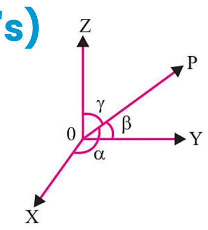
Three Dimensional Geometry

1. Direction Cosines of A Line (Dc's)

The direction cosines are generally denoted by l, m, n.

Hence, l = cos α, m = cos β, n = cos γ

Note that $l^2 + m^2 + n^2 = 1$



2. Direction Ratio of A Line (Dr's)

- Any three numbers a, b and c proportional to the direction cosines l, m and n, respectively are called direction ratios of the line.
- The direction ratios of a line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $(x_2 - x_1)$, $(y_2 - y_1)$, $(z_2 - z_1)$
- $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$
- $l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}$ and $n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$

3. Equation of Line

1. Equation of a line through a given point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$:

In vector form, $\vec{r} = \vec{a} + \lambda\vec{b}$

In cartesian form, $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$ Here, a, b, c are also the direction ratios of the line.

2. Equation of a line passing through two given points with position vectors $\vec{a}$ and $\vec{b}$:

In vector form, $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

In cartesian form, $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$ where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ & $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

4. Angle Between Two Lines

In vector form, The angle between two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given as:

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

In cartesian form, The angle between two lines :

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is $\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$

- If two lines are perpendicular, then $\vec{b}_1 \cdot \vec{b}_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$
- If two lines are parallel, then $\vec{b}_1 = \lambda\vec{b}_2$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

5. Shortest Distance Between Two Lines

1. Distance Between Parallel Lines the shortest distance between parallel lines

$L_1 : \vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $L_2 : \vec{r} = \vec{a}_2 + \mu\vec{b}$ is $d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$

2. Distance Between Two Skew Lines In vector form, The distance between two skew lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

In cartesian form,
The distance between two skew lines :

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is:

$$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2 + (c_1a_2-c_2a_1)^2 + (a_1b_2-a_2b_1)^2}}$$

6. Equation of A Plane In Normal Form

Vector Form

$\vec{r} \cdot \hat{n} = d$

Here $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{n}$ is the unit vector along the normal from origin to the plane.

d is perpendicular distance of the plane from the origin.

Cartesian Form

$lx + my + nz = d$

where ℓ, m, n are the direction cosines of $\hat{n}$ (unit vector along the normal from origin to the plane).

7. Equation Of A Plane Perpendicular To A Given Vector And Passing Through A Given Point

Vector Form

Let a plane pass through a point with position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$. Then its equation is given as: $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

Cartesian Form

Let a plane pass through a point $(x_1, y_1 z_1)$ & the direction ratio of the vector perpendicular to the plane be A, B, C. Then its equation is given as:

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

8. Equation Of A Plane Passing Through Three Non-Collinear Points

Vector Form

$[\vec{r}\vec{b}\vec{c}] + [\vec{r}\vec{a}\vec{b}] + [\vec{r}\vec{c}\vec{a}] = [\vec{a}\vec{b}\vec{c}]$ or $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$

where, $\vec{a}, \vec{b}, \vec{c}$ are the position vector of three given noncollinear points through which the plane passes.

Cartesian Form

The equation of plane passing through three noncollinear points Y with coordinates, (x_1, y_1, z_1) (x_2, y_2, z_2) & (x_3, y_3, z_3) is given as:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

9. Intercept Form of The Equation of A Plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Where a, b, c are the intercepts made by the plane on x, y & z axes respectively.

10. Plane Passing Through The Intersection Of Two Given Planes

Vector Form

Equation of plane passing through the point of intersection of two planes

$\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is given as:

$\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2) = d_1+\lambda d_2$

Cartesian Form

$\vec{n}_1 = A_1\hat{i}+B_1\hat{j}+C_1\hat{k}$

$\vec{n}_2 = A_2\hat{i}+B_2\hat{j}+C_2\hat{k}$ and $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

therefore its cartesian equation is:

$(A_1x+B_1y+C_1z-d_1)+\lambda(A_2x+B_2y+C_2z-d_2)=0$

12. Angle Between Two Planes

Vector Form: The angle between two planes $\vec{r}\cdot\vec{n}=d_1$ & $\vec{r}\cdot\vec{n}=d_2$ is given as:

$$\cos\theta = \left|\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$$

Cartesian Form: The angle between two planes

$a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is given as

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

- If two planes are perpendicular, then $\vec{n}_1\cdot\vec{n}_2=0$ or $a_1a_2+b_1b_2+c_1c_2=0$
- If two planes are perpendicular, then $\vec{n}_1=\lambda\vec{n}_2$ or $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

11. Coplanarity of Two Lines

Vector Form

Two lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ are coplanar, if $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0$

Cartesian Form

Two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ & $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are coplanar, if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

13. Distance Of A Point From A Plane

Vector Form

Distance of a point with position vector $\vec{a}$ from a plane $\vec{r}\cdot\vec{n}=d$ is given as: $\frac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}$

Cartesian Form

Distance of a point (x_1,y_1,z_1) from a plane: $ax+by+cz=d$ is given as :

$$\left|\frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right|$$

14. Angle Between A Line And A Plane

Vector Form

Angle between a line

$\vec{r}=\vec{a}+\lambda\vec{b}$ and a plane $\vec{r}\cdot\vec{n}=d$ is

$$\sin\theta = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$$

Cartesian Form

Angle between a line $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and a plane

$a_2x+b_2y+c_2z=d$ is given as:

$$\sin\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

- If line is perpendicular to the plane, then $\vec{n}=\lambda\vec{b}$ or $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$
- If line is parallel to the plane, then $\vec{n}\cdot\vec{b}=0$ or $a_1a_2+b_1b_2+c_1c_2=0$