DIFFERENTIAL EQUATION

1 Order Of Differential Equation

The order of a differential equation is the order of the highest derivative occuring in the differential equation.
For example

$\frac{d^2y}{dx^2}+y=0$ is a second order differential equation

$\left(\frac{d^3y}{dx^3}\right)+x^2\left(\frac{d^2y}{dx^2}\right)^3=0$ is a third order differential equation.

2 Degree Of Differential Equation

The degree of a differential equation is the highest degree of the highest derivative occuring in the differential equation when it is a polynomial of the differential coefficients i.e., differential coefficients free from radicals & fractions.

For example

Since, $\frac{d^3y}{dx^3}+x^2\left(\frac{d^2y}{dx^2}\right)^3=0$ as order = 3

its degree = 1, as $\frac{d^3y}{dx^3}$ has power 1.

3 Differential Equations With Variables Separable

$\frac{dy}{dx}=h(y)\cdot g(x)$

Separating the variables, we have $\frac{dy}{h(y)}=g(x)\cdot dx$

Integrate both sides $\int\frac{dy}{h(y)}=\int g(x)\cdot dx$

4 Reducible to the separate variable type

$\frac{dy}{dx}=f(ax+by+c)$ is solved by putting ax + by + c = t, etc

5 Homogenous differential equation

(i) P(x, y)dx + Q(x, y)dy = 0 is called homogenous, if P & Q are homogenous functions of the same degree on x & y. Reducible to $y'=f\left(\frac{y}{x}\right)$

substitute y = xu, u is unknown function. The equation is transformed to an equation with variable separables.

(ii) $\frac{dy}{dx}=f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$, $a_1b_2 - a_2b_1 \neq 0$, then substitute x = u + h, y = v + k

if $a_1b_2 - a_2b_1 = 0$, $u = a_1x + b_1y$ transforms into a variable separable form.

- P(x, y) function is homogenous of degree n, if for any real t, $P(tx, ty) = t^n(P(x, y))$.
- A differential equation of the form $\frac{dy}{dx}=f(x,y)$ is homogeneous, if f(x, y) is a homogeneous function of degree zero i.e., $f(tx, ty) = t^0.f(x,y)$

6 Exact differential equation

M(x, y)dx + N(x, y)dy = 0 is exact if its LHS expression is the exact differential of some function u(x, y).

du = Mdx + Ndy

Then solution is u(x, y) = c.

The sufficient condition for the differential Mdx + Ndy = 0 to be exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

The solution of Mdx + Ndy = 0 is

$\int_{y-\text{ constant}} Mdx+\int(\text{terms of N not containing x})dy=c$, provided

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

7 Linear Differential Equations

A differential equation of the form $\frac{dy}{dx}+Py=Q$

where P & Q are constants or functions of only, is known as a First Order Linear Differential Equation.

$y(I.F)=\int(Q\times I.F)dx+c$

$\frac{dx}{dy}+P'x=Q'$

$I.F=e^{\int Pdy}$

$x(I.F)=\int\left(Q'\times I.F\right)dy+c$

8 After linear differential equation

Solution by inspection

(1) d(xy) = xdy + ydx

(2) $d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}$

(3) $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$

(4) $d\left(\frac{x^2}{y}\right)=\frac{2xydx-x^2dy}{y^2}$

(5) $d\left(\frac{y^2}{x}\right)=\frac{2xydy-y^2dx}{x^2}$

(6) $d\left(\frac{x^2}{y^2}\right)=\frac{2xy^2dx-2x^2ydy}{y^4}$

(7) $d\left(\frac{y^2}{x^2}\right)=\frac{2x^2ydy-2xy^2dx}{x^4}$

(8) $d\left(\tan^{-1}\frac{x}{y}\right)=\frac{ydx-2dy}{x^2+y^2}=\frac{d(x/y)}{1+(x/y)^2}$

(9) $d\left(\tan^{-1}\frac{y}{x}\right)=\frac{xdy-ydx}{x^2+y^2}=\frac{d(y/x)}{1+(y/x)^2}$

(10) $d[\ln(xy)]=\frac{xdy+ydx}{xy}$

(11) $d\left[\ln\left(\frac{x}{y}\right)\right]=\frac{ydx-xdy}{xy}$

(12) $d[\ln(y/x)]=\frac{xdy-ydx}{xy}$

(13) $d\left[1/2\ln\left(x^2+y^2\right)\right]=\frac{xdx+ydy}{x^2+y^2}$

(14) $d(-1/xy)=\frac{xdy+ydx}{x^2y^2}$

(15) $d\left(\frac{e^x}{y}\right)=\frac{ye^xdx-e^xdy}{y^2}$

(16) $d\left(\frac{e^y}{x}\right)=\frac{xe^ydy-e^ydx}{x^2}$

(17) $\frac{d[f(x,y)]^{1-n}}{1-n}=\frac{f'(x,y)}{(f(x,y))^n}$

9 Bernoulli's equation

$\frac{dy}{dx}+Py=Qy^n$ dividing $y^n \rightarrow y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$ ...(i)

$y^{1-n}=z$

(i) $\frac{dz}{dx}+(1-n)Pz=(1-n)Q$

Solution is

$ze^{\int(1-n)Pdx}=\int\left\{(1-n)Q\cdot e^{\int(1-n)Pdx}\right\}dx$

10 Orthogonal Trajectory

Any curve, which cuts every member of a given family of curves at right angles, is called an orthogonal trajectory of the family.

Procedure for finding Orthogonal Trajectory :

(i) Let f(x, y, c) = 0 is the equation of family.

(ii) Differentiate f = 0, w.r.t. x & eliminate c.

(iii) Substitute $-\frac{dx}{dx}$ for $\frac{dy}{dx}$ That is the differential equation of OT. Now, solve it to get OT.

11 Clairaut's equation

Form y = px + f(p)

Method. Differentiate w.r.t. x, we get

$$\{x + f'(p)\}\frac{dp}{dx} = 0$$

$\therefore$ **p = c or f '(p) + x = 0**

When p = c, the general solution is y = cx + f(c) which gives a family of straight lines

When f '(p) + x = 0, eliminating p from y = px + f(p) and f '(p) + x = 0 we get a solution which is a curve (without any arbitrary constant) touching all the lines given by y = cx + f(c). This solution is called the singular solution.

12 Facts from cartesian curve

(i) Slope of tangent at any point $P(x, y) = \frac{dy}{dx}$

(ii) Equation of tangent PQ at (x, y) is $Y - y = \frac{dy}{dx}(X - x)$

(iii) Equation of normal PR at (x, y) is $Y - y = -\frac{dx}{dy}(X - x)$

(iv) Length of tangent PQ at $(x, y) = \left| y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} \right|$

(v) Length of normal PG at $(x, y) = \left| y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right|$

(vi) Length of subtangent QM at $(x, y) = \left| y \cdot \frac{dx}{dy} \right|$

(vii) Length of subnormal MR at $(x, y) = \left| y \cdot \frac{dy}{dx} \right|$