution of X given Y as follows. The conditional rv $X|Y \sim F_{X|Y}$ with conditional pmf or pdf for $X|Y = y$ given by

$$f(x|y) = \frac{f(x,y)}{f(y)}.$$