DETERMINANT

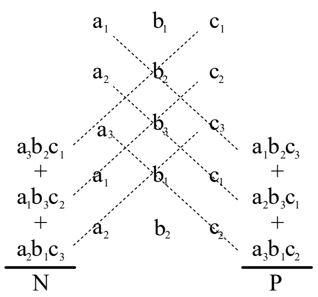
1. DETERMINANT OF A SQUARE MATRIX OF ORDER TWO AND THREE

Expansion of two order: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$

Expansion of three order: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

2. SARRUS RULE

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$



$\Rightarrow \Delta = P - N$

3. PROPERTIES OF DETERMINANTS

(i) The value of a determinant remains unchanged if its rows and columns are interchanged.

(ii) If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

(iii) If any two rows (or columns) of a determinant are identical, then the value of determinant is zero.

(iv) If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

(v) If some or all the elements of a row or column of a determinant are expressed as a sum of two (or more) terms, then the determinant can be expressed as a sum of two (or more) determinants.

(vi) If the equimultiples of corresponding elements of other row (or column) are added to each element of any row or column of a determinant, then the value of the determinant remains the same.

(vii) $|A^T| = |A|$, where A^T = transpose of A

(viii) If $A = [a_{ij}]_{3\times3}$, then $|kA| = k^3|A|$.

(ix) The determinant of the product of matrices is equal to product of their respective determinants, i.e., $|AB| = |A||B|$, where A& B are square matrices of same order

(x) $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{vmatrix} = a_1b_2c_3$

4. USE OF DETERMINANTS IN CO-ORDINATE GEOMETRY

(i) Area of triangle, whose vertices are (x_r, y_r) $\Delta = \left|\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$

(ii) If $a_rx + b_ry + c_r = 0$ are the sides of a triangle, then the area =

$\left|\dfrac{1}{2c_1c_2c_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2\right|$ C_1, C_2, C_3 are cofactors of c_1, c_2, c_3.

(iii) Equation of a straight line passing through two points

(x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(iv) If three lines $a_rx + b_ry + c_r = 0$ are concurrent, then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

(v) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight line then $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(vi) The equation of circle through three non-collinear points (x_r, y_r) is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

5. MINOR AND COFACTOR OF AN ELEMENT OF A DETERMINANT

Minor: The determinant that is left by cancelling the row and column intersecting at a particular element of a determinant is called the minor of that element of the determinant. Minor of an element a_{ij} of a determinant is denoted by M_{ij}.

Cofactor: The cofactor of an element a_{ij} of a determinant is denoted by A_{ij} (or Cij) and is equal to $(-1)^{i+j} M_{ij}$.

6. ADJOINT OF A MATRIX

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where A_{ij} is the cofactor of a_{ij}

If A be any given square matrix of order n, then A(adj A)= (adj A A=|A|I where I is the identity matrix of order n.

7. PROPERTIES OF ADJOINT OF A MATRIX

If A be any given square matrix of order n, we can define the following:

- A(adj A) = adj A) A = AI, where I is the identity matrix of order n
- For a zero matrix 0, adj(0) = 0
- For an identity matrix I, adj(I) = I
- For any scalar k, $\operatorname{adj}(kA) = k^{n-1}\operatorname{adj}(A)$
- $\operatorname{adj}(A^T) = (\operatorname{adj} A)^T$
- det(adj A), i.e. $|\operatorname{adj} A| = (\det A)^{n-1}$
- If A is an invertible matrix and A^{-1} be its inverse, then : adj A = $(\det A)^{A-1}$ is invertible with inverse $(\det A)^{-1}$ $A\operatorname{adj}(A^{-1}) = (\operatorname{adj} A)^{-1}$

  Suppose A and B are two matrices of order n, then adj(AB) = (adj B)(adj A)
- For any non-negative integer p, $\operatorname{adj}(A^p) = (\operatorname{adj} A)^p$

  If A is invertible, then the above formula also holds for negative k.

8. SINGULAR AND NON-SINGULAR MATRICES

A square matrix A is said to be singular if |A| = 0, otherwise it is called non-singular matrix.

If A & B are non-singular matrix of same order, then AB & BA are also non-singular matrices of same order.

9. INVERSE OF A MATRIX

If A and B are two matrices such that AB=I=BA

then B is called the inverse of A and it is denoted by A^{-1}

Also, $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$, if $|A| \neq 0$

10. PROPERTIES OF INVERSE

If A and B are the non-singular matrices, then the inverse matrix should have the following properties

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = A^{-1}B^{-1}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
- $(A_1 A_2 \ldots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1} \ldots A_2^{-1}A_1^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(kA)^{-1} = (1/k)A^{-1}$
- $AB = I_n$, where A and B are inverse of each other.

11. PRODUCT OF TWO DETERMINANTS

Let $\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

Then product of Δ_1 and Δ_2 is defined as

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1x_1 + a_2x_2 + a_3x_3 & a_1y_1 + a_2y_2 + a_3y_3 & a_1z_1 + a_2z_2 + a_3z_3 \\ b_1x_1 + b_2x_2 + b_3x_3 & b_1y_1 + b_2y_2 + b_3y_3 & b_1z_1 + b_2z_2 + b_3z_3 \\ c_1x_1 + c_2x_2 + c_3x_3 & c_1y_1 + c_2y_2 + c_3y_3 & c_1z_1 + c_2z_2 + c_3z_3 \end{vmatrix}$$

12. DIFFERENTIATION OF A DETERMINANT

Let $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix}$

then $\Delta'(x)$

$$= \begin{vmatrix} f'_1(x) & g'_1(x) & h'_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f'_2(x) & g'_2(x) & h'_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f'_3(x) & g'_3(x) & h'_3(x) \end{vmatrix}$$

13. INTEGRATION OF DETERMINATION

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ p & q & r \\ l & m & n \end{vmatrix}$, where p, q, r, l, m and n are constants.

Then $\int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)dx \\ p & q & r \\ l & m & n \end{vmatrix}$

14. USE OF SUMMATION

If $f(r) = \begin{vmatrix} r & r^2 & r^3 \\ p & q & t \\ 1 & 2 & 3 \end{vmatrix}$ where p, q, t are constants, then $\sum_{r=1}^{n} f(r) = \begin{vmatrix} \sum_{r=1}^{n} r & \sum_{r=1}^{n} r^2 & \sum_{r=1}^{n} r^3 \\ p & q & t \\ 1 & 2 & 3 \end{vmatrix}$

15.

$a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$

- Consistent system of equation has solution
  - Unique solution: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (Equation represents intersecting lines)
  - Infinite solution: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (Equation represents coincident lines)
- Inconsistent system of equation has no solution: $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ (Equation represents parallel disjoint lines)

16.

Check value of Δ

- $\Delta \neq 0$
  - Consistent system and has unique solution
  - $x = \frac{\Delta_1}{\Delta}; y = \frac{\Delta_2}{\Delta}; z = \frac{\Delta_3}{\Delta}$
- $\Delta = 0$
  - Check the values of Δ_1, Δ_2 and Δ_3
    - At least one of Δ_1, Δ_2 and Δ_3 is not zero
      - Inconsistent system
    - $\Delta_1 = \Delta_2 = \Delta_3 = 0$
      - Put z = t and solve any two equations to get the values of x and y in terms of t.

17.

1. Symmetric determinant

The elements situated at equal distance from the diagonal are equal both in magnitude and sign.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$$

2. Skew symmetric determinant

All the diagonal elements are zero and the elements situated at equal distance form the diagonal are equal in magnitude but opposite in sign. The value of a skew symmetric determinant of odd order is zero.

$$\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$$

3. Circulant determinant:

The elements of the rows (or columns) are in cyclic arrangement

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

4. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$

5. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$

6. $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca).$