MAI-102 TUT-5

1

a) $X \sim$ Binomial Distribution $(n=25, p=0.2)$

$$p(x) = {}^{n}C_{x}\, p^{x}(1-p)^{n-x} \qquad x=0,1,\dots n$$

$$\mu = np = 5$$

$$\sigma^2 = np(1-p) = 25 \times \frac{1}{5} \times \frac{4}{5} = 4$$

$$\mu - 2\sigma = 5 - 2*2 = 1$$

$$P(X<1) = P[0] = (1-p)^{25} = (0.8)^{25}$$

b) $X \sim$ Binomial Distribution $(n=900, p=0.5)$

$$P(x) = {}^{n}C_{x}\, p^{x}(1-p)^{n-x}$$

$$\mu = np = 450$$

$$\sigma^2 = np(1-p) = 900 \times \frac{1}{2} \times \frac{1}{2} = 225$$

$$P(360 < X < 540) = P(|X-450| < 90)$$

$$= P(|X-450| < k\sigma)$$

$$k\sigma = 90$$

$$k = \frac{\cancel{90}\,6}{\cancel{15}} = 6$$

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

$\therefore P(|X-450|<90) \geq 1-\frac{1}{36} = \frac{35}{36}$

$\therefore P(|X-450|<90) \geq \frac{35}{36}$

c) total no of ways $= 6^7$

~~no of ways of distribution of 1, 1, 2, 3, 4, 5, 6 over 7 tables~~

$\cancel{\therefore P(E) = \frac{7!/(2!)}{6^7}}$

E_1 = event that a permutation of 1, 1, 2, 3, 4, 5, 6, 7 occurs

E_2 = event that 1 appears twice

$$P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{\frac{7!}{2!} \times \left(\frac{1}{6}\right)^7}{\frac{1}{5!}\,\frac{7}{2}\ \frac{5^5}{6^7}}$$

$$= \frac{4!}{5^4}$$

$$= \frac{24}{625}$$

d) $X \sim$ Binomial Distribution $(n = n,\ p = \frac{1}{4})$

$P(x) = {}^nC_x\, p^x (1-p)^{n-x}$

$\cancel{P(1) = n\,p\,(1-p)^{n-1}}$

$\cancel{P(1) \geq 0.8}$

$n \cdot \frac{1}{4} \times \left(\frac{3}{4}\right)^{n-1} \geq \cancel{0.8}\ 3.2$

$2/x$ $\quad \frac{9}{16}\,8$ $\quad \cancel{m^2\left(\frac{3}{4}\right)^{m-1} \ge \left(\frac{3}{4}\right)^{m}}$ $\quad \left(\frac{3}{4}\right)^{m}$

$$\cancel{m\left(\frac{3}{4}\right)^{m-1} \ge 3.2}$$

$$\cancel{m\left(\frac{3}{4}\right)^{m} \ge 2.4}$$

probability of atleast 1 success $= 1 - P(0)$

$$= 1 - (1-p)^m \ge 0.8$$

$$0.2 \ge \left(\frac{3}{4}\right)^m$$

$$\boxed{m \ge 6}$$

2) $E[X] = 10 \qquad \sigma_X = 3$

if $X \sim$ Negative Binomial (p, r)

$$\cancel{\frac{r}{p} = 10 \qquad \frac{r}{p}\frac{(1-p)}{p} = 9}$$

$$\cancel{\frac{1-p}{p} = 0.9}$$

$$\cancel{1.9\,p = 1}$$

$$E[X] = \frac{r}{p}(1-p) \qquad \sigma^2 = \frac{r}{p}\left(\frac{1-p}{p}\right)$$

$$\frac{r}{p}(1-p) = 10 \Rightarrow \sigma^2 = \frac{10}{p} \Rightarrow p = \frac{10}{9} > 1$$

NOT POSSIBLE

$(0.5)^n$
$n(0.5)^{n-1}$

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3) $p = 0.75 = \frac{3}{4}$

(i) $(1-p)^4 p = \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)$

(ii) $^7C_4 (p)^4 (1-p)^3 \times p$

$= \frac{7 \times \cancel{6} \times 5}{\cancel{6}} \times \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^3$

$= 35 \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^3$

(iii) $= p\left(1 - \left(^9C_0 (1-p)^0 p^9 + {^9C_1} (1-p)^1 p^8 + {^9C_2} (1-p)^2 p^7 + {^9C_3} (1-p)^3 p^6\right)\right)$

$\cancel{= \frac{3}{4}\left(1 - \left(\left(\frac{3}{4}\right)^9 + 9 \times \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^8 + {^9C_2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^7\right.\right.}$

4) $X \sim$ N Binomial $(p = 0.8, r)$

x = no of launch attempts after which r th success occurs

$P(x) = {^{x-1}C_{r-1}} (p)^r (1-p)^{x-r}$

a) $\cancel{(i)\ P(X=6) = {^5C}}$ $\quad r = 3$

(i) $P(X=6) = {^5C_2} (p)^3 (1-p)^3 = 10 \times (0.8)^3 \left(\frac{1}{5}\right)^3$

(ii) $P(X=3) + P(X=4) + P(X=5)$

$= \sum_{x=3}^{5} {^{x-1}C_2} (0.8)^3 (0.2)^{x-3}$

7/10

b) Y = rv cost of X attempts for 3 successes

$$Y = (X-3)(50,000) + X(5,00,000)$$

$$= X(5,50,000) - 1,50,000$$

$$E[Y] = (5,50,000) * E[X] - 1,50,000$$

$$E[X] = \frac{r}{p} = \frac{3}{0.8}$$

$$\therefore E[Y] = ₹1,912,500$$

5) $X \sim$ geometric distribution

$$P(x) = p(1-p)^{x-1}$$

$$P(X \geq k+j \mid X \geq k) = \frac{P((X \geq k+j) \cap (X \geq j))}{P(X \geq k)}$$

since $j \geq 0$

$$\{(X \geq k+j)\} \cap \{(X \geq j)\} = \{X \geq k+j\}$$

$$\cancel{= \frac{P(X \geq k+j)}{P(X \geq k)}} \; \cancel{= \frac{p(1-p)^{k+j-1}}{p(1-p)^{k-1}}}$$

$$\cancel{= p(1-p)^{j}} \; \cancel{= P(X \geq j}$$

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$$P(X \geq \alpha) = p(1-p)^{\alpha-1} + p(1-p)^{\alpha} + p(1-p)^{\alpha+1} \ldots$$

$$= \frac{\cancel{p}(1-p)^{\alpha-1}}{\cancel{p}} = (1-p)^{\alpha-1}$$

$$\frac{P(X \geq k+j)}{P(X \geq k)} = \frac{(1-p)^{k+j-1}}{(1-p)^{k-1}} = (1-p)^{j-1}$$

$$= P(X \geq j)$$

HP

6)

a) $X \sim$ Poisson Distribution

$$P(X) = e^{-\lambda}\left(\frac{\lambda^{x}}{x!}\right)$$

(i) $P(X=1) = P(X=2)$

$$\cancel{e^{-\lambda}}\frac{\lambda}{1!} = \cancel{e^{-\lambda}}\frac{\lambda^2}{2!}$$

$$\boxed{\lambda = 2}$$

(i) $P(X=4) = e^{-\lambda}\frac{\lambda^4}{4!} = e^{-2}\frac{\cancel{16}\,2}{\cancel{24}\,3} = \frac{2}{(3e^2)}$

(ii) $P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2)$

$$= 1 - e^{-2} - e^{-2}(2) - e^{-2}(2)$$

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b) $X \sim$ Poisson (λ)

$$P(X) = e^{-\lambda} \frac{\lambda^x}{x!}$$

$$E[X] = \lambda$$

$$\lambda = 4$$

Probability that he will sell 0 items $= P(0) = e^{-4}$

$Y \sim$ Binomial dist ($n = 25, p = e^{-4}$)

Y = no of days out of 25 days he will sell none

$$E[Y] = np = 25e^{-4}$$

7) X = no of errors in a single page

$X \sim$ Poisson (λ)

$$E[X] = \lambda = \frac{270}{675} = 0.4$$

(i) $P(X>2) = 1 - P(0) - P(1) - P(2)$

$= 1 - e^{-\lambda} - e^{-\lambda}(\lambda) - e^{-\lambda}\frac{\lambda^2}{2}$

where $\lambda = 0.4$

(ii) $= {}^5C_3 (P(0))^3 (1 - P(0))^2$

$= 10 \times (e^{-\lambda})^3 (1 - e^{-\lambda})^2$ where $\lambda = 0.4$

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(iii) $= {}^5C_2 \,(P(1))^2 (P(0))^3 + {}^5C_1 \,(P(2))(P(0))^4$

$= 10 \times (e^{-\ell}\ell)^2 e^{-3\ell} + {}^5C_1 \, e^{-\ell}\frac{\ell^2}{2} e^{-4\ell}$

$= e^{-5\ell}\left(10\ell^2 + \frac{5\ell^2}{2}\right)$

$= \frac{25}{2}\ell^2 e^{-5\ell}$ where $\ell = 0.4$

8) ~~Probability that~~ $X \sim$ Binomial ~~distribution~~ dist $(n = 10{,}000,\ p = 0.0004)$

$P(X > 5) = 1 - (P(0) + P(1) + P(2) + P(3) + P(4) + P(5))$

$= 1 - \sum_{x=0}^{5} {}^{10000}C_x (p)^x (1-p)^{10{,}000-x}$

where $p = 0.0004$, $n = 10{,}000$

By poisson limit theorem as n is very lage we can model X as Poisson Distribution

$X \sim \text{Poisson}(\ell)$

$\ell = np = 4$

$P(X > 5) = 1 - \sum_{0}^{5} e^{-\ell}\frac{\ell^x}{x!}$

$= 1 - e^{-4}\left(1 + \frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right)$

$= 1 - e^{-4}\left(\frac{643}{15}\right) = 0.215$

$\frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$ | ${}^nC_x\, p^x (1-p)^{n-x} = e^{-np} \frac{(np)^x}{x!}$

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9) $X \sim$ Poisson Distribution (λ)

$$\lambda = 0.5 \times 10 = 5$$

$$P(x) = e^{-5} \frac{5^x}{x!}$$

p = probability that no more than 2 particles are emitted

$= P(0) + P(1) + P(2)$

$= e^{-5}\left(1 + 5 + \frac{25}{2!}\right)$

~~Probability that atleast 2 intervals emit no more than 2 particles =~~

$\cancel{= \sum_{x=2}^{8} {}^5C_x (p)^x (1-p)^{5-x}}$

$$= 1 - \sum_{x=0}^{x=1} {}^7C_x (p)^x (1-p)^{7-x}$$

$$= 1 - (1-p)^7 - 7p(1-p)^6$$

$$= 1 - (1-p)^6 \{1 - p - 7p\}$$

$$= 1 - (1-p)^6 \{1 - 8p\}$$

where $p = \frac{37}{2} e^{-5}$

$9\left(\frac{2}{3}+\frac{e^t}{3}\right)^8 \frac{e^t}{3}$

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10)

a) $E[e^{tX}] = \left(\frac{2}{3} + \frac{e^t}{3}\right)^9 = g(t)$

$P(|X-\mu| < 2\sigma)$

$\mu = g'(t)\Big|_{t=0} = 9\left(\frac{2}{3} + \frac{e^t}{3}\right)^8 \frac{1}{3}\Big|_{t=0}$

$= 3$

$E[X^2] = g''(t)\Big|_{t=0} = \cancel{e^t\Big|_{t=0} = 1}$

$\cancel{= 3 \times 8 \left(\frac{2}{3} + \frac{e^t}{3}\right)^7 \frac{e^t}{3}\Big|_{t=0} = 8}$

$\cancel{\sigma^2 = E[X^2] - \mu^2 = 8 - 9}$

$= \frac{d}{dt}\left(3\left(\frac{2}{3} + \frac{e^t}{3}\right)^8 e^t\right)\Big|_{t=0}$

$\cancel{= 3\left(\frac{2}{3} + \frac{e^t}{3}\right) e^t}$

$= 3\left(\frac{2}{3} + \frac{e^t}{3}\right)^8 e^t + 24\left(\frac{2}{3} + \frac{e^t}{3}\right)^7 \frac{e^t}{3} e^t$

$= 3 + \cancel{\frac{24}{3}}\,8 = 11$

$\sigma^2 = E[X^2] - \mu^2 = 11 - 9 = 2$

1.414

2.8.. 1, 2, 3, 4, 5 5.8

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$P(|X-3| < 2\sqrt{2})$

$= P(1) + P(2) + P(3) + P(4) + P(5)$

$= \cancel{(1 - P(0) - P(6) - P(7) - P(8) - P(9))}$

$$= \sum_{x=1}^{5} {}^{9}C_{x} \left(\frac{1}{3}\right)^{x} \left(\frac{2}{3}\right)^{9-x}$$

b) $E[e^{tx}] = e^{100(e^t-1)}$

$\Rightarrow X \sim$ Poission distribution(λ)

$e^{\lambda(e^t-1)} = e^{100(e^t-1)}$

$\underline{\lambda = 100}$

$\mu = 100 = \lambda$

$\sigma^2 = 100 = \lambda$

$\Rightarrow \sigma = 10$

$P(|X - 100| < 25) = P(|X - \mu| < 2.5\sigma)$

By chebyshev ineq.

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

$\therefore P(|X-100| < 25) \geq 1 - \frac{1}{(2.5)^2} = \frac{21}{25}$

$\therefore P(|X-100| < 25) \geq \frac{21}{25}$

11) $f_x(x) = \begin{cases} 1/3 & -1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$

$$MGF = E[e^{tx}] = \int_{-1}^{2} f_x(x)\, e^{tx}\, dx$$

$$= \frac{1}{3}\int_{-1}^{2} e^{tx}\, dx$$

$$g(t) = \frac{1}{3t}\left(e^{2t} - e^{-t}\right)$$

$$\cancel{E[X] = \frac{d(g(t))}{dt}\Big|_{t=0} =}$$

$$\mu = E[x] = \int_{-1}^{2} x\,\frac{1}{3}\, dx = \frac{1}{\cancel{3}\times 2}\,\cancel{3} = \frac{1}{2}$$

$$E[x^2] = \int_{-1}^{2} x^2\,\frac{1}{3}\, dx = \frac{1}{3\times 3}\times 9 = 1$$

$$\sigma^2 = E[x^2] - (E[x])^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\sigma = \frac{\sqrt{3}}{2}$$

12) $P(X \geq 900 \mid X \geq 300) = P(X \geq 600)$

(Memoryless property)

$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & x > 0 \\ 0 & \end{cases}$

$\mu = \theta = 500$

$$P(X \geq 600) = \int_{600}^{\infty} f(x)\, dx = \int_{600}^{\infty} \frac{1}{\theta} e^{-x/\theta}\, dx = \frac{1}{\theta}(-\theta)\, e^{-\frac{600}{\theta}}$$

$\cancel{e^{300}}$

$$\cancel{P(X \geq 300) = e^{-\frac{300}{500}} = e^{-3/5}}$$

$$P(X \geq 600) = e^{-\frac{600}{500}} = e^{-6/5}$$

$$\Rightarrow P(X \geq 900 \mid X \geq 300) = e^{-6/5}$$

13) A ~ rv denoting time taken by 1st programmer to finish

B ~ rv denoting time taken by 2nd programmer to finish

A ~ Uniform (a = 1 hr, b = 4 hr)

B ~ gamma ($\alpha = 5$, $\beta = 1$ hr)

$$f_A(x) = \frac{1}{b-a}$$

$$F_A(x) = \int_0^x \frac{1}{b-a}\,dx = \frac{x}{b-a} = \frac{x}{3}$$

$$F_A(2) = \frac{2}{3}$$

$$f_B(x) = \frac{1}{\Gamma\alpha\,\beta^\alpha} x^{\alpha-1} e^{-x/\beta} = \frac{1}{24 \cdot 1} x^4 e^{-x}$$

$$\cancel{F_B(x) = \int_0^x f_B(x)\,dx = \frac{1}{\Gamma\alpha\,\beta^\alpha} x^{\alpha-1} e^{-x/\beta}}$$

$$F_B(x) = \int_0^x \frac{1}{24} x^4 e^{-x}\,dx$$

$$F_B(2) = \int_0^2 \frac{1}{\cancel{24}_{8}} \times \frac{2}{3} \times \frac{\cancel{2}}{2}\, x^4 e^{-x}\,dx$$

$$= \frac{2}{3} \int_0^2 \frac{1}{16} x^4 e^{-x}\,dx$$

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$$= \frac{2}{3} \quad \frac{\int_0^2 x^4 e^{-x}\, dx}{16}$$

$$I = \int_0^2 x^4 e^{-x}\, dx < \int_0^2 x^4 dx$$

$$< \frac{32}{5}$$

$$I < \frac{32}{5}$$

$$\Rightarrow I < 16 \Rightarrow \frac{I}{16} < 1$$

$$\therefore \frac{2}{3} \cdot \frac{I}{16} < \frac{2}{3}$$

$$\therefore F_B(2) < F_A(2)$$

$\therefore$ A has a higher chance to finish before 11 AM

14) a) $X \sim$ exponential ~~distributio~~ distribution
X = survival time

~~PDF~~ $f(x) = \frac{1}{\beta} e^{-x/\beta}$

$$\mu = \beta = 10$$

$$P(X \geq 8) = 1 - P(X < 8) = 1 - \int_0^8 f_X(x)\, dx$$

$$= \int_0^8 \frac{1}{\beta} e^{-x/\beta}\, dx$$

$$= 1 - \frac{1}{\beta}(-\beta)\left(e^{-8/\beta} - 1\right) = 1 - (1 - e^{-0.8}) = e^{-0.8}$$

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b) $\mu = \lambda = 10$

$\sigma^2 = \lambda^2 = 100$

$\Rightarrow \sigma = 10$

$$P(X \geq 10 + 2(10)) = \int_{30}^{\infty} f_x(x)\,dx$$

$$= e^{-3}$$

a) (continuation)

$$P(X \leq 12) = \int_0^{12} f_x(x)\,dx$$

$$= \frac{1}{\lambda}\int_0^{12} e^{-x/\lambda}\,dx$$

$$= (-1)\left(e^{-12/10} - 1\right)$$

$$= 1 - e^{-1.2}$$

$$\cancel{P(8 \leq X \leq 12) = P(X \leq 12) - P(X \geq 8)}$$

$$\cancel{= (1 - e^{-1.2}) - e^{-0.8}}$$

$$\cancel{= 1 - e^{-1.2} - e^{-0.8}}$$

$$P(8 \leq X \leq 12) = \int_8^{12} f_x(x)\,dx = e^{-0.8} - e^{-1.2}$$

15)

a) $X \sim \text{exponential}(\beta)$

$$f_x(x) = \frac{1}{\beta} e^{-x/\beta}$$

$$P(X > x+y \mid X > x) = P(X > y)$$

$$P(X > x+y \mid X > x) = \frac{P((X > x+y) \cap (X > x))}{P(X > x)}$$

$$P(X > \alpha) = \int_{\alpha}^{\infty} f_x(x)\,dx = \frac{1}{\beta} \int_{\alpha}^{\infty} e^{-x/\beta}\,dx$$

$$= \frac{1}{\cancel{\beta}}(\cancel{\beta})\, e^{-\alpha}$$

$$= e^{-\alpha}$$

$$P((X > x+y) \cap (X > x)) = P(X > x+y) \quad \because y > 0$$

$$= \frac{e^{-(x+y)}}{e^{-x}} = e^{-y} = P(X > y)$$

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b) $E[X^m] = (m+1)!\, 2^m \quad m = 1, 2, 3 \cdots$

$$\text{mgf} = E[e^{tX}] = E[1] + E[X]\, t + \frac{E[X^2]}{2!} t^2 + \frac{E[X^3]}{3!}$$

$$= \sum_{k=0}^{\infty} E[X^k] \frac{t^k}{k!} = \sum (k+1)!\, 2^k \frac{t^k}{k!} = \sum^{\infty} (k+1)\, 2^k\, t$$

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$$= \sum_{k=0}^{\infty} (k+1)(2t)^{k} = \sum_{k=0}^{\infty} (k+1)(2t)^{k}$$

Let $S = \sum_{k=0}^{\infty} (k+1)(2t)^{k}$

$$S(2t) = \sum_{k=0}^{\infty} (k+1)(2t)^{k+1}$$

$$S - S(2t) = 1 + (2t) + (2t)^2 + (2t)^3 \cdots$$

$$= \frac{1}{1-2t}$$

$$\therefore S = \frac{1}{(1-2t)^2}$$

$$\therefore E[e^{tx}] = (1-2t)^{-2}$$

$X \sim \text{gamma}(\beta = 2, \alpha = 2)$

$$\therefore f_X(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & 0 < x \\ 0 & \text{otherwise} \end{cases}$$

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c) $$f_X(x) = \frac{1}{\beta^2} x e^{-x/\beta}$$

$$\frac{d}{dx} f_X(x) = \frac{1}{\beta^2}\left(e^{-x/\beta} + x e^{-x/\beta}\left(-\frac{1}{\beta}\right)\right) = 0$$

$$1 = \frac{x}{\beta}$$

$$x = \beta$$

$\therefore$ if mode $= 2$
$\beta = 2$

$$X \sim \text{gamma}(\alpha = 2, \beta = 2)$$
$$\sim \chi^2(k=4)$$

16) $$Y \sim \chi^2(8) = \text{gamma}(4, 2)$$

$$\mu_Y = E[Y] = \alpha\beta = 8$$

$$\sigma^2_Y = \alpha\beta^2 = 16$$
$$\sigma_Y = 4$$

$$X \sim \text{uniform}(b, c)$$

$$E[X] = \frac{b+c}{2} = 8 \; ; \; \sigma_X^2 = \frac{(b-c)^2}{12} \cancel{\frac{b^2+c^2+bc}{3}} = 16$$

$$b + c = 16 \quad \cdots (i)$$

$\Rightarrow \cancel{(b+c)^2 - bc = 48}$

$$(b-c)^2 = 16 \times 12$$

$$b - c = 8\sqrt{3}$$

$$b = 8 - 4\sqrt{3}$$
$$c = 8 + 4\sqrt{3}$$

17)

a) $X \sim N(\mu, \sigma^2)$

$$Z = \frac{X-\mu}{\sigma}$$

$$Z \sim N(0,1)$$

$$P(-b < Z < b) = F_Z(b) - F_Z(-b) = 0.90$$

$$F_Z(b) - (1 - F_Z(b)) = 0.90$$

$$2F_Z(b) = 1.9$$

$$F_Z(b) = 0.950$$

$$b = 1.645 \text{ (from table)}$$

b) $X \sim N(\mu, \sigma^2)$ $P(X<89) = 0.90$

$P(X<94) = 0.95$

$$Z = \frac{X-\mu}{\sigma}$$

$$X = Z\sigma + \mu$$

$$P(X<89) = P(Z\sigma + \mu < 89) = P\left(Z < \frac{89-\mu}{\sigma}\right) = 0.9$$

$$\frac{89-\mu}{\sigma} = 1.285$$

$$89 = (1.285)\sigma + \mu \quad \dots (i)$$

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similarly,

$$94 = (1.645)\,\sigma + \mu \quad \text{--- (ii)}$$

sub (i) from (ii)

$$0.36\,\sigma = 5$$

$$\sigma = \frac{500}{36} = 13.88$$

$$\mu = 94 - (1.645)\,\sigma$$
$$\mu = 71.1674$$

c)

$$f(x) = c\,2^{-x^2}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$c\int_{-\infty}^{+\infty} 2^{-x^2}\,dx = c\int_{-\infty}^{+\infty} e^{-[\ln(2)]x^2}\,dx$$

$$= c\,\frac{\sqrt{\pi}}{\sqrt{\ln(2)}}$$

$$\therefore c = \sqrt{\frac{\ln(2)}{\pi}}$$

d) $$\int_{2}^{3} e^{-2(x-3)^2}\,dx = I$$

say for an rv $X \sim N\left(\mu = 3, \sigma = \frac{1}{2}\right)$

$$f_x(x) = \frac{\sqrt{2}}{\sqrt{\pi}}\, e^{-2(x-3)^2}$$

$$\therefore\ I = \sqrt{\frac{\pi}{2}}\left(F_x(3) - F_x(2)\right)$$

$$Z = \frac{X-\mu}{\sigma}$$

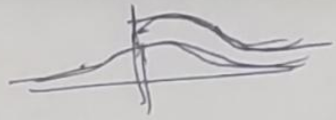
$$X = Z\sigma + \mu$$

$$3 = Z\sigma + \mu \Rightarrow Z = 0$$
$$2 = Z\sigma + \mu \Rightarrow Z = -2$$

$$\therefore\ F_x(3) - F_x(2) = F_x(0) - F_x(-2)$$

$$= 0.5 - 0.0228 \text{ (Table)}$$
$$= 0.4772$$

$$\therefore\ I = 0.598081$$



$\int x f(x) - E[x]$

$\int |x| f(x) =$

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e) $E[e^{tx}] = e^{3t + 8t^2}$

Comparing with MGF of normal distri

$$e^{\mu t + \frac{1}{2}t^2\sigma^2} = e^{3t+8t^2}$$

$$\mu = 3$$

$$\frac{\sigma^2}{2} = 8$$

$$\sigma = 4$$

$$\therefore X \sim N(\mu = 3, \sigma = 4)$$

$$Z = \frac{X-3}{4} = \frac{X-\mu}{\sigma}$$

if $x = -1$

$z = -1$

if $x = 9$

$z = 1.5$

$\therefore \cancel{P(9) - P(-1) = F_X(9)}$

$$P(-1 < X < 9) = F_X(9) - F_X(1)$$
$$= F_Z(1.5) - F_Z(-1)$$
$$= 0.9332 - 0.1587$$
$$= 0.7745$$

f) $X \sim N(\mu, \sigma^2)$ TP: $E(|X-\mu|) = \sigma\sqrt{\frac{2}{\pi}}$

$$z = \left|\frac{x-\mu}{\sigma}\right|$$

$$E\left[\left|\frac{x-\mu}{\sigma}\right|\right] = E[|z|]$$

$$f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$E[|z|] = 2\int_0^\infty z\, f_z(z)\, dz$$

$$= 2\int_0^\infty \frac{z}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$k = \frac{z}{\sqrt{2}}$$

$$dk = \frac{dz}{\sqrt{2}}$$

$$= 2\int_0^\infty \frac{\cancel{\sqrt{2}}\, k}{\sqrt{\cancel{2}\pi}} e^{-k^2} \sqrt{2}\, dk$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty k e^{-k^2}\, dk$$

$$f = k^2$$

$$df = 2k\, dk$$

$$= \frac{\cancel{2}\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-f} \frac{df}{\cancel{2}}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}$$

$$\therefore E\left[\frac{|x-\mu|}{\sigma}\right] = \sqrt{\frac{2}{\pi}}$$

$$\therefore E[|x-\mu|] = \sigma\sqrt{\frac{2}{\pi}}$$

$-1.9 \quad -1.45 \quad 0.07635$

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g) $X \sim N(1,4)$

$P(1 < x^2 < 9) = P(1 < x < 3) + P(-3 < x < -1)$

$\mu = 1 \quad \sigma = 2$

$Z = \frac{x-\mu}{\sigma} = \frac{x-1}{2}$

if $x = 1 \Rightarrow Z = 0$

$x = 3 \Rightarrow Z = 1$

$x = -1 \Rightarrow Z = -1$

$x \Rightarrow -3 \Rightarrow Z = -2$

$P(1 < x < 3) = F_x(3) - F_x(1) = F_Z(1) - F_Z(0)$

$= 0.8413 - 0.5$

$= 0.3413$

$P(-3 < x < -1) = F_x(-1) - F_x(-3) = F_Z(-1) - F_Z(-2)$

$= +0.1587 - 0.0228$

$= 0.1359$

$\therefore P(1 < x^2 < 9) = 0.3413 + 0.1359$

$= 0.4772$

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a) $X \sim N(\mu = 119\ km/hr, \sigma = 13.1 km/hr)$

$P(100 < X < 120) = P(100 < Z\sigma + \mu < 120)$

$= P(-1.45 < Z < 0.076335)$

$= F_Z(0.076335) - F_Z(-1.45)$

$= 0.5279 - 0.0735$

$= 0.4544$

$\underline{0.1}$

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b) $1 - F_Z(z) = 0.1$

$F_Z(z) = 0.9$

$z = 1.285 = \dfrac{x - \mu}{\sigma}$

$\therefore x = 135.78$ kmph

c) $P(x > 100) = 1 - F_X(\overset{100}{\bullet})$

$= 1 - F_Z(-1.45)$

$= 1 - 0.0735$

$= 0.9265$

$\therefore$ 92.65% of vehicles exceeded 100 km/hr

d) P(atleast one is not exceeding speed limit) $= 1 -$ P(all exceeding speed limit)

$= 1 - (0.9265)^5$

$= 0.3173$

e) $\cancel{P(X > 70) = 1 - F_X(70)}$

$\cancel{x = 70 \Rightarrow z = 3.740}$

$\cancel{= 1 - F_Z(3.740)}$

$\cancel{= 1 -}$

70 miles / hr = 112.65 km/hr

$P(X > 112.65) = P(z > -0.484) = 1 - F_Z(-0.484)$

$= 0.684$

19) $X \sim N(\mu = 200,\ \sigma = 35)$

a) $P(X \leq 250) = F_X(250)$

$$Z = \frac{X - \mu}{\sigma}$$

if $x = 250 \Rightarrow Z = 1.42857$

$\therefore P(X \leq 250) = F_Z(1.42857) = 0.92220$

b) $P(300 < X < 400)$

$P(2.85 < Z < 5.71) = F_Z(5.71) - F_Z(2.85)$

$\approx 1 - 0.9978$

$= 0.0022$

c) $P(|X-\mu| \geq 1.5\sigma)$

$= P\left(\left|\frac{X-\mu}{\sigma}\right| \geq 1.5\right)$

$= P(|Z| \geq 1.5)$

$= P(Z \geq 1.5) + P(Z \leq -1.5)$

$= 1 - F_Z(1.5) + F_Z(-1.5)$

$= 1 - F_Z(1.5) + 1 - F_Z(1.5)$

$= 2 - 2F_Z(1.5) = 0.1336$

20) $X \sim N(\mu = 3cm, \sigma = 0.1cm)$

$Y \sim N(\mu = 3.04cm, \sigma = 0.02cm)$

$x = P(2.9cm < X < 3.1cm) = F_x(3.1) - F_x(2.9)$

$z = \frac{x - \mu}{\sigma}$ $= F_z(1) - F_z(-1)$

$= 0.8413 - 0.1587$

$= 0.6826$

$y = P(2.9 < Y < 3.1) = F_y(3.1) - F_y(2.9)$

$= F_z(3) - F_z(-7)$

$= 0.9987$

$\therefore y > x$

$\therefore$ machine Y is more likely to produce acceptable cork.