Review Paper: Coordinated Motion Design on Lie Groups

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***Abstract*—The present paper proposes a unified geometric framework for coordinated motion on Lie groups. It first gives a general problem formulation and analyzes ensuing conditions for coordinated motion. Then, it introduces a precise method to design control laws in fully actuated and underactuated settings with simple integrator dynamics. It thereby shows that coordination can be studied in a systematic way once the Lie group geometry of the configuration space is well characterized. Applying the proposed general methodology to particular examples allows to retrieve control laws that have been proposed in the literature on intuitive grounds. A link with Brockett's double bracket flows is also made. The concepts are illustrated on $SO(3)$, $SE(2)$ and $SE(3)$.**

***Index Terms*—Cooperative systems, distributed control, motion planning, Lie groups, geometric control.**

I. Introduction

RECENT efforts focus on coordinating swarms of autonomous agents on nonlinear manifolds, such as oscillators on circles, spacecraft formations, and mechanical system networks. Consensus algorithms on vector spaces efficiently coordinate agents, but nonlinear manifold scenarios require different approaches.
The paper proposes the following contributions:

- A unified geometric framework for coordinated motion on Lie groups.
- General problem formulation and conditions for coordinated motion defined using the first principles of symmetry and its three variants. left-invariant-coordination (LIC), right-invariant-coordination (RIC), bi-invariant-coordination (BIC).
- Investigates how BIC restricts compatible relative positions independent of system dynamics.
- A precise method to design control laws for fully actuated and underactuated settings with integrator dynamics.
- Control laws based on standard vector space consensus algorithms are given that achieve the easier tasks of RIC and fully actuated LIC for any initial condition on general Lie groups.
- Shown to apply to the practically most relevant problem of left-invariant coordination of underactuated agents.
- It shows coordination can be studied systematically once the Lie group geometry of the configuration space is characterized.
- Showcases an organized method of finding control laws for examples that previously had been designed on an intuitive basis.

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A. Conceptual Understanding

1) Symmetry (or invariance): Refers to the consistent behavior of a system under certain transformations or operations. When a system exhibits symmetry, its properties remain unchanged despite specific changes.
When coordinating a swarm, practical control laws act as internal forces within the group. These forces don't depend on an external reference frame, avoiding arbitrary choices.
Independence from the reference frame ensures invariance ie. properties remain unchanged even after applying the same transformation to all agents.

2) The Consensus Problem: Consensus algorithms in swarm intelligence aim to unify all agents' understanding of the system's state, despite their differing initial states.
They ensure convergence to a shared value, even amidst uncertainties. Agents exchange information with neighbors, iteratively updating their state estimates. Over time, these discrepancies lessen, leading to a common estimate. This is vital when global knowledge is unavailable and decisions rely on local information. Thus, consensus algorithms allow the swarm to operate as a unified, intelligent entity.
The proposed controller architecture in this study consists of two steps, firstly applying the consensus algorithm (for reaching the desired velocity) and adding to it a position controller derived from geometric Lyapunov functions (for relative position control).

B. Basic Terminologies

1) Coordinated motion: Refers to a situation where a swarm of agents move in an organized way with fixed relative positions.

2) Synchronization: Refers to a special case of coordination in a swarm where all agents reach a common state on the given manifold (ie. fixed relative position and orientation)

3) Objective: The primary goal is to achieve coordinated motion within the swarm, ie. it maintains its relative positions on the lie group.
Although some applications require motion particular relative positions which are more efficient, that is not the objective of this study.

4) Application details: The present work focuses on coordinated motion while conserving relative positions on the Lie group. Although the actual relative positions are not actively controlled, combining the results with invariant relative position control algorithms allows achieving desired configurations (specific relative positions) and stable coordinated motion (fixed relative positions).
Motion of the fixed relative positions together on the Lie group can be viewed as "orthogonal" (independent) to driving the agents towards particular relative positions on the Lie group.

5) Assumptions: Simplified first-order dynamics with affine control are assumed for individual agents.
The Lie algebra $\mathfrak{g}$ of G is always endowed with the Euclidean Metric.
The initial desired left-invariant velocity of all agents in the system $\eta_k^l(0) \in \mathcal{C} = \{a + Bu : u \in \mathbb{R}^m\}$
The Adjoint Orbit ,$O_{\mathcal{C}}$, of the whole space spanned by the control input $u \in \mathcal{C}$ is a convex set.

II. THE GEOMETRY OF COORDINATION

List of Variables

Variable key	Represents
N	Number of agents
G	Lie Group
e	Identity element of G
TG_g	Tangent space of $g \in G$
$\mathfrak{g}$	Lie algebra of G, TG_e
$g_k(t) \in G$	Position of agent k at time t
g_k^{-1}	Group inverse of g_k
$L_h : g \mapsto hg$	Left translation on G
$R_h : g \mapsto gh$	Right translation on G
$\lambda_{jk}(t)$	Left invariant position of j wrt g
$\rho_{jk}(t)$	Right invariant position of j wrt g
$[,]$	Lie Bracket on $\mathfrak{g}$
$Ad_g : \mathfrak{g} \mapsto \mathfrak{g}$	Adjoint, input $g \in G$, maps $\eta \in \mathfrak{g}$
$\xi_k^l \in \mathfrak{g}$	Left invariant velocity of agent k
$\xi_k^r \in \mathfrak{g}$	Right invariant velocity of agent k
O_ξ	Adjoint Orbit of $\xi \in \mathfrak{g}$
L_{h*}: $\mathfrak{g} \to TG_{hg}$	Left Trivialization on G by $h \in G$
R_{h*}: $\mathfrak{g} \to TG_{gh}$	Right Trivialization on G by $h \in G$
ad_ξ	Kernel of the Adjoint map
$CM_\xi(\xi)$	Isotropy subgroup of ξ
cm_ξ	Isotropy Lie algebra of ξ, $\mathrm{cm}_\xi \subseteq ad_\xi$
$\exp(\eta t) : \mathfrak{g} \to G$	Exponential map of group G

A. Relative Positions and Coordinations

Definition 1: The left-invariant relative position on G of agent j with respect to agent k is $\lambda_{jk} = g_k^{-1} g_j$. The right-invariant relative position on G of j with respect to k is $\rho_{jk} = g_j g_k^{-1}$.
λ_{jk} is invariant under left multiplication $(hg_k)^{-1}(hg_j) = g_k^{-1} g_j \ \forall h \in G$, similarly for ρ_{jk}.
These definitions lead to Left invariant coordination (LIC) and Right invariant coordination (RIC).

Definition 2: Left-invariant coordination (LIC) means constant left-invariant relative positions $\lambda_{jk}(t) = g_k^{-1} g_j$, resp. right-invariant coordination (RIC) means constant right-invariant relative positions $\rho_{jk} = g_j g_k^{-1}$ $\forall$ pair of agents j, k. Biinvariant coordination (BIC) means simultaneous LIC and RIC: $g_k^{-1} g_j$ and $g_j g_k^{-1}$ are both constant for all j,k
Synchronization, as discussed earlier is a special type of Biinvariant coordination where in all agents are at the same point on G: $g_k(t) = g_j(t) \forall j, k$

B. Velocities and Coordination

Definition 3: Denote by $\mathfrak{g}$ the Lie algebra of G, i.e., its tangent space at identity e. This paper always considers $\mathfrak{g}$ endowed with the Euclidean metric. Denote by [,] the Lie bracket on $\mathfrak{g}$. Let $L_{h*} : TG_g \to TG_{hg}$ and $R_{h*} : TG_g \to TG_{gh}$ be the maps on tangent spaces induced by L_h and R_h respectively. Let $Ad_g = R_{g-1*} L_{g*} : \mathfrak{g} \to \mathfrak{g}$ denote the adjoint representation.

Definition 4: Left-invariant velocity $\xi_k^l \in \mathfrak{g}$ and right-invariant velocity $\xi_k^r \in \mathfrak{g}$ of agent k are defined by
$\xi_k^l(\tau) = L_{g-1}(\tau) * \left(\frac{d}{dt} g_k(t) \Big|_{t=\tau} \right)$ and
$\xi_k^T(\tau) = R_{g-1}(\tau) * \left(\frac{d}{dt} g_k(t) \Big|_{t=\tau} \right)$.
Indeed, $g_k(t)$ and $L_h g_k(t)$ (resp. $R_h g_k(t)$) have the same leftinvariant (resp. right-invariant) velocity $\xi_k^l(t)$ (resp. $\xi_k^\xi(t)$), for any fixed $h \in G$. Note the important equality

$$\xi_k^r = Ad_{g_k} \xi_k^l. \tag{1}$$

Adjoint orbit of $\xi \in \mathfrak{g}$ is set $O_\xi = \{Ad_g \xi : g \in G\} \subseteq \mathfrak{g}$.

Proposition 1: Left-invariant coordination corresponds to equal right-invariant velocities $\xi_j^r = \xi_k^r \forall j, k$. Right-invariant coordination corresponds to equal left-invariant velocities $\xi_j^l = \xi_k^l \forall j, k$
Proof:
For λ_{jk} constant implies $\frac{d}{dt}\lambda_{jk} = 0$
$\frac{d}{dt}\left(g_k^{-1} g_j\right) = L_{g_k^{-1}*} \frac{d}{dt} g_j + R_{g_j *} \frac{d}{dt} g_k^{-1}$.
But if $\frac{d}{dt} g_k = L_{g_k *} \xi_k^l$, then $\frac{d}{dt} g_k^{-1} = -L_{g_k{}^{-1}*} Ad_{g_k} \xi_k^l$.
Thus $\frac{d}{dt}\left(g_k^{-1} g_j\right) = L_{g_k^{-1} g_j *} \xi_j^l - L_{g_k^{-1}*} R_{g_j *} Ad_{g_k} \xi_k^l$
$= L_{g_k^{-1} g_j *} Ad_{g_j}^{-1} \left(Ad_{g_j} \xi_j^l - Ad_{g_k} \xi_k^l \right)$.
Since $L_{g_k^{-1} g_j *}$ and $Ad_{g_j^{-1}}$ are invertible,
$\frac{d}{dt}(\lambda_{jk}) = 0 \Leftrightarrow Ad_{g_j} \xi_j^l = Ad_{g_k} \xi_k^l \Leftrightarrow \xi_j^r = \xi_k^r$.
The proof for right-invariant coordination is strictly analogous.

Proposition 2: Biinvariant coordination on a Lie group G is equivalent to the following condition in the Lie algebra $\mathfrak{g}$:

$$\forall k = 1 \ldots N, \xi_k^l = \xi^l \in \bigcap_{i,j} \ker \left(Ad_{\lambda_{ij}} - Id \right)$$

or equivalently

$$\xi_k^r = \xi^r \in \bigcap_{i,j} \ker \left(Ad_{\rho_{ij}} - Id \right).$$

Proof:
RIC requires $\xi_k^l = \xi_j^l \forall j, k$; denote the common value of the ξ_k^l's by ξ^l.

Then LIc requires $Ad_{g_k}\xi^l = Ad_{g_j}\xi^l \Leftrightarrow \xi^l = Ad_{\lambda_{jk}}\xi^l \forall j,k$.
The proof is similar with ξ^r

Proposition 3: Let $CM_\xi := \{g \in G : Ad_g\xi = \xi\}$.
a. For every $\xi \in \mathfrak{g}, CM_\xi$ is a subgroup of G.
b. The Lie algebra of CM_ξ is the kernel of $ad_\xi = [\xi, \,]$, i.e., $\mathrm{cm}_\xi = \{\eta \in \mathfrak{g} : [\xi, \eta] = 0\}$.
Proof:
$Ad_e\xi = \xi \forall \xi$ *since* Ad_e *is the identity operator.*
$Ad_g\xi = \xi$ *implies* $Ad_{g-1}\xi = \xi$ *by inversion of the relation.*
Moreover, if $Ad_{g_1}\xi = \xi$ *and* $Ad_{g_2}\xi = \xi$,
then $Ad_{g_1g_2}\xi = Ad_{g_1}Ad_{g_2}\xi = Ad_{g_1}\xi = \xi$.
Thus CM_ξ *satisfies all group axioms and hence* $CM_\xi \leqslant G$.
This proves part a
Let $g(t) \in CM_\xi$ *with* $g(\tau) = e$ *and* $\frac{d}{dt}g(t)\big|_\tau = \eta$.
Then $\eta \in \mathbf{cm}_\xi =$ *the tangent space to* CM_ξ *at e.*
For constant ξ, $Ad_g(t)\xi = \xi$ *implies* $\frac{d}{dt}(Ad_g(t))\,\xi = 0$, *with the basic Lie group property* $\frac{d}{dt}(Ad_g(t))\big|_\tau = ad_\eta$. *Therefore* $[\eta, \xi] = 0$ *is necessary.*
It is also sufficient since, for any η *such that* $[\eta, \xi] = 0$,
the group exponential curve $g(t) = \exp(\eta t)$ *belongs to* CM_ξ.
From the Baker-Campbell-Hausdorff formula
Hence also proving part b.
CM_ξ and cm_ξ are called the isotropy subgroup (Stabilizer subgroup) and isotropy Lie algebra of ξ.

C. Key Takeaways and the link with consensus

Proposition 1 shows that coordination on the Lie group G is equivalent to consensus in the vector space $\mathfrak{g}$. Biinvariant coordination requires simultaneous consensus on ξ_k^l and ξ_k^T; but the latter are not independent, they are linked through (1) which depends on the agents' positions.
Proposition 2 shows that biinvariant coordination puts no constraints on the relative positions when the group is Abelian, since $Ad_{\lambda_{jk}} = Id \forall \lambda_{jk}$
In contrast, on a general Lie group, biinvariant coordination with non-zero velocity can restrict the set of possible relative positions as follows.

From Propositions 2 and 3, one method to obtain a biinvariantly coordinated motion on G is to
(1) choose ξ^l in the vector space $\mathfrak{g}$ and set $\xi_k^l = \xi^l \forall k$
(2) position the agents on G such that $\lambda_{jk} \in CM_{\xi l}$ for pairs j, k corresponding to the edges of an undirected tree graph; the Lie group property of CM_{ξ^l} then ensures that $\lambda_{jk} \in CM_{\xi^l}$ for all pairs j, k. The same can be done with ξ^r and the ρ_{jk}. Note that a swarm at rest $\left(\xi_k^l = \xi_k = 0 \forall k\right)$ is always biinvariantly coordinated.

Remark 1: In many applications involving coordinated motion, reaching a particular configuration, i.e., specific values of the relative positions, is also relevant. Specific configurations are defined as extrema of a cost function in [38]. Imposing relative positions in the (intersection of) set(s) CM_ξ for some ξ can be another way to classify specific configurations; unlike [38], it works for non-compact Lie groups. For compact groups, there seems to be no connection between configurations characterized through CM_ξ and those defined by [38].

Remark 2: One can also first fix relative positions λ_{jk} and then characterize the set of velocities ξ compatible with biinvariant coordination. For non-Abelian groups and a sufficiently large number N of agents, this set generically reduces to $\xi = 0$.

III. COORDINATION BY CONSENSUS IN THE LIE ALGEBRA

A. Control Setting

Left-invariant systems on Lie groups appear naturally in many physical systems, such as rigid bodies in space and cart-like vehicles. Motivated by examples like 2-axes attitude control and steering control on $SE(2)$ or $SE(3)$, this paper considers left-invariant dynamics with affine control.

$$\frac{d}{dt}g_k = L_{g_k*}\xi_k^l \quad \text{with} \quad \xi_k^l = a + Bu_k; \;\; k = 1...N \tag{2}$$

The Lie algebra $\mathfrak{g}$ is identified with $\mathbb{R}^n$
$a \in \mathbb{R}_n$ is a constant drift velocity
$B \in \mathbb{R}^{n\times m}$ has full column rank and specifies full range of control; without loss of generality column vectors of B are assumed to be orthonormal.
control $u_k \in \mathbb{R}^m$
Let the set of all assignable ξ_k^l be $\mathcal{C} = \{a+Bu : u \in \mathbb{R}^m\}$. For fully actuated agent $n = m$, (2) simplifies to $\frac{d}{dt}g_k = L_{g_k*}u_k$ without loss of generality. We consider Lie algebra $\mathfrak{g}$ to be endowed with the Euclidean Metric.
Feedback control laws must be functions of variables which are compatible with the settings of the problem, ie. left-invariant.
In terms of left invariant variables, LIC corresponds to fixed (LI) positions, while RIC corresponds to equal (LI) velocities.

A note in communication links:
In practical situations communication between all agents cannot be assumed. The information flow among agents is modeled by a restricted set of communication links;
$j \rightsquigarrow k$: denotes j sends information to k
The communication topology is asscoaited to graph $\mathbb{G}$.
$\mathbb{G}$ is undirected if $k \rightsquigarrow j \Leftrightarrow j \rightsquigarrow k$
$\mathbb{G}$ is uniformly connected if $\exists\ \delta > 0$ and $T > 0$ such that $\forall t$, taking union of links appearing for at least δ in time span $[t, t+T]$, there is a directed path $k \rightsquigarrow a \rightsquigarrow b \ldots \rightsquigarrow j$ from k to every other agent j.

B. Right-Invariant Coordination

$$\text{RIC Requires: } \xi_k^l = \xi_j^l, \;\forall j, k$$

From (2) this implies equal $u_k \forall k$; there's no condition on relative positions which can vary arbitrarily.
This problem is solved using the vector space consensus algorithm.

$$\text{From Consensus: } \frac{d}{dt}\xi_k^l = \sum_{j\rightsquigarrow k}(\xi_j^l - \xi_k^l), \;\; k = 1\ldots N. \tag{3}$$

From (2), it translates to

$$\tfrac{d}{dt}u_k = \Sigma_{j\rightsquigarrow k}(u_j - u_k)$$

If $\mathbb{G}$ is uniformly connected, the consensus algorithm converges exponentially to acheive $\xi_k^l = \xi_j^l\ \forall j, k$.

Summarizing: Asymptotic RIC is ensured for any initial u_k, any realtive positions λ_{jk} (these have no influence). Agent k relies on the left invariant velocity ξ_j^l of $j \rightsquigarrow k$.

For a time-invariant and undirected communication graph $\mathbb{G}$, (3) is a gradient descent for the disagreement cost function $V_r = \Sigma_k \Sigma_{j \rightsquigarrow k} \parallel \xi_k^l - \xi_j^l \parallel^2$, with a Euclidean metric in $\mathfrak{g}$.

C. Left-Invariant Coordination

$$\text{LIC Requires: } \xi_k^r = \xi_j^r, \ \forall j, k$$

$$\text{From Consensus: } \frac{d}{dt}\xi_k^r = \sum_{j \rightsquigarrow k} (\xi_j^r - \xi_k^r), \ k = 1 \dots N. \quad (4)$$

Using (1) in terms of left invariant variables.

$$\tfrac{d}{dt}(Ad_{g_k}\xi_k^l) = \textstyle\sum_{j \rightsquigarrow k}(Ad_{g_j}\xi_j^l - Ad_{g_k}\xi_k^l)$$

$$\frac{d}{dt}(Ad_{g_k})\xi_k^l + (Ad_{g_k})\frac{d}{dt}\xi_k^l = \sum_{j \rightsquigarrow k}(Ad_{g_j}\xi_j^l - Ad_{g_k}\xi_k^l)$$

$$\text{But, } \frac{d}{dt}(Ad_{g_k})\xi_k^l = Ad_{g_k}[\xi_k^l, \xi_k^l] = 0 \quad \text{(AP1)}$$

$$\frac{d}{dt}\xi_k^l = \sum_{j \rightsquigarrow k}(Ad_{g_k^{-1} g_j}\xi_j^l - \xi_k^l), \ k = 1 \dots N. \quad (5)$$

If $\mathbb{G}$ is uniformly connected, (5) converges ensuring global exponential coordination.
But in equilibrium on an underactuated setting,

$$Ad_{\lambda_{jk}}(a + Bu_j) = a + Bu_k \ \forall j, k \quad (6)$$

Here, arbitrary relative positions of agents might lead to no possible solution $(u_1, u_2 \dots, u_N)$, This problem lead to further study on underactuated setting in LIC.

For BIC at equilibrium the conditions (3) and (5) translate to

$$Ad_{\lambda_{jk}}(a + Bu_k) = a + Bu_k \ \forall j, k \quad (7)$$

This further constrains the relative positions
The disagreement cost funciton associated to (4) ,
$V_l = \Sigma_k \Sigma_{j \rightsquigarrow k} \parallel Ad_{g_k}\xi_k^l - Ad_{g_j}\xi_j^l \parallel^2$
This cost function is neither left nor, right invariant due to presence of position g_k, hence (5) cannot be left-invariant gradient of V_l. This is crucial because BIC aims to achieve agreement on motion regardless of the starting positions.

Nevertheless, let $\mathcal{G}_u$ be the subclass of compact groups with unitary adjoint representation, i.e.,
$\parallel Ad_g \xi \parallel = \parallel \xi \parallel \ \ \forall g \in G, \forall \xi \in \mathfrak{g}$.
Then we can define a bivariant Reimannian metric on G if and only if $G \in \mathcal{G}_u$. Hence now can define the metric,
$V_l = \Sigma_k \Sigma_{j \rightsquigarrow k} \parallel \xi_k^l - Ad_{g_k^{-1} g_j}\xi_j^l \parallel^2$
And for fixed undirected graph $\mathbb{G}$, (5) is the gradient descent for V_l.

IV. Control Design: Fully Actuated Biinvariant Coordination

A. Biinvariant Coordination on General Lie Groups

1) Objective: To obtain an autonomous, left-invariant algorithm for biinvariant coordination by achieving RIC and LIC simultaneously.

STEP I: Assume a given right-invariant velocity ξ^r, st. LIC is ensured for all agents if all agents apply $\xi_k^l = Ad_{g_k}^{-1}\xi_r$

STEP II: To achieve RIC, write a general controller

$$\xi_k^l = \eta_k^l + q_k, \quad k = 1 \dots N \quad (8)$$

q_k: control neccessary to achieve relative position.
η_k^l: desired velocity, thus $\eta_k^l = Ad_{g_k}^{-1}\xi_r$.
Only designing q_k remains to achieve BIC. For fixed undirected communication graph $\mathbb{G}$ define,

$$V_{tr}(g_1, g_2 \dots g_N = \tfrac{1}{2}\textstyle\sum_k \sum_{j \rightsquigarrow k} \parallel \eta_k^l - \eta_j^l \parallel^2$$

V_{tr} characterizes the distance from RIC assuming that every agent has velocity $\xi_k^l = Ad_{g_k}^{-1}\xi^r$.

$$\text{Since, } (\tfrac{d}{dt}Ad_{g_k}^{-1})\eta = -[\xi_k^l, Ad_{g_k}^{-1}\eta] \ \forall \eta \in \mathfrak{g} \quad \text{(AP2)}$$

Hence variation of v_{tr} due to motion of g_k is

$$\frac{d}{dt}V_{tr} = 2\sum_k \sum_{j \rightsquigarrow k}(\eta_k^l - \eta_j^l) \cdot [\eta_k^l, \xi_k^l] \quad (9)$$

$\{\cdot\}$: canonical scalar product in $\mathfrak{g}$, with Euclidean Metric.

STEP III: Now define $\langle , \rangle$ such that
$\xi_1 \cdot \langle \xi_2, \xi_3 \rangle + [\xi_1, \xi_2] \cdot \xi_3 = 0 \forall \xi_1, \xi_2, \xi_3 \in \mathfrak{g}$.
Then (9) rewrites $\frac{d}{dt}V_{tr} = 2\sum_k \sum_{jwk} \langle \eta_k^l, \eta_k^l - \eta_j^l \rangle \cdot q_k$

$$\text{We choose } \ q_k = -\left\langle \eta_k^l, \sum_{j \rightsquigarrow k}\left(\eta_k^l - \eta_j^l\right) \right\rangle \quad (10)$$

ensuring that V_{tr} is non-increasing along the solutions:

$$\frac{d}{dt}V_{tr} = -2\sum_k \sum_{j \rightsquigarrow k}\left\langle \eta_k^l, \sum_{j \rightsquigarrow k}\left(\eta_k^l - \eta_j^l\right) \right\rangle^2 \leq 0.$$

STEP IV: It remains to replace the reference velocity ξ^r by estimates on which the agents progressively agree.
Since the goal is to define a common right-invariant velocity in $\mathfrak{g}$, it is natural to proceed as in Section III-C and use the consensus algorithm

$$\frac{d}{dt}\eta_k^r = \sum_{j \rightsquigarrow k}\left(\eta_j^r - \eta_k^r\right) \quad (11)$$

writing in terms of left-invariant velocities using AP1 & AP2

$$\frac{d}{dt}\eta_k^l = \sum_{j \rightsquigarrow k}\left(Ad_{\lambda_{jk}}\eta_j^l - \eta_k^l\right) - \left[\xi_k^l, \eta_k^l\right], k = 1 \dots N. \quad (12)$$

Summarizing: The overall controller is the cascade of a consensus algorithm to agree on a desired velocity for LIC, and a position controller designed to decrease a natural distance to RIC. To implement the controller, agent k must receive from communicating agents $j \rightsquigarrow k$ their relative positions λ_{jk} and the values of their left-invariant auxiliary variables η_j^l

Theorem 1: Consider N fully actuated agents communicating on a fixed, undirected graph $\mathbb{G}$ and evolving on Lie group G according to $\frac{d}{dt}g_k = L_{g_k*}\xi_k^l$ with controller (8), (10), (12).
(i) For any initial conditions $\eta_k^l(0)$, the $\eta_k^r(t) = Ad_{g_k}\eta_k^l(t)$ exponentially converge to $\overline{\eta^r} := 1/N\sum_k \eta_k^r(0)$.
(ii) Define

$$\overline{V_{tr}}\,(g_1, g_2, \dots g_N) := \frac{1}{2}\sum_k \sum_{j\rightsquigarrow k} \left\| Ad_{g_k}^{-1}\overline{\eta^r} - Ad_{g_j}^{-1}\overline{\eta^r} \right\|^2$$

All solutions converge to the critical set of $\overline{V_{tr}}$. In particular, left-invariant coordination is asymptotically achieved for all initial conditions.
(iii) Biinvariant coordination is (at least locally) asymptotically stable Proof: This theorem is regarding convergence of (8), (10), (12).. Therefore, (i) simply restates a well-known convergence result for consensus algorithms in vector spaces on fixed undirected graphs

B. Biinvariant Coordination on Lie Groups With a Biinvariant Metric

As seen in Section-III C, we define $\mathcal{G}_u$ to be a subclass of compact groups with unitary adjoint representation, with $V_l = \Sigma_k \Sigma_{j\rightsquigarrow k} \parallel \xi_k^l - Ad_{g_k^{-1}g_j}\xi_j^l \parallel^2$ that can be used for left invariant control design.

Case 1: Consider defining $V_t = V_r + V_l$, deriving a gradient descent for the disagreement cost V_t of the form $\frac{d}{dt}\xi_k^l = f(\xi_k^l, \{\xi_j^l, g_k^{-1}g_j : j \rightsquigarrow k\})$
However the simulations of the control law for $SO(n)$ always converges to $\xi_k^l = 0\ \forall k$. A possible explanation for this behavior is that the gradient controls velocities, not explicitly positions, while it was shown in Section II that BIC at non-zero velocity involves restrictions on compatible positions.

Case 2: Since we have a biinvariant metric, it allows to switch the roles of LIC and RIC in Section IV-A .
Implying using a consensus algorithm to define a common left-invariant velocity for RIC, and a cost function to drive positions to LIC.

Following same steps as before:
Define common left invariant velocity ξ^l
Define RIC consensus algorithm on auxiliary variables

$$\frac{d}{dt}\eta_k^l = \sum_{j\rightsquigarrow k}(\eta_j^l - \eta_k^l),\ \ k = 1\dots N. \tag{13}$$

Define the disagreement cost function
$V_t l(g_1, g_2 \dots, g_N) = \frac{1}{2}\sum_k \sum_{j\rightsquigarrow k} \parallel Ad_{g_k}\eta_k^l - Ad_{g_j}\eta_j^l \parallel^2$
$= \frac{1}{2}\sum_k \sum_{j\rightsquigarrow k} \parallel \eta_k^l - Ad_{g_k^{-1}g_j}\eta_j^l \parallel^2$
For LIC and proceeding as in previous subsection, we get

$$\text{Control}:\ \ q_k = -\left\langle \eta_k^l, \sum_{j\rightsquigarrow k}\left(\eta_k^l - Ad_{g_k^{-1}g_j}\eta_j^l\right)\right\rangle \tag{14}$$

And the analogous theorem;

Theorem 2: Consider N fully actuated agents communicating on a connected, fixed, undirected graph $\mathbb{G}$ and evolving on Lie group $G \in \mathcal{G}_u$ according to $\frac{d}{dt}g_k = L_{g_k*}\xi_k^l$ with controller (8), (13), (14).
(i) For any initial conditions $\eta_k^l(0)$, the $\eta_k^l(t)$ exponentially converge to $\overline{\eta^l} := 1/N\sum_k \eta_k^l(0)$.
(ii) Define

$$\overline{V_{tl}}\,(g_1, g_2, \dots g_N) := \frac{1}{2}\sum_k \sum_{j\rightsquigarrow k} \left\| Ad_{g_k}^{-1}\overline{\eta^l} - Ad_{g_j}^{-1}\overline{\eta^l} \right\|^2$$

All solutions converge to the critical set of $\overline{V_{tl}}$. In particular, left-invariant coordination is asymptotically achieved for all initial conditions.
(iii) Biinvariant coordination is (at least locally) asymptotically stable
Proof is similar to Theorem 1.

1) Further Implications: Advantage of theorem 2 over theorem 1 is that it can directly be applied to underactuated systems.
With a valid consensus velocity $\xi^l \in \mathcal{C} = \{a + Bu : u \in \mathbb{R}^m$. Provided that $\eta_k^l(0) \in \mathcal{C}\forall k$. The only change in underactuated systems is that the control q_k is now used after taking it's projection on the control range B.

$$\xi_k^l = a + Bu_k = \eta_k^l + BB^T q_k$$

When ξ^l is is asymptotically defined with (13), the convergence argument for asymptotically autonomous systems must be extended to projections of gradient systems; a general proof of this technical issue is lacking in the present paper. It is the only reason to restrict Theorem 2 to fully actuated agents.

V. CONTROL DESIGN: UNDERACTUATED LEFT-INVARIANT COORDINATION

Here the role of the cost function is no longer to add a second level of coordination, but to fulfill the underactuation constraints.The present section explicitly considers the most general setting of possibly directed and time-varying interconnection graph $\mathbb{G}$.

A. Left-Invariant Coordination of Underactuated Agents

Analogous to the previous section, the control design is split into finding a feasible right-invariant velocity by a consensus algorithm. Then The corresponding left-invariant velocity is enforced by a Lyapunov-based feedback that decreases its distance from $\mathcal{C} = \{a + Bu : u \in \mathbb{R}^m\}$
From the consensus algorithm we get a feasible right-invariant velocity, that is a vector ξ^r in the set

$$O_c := \{Ad_g\xi : \xi \in \mathcal{C} \text{ and } g \in G\}$$

If O_c is convex, then it is sufficient to initialize the consensus algorithm (12) with $\eta_k^l(0) \in \mathcal{C}$.
When O_c is not convex, the consensus algorithm must be adapted and the present paper has no general method.

Define $d(\eta, \mathcal{C})$: Euclidean distance in $\mathfrak{g}$ from η to the set $\mathcal{C}$.
Let $\Pi_{\mathcal{C}}(\eta)$ be the projection of η on $\mathcal{C}$;
since $\mathcal{C}$ is convex, $\forall \eta\ \Pi_{\mathcal{C}}(\eta)$ is the unique point in $\mathcal{C}$ such that $d(\eta, \mathcal{C}) = d(\eta, \Pi_{\mathcal{C}}(\eta)) = \| \eta - \Pi_{\mathcal{C}}(\eta) \|$
Following the same steps as in Section IV-A,
define $\eta_k^l := Ad_{g_k}^{-1} \xi^r$. Writing

$$\xi_k^l = a + Bu_k = \Pi_{\mathcal{C}}\left(\eta_k^l\right) + Bq_k, \quad k = 1 \ldots N, \tag{15}$$

Next: to design $q_k \in \mathbb{R}^m$ such that asymptotically, g_k is driven to a point where $\eta_k^l \in \mathcal{C}$ and q_k converges to 0 ; this would asymptotically ensure LIC. For each individual agent k, write the cost function

$$V_k\left(g_k\right) = \frac{1}{2}\left\|Ad_{g_k}^{-1} \xi^r - \Pi_{\mathcal{C}}\left(Ad_{g_k}^{-1} \xi^r\right)\right\|^2 = \frac{1}{2}\left\|\eta_k - \Pi_{\mathcal{C}}\left(\eta_k^l\right)\right\|^2$$

V_k characterizes the distance from η_k to $\mathcal{C}$, that is the distance from LIC assuming that every agent implements $\xi_k^l = \Pi_{\mathcal{C}}\left(Ad_{g_k}^{-1} \xi^r\right)$. The time variation of V_k due to motion of g_k is

$$\frac{d}{dt} V_k = \left(\eta_k^l - \Pi_{\mathcal{C}}\left(\eta_k^l\right)\right) \cdot \left[\eta_k^l, \Pi_{\mathcal{C}}\left(\eta_k^l\right) + Bq_k\right] \tag{16}$$

It must hold $(\eta - \Pi_{\mathcal{C}}(\eta)) \cdot [\eta, \Pi_{\mathcal{C}}(\eta)] \leq 0 \forall \eta \in O_{\mathcal{C}}$;
Then (16) implies $\frac{d}{dt} V_k \leq f\left(\eta_k^l\right) \cdot q_k$, where

$$f\left(\eta_k^l\right) = B^T\left\langle\eta_k^l, \left(\eta_k^l - \Pi_{\mathcal{C}}\left(\eta_k^l\right)\right)\right\rangle \tag{17}$$

when identifying $\mathfrak{g}$ with $\mathbb{R}^n$, and a natural control is

$$q_k = -f\left(\eta_k^l\right), \quad k = 1 \ldots N. \tag{18}$$

Note that when $O_{\xi^r} \subseteq \mathcal{C}$, the position control q_k is unnecessary and vanishes, yielding simply $\xi_k^l = Ad_{g_k}^{-1} \xi^r \forall t$. Hence we get a similar control structure as previously found.

Since agents only interact through the consensus algorithm, not through the cost function, a connected fixed undirected graph is not required: $\mathbb{G}$ can be directed and time-varying, as long as it remains uniformly connected.

A general characterization of the behavior of solutions of the closed-loop system is more difficult here because the position controller is not a gradient anymore. A crucial step for which the present paper proposes no explicit general solution is the design of an appropriate consensus algorithm on auxiliary variables. The other assumptions in the following result can be readily checked for any particular case.

Theorem 3: Consider N underactuated agents communicating on a uniformly connected graph $\mathbb{G}$ and evolving on Lie group G according to $\frac{d}{dt} g_k = L_{g_k} * \xi_k^l$ with controller (15), (18) where f is defined in (17), assuming that $\forall \eta \in O_{\mathcal{C}}$, it holds $(\eta - \Pi_{\mathcal{C}}(\eta)) \cdot [\eta, \Pi_{\mathcal{C}}(\eta)] \leq 0$. Assume that an appropriate consensus algorithm drives the arbitrarily initiated $\eta_k, k = 1 \ldots N$, such that they exponentially agree on $Ad_{g_k} \eta_k \rightarrow \xi^r \in O_{\mathcal{C}} \forall k$, independently of the agent motions $g_k(t)$.
(i) If the agents are controllable, then LIC is locally asymptotically stable.
(ii) If, for any fixed $\eta_k = \xi^r$, bounded V_k implies bounded η_k', and $f\left(\eta_k\right) \rightarrow 0$ implies $g_k \rightarrow \left\{g : f\left(Ad_g^{-1} \eta_k^r\right) = 0\right\}$, then all agent trajectories on G converge to the set where $f\left(Ad_{g_k}^{-1} \xi^r\right) = 0$.

VI. Conclusion

This paper proposes a geometric framework for coordination on general Lie groups and methods for the design of controllers driving a swarm of underactuated, simple integrator agents towards coordination.

Following the numerous results about coordination on particular Lie groups, various directions are still open to extend the general framework of the present paper. A first case often encountered in practice is to stabilize specific relative positions of the agents ("formation control").

In the present paper, relative positions of the agents are asymptotically fixed but arbitrary. The requirement of synchronization (most prominently, "attitude synchronization" on $SO(3)$) also fits in this category. A second important extension would be to consider more complex dynamics, like those of mechanical systems.

VII. References

[1] Coordinated Motion Design on Lie Groups