Assignment 5: Quesion 1

April 14, 2015

1. This question is inspired from one of the questions that was asked in class. We will prove why the value of the coherence between $m \times n$ measurement matrix $\mathbf{\Phi}$ (with all rows normalized to unit magnitude) and $n \times n$ orthonormal representation matrix $\mathbf{\Psi}$ must lie within the range $(1, \sqrt{n})$. Recall that the coherence is given by the formula $\mu(\mathbf{\Phi}, \mathbf{\Psi}) = \sqrt{n}\text{max}_{i,j\in\{0,1,\ldots,n-1\}}|\mathbf{\Phi^i}^t\mathbf{\Psi_j}|$. Proving the upper bound should be very easy for you. To prove the lower bound, proceed as follows. Consider a unit vector $\mathbf{g} \in \mathbb{R}^n$. We know that it can be expressed as $\mathbf{g} = \sum_{k=1}^n \alpha_k \mathbf{\Psi_k}$ as $\mathbf{\Psi}$ is an orthonormal basis. Now prove that $\mu(\mathbf{\Psi}, \mathbf{g}) = \text{max}_{i\in\{0,1,\ldots,n-1\}}\frac{|\alpha_i|}{\sum_{j=1}^n \alpha_j^2}$. Exploiting the fact that $\mathbf{g}$ is a unit vector, prove that the minimal value of coherence is given by $\mathbf{g} = \sqrt{1/n}\sum_{k=1}^n \mathbf{\Psi_k}$ and hence the minimal value of coherence is 1. [3 points]

Answer The coherence is given by the formula

$$\mu(\mathbf{\Phi}, \mathbf{\Psi}) = \sqrt{n}\text{max}_{i,j\in\{0,1,\ldots,n-1\}}|\mathbf{\Phi^i}^t\mathbf{\Psi_j}| \tag{1}$$

$$\therefore \mu(\mathbf{\Phi}, \mathbf{\Psi}) = \sqrt{n}\text{max}_{i,j\in\{0,1,\ldots,n-1\}}|\cos\theta_{i,j}| \quad \text{where } \theta_{i,j} \text{ is the angle between } \mathbf{\Phi^i} \text{ and } \mathbf{\Psi_j} \tag{2}$$

$$\cos\theta_{i,j} \leq 1 \tag{3}$$

$$\therefore \text{max}_{i,j\in\{0,1,\ldots,n-1\}}|\cos\theta_{i,j}| \leq 1 \tag{4}$$

$$\therefore \mu(\mathbf{\Phi}, \mathbf{\Psi}) \leq \sqrt{n} \tag{5}$$

Hence, upper bound is proven.
To prove the lower bound, consider a unit vector $\mathbf{g} \in \mathbb{R}^n$.
Since $\mathbf{\Psi}$ is an orthonormal basis, $\mathbf{g}$ can be expressed as $\mathbf{g} = \sum_{k=1}^n \alpha_k \mathbf{\Psi_k}$.

$$
\begin{aligned}
\mu(\mathbf{\Psi},\mathbf{g}) &= \sqrt{n}\text{max}_{i\in\{0,1,\dots,n-1\}}|\mathbf{g}^t\mathbf{\Psi_i}| && (6)\\
&= \sqrt{n}\text{max}_{i\in\{0,1,\dots,n-1\}}|(\sum_{k=1}^{n}\alpha_k\mathbf{\Psi_k})^t\mathbf{\Psi_i}| && (7)\\
&= \sqrt{n}\text{max}_{i\in\{0,1,\dots,n-1\}}|\sum_{k=1}^{n}\alpha_k(\mathbf{\Psi_k}^t\mathbf{\Psi_i})| && (8)\\
&= \sqrt{n}\text{max}_{i\in\{0,1,\dots,n-1\}}|\alpha_i| \quad \because \mathbf{\Psi}\text{ is orthonormal and }\mathbf{g}\text{ is a unit vector} && (9)\\
\sum_{j=1}^{n}\alpha_j^2 &= 1 \quad \because \mathbf{g}\text{ is a unit vector} && (10)\\
\text{Let, max}_{i\in\{0,1,\dots,n-1\}}|\alpha_i| &= \lambda && (11)\\
\therefore \sum_{j=1}^{n}\alpha_j^2 &\le n\lambda^2 \quad \text{with equality when }|\alpha_i| = \lambda\text{ for all }i && (12)\\
\therefore 1 &\le n\lambda^2 && (13)\\
\therefore \lambda &\ge \frac{1}{\sqrt{n}} && (14)\\
\therefore \text{max}_{i\in\{0,1,\dots,n-1\}}|\alpha_i| &\ge \frac{1}{\sqrt{n}} && (15)\\
\therefore \mu(\mathbf{\Psi},\mathbf{g}) &\ge 1 && (16)
\end{aligned}
$$

Therefore, $\mu(\mathbf{\Psi},\mathbf{g}) \ge 1$ when $|\alpha_i| = \lambda$ for all i.
i.e. $|\alpha_i| = \lambda = \frac{1}{\sqrt{n}}$
i.e. $\mathbf{g} = \sqrt{1/n}\sum_{k=1}^{n}\mathbf{\Psi_k}$.