Assignment 1: Question 2

Question: In this exercise, we will prove the orthocenter theorem pertaining to the vanishing points Q, R, S of three mutually perpendicular directions OQ, OR, OS, where O is the pinhole (origin of camera coordinate system). Let the image plane be $Z = f$. Recall that two directions v_1 and v_2 are orthogonal if $v_1^T v_2 = 0$. One can conclude that OS is orthogonal to $OR - OQ$ (why?). Also the optical axis Oo (where o is the optical center) is orthogonal to $OR - OQ$ (why?). Hence the plane formed by triangle OSo is orthogonal to $OR - OQ$ and hence line oS is perpendicular to $OR - OQ = QR$ (why?). Likewise oR and oQ are perpendicular to QS and RS. Hence we have proved that the altitudes of the triangle QRS are concurrent at the point o. QED. Now, in this proof, I considered the three perpendicular lines to be passing through O. How will you modify the proof if the three lines did not pass through O? [4 points]

Answer:
Since OR and OQ are orthogonal to OS,

$$OS^T OR = 0$$

and

$$OS^T OQ = 0$$

Consider,

$$\begin{aligned} OS^T(OR - OQ) &= OS^T OR - OS^T OQ \\ &= 0 \end{aligned}$$

Therefore, OS is orthogonal to $OR - OQ$.

Q and R lie on the image plane and the optical axis Oo is perpendicular to the image plane.
Therefore, Oo is perpendicular to $OR - OQ = QR$.

This implies that the plane defined by Oo and OS is orthogonal to $OR - OQ$. This is the plane formed by the triangle OSo.
Therefore, oS is perpendicular to $OR - OQ = QR$.

Similarly, oR and oQ are perpendicular to QS and RS.
Hence altitudes of the triangle QRS are concurrent at the point o.

Suppose, the lines OQ, OR and OS did not pass through O but passed through another point O'.
Now, if we translate $O'Q, O'R$ and $O'S$ to O, the vanishing points do not change. ($\because$ definition of vanishing points)
We can now apply the above procedure to prove the orthocenter property.