CONTINUITY AND DIFFERENTIABILITY

01 Continuity

Continuity of a Function at a Point

Suppose f is a real function on a subset of the real numbers & let c be a point in the domain of f. Then f is continuous at c if

$\lim_{x\to c} f(x) = f(c)$

Continuity of a Function in an Interval

Suppose f is a function defined on a closed interval [a, b], then for f to be continuous, it needs to be continuous at every point in [a, b] including the end points a & b.

Continuity of f at a, $\lim_{x\to a^+} f(x) = f(a)$

Continuity of f at b, $\lim_{x\to b^-} f(x) = f(b)$

A function which is not continuous at point x=c is said to be discontinuous at that point

02 Algebra of Continuous Functions

Theorem 1: Suppose f & g be two real functions continuous at a real number c, Then

(1) f + g is continuous at x=c

(2) f - g is continuous at x=c

(3) f.g is continuous at x=c

(4) f/g is continuous at x=c, (provided $g(c)\neq 0$)

Theorem 2: Suppose f & g are real valued functions such that (fog) is defined at c. If g is continuous at c& if f is continuous at g(c), then (fog) is continuous at c.

03 Differentiability

A function f is said to be differentiable at a point c in its domain, if its left hand & right hand derivatives exist at c are equal.

Here at x = c,

Left Hand Derivative,

$$\text{L.H.D.} = \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h} = Lf'(c)$$

Right Hand Derivative,

$$\text{R.H.D.} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = Rf'(c)$$

Theorem: If a function f is differentiable at a point c, then it is also continuous at that point. Therefore, every differentiable function is continuous, but the converse is not true.

04 Algebra of Derivatives

Let u, v be the functions of X.

(1) Sum and Difference Rule $(u \pm v)' = u' \pm v'$

(2) Leibnitz or Product Rule $(uv)' = uv' + u'v$

(3) Quotient Rule $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

05 Chain Rule

If y is a function of u, u is a function of v & v is a function of x.

Then, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$

06 Implicit Functions

An equation of the form f(x, y) = 0 in which y is not expressible in terms of x is called an implicit function of x & y.

Derivative of Implicit Functions

Let y=f(x, y), where f(x, y) be an implicit function of x & y.

Firstly differentiate both sides of equation w.r.t x

Then take all terms involving $\frac{dy}{dx}$ on L.H.S. & remaining terms on R.H.S. to get the required value.

07 Differentiation of Inverse Trigonometric Functions

f(x)	f′(x)	Domain of f′
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$	(-1,1)
$\cos^{-1}x$	$\frac{-1}{\sqrt{1-x^2}}$	(-1,1)
$\tan^{-1}x$	$\frac{1}{1+x^2}$	R
$\cot^{-1}x$	$\frac{-1}{1+x^2}$	R
$\sec^{-1}x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\lvert x\rvert>1$
$\text{cosec}^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\lvert x\rvert>1$

08 Logarithmic Differentiation

Logarithmic Differentiation is a very useful technique to differentiate functions of the form $f(x)=[u(x)]^{v(x)}$, where f(x) & u(x) are positive.

We apply logarithm (to base) on both sides to the above equation & then differentiate by using chain rule, in this way we can find f' (x). This process is called logarithmic

$$\frac{d}{dx}\left(e^x\right) = e^x,\ \frac{d}{dx}(\log x) = \frac{1}{x}\ \&\ \frac{d}{dx}a^x = a^x \log a$$

09 Derivatives of Functions In Parametric Form

The set of equations x = f(t), y = g(t) is called the parametric form of an equation.

Here, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ or $\frac{g(t)}{f(t)}$

Here, $\frac{dy}{dx}$ is expressed in terms of parameter only without directly involving the main variables.

10 Second Order Derivative

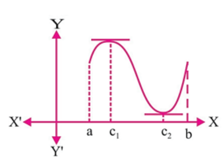
Let y = f(x), then $\frac{dy}{dx} = f'(x)$

If f′(x) is differentiable, then we may differentiate it again w.r.t. x & get the second order derivative represented by:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) \text{ or } \frac{d^2y}{dx^2} \text{ or } f''(x) \text{ or } D^2y \text{ or } y'' \text{ or } y_2$$

11 Rolle's Theorem

If f: [a, b]→R is continuous on [a, b]& differentiable on (a, b) such that f(a) = f(b), then there exists some c in (a, b) such that f' (c) = 0

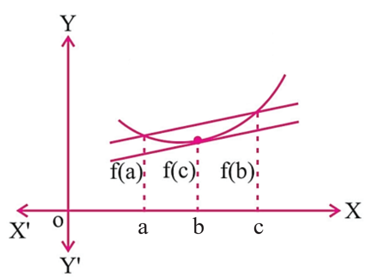


In the above graph, the slope of tangent to the curve at least at one point becomes zero. The slope of tangent at any point on the graph of y = f(x) is nothing but the derivative of f(x) at that point.

12 Mean Value Theorem

If f: [a, b]→R is continuous on [a, b] & differentiable on (a, b). Then there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



The Mean value Theorem states that there is a point c in (a, b) such that the slope of the tangent at (c, f(c)) is same as the slope of the secant between (a, f(a)) and (b, f(b)) or there is a point c in (a, b) such that the tangent at (c, f(c) is parallel to the secant between (a, f(a) & (b, f(b))