LIMITS

1. Right hand and Left hand Limit

To evaluate $\lim_{x \to a^+} f(x)$

1. Put x = a + h in f (x) to get $\lim_{h \to 0} f(a+h)$
2. Take the limit as h $\to$ 0.

To evaluate $\lim_{x \to a^-} f(x)$.

1. Put x = a – h in f (x) to get $\lim_{h \to 0} f(a-h)$.
2. Take the limit as h $\to$ 0.

2. Some Useful Limits

(1) $\lim_{x \to a} \frac{x^n - a^n}{x - a}$ = na^{n-1}, where n $\in$ Q, the set of rational numbers.

(2)

(i) $\lim_{n \to \infty} a^n = \begin{cases} \infty, & if\ a>1 \\ 1, & if\ a=1 \\ 0, & if\ -1<a<1 \\ does\ not\ exist, & if\ a \le -1 \end{cases}$

(ii) $\lim_{x \to \infty} \frac{a_0x^p + a_1x^{p-1} + \ldots\ldots + a_{p-1}x + a_p}{b_0x^q + b_1x^{q-1} + \ldots\ldots + b_{q-1}x + b_q}$

$= \begin{cases} \frac{a_0}{b_0}, & if\ p=q \\ 0, & if\ p<q \\ \infty, \frac{a_0}{b_0} > 0 & if\ p>q \end{cases}$

(3) Trigonometric Limits

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(ii) $\lim_{x \to 0} \frac{\cos x}{x} = 1$

(iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(iv) $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

(v) $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

(vi) $\lim_{x \to 0} \frac{\sin x^0}{x} = \frac{\pi}{180}$.

Some Useful Expansions

(i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ to ∞

(ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ to ∞

(iii) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$ to ∞

(iv) $\sin^{-1} x = x$

$+ \frac{1^2.x^3}{3!} + \frac{1^2.3^2}{5!}X^5 + \frac{1^2.3^2.5^2}{7!}x^5 + \ldots$ to ∞

(v) $(\sin^{-1} x)^2$

$= \frac{2}{2!}x^2 + \frac{2.2^2}{4!}x^4 + \frac{2.2^2.4^2}{6!}x^6 + \ldots$ to ∞

(vi) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ to ∞.

(4) Exponential and Logarithmic Limits

(i) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(ii) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$, a > 0

(iii) $\lim_{x \to 0} \frac{a^x - b^x}{x} = \log_e\left(\frac{a}{b}\right)$; a, b > 0

(iv) $\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = n$

(v) $\lim_{n \to 0} \left(1 + \frac{1}{n}\right)^n = e$

(vi) $\lim_{h \to 0} (1+ah)^{1/h} = e^a$

(vii) $\lim_{x \to \infty} \frac{\log x}{x^m} = 0$, (m > 0)

(viii) $\lim_{x \to 0} \frac{\log_a(1+x)}{x} = \log_a e$, (a > 0, a $\neq$ 1)

(ix) $\lim_{x \to 0} \left(1 + \frac{a}{x}\right)^x = e^a$

(x) $\lim_{x \to 0} \left(1 + \frac{1}{f(x)}\right)^{f(x)}$ = e, where f (x) $\to \infty$ as x $\to$ 0.

(xi) $\lim_{x \to 0} (1+f(x))^{1/f(x)}$ = e, where f(x) $\to$ 0 as x $\to$ 0

Some Useful Expansions

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to ∞

(ii) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$ to ∞

(iii) $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ to ∞, $-1 < x \le 1$

(iv) $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$ to ∞, $-1 \le x < 1$

(v) $a^x = e^{x\log a} = 1 + x\log a + \frac{(x\log a)^2}{2!} + \ldots$ to ∞.

(vi) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}X^2 + \ldots$ to ∞, $-1 < x < 1$,

Note

(i) If $\lim_{x\to a} f(x) = A > 0$ and $\lim_{x\to a} g(x) = B$, then $\lim_{x\to a}[f(x)]^{g(x)} = A^B$.

(ii) If $\lim_{\to 0} f(x) = 1$ and $\lim_{x\to 0} g(x) = \infty$, then

$$\lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}.$$

3. EVALUATION OF LIMITS USING L'HOSPITAL'S RULE

(i) $\left(\frac{0}{0}\right)$ form: If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, provided the limit on the R.H.S. exists.

(ii) $\left(\frac{\infty}{\infty}\right)$ form: If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, provided the limit on the R.H.S. exists.

Note: That sometimes we have to repeat the process if the form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ again.