VECTOR ALGEBRA

01 Types Of Vectors

1. Zero Vector : A vector whose initial and terminal points coincide, It has zero magnitude.
2. Unit Vector: A vector whose magnitude is unity. The unit vector in the direction of $\vec{a}$ is denoted as $\hat{a}$.
3. Coinitial Vectors : Two or more vectors having the same initial point.
4. Collinear Vectors: Two or more vectors are collinear, if they are parallel to the same line irrespective of their magnitude.
5. Equal Vectors: Two vectors are said to be equal, if they have same magnitude & direction regardless of the position of their initial points.
6. Negative of a vector : A vector whose magnitude is the same as that of the given vector, but the direction is opposite to that of it.
7. Position Vector: Let O be the origin & P(X,Y,Z) be a point with respect to the origin O. Then the vector called the position vector of the point P with respect to O. $\overline{OP}$ is

$$|\overline{OP}| = \sqrt{x^2+y^2+z^2}$$

- Direction angles: The angles made by $\overline{OP}$ with positive direction of x, y, & z-axes (say α, β & γ respectively).
- Directions cosines: the cosine value of these angles i.e., $\cos\alpha$, $\cos\beta$ & $\cos\gamma$ of $\overline{OP}$ denoted by l, m & n respectively.

02 Properties of Vector Addition

(i) For any two vectors $\vec{a}$ & $\vec{b}$, $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative property)
(ii) For any three vectors $\vec{a}, \vec{b}$, & $\vec{c}$ $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (Associative property)

03 Multiplication Of A Vector By A Scalar

if $\vec{a}$ is multiplied by scalar m then the product $m\vec{a}$ is a vector whose magnitude is $|m|$ times that of $\vec{a}d$ & direction is same as $\vec{a}$ if m is positive where as opposite to that of $\vec{a}$ if m is negative.

- $m(\vec{a})=(\vec{a})m$
- $(m+n)\vec{a}=m\vec{a}+n\vec{a}$
- $m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$
- $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}.$

04 Dot or Scalar Product of Vectors

Dot product of two vectors $\vec{a}$ & $\vec{b}$ inclined at an angle θ is $(\vec{a}\cdot\vec{b})=|\vec{a}||\vec{b}|\cos\theta$

- $\vec{a}\cdot\vec{b}\in R$
- $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
- $(x\vec{a})\cdot\vec{b}=x(\vec{a}\cdot\vec{b})=\vec{a}\cdot(x\vec{b})$
- $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
- If $\vec{a}$ & $\vec{b}$ perpendicular, $\vec{a}\cdot\vec{b}=0$
- $\vec{a}\cdot\vec{b}<0$ iff angle between $\vec{a}$ & $\vec{b}$ is obtuse.
- $\hat{i}\cdot\hat{i}=1$, $\hat{j}\cdot\hat{j}=1$, $\hat{k}\cdot\hat{k}=1$, $\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{i}\cdot\hat{k}=0$
- If two vectors have same direction then $\cos\theta=1\Rightarrow\vec{a}\cdot\vec{b}=ab$
- If two vectors have opposite direction then $\cos\theta=-1\Rightarrow\vec{a}\cdot\vec{b}=-ab$
- If $\hat{a}$ & $\hat{b}$ are unit vectors, $\hat{a}\cdot\hat{b}=\cos\theta$
- $\vec{a}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ $\vec{a}\cdot\vec{b}=a_1b_1+b_1b_2+c_1b_3$.
- Projection of a vector $\vec{b}$ on the other vector $\vec{a}$ is given by $\vec{b}\cdot\hat{a}$ or $\vec{b}\left(\frac{\vec{a}}{|\vec{a}|}\right)$
- A vector in the direction of the bisector of the angle between the two rectors $\vec{a}$ & $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}$
- Bisector of the interior angle between two vectors $\vec{a}$ & $\vec{b}$ is $\lambda\left(\frac{\vec{a}}{a}+\frac{\vec{b}}{b}\right)$ i.e., $\lambda(\hat{a}+\hat{b})$ where $\lambda\in R^+$ & Bisector of the interior angle is $\lambda\left(\frac{\vec{a}}{a}-\frac{\vec{b}}{b}\right)$, is $\lambda(\hat{a}-\hat{b})$

05 Cross product

Let $\vec{a}$ & $\vec{b}$ be two non-zero vectors inclined at an angle θ
Then, vector product is defined as $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\hat{n}$
where, $\hat{n}$ is a unit vector perpendicular to both vectors $\vec{a}$ & $\vec{b}$ such that $\vec{a},\vec{b}$ & $\hat{n}$ form a right handed system.

- **Lagrange's Identity:**
For any two vectors $\vec{a},\vec{b}$
$(\vec{a}\times\vec{b})^2=a^2b^2-(\vec{a}\cdot\vec{b})^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}$

- **Formulation of vector product in terms of scalar product:** The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that
$|\vec{c}|=\sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}$
$\vec{c}\cdot\vec{a}=\vec{c}\cdot\vec{b}=0$
$\vec{a},\vec{b},\vec{c}$ form a right-handed system.

- **Remarks**

(a) $\vec{a}\times\vec{b}$ is a vector.
(b) If $\vec{a}$ & $\vec{b}$ are nonzero vectors, then $\vec{a}\times\vec{b}=0\Leftrightarrow\vec{a}\parallel\vec{b}$
(c) For mutually perpendicular unit vectors $\hat{i},\hat{j},\hat{k}$,
$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$
(d) $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$
(e) If $\vec{a}$ & $\vec{b}$ represent the adjacent sides of a triangle then its area is $\frac{1}{2}|\vec{a}\times\vec{b}|$
(f) If $\vec{a}$ & $\vec{b}$ represent the adjacent sides of a parallelogram then the area is $|\vec{a}\times\vec{b}|$
(g) $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$
(h) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $|\vec{a}\times\vec{b}|=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$
(i) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n}=\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

- **Vector area:**
- If $\vec{a},\vec{b}$ & $\vec{c}$ are the position vectors of 3 points then area of $\triangle$ ABC $=\frac{1}{2}[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}]$ A, B, C are collinear iff $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0.$
- Area of any quadrilateral whose diagonal vectors are d_1 & d_2 is given by $\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$

06 Vector Triple Product:

Vector Triple Product of $\vec{a},\vec{b},\vec{c}$ is $\vec{a}\times(\vec{b}\times\vec{c})$.
It is a vector perpendicular to the plane containing $\vec{a}$ & $\vec{b}\times\vec{c}$ lying in the plane of $\vec{b}$ & $\vec{c}$

- $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$
- $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$
- $(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$

07 Test of Collinearity

$x\vec{a}+y\vec{b}+z\vec{c}=0$ [x, y, z scalars, x + y + z = 0]

08 Test of Coplanarity

$x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d}=0$ [x, y, z, w scalars, x + y + z + w = 0]

09 Reciprocal system of Vectors

If $\vec{a},\vec{b},\vec{c}$ & $\vec{a}',\vec{b}',\vec{c}'$ are two sets of noncoplanar vectors such that $\vec{a}\cdot\vec{a}'=\vec{b}\cdot\vec{b}'=\vec{c}\cdot\vec{c}'=1$ then the two systems are called reciprocal systems.

$$\vec{a}'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]},\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]},\ \vec{c}'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$

10 Scalar Triple Product/Box Product: $[\vec{a}\ \vec{b}\ \vec{c}]$

Box product of $\vec{a},\vec{b},\vec{c}$ is $(\vec{a}\times\vec{b})\cdot\vec{c}=abc\sin\theta\cos\phi$

$\theta\rightarrow$ angle between $\vec{a}$ & $\vec{b}$

$\phi\rightarrow$ angle between $\vec{a}\times\vec{b}$ and $\vec{c}$

Box product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a},\vec{b},\vec{c}$

$$V=[\vec{a}\ \vec{b}\ \vec{c}]$$

- $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$
- $[\vec{a}\vec{b}\vec{c}]=[\vec{b}\vec{c}\vec{a}]=[\vec{c}\vec{a}\vec{b}]$.
- $\vec{a}\cdot(\vec{b}\times\vec{c})=-\vec{a}\cdot(\vec{c}\times\vec{b})$.
- $[\vec{a}\vec{b}\vec{c}]=-[\vec{a}\vec{c}\vec{b}]$.
- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ then $[\vec{a}\ \ \vec{b}\ \ \vec{c}]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$
- $\vec{a},\vec{b},\vec{c}$ are coplanar if $[\vec{a}\ \vec{b}\ \vec{c}]=0$
- If $\vec{a},\vec{b},\vec{c}$ are non-coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}]>0$ for right handed system & $[\vec{a}\ \vec{b}\ \vec{c}]<0$ for left handed system.
- $[\hat{i}\ \hat{j}\ \hat{k}]=1$
- $[k\vec{a}\ \ \vec{b}\ \ \vec{c}]=k[\vec{a}\ \ \vec{b}\ \ \vec{c}]$
- $[(\vec{a}+\vec{b})\ \ \vec{c}\ \ \vec{d}]=[\vec{a}\ \ \vec{c}\ \ \vec{d}]+[\vec{b}\ \ \vec{c}\ \ \vec{d}]$

11 Direction cosines & Direction Ratios

- If $\vec{a}$ makes angles of α,β,γ with the direction of x, y, z axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction cosines which $\vec{a}$ is are usually denoted by l, m, n.
- Any three members a,b,c proportional to the direction cosines of a line are called direction ratios

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$$

$$l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}};\quad m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}},\ l^2+m^2+n^2=1$$

12 Vector Equation of a Line

- Parametric vector equation of a line passing through two points $A(\vec{a})$ & $B(\vec{b})$ is $\vec{r}=\vec{a}+t(\vec{b}-\vec{a})$
- If line passes through the point $A(\vec{a})$ & is parallel to $\vec{b}$, then its equation is $\vec{\gamma}=\vec{a}+t\vec{b}$
- Equation of the bisectors of the angle between the lines, $\vec{r}=\vec{a}+\lambda\vec{b}$ & $\vec{r}=\vec{a}+\mu\vec{c}$ is $\vec{r}'=\vec{a}+t(\vec{b}+\vec{c})$ & $\vec{r}=\vec{a}+p(\vec{c}-\vec{b})$

13 Shortest distance between two lines

If two lines are $\vec{r}_1=\vec{a}_1+k\vec{b}$ & $\vec{r}_2=\vec{a}_2+k\vec{b}$

then $\alpha=\left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$

14 Equation of Plane

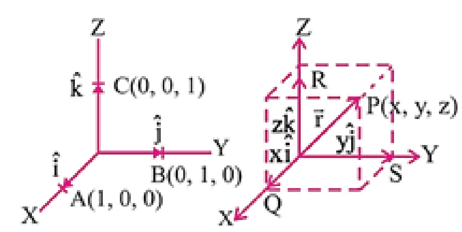
$(\vec{r}-\vec{r}_0)\cdot\vec{n}=0$ containing the point with position vector $\vec{r}_0$, where $\vec{n}$ is a vector normal to the plane.

$\vec{r}\cdot\vec{n}=d$ general equation

15 Projection & Component

- Projection of $\vec{a}$ along $\vec{b}=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$
- Component of $\vec{a}$ along $\vec{b}=\frac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}$
- Projection of $\vec{a}\perp\vec{b}=\frac{|\vec{a}\times\vec{b}|}{|\vec{b}|}$
- Component of $\vec{a}\perp\vec{b}=\vec{a}-\frac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}$

16 Component Of Vector



$\overline{OA},\overline{OB}$ & $\overline{OC}$ are unit vectors along x,y & z axes respectively, denoted by $\hat{i},\hat{j}$ & $\hat{k}$ respectively Position Vector of with reference to O is given by:

$$\overline{OP}(\text{ or }\vec{\mathbf{r}})=x\hat{i}+y\hat{j}+z\hat{k}.$$

This form of any vector is called its component form.

Also, $(\overline{OP})=|\vec{r}|=\sqrt{x^2+y^2+z^2}$

17 Vector Joining Two Points

Let $A(x_1,y_1,z_1)$ & $B(x_2,y_2,z_2)$ be any two points in the space, then

$$\overline{OA}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\ \&\ \overline{OB}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$$

$$\therefore\ \overline{AB}=\overline{OB}-\overline{OA}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$$

$$|\overline{AB}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

18 Section Formulae

The position vector of a point R dividing a line segment joining the points points P & Q whose position vectors are $\vec{a}$ & $\vec{b}$ respectively, in the ratio m : n

(i) internally, is given by $\frac{m\vec{b}+n\vec{a}}{m+n}$

(ii) externally, is given by $\frac{m\vec{b}-n\vec{a}}{m-n}$

The position vector of the middle point of PQ is given by $\frac{1}{2}(\vec{a}+\vec{b})$

19 Scalar product of four vectors

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}.\vec{c} & \vec{b}.\vec{c}\\ \vec{a}.\vec{d} & \vec{b}.\vec{d}\end{vmatrix}$$

20 Vector product of four vectors

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$$