APPLICATION OF DERIVATIVES

1. Common application of derivatives.

- Finding Rate of Change of a Quantity
- Finding the Approximation Value
- Finding the equation of a Tangent and Normal To a Curve
- Finding Maxima and Minima, and Point of Inflection
- Determining Increasing and Decreasing Functions

(2) Approximations

Assume we have a function y = f(x), which is defined in the interval [a, a + h], then the average rate of change in the function in the given interval is (f(a + h) − f(a))/h

Now using the definition of derivative, we can write

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

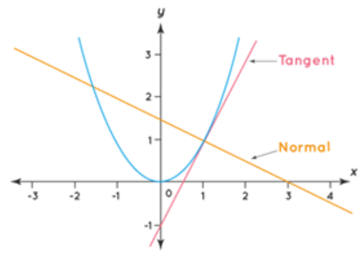
which is also the instantaneous rate of change of the function f(x) at a. Now, for a very small value of h, we can write

$$f'(a) \approx (f(a+h) - f(a))/h$$

(3) Equations of tangent & Normal

- Equation of tangent at (x_1, y_1): $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$
- Equation of Normal at (x_1, y_1):

$$y - y_1 = \left(-\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$



(4) Increasing decreasing functions.

Properties of monotonic functions :

(1) If f(x) is continuous on [a, b] such that f' (c) ≥ 0 for each c, then f(x) is monotonically increasing function. Similar definition goes for monotonically decreasing function.

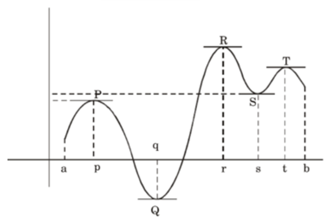
(2) If f(x) is strictly increasing function on [a, b] then $f^{-1}(x)$ exists & is also strictly increasing on [a, b]. Similar result follows for strictly decreasing functions.

(3) If f(x) & g(x) are two continuous & differentiable functions, then we can relate fog (x) & gof (x) by the following table

\+ denotes increasing function
− denotes decreasing function

f(x)	g(x)	fog/gof
+	+	+
+	−	−
−	+	−
−	−	+

(5) Maxima & Minima



Point P, R T are points of local maxima
Point Q, S are point of local minima

Global maxima, global minima

Maximum = max $\{f(a), f(c_1), f(c_2)....f(c_n), f(b)\}$

Minimum = min $\{f(a), f(c_1), f(c_2)....f(c_n), f(b)\}$

$c_1\ c_2....\ c_n$ are n critical points.

(6) Test Of Local Maxima & Minima

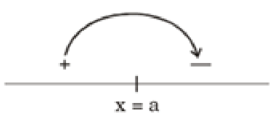
First Derivative Test:

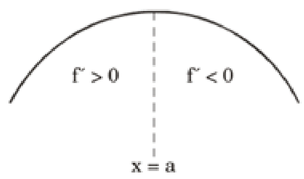
Step 1 : Find the critical points of the function by putting f'(x) = 0

Step 2 : For each of the critical points obtained in step 1 do the following :

Case 1 : x = a is local maxima

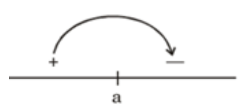
If f'(x) changes from + to - as x passes through a

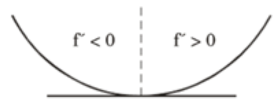




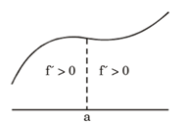
Case 2 : x = a is local minima

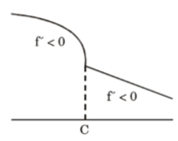
if the sign of f'(x) changes from - to + as x passes through a





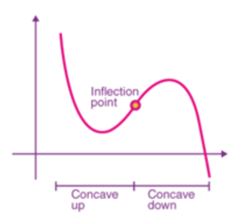
Case 3 : There is no sign change across a this means that x = a is neither a point of maxima nor minima.





(7) Point of inflection

A point of inflection is point where the curve changes its shape from convex to concave or from concave to convex.



(8) Higher Order Test

Let f be a differentiable function on interval I & let c be any point in the domain of f such that

(1) $f'(c)=f''(c)=f'''(c)\ldots=f^{n-1}(c)=0$ and

(2) $f^{n}(c)\neq 0$ and exists.

then if n is even $\begin{cases} f^{n}(c)<0 \Rightarrow x=c \text{ is a local maxima} \\ f^{n}(c)>0 \Rightarrow x=c \text{ is a local minima} \end{cases}$

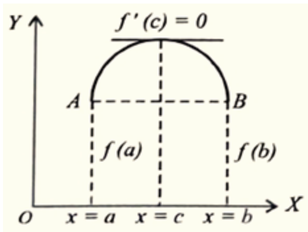
(9) Rolle's Theorem

If a function f defined on the closed interval [a, b], is
1. continuous on [a, b],
2. derivable on (a, b) and
3. f (a) = f (b), then there exists atleast one real number c between a and b (a < c < b) such that f '(c) = 0.

Geometrical interpretation

Let the curve y = f (x), which is continuous on [a, b] and derivable on (a, b), be drawn.

The theorem states that between two points with equal ordinates on the graph of f, there exists atleast one point where the tangent is parallel to x-axis.



Algebraic interpretation

Between two zeros a and b of f (x) (i.e., between two roots a and b of f (x) = 0) there exists atleast one zero of f '(x).

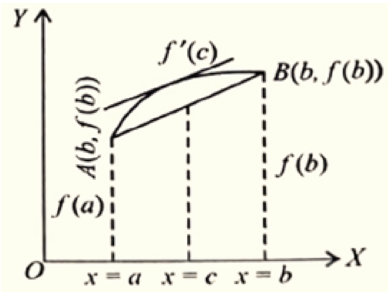
Key Points to Remember

1. The value of c may not be unique i.e., there can be more than one such c.
2. Every polyonomial function is continuous and differentiable for all real x.
3. The function log x is continuous on (0, ∞).
4. |x − a| is not differentiable at x = a (e.g., |x| is not differentiable at x = 0).
5. If the derivative of a function has finite and unique value on an interval, then the function is derivable on that interval.

(10) Lagrange's Mean Value Theorem

If a function f defined on the closed interval [a, b], is
1. continuous on [a, b] and
2. derivable on (a, b), then there exists atleast one real number c between a and b (a < c < b) such that

$$f(c) = \frac{f(b) - f(a)}{b - a}$$



Geometrical interpretation
The theorem states that between two points A and B on the graph off there exists atleast one point where the tangent is parallel to the chord AB.

(11) Parametric coordinates

(1) $x^{2/3} + y^{2/3} = a^{2/3}$: x = a $\cos^3\theta$, y = a $\sin^3\theta$.

(2) $\sqrt{x}+\sqrt{y}=\sqrt{a}$: x = a $\cos^4\theta$, y=a $\sin^4\theta$

(3) $\frac{x^n}{a^n}+\frac{y^n}{b^n}=1$: x = a $(\cos\theta)^{2/n}$,y = b$(\sin\theta)^{2/n}$

(4) $c^2(x^2 + y^2) = x^2y^2$: x = c sec θ , y = c cosec θ

(5) $y^2 = x^3$: x = t^2 , y = t^3

(12) Angle of intersection of two Curves

The angle between the tangents to the two curves at their point of intersection.

Let c_1 & c_2 be two curves.

$$m_1 = \tan\theta_1 = \left(\frac{dy}{dx}\right)_{c_1}$$

$$m_2 = \tan\theta_2 = \left(\frac{dy}{dx}\right)_{c_2}$$

$$\text{Angle of intersection, } \theta = \tan^{-1}\left|\frac{\left(\frac{dy}{dx}\right)_{c_1}-\left(\frac{dy}{dx}\right)_{c_1}}{1+\left(\frac{dy}{dx}\right)_{c_1}\left(\frac{dy}{dx}\right)_{c_2}}\right|$$

(13) Orthogonal Curves

If the angle of intersection of two curves is a right angle, the two curves are said to be orthogonal. if the curves are orthogonal,

$$\left(\frac{dy}{dx}\right)_{c_1}\left(\frac{dy}{dx}\right)_{c_2} = -1$$

(14) Subtangent & Subnormal

$$\text{Length of Tangent} = \left|y_1\sqrt{1+\left(\frac{dx}{dy}\right)^2_{x_1,y_1}}\right|$$

$$\text{Length of Normal} = \left|y_1\sqrt{1+\left(\frac{dy}{dx}\right)^2_{x_1,y_1}}\right|$$

$$\text{Length of Subtangent} = \left|y_1\left(\frac{dx}{dy}\right)x_1, y_1\right|$$

(Projection of tangent)

$$\text{Length of subnormal} = \left|y_1\left(\frac{dy}{dx}\right)_{x_1,y_1}\right|$$

(Projection of normal)

(15) Leibnitz-rule

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)dt\right] = f(\Psi(x))\left\{\frac{d}{dx}\Psi(x)\right\} - f(\phi(x))\left\{\frac{d}{dx}\phi(x)\right\}$$

(16)Extrema of discontinuous Functions

1. Minimum of discontinuous Functions :

For Minimum at x = a
f(a) ≤ f(a + h)
f(a) ≤ f(a − h)

2. Maximum of discontinuous Functions:

For maximum at x = a
f(a) ≥ f(a + h)
f(a) ≥ f(a − h)

3. Neither Maximum nor minimum exists: