Assignment 1: Question 5

Question: Consider two sets of corresponding points $\{\mathbf{p}_{1i} = (x_{1i}, y_{1i})\}_{i=1}^{n}$ and $\{\mathbf{p}_{2i} = (x_{2i}, y_{2i})\}_{i=1}^{n}$. Assume that each pair of corresponding points is related as follows: $\mathbf{p}_{2i} = \alpha\mathbf{R}\mathbf{p}_{1i} + \mathbf{t} + \eta_i$ where $\mathbf{R}$ is an unknown rotation matrix, $\mathbf{t}$ is an unknown translation vector, α is an unknown scalar factor and η_i is a vector (unknown) representing noise. Explain how you will extend the method we studied in class for estimation of $\mathbf{R}$ to estimate α and $\mathbf{t}$ as well. Derive all necessary equations (do not merely guess the answers). [5 points]

Answer:
We seek to minimize,

$$
\begin{aligned}
E(\boldsymbol{R}, \boldsymbol{t}, \alpha) &= ||\boldsymbol{P_2} - (\alpha\boldsymbol{R}\boldsymbol{P_1} + \boldsymbol{t})||_F^2 \\
&= \sum_{i=0}^{N} ||\boldsymbol{p}_{2i} - (\alpha\boldsymbol{R}\boldsymbol{p}_{1i} + \boldsymbol{t})||_F^2 \\
\frac{\partial E}{\partial \boldsymbol{t}} &= 2\sum_{i=0}^{N} (\boldsymbol{p}_{2i} - \alpha\boldsymbol{R}\boldsymbol{p}_{1i} - \boldsymbol{t})(-1) \\
0 &= 2\sum_{i=0}^{N} (\boldsymbol{p}_{2i} - \alpha\boldsymbol{R}\boldsymbol{p}_{1i} - \hat{\hat{\boldsymbol{t}}})(-1) \\
N\hat{\boldsymbol{t}} &= \sum_{i=0}^{N} \boldsymbol{p}_{2i} - \alpha\boldsymbol{R}\sum_{i=0}^{N} \boldsymbol{p}_{1i} \\
\hat{\boldsymbol{t}} &= \bar{\boldsymbol{p}_2} - \alpha\boldsymbol{R}\bar{\boldsymbol{p}_1}
\end{aligned}
$$

$$
\begin{aligned}
E(\boldsymbol{R}, \hat{\boldsymbol{t}}, \alpha) &= ||\boldsymbol{P_2} - (\alpha\boldsymbol{R}\boldsymbol{P_1} + \hat{\boldsymbol{t}})||_F^2 \\
E(\boldsymbol{R}, \hat{\boldsymbol{t}}, \alpha) &= ||\boldsymbol{P_2} - (\alpha\boldsymbol{R}\boldsymbol{P_1} + \bar{\boldsymbol{P}}_2 - \alpha\boldsymbol{R}\bar{\boldsymbol{P}}_1)||_F^2 \\
E(\boldsymbol{R}, \alpha) &= ||\boldsymbol{P_2'} - (\alpha\boldsymbol{R}\boldsymbol{P_1'})||_F^2
\end{aligned}
$$

where, $\boldsymbol{P_2'} = \boldsymbol{P_2} - \bar{\boldsymbol{P}}_2$ and $\boldsymbol{P_1'} = \boldsymbol{P_1} - \bar{\boldsymbol{P}}_1$.

$$
\begin{aligned}
E(\boldsymbol{R}, \alpha) = & \quad ||\boldsymbol{P_2'} - (\alpha\boldsymbol{R}\boldsymbol{P_1'})||_F^2 && (1) \\
= & \quad trace((\boldsymbol{P_2'} - \alpha\boldsymbol{R}\boldsymbol{P_1'})^T(\boldsymbol{P_2'} - \alpha\boldsymbol{R}\boldsymbol{P_1'})) && (2) \\
= & \quad trace(\boldsymbol{P_2'}^T\boldsymbol{P_2'} - \alpha\boldsymbol{P_2'^T}\boldsymbol{R}\boldsymbol{P_1'} - \alpha(\boldsymbol{R}\boldsymbol{P_1'})^T\boldsymbol{P_2'} + \alpha^2(\boldsymbol{R}\boldsymbol{P_1'})^T\boldsymbol{R}\boldsymbol{P_1'}) && (3) \\
= & \quad trace(\boldsymbol{P_2'}^T\boldsymbol{P_2'} - 2\alpha\boldsymbol{P_2'^T}\boldsymbol{R}\boldsymbol{P_1'} + \alpha^2\boldsymbol{P_1'}^T\boldsymbol{P_1'}) && (4) \\
= & \quad trace(\boldsymbol{P_2'}^T\boldsymbol{P_2'}) - 2\alpha trace(\boldsymbol{P_2'^T}\boldsymbol{R}\boldsymbol{P_1'}) + \alpha^2 trace(\boldsymbol{P_1'}^T\boldsymbol{P_1'}) && (5) \\
& && (6)
\end{aligned}
$$

Assume α to be constant for the estimation of $\boldsymbol{R}$.
Then, minimizing $E(\boldsymbol{R})$ is equivalent to maximizing $trace(\boldsymbol{P_2'^T R P_1'})$. (We consider $\alpha > 0$, since $\alpha < 0$ causes reflection.)

$$
\begin{aligned}
trace(\boldsymbol{P_2'^T R P_1'}) &= trace(\boldsymbol{R P_1' P_2'^T}) \\
\boldsymbol{P_1' P_2'^T} &= \boldsymbol{U S V^T} \\
trace(\boldsymbol{P_2'^T R P_1'}) &= trace(\boldsymbol{R U S V^T}) \\
&= trace(\boldsymbol{S V^T R U}) \\
&= trace(\boldsymbol{S X})
\end{aligned}
$$

where $\boldsymbol{X}$ is orthogonal.
This expression is maximized if $\boldsymbol{X}_{ii} = 1$ all along its diagonal.
As $\boldsymbol{X}$ is orthonormal, we must have $\boldsymbol{X} = \boldsymbol{I}$
Therefore,

$$\boldsymbol{R} = \boldsymbol{V U^T}$$

From 5,

$$
\begin{aligned}
E(\boldsymbol{R}, \alpha) &= trace(\boldsymbol{P_2'}^T \boldsymbol{P_2'}) - 2\alpha trace(\boldsymbol{P_2'^T R P_1'}) + \alpha^2 trace(\boldsymbol{P_1'}^T \boldsymbol{P_1'}) \\
E(\boldsymbol{R}, \alpha) &= trace(\boldsymbol{P_2'}^T \boldsymbol{P_2'}) - 2\alpha trace(\boldsymbol{P_2'^T V U^T P_1'}) + \alpha^2 trace(\boldsymbol{P_1'}^T \boldsymbol{P_1'}) \\
\frac{\partial E}{\partial \boldsymbol{\alpha}} &= -2 trace(\boldsymbol{P_2'^T V U^T P_1'}) + 2\alpha trace(\boldsymbol{P_1'}^T \boldsymbol{P_1'}) \\
0 &= -2 trace(\boldsymbol{P_2'^T V U^T P_1'}) + 2\alpha trace(\boldsymbol{P_1'}^T \boldsymbol{P_1'}) \\
\alpha &= \frac{trace(\boldsymbol{P_2'^T V U^T P_1'})}{trace(\boldsymbol{P_1'}^T \boldsymbol{P_1'})}
\end{aligned}
$$