Differentiation

1. DERIVATIVE OF A FUNCTION

Derivative at a Point

The value of $f'(x)$ obtained by putting x = a, is called the derivative of $f(x)$ at x = a and it is denoted by $f'(a)$ or $\left\{\frac{dy}{dx}\right\}_{x=a}$.

2. Standard Derivatives

The following formulae can be applied directly for finding the derivative of a function:

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
7. $\frac{d}{dx}(e^x) = e^x$
8. $\frac{d}{dx}(a^x) = a^x \log_e a, a > 1$
9. $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$
10. $\frac{d}{dx}(x^n) = nx^{n-1}$
11. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
12. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
13. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, -\infty < x < \infty$
14. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
15. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$
16. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, -\infty < x < \infty$
17. $\frac{d}{dx}(|x|) = \frac{|x|}{x}$ or $\frac{x}{|x|}, x \neq 0.$

3. RULES FOR DIFFERENTIATION

1. The derivative of a constant function is zero, i.e., $\frac{d}{dx}(c) = 0.$
2. The derivative of constant times a function is constant times the derivative of the function, i.e.,

   $$\frac{d}{dx}\{c.f(x)\} = c\frac{d}{dx}\{f(x)\}.$$
3. The derivative of the sum or difference of two function is the sum or difference of their derivatives, i.e.,

   $$\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}.$$

Note: In general, if $f_1(x), f_2(x), \ldots f_n(x)$ are n differentiable functions, then we have

$$\frac{d}{dx}[f_1(x) \pm f_1(x) \pm \ldots \pm f_n(x)] = \frac{d}{dx}[f_1(x)] \pm \frac{d}{dx}[f_2(x)] \pm \ldots \pm \frac{d}{dx}[f_n(x)].$$

PRODUCT RULE OF DIFFERENTIATION

If $f(x)$ and $g(x)$ are differentiable functions of x, then

$$\frac{d}{dx}[f(x)\,g(x)] = f(x)\,g'(x) + g(x)\,f'(x).$$

NOTE

$$\frac{d}{dx}[f(x)\,g(x)\,h(x)] = f(x)\,g(x)\,h'(x) + f(x)\,g'(x)\,h(x) + f'(x)\,g(x)\,h(x)$$

Quotient Rule of Differentiation

If $f(x)$ and $g(x)$ are two differentiable functions of x, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}.$$

Differentiation of a Function (Chain Rule)

If y is a differentiable function of u and u is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Key Points on Chain Rule

1. The chain rule can be extended further as:

If y is a function of u, u is a function of v and v is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$ and so on.

2. If $y = u^n$, where u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = nu^{n-1} \times \frac{du}{dx}.$$

$$\left[\because \frac{dy}{du} = nu^{n-1}\right]$$

4. DERIVATIVE OF PARAMETRIC FUNCTIONS

If x = f (t) and y = g (t), then

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{f'(t)}{g'(t)}.$$

And $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}=\frac{d}{dt}\left(\frac{dy}{dx}\right)/\frac{dx}{dt}.$

5. DIFFERENTIATION OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

If y = f (x) and z = g(x), then in order to find the derivative of f (x) w.r.t. g(x), we use the formula

$$\frac{dy}{dz}=\frac{dy/dx}{dz/dx}=\frac{f'(x)}{g'(x)}$$

6. LOGARITHMIC DIFFERENTIATION

Properties of Logarithms

1. $\log_e (mn) = \log_e m + \log_e n$
2. $\log_e \left(\frac{m}{n}\right) = \log_e m - \log_e n$
3. $\log_e (m)^n = n \log_e |m|$
4. $\log_e e = 1$
5. $\log_n m = \frac{\log_e m}{\log_e n}$
6. $\log_n m . \log_m n = 1.$

Shorter Methods of Finding the Derivative of a Logarithmic Function

If y = $[f(x)]^{g(x)}$, then to find $\frac{dy}{dx}$, in addition to the method discussed above, we can also apply any of the following two methods:

Method 1

Step 1. Express y = $[f(x)]^{g(x)} = e^{g(x) \log f(x)}$

$[\because a^x = e^{x \log a}]$

Step 2. Differentiate w.r.t. x to obtain $\frac{dy}{dx}$

Method 2

Step 1. Evaluate

A = Differential coefficient of y treating f (x) as constant.

Step 2. Evaluate

B = Differential coefficient of y treating g(x) as constant.

Step 3. $\frac{dy}{dx}$ = A + B.

7. DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

Important Substitutions to Reduce the Function to a Simpler Form

Expressions	Substitutions
$\sqrt{a^2-x^2}$	Put x = a sin θ or x = a cos θ
$\sqrt{x^2-a^2}$	Put x = a sec θ or x = a cosec θ
$\sqrt{a^2+x^2}$	Put x = a tan θ or x = a cot θ
$\frac{a-x}{a+x}$ or $\frac{a+x}{a-x}$	Put x = a tan θ
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	Put x = a cos θ
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	Put $x^2 = a^2 \cos \theta$

8. DIFFERENTIATION OF A FUNCTION GIVEN IN THE FORM OF A DETERMINANT

If y = $\begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, then

$$\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ p'(x) & q'(x) & r'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Note: The differentiation of a determinant can be done in columns also.