GROUP THEORY

Maths

15/April/20

Commutative Property : $a*b = b*a$

Associative Property : $a*(b*c) = (a*b)*c$.

Closure Property : $a*b \in S$
$a \in S$ & $b \in S$.

Extensive of Identity : $\boxed{a * e = a}$
→ Identity Element.
Addition Identity = 0 & Multiplicative = 1

Inverse Element : $1 \to -1$, $2 \to 2$, ∴ $\boxed{a * a^{-1} = e}$

Addition Inverse : $1 \to -1$
Multiplication Inverse : $2 \to 1/2$

Algebraic Structure :

1] SEMI-GROUP :

(G,*) is a semi group
if 1) Closure Property : $a, b \in G \Rightarrow a*b \in G$.
2) Associativity : $a*(b*c) = (a*b)*c$

2] MonoID : (G,*) → semigroup
① Closure property
② Associativity
③ Ex~~ten~~istence of Identity Element.

3) Group
(a) Closure
(b) Associativity
(c) Existence of Identity
(d) Existence of ~~Ex~~ Inverse.

4) Abelian Group
(a) Closure
(b) Associativity
(c) Existence of Identity
(d) Existence of Inverse.
(e) Commutative Property

NOTE: Matrix Multiplication is not commutative property

Q) Show that $G = \{-1, 1, i, -i\}$ is abelian group for matrix multiplication.

Finite Group :
Infinite Group :

	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	+1
-i	-i	i	+1	-1.

(a) Satisfies closure Property
(b) Associativity Property.
(c) 1 is identity
(d) Inverse element exists
(e) Commutative Property

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Q) $Z_5 = \{0, 1, 2, 3, 4\}$ is abelian group

$$a +_5 b = \begin{cases} a+b & ; a+b < 5 \\ a+b-5 & ; a+b \geq 5 \end{cases}$$

$$3 +_5 4 = 7 - 5 = 2$$

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	1
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(a) Closure Property is satisfied.
(b) Associative Property is satisfied.
(c) 0 is the identity element
(d) It is inverse
(e) $a+b = b+a$ $\therefore$ It is Commutative Property.

Q) Show that $a * b = \frac{ab}{2}$, $\forall a, b \in Q^+$ is an abelian group?

Ans) ① Closure property : let $a, b \in Q^+$

$a * b = \frac{ab}{2} \in Q^+$

② Associativity :

$a, b, \in Q^+$

$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{abc}{4}$

$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{abc}{4}$

③ Existence of Identity :

$$\boxed{\begin{aligned} a * e &= a \\ \frac{\cancel{a}e}{2} &= \cancel{a} \\ \boxed{e = 2} \end{aligned}}$$

$e = 2 \in Q^+$

s.t. $a * 2 = \frac{a\cancel{2}}{\cancel{2}} = a$

④ Existence of Inverse :

$$\boxed{\begin{aligned} a * b &= e \\ \frac{ab}{2} &= 2 \\ b &= 4/a \end{aligned}}$$

$\forall a \in Q^+$.

$\exists \frac{4}{a} \in Q^+$

s.t. $a \times \frac{4}{4} = \frac{a\,4/2}{2} = a$

⑤ Commutativity

$a * b = \frac{ab}{2}$ & $b * a = \frac{ba}{2}$

Q) S is a set of real no. except -1. Then show that $(S, \circ)$ is a group where $\circ$ is operation defined as

$a \circ b = a + b + ab$, $a, b \in S$.

$1 \circ 2 = 1 + 2 + 1.2 = 5$

$5 \circ 10 = 5 + 10 + 5.10 = 65$

Ans ① **Closure:** $a, b \in S \{a, b \neq -1\}$

$a \circ b \neq -1$

Let $a \circ b = -1 \Rightarrow a + b + ab = -1$

$(1 + a) + b(1 + a) = 0$

$(1 + a) + (b + 1) = 0$

$a = -1$ & $b = -1$

contradiction

$\therefore a \circ b \neq -1$

② $a \circ b = a + b + ab$

$a \circ (b \circ c) = a \circ (b + c + bc)$

$= a + b + c + bc + ab + ac + abc$

$(a \circ b) \circ c = (a + b + ab) \circ c$

$= a + b + c + ab + c(a + b + ab)$

$= a + b + c + ab + bc + ac + abc$

$\therefore$ it is satisfied.

③ $(a \circ e) = a$

$\cancel{a} + e + ae = \cancel{a}$

$e(1 + a) = 0$

$\boxed{e = 0}$

existence of identity $e = 0 \in Q^+$ is identity element

$a \circ 0 = a + 0 + a.0 = a$

④ **Existence of Inverse:**

$a \circ b = e = 0$

$a + b + ab = 0$

$b(1 + a) = a$

$b = \frac{-a}{1 + a}$

$\forall a \in S$

$\exists \frac{-a}{1+a} \in S$

is inverse element.

$a \circ \left(\frac{-a}{1+a}\right) = a + \left(\frac{-a}{1+a}\right) + a\left(\frac{-a}{1+a}\right)$

$= \frac{a - a^2 - a - a^2}{1 + a} = 0$

17/04/20 **Group Theory**

$\boxed{f \circ g \neq g \circ f}$

Ex: 5

Q) $\{f_1, f_2, \ldots f_6\}$ $(g \circ f)$

$f_1 = z$; $f_2 = \frac{1}{z}$; $f_3 = 1 - z$; $f_4 = \frac{z}{z-1}$

$f_5 = \frac{1}{1-z}$; $f_6 = \frac{z-1}{z}$

$(f_1 \circ f_2)_z = f_1[f_2(z)]$
$= f_1\left(\frac{1}{z}\right) = z$

$f_3 \circ f_4 = f_3[f_4(z)]$
$=$

	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_6	f_1	f_5	f_4	f_2
f_4	f_4	f_5	f_6	f_1	f_2	f_3
f_5	f_5	f_4	f_2	f_3	f_6	f_1
f_6	f_6	f_3	f_4	f_2	f_1	f_5

1. Closure
2. Associative
3. Identity : f_1
4. Inverse : exists
5. It is not commutative.

$\therefore$ It is not an abelian group, but it is a group.

Ex: 7) $G = \{(a, b),\ a, b \in R,\ a \neq 0\}$ is non abelian group for $\circ$ defined as $(a,b) \circ (c,d) = (ac, bc+d)$

Ans) ① $(a,b), (c,d) \in G$
$a, b, c, d \in R$ $(a, b, c, d \neq 0)$
$(ac, bc+d) \in R$
$(ac, bc+d) \in G$, $(ac, bc+d \neq 0)$
$\therefore$ It is Closure.

② $(a,b) \circ [(c,d) \circ (e,f)] = [(a,b) \circ [c,d]] \circ (e,f)$

$\Rightarrow (a,b) \circ [ce, de+f] = [ac, bc+d] \circ (e,f)$
$\Rightarrow [ace, (bce+de+f)] = (ace, bce+de+f)$
$\therefore$ It is associative.

③ $ae = a$
$(a,b) \circ (e_1, e_2) = (a,b)$
$ae_1 = a$ $\quad be_1 + e_2 = b$
$\boxed{e_1 = 1}$ $\quad e_2 = b - b$
$\boxed{e_2 = 0}$

$\therefore (1, 0) = (e_1, e_2)$
$\exists\ (1,0) \in G$
s.t. $(a,b) \circ (1,0) = (a, b.1+0)$
$\exists (a,b)$
$(1,0) \circ (a,b) = (a,b)$

4. **Existence of inverse :** $(a, b) \circ (c, d) = (1, 0)$

$ac, bc + d = 1, 0$

$ac = 1, \quad bc + d = 0$

$c = \frac{1}{a}, \quad \frac{b}{a} + d = 0$

$d \Rightarrow -b/a$

$\forall (a, b) \in G \Rightarrow \exists \left(\frac{1}{a}, \frac{-b}{a}\right) \in G.$

$(a, b) \circ \left(\frac{1}{a}, \frac{-b}{a}\right) = a \cdot \frac{1}{a}, \; b * \frac{1}{a} - \frac{b}{a} \Rightarrow (1, 0)$

Becoz composite fun^ is not commutative, therefore it is a non-abelian group.

Properties :

Properties ① Identity element is always unique.

Proof :

let G be a group

let e_1, e_2 be two identity element

let $a \in G$

If e_1 is identity $\Rightarrow ae_1 = a$

If e_2 is identity $\Rightarrow ae_2 = a$

$\Rightarrow ae_1 = ae_2 \Rightarrow e_1 = e_2$

$\therefore$ Identity is always unique.

Property ② Inverse element of an element is always unique.

proof

let G be a group & let $a \in G$

Also let $b, c \in G$

such that $a^{-1} = b, \; a^{-1} = c.$

i.e; a have two inverse element

Now $a^{-1} = b \Rightarrow ab = e = ba$

$a^{-1} = c \Rightarrow ac = e = ca$

$(ba = e) * c$

$cba = ce = c$

$cab = c$ [associative Property $ab = ba$]

$eb = c$

$be = c$ [associative $eb = be$]

$\boxed{b = c}$

$\therefore$ The inverse element of an element is always unique.

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1. Identity is unique.
2. Inverse of each element is unique.

Property 3

If G is group & $a, b \in G$,

(i) $(a^{-1})^{-1} = a$

(ii) $(ab)^{-1} = b^{-1}a^{-1}$.

(i) $a \in G$ and G is group

$\Rightarrow a^{-1} \in G$ [Existence of inverse]

such that $aa^{-1} = e = a^{-1}a$

$a^{-1}a = e = aa^{-1}$

$\boxed{(a^{-1})^{-1} = a}$

(ii) $(ab)^{-1} = b^{-1}a^{-1}$

$\Rightarrow (ab)(b^{-1}a^{-1}) = e$

$\Rightarrow (b^{-1}a^{-1})(ab) = e$

$a \in G,\ b \in G \Rightarrow a^{-1}, b^{-1} \in G$

a, a^{-1}, b, b^{-1}, ab all are elements of G.

$(ab)\cdot(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$ [Associative law]

$= a(e)a^{-1}$ [Existence of Inverse]

$= aa^{-1}$ [$a \times a^{-1} = e$]

$= e$ (Identity)

$(b^{-1}a^{-1})\cdot(ab) = b^{-1}(a^{-1}a)b$

$= b^{-1}eb$

$= b^{-1}b$

$= e$.

$\Rightarrow (ab)(b^{-1}a^{-1}) = e = (b^{-1}a^{-1})(ab)$

Property 4 : Cancellation Laws → Left Cancellation law, → right Cancellation Law

$\Rightarrow a, b, c \in G$ and G is group.

then (i) $ab = ac \Rightarrow b = c$ [Left cancellation law]

(ii) $ba = ca \Rightarrow b = c$ [right cancellation law]

Proof : $a, b, c \in G$

& G is group. $\Rightarrow a^{-1} \in G$

$ab = ac$

$a^{-1}(ab) = a^{-1}(ac)$ [associative]

$(a^{-1}a)b = (a^{-1}a)c$ [associative]

$eb = ec$ [inverse]

$b = c$ [Identity]

(ii) $ba = ca$

$(ba)a^{-1} = (ca)a^{-1}$ [Association]

$b(aa^{-1}) = c(aa^{-1})$ [Associative]

$be = ce$ [inverse]

$b = c$ [Identity]

* definitions :- order of group and order of element.

$(G, *)$

↳ $O(G) \Rightarrow$ order of group

n is the order of a

$G = \{1, -1, i, -i\} \Rightarrow \boxed{e = 1}$ $\Rightarrow \boxed{a^n = e}$

$O(G) = 4$

$O(1) = (1)^1 = 1$

$O(-1) = (-1)^2 = 1 \Rightarrow e = O(-1) = 2$

$O(i) = (i)^4 = 1 \Rightarrow O(i) = 4$

$O(-i) = (-i)^4 = 1 \Rightarrow O(-i) = 4$

(i) Identity Ka order will always be 1.

(ii) ~~O(e)~~ $\boxed{O(\text{element}) \leq O(\text{Group})}$

(iii) Order of $a = O(a) = n \Rightarrow a^n = e$.

also $a^m = e$

then m is a multiple of n.

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Subgroup

If $(G, *)$ is a group and $H \subseteq G$. Then H is a subgroup of G if

(i) H is closed for composition $*$ $a, b \in H \Rightarrow a * b \in H$.

(ii) H is a group for induced composition.

(i) Identity of G & H are same

(ii) inverse of each element s

(iii) if $O(a) \Rightarrow H$ is G and $O(a) \Rightarrow H$

Types of Subgroup
- Proper Sub Group
- Im Proper Sub Group or Trivial Subgroup

$(G, *)$ is a group

(i) G itself

(ii) $\{e\}$

• Proper Sub Group

$G = \{Z, +\}$

$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$e = 0$,

$H = \{2n, n \in I^+\}$

$n = \{0, 2, 4, 6, 8, \ldots\}$

$\boxed{H \subset G}$

(i) Closure

(ii) Identity Element = 0

(iii) Inverse

(iv) Associative

$\therefore$ $\boxed{H \subseteq G}$

So H is a proper Sub Group.

A non void subset H of group $(G, *)$ is a subgroup if and only if $a, b \in H \Rightarrow ab^{-1} \in H$

H is a subgroup of G $\Leftrightarrow$ $(a, b^{-1} \in H \Rightarrow ab^{-1} \in H)$

Sol^n: let H is a subgroup of $(G, *)$

$\Rightarrow$ H is a group.

Now, $a, b \in H \Rightarrow b^{-1} \in H$ [Existence of Inverse]

$a \in H, b^{-1} \in H \Rightarrow ab^{-1} \in H$ [Closure Property]

$\Rightarrow a, b \in H \Rightarrow ab^{-1} \in H$

let condition is true,

$\Rightarrow a, b \in H \Rightarrow ab^{-1} \in H$

Now, we show H is a subgroup of G.

$\Rightarrow$ H is a group itself.

$a \in H, a \in H \Rightarrow aa^{-1} \in H$ [By condition]

$e \in H$ [$\therefore$ identity exist in H]

$e \in H, a \in H \Rightarrow ea^{-1} \in H$ [By condition]

$\therefore a^{-1} \in H$ [Existence of Inverse]

Let $a, b \in H \Rightarrow b \in H$

$b^{-1} \in H$

$\Rightarrow a, b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H$

$\Rightarrow ab \in H$ (closure Property)

And it is always associative as $H \subset G$.

$\Rightarrow H$ is a group itself.

Theorem :

The intersection of any two subgroup of G is again a subgroup.

Sol^n Let $(G, *)$ be a group & H_1, H_2 be its two subgroup.

we have to show $H_1 \cap H_2$ is also a subgroup of G.

(i) $a, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in (H_1 \cap H_2)$

(ii) $H_1 \cap H_2$ is non void.

(2) $\Rightarrow e \in H_1, e \in H_2$ [Where e is identity of G, H_1, H_2]

$\Rightarrow e \in H_1 \cap H_2$

$\Rightarrow H_1 \cap H_2 \neq \phi$

(1) $\Rightarrow a, b \in H_1 \cap H_2 \Rightarrow a, b \in H_1$ and $a, b \in H_2$.

$\Rightarrow ab^{-1} \in H_1$ and $ab^{-1} H_2$ (Condition)

$\Rightarrow ab^{-1} \in H_1 \cap H_2$

$\therefore H_1 \cap H_2$ is a subgroup of G.

Theorem

The union of any two subgroup of G is not necessarily subgroup.

Proof : Let $G = \{Z, +\}$ is a group.

$H_1 = \{2n, n \in Z\}$

$H_2 = \{3n, n \in Z\}$

H_1 & H_2 are two subgroup of G.

$H_1 \cup H_2 = \{-6, -4, -3, 2, 0, 2, 3, 4, 6\}$

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$2, 3 \in H_1 \cup H_2$

$2 + 3 = 5 \notin H_1 \cup H_2 \Rightarrow$ it does not satisfy closure property. Here it is not a subgroup of G.

—x——x——x——x——x——x—

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subGroup $\Leftrightarrow a, b \in H \Rightarrow ab^{-1} \in H$

G

$H \subset G$

subgroup

(i) $H \neq \phi$

(ii) H itself is a group

closure, asso., e, a^{-1}

If $a, b \in H$
$\Rightarrow ab^{-1} \in H$

Proper

Improper | trivial
G {e}

Theorem

The union of two subgroup is a subgroup if one is contained in another.

Proof:

Let $(G, *)$ be a group and H_1 & H_2 be two subgroup we have to show,

$H_1 \cup H_2$ is a subgroup $\Leftrightarrow H_1 \subset H_2$ or $H_2 \subset H_1$.

(i) Let $H_1 \subset H_2$ or $H_2 \subset H_1$

If $H_1 \subset H_2 \Rightarrow H_1 \cup H_2 = H_2$ & H_2 is subgroup.

$\Rightarrow H_1 \cup H_2$ is also a subgroup

$H_2 \subset H_1 \Rightarrow H_1 \cup H_2 = H_1$ & H_1 is subgroup

$\Rightarrow H_1 \cup H_2$ is also a subgroup.

(ii) Let $H_1 \cup H_2$ is a subgroup

$\Rightarrow H_1 \subset H_2$ or $H_2 \subset H_1$.

Let $H_1 \not\subset H_2 \Rightarrow \exists\, a \in H_1$ such that $a \notin H_2$.

Let $b \in H_2$

$a \in H_1, b \in H_2 \Rightarrow a, b \in H_1 \cup H_2$ [subgroup $= H_1 \cup H_2$ $\Rightarrow H_1 \cup H_2$ is a group $\Rightarrow$ satisfies closure]

$\Rightarrow ab \in H_1 \cup H_2$

~~$b \in H_2 \Rightarrow b^{-1} \in H_2$ [Existence of Identity]~~

~~$ab \in H_1 \cup H_2$, $b^{-1} \in H_1 \cup H_2$~~

$\Rightarrow ab \in H_1$ or $ab \in H_2$

If $ab \in H_2, b \in H_2 \Rightarrow (ab)b^{-1} \in H_2 \Rightarrow a(bb^{-1}) \in H_2$

$\Rightarrow ae \in H_2 \Rightarrow a \in H_2$

$\Rightarrow$ a contradiction $\Rightarrow ab \notin H_2$

$\Rightarrow ab \in H_1, a \in H_1 \Rightarrow a^{-1}(ab) \in H_1 \Rightarrow (a^{-1}a)b \in H_1$

$\Rightarrow eb \in H_1 \Rightarrow b \in H_1$.

$\Rightarrow H_2 \subset H_1$

We have show that G is a group and center of group G is given by $Z(G) = \{x \in G \mid gx = xg \ \forall g \in G\}$ is a subgroup.

$\boxed{eg = e = ge}$

Proof : let $x_1, x_2 \in Z(G)$

$gx_1 = x_1 g, \quad gx_2 = x_2 g$

$\exists e \in G$ such that $eg = ge \ \forall g \in G$

$\Rightarrow e \in Z(G)$

$\Rightarrow Z(G) \neq \phi$

$gx_2 = x_2 g \Rightarrow x_2^{-1}(gx_2)x_2^{-1} = x_2^{-1}(x_2 g)x_2^{-1}$

$= x_2^{-1} g (x_2 x_2^{-1}) = (x_2^{-1} x_2)(g x_2^{-1})$

$= x_2^{-1} g e = e g x_2^{-1}$

$= x_2^{-1} g = g x_2^{-1}$

$\Rightarrow x_2^{-1} \in Z(G)$

$(x_2 x_2^{-1})g = x_2(x_2^{-1} g) = x_1 (g x_2^{-1}) = (x_1 g)x_2^{-1}$

$= g(x_1)x_2^{-1} = g(x_1 x_2^{-1})$

$x_1 x_2^{-1} \in Z(G)$

$Z(G)$ is a subgroup.

Q) subgroup $= \{a, b \in H \Rightarrow ab^{-1} \in H\}$

$Z(G) = \{x \in G \mid gx = xg \ \forall g \in G\}$

$G = \{a + b\sqrt{2} \mid a, b \in Q\}$ is a group w.r.t. addition}

(i) associative (✓) (iii) identity element, $e \Rightarrow 0$

(ii) Closure (✓) (iv) $a^{-1} \Rightarrow (-a - b\sqrt{2})$

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Q) If $a, b \in G$, then we have to show $ax = b$, $ya = b$ {have unique solution}

proof : $a \in G \Rightarrow a^{-1} \in G$ [Existence of ~~identity~~ Inverse]

$a^{-1} \in G, b \in G \Rightarrow a^{-1}b \in G$ [closure Property]

$ax = a(a^{-1}b) = (aa^{-1})b = eb = b$

$x \to a^{-1}b$ is solution of eq^n $ax = b$.

If it is not unique, $\Rightarrow$ let x_1, x_2 are sol^n of $ax = b$

$ax_1 = b, \quad ax_2 = b$

$ax_1 = ax_2$

$x_1 = x_2$ (left cancellation law)

$\therefore$ sol^n is unique.

(ii) $a^{-1} \in G, b \in G$

$\Rightarrow ba^{-1} \in G$ [closure Property]

Now, $\boxed{ya = b}$

$(ba^{-1})a = b(a^{-1}a) = b(e) = b$

$\Rightarrow ba^{-1}$ is solⁿ of eqⁿ $ya = b$

Now, we show that this is unique.

let $y_1 y_2$ be solⁿ of $ya = b$

$y_1 a = b, \; y_2 a = b$

$y_1 a = y_2 a$

$y_1 = y_2$ [Right cancellation law]

$\Rightarrow$ solⁿ will be unique.

* Cyclic Group : a group which can be generated through a single element.

ex: $G = \{1, -1, i, -i\}$ {Multiplication Composition}

$(i)^4 = 1$

$((i)^2) = -1$

$(i)^1 = i$

$-(i)^3 = -i$ {i is generator}

$\therefore$ G is a cyclic group.

Ex:-2 $(z^*, +)$ {Addition Composition}

{'1' is generator}

* A group G is a cyclic group, if there exits a element $a \in G$ such that $G = [a]$ i.e; every element can be written as power of a(x) when composition of addition, we add no. of add (+)

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Properties of a cyclic group

(i) Every cyclic Group is abelian

(ii) If 'a' is a generator of a cyclic group G, then a^{-1} is also its generator.

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3) The order of finite cyclic group is equal to order of its generator.

$G = \{1, -1, i, -i\}$ $\qquad o(G) = o(a)$

$o(G) = 4$

$o(i) = 4$

$i^4 = i^2 \times i^2$

$= -1 \times -1$

$= 1 = e$

Proof :

$G = [a] \qquad a \in G \quad o(a) = n \Rightarrow a^n = e$

H is a sub group of G whose order is n.

$o(G) = m$

Case ① : $m \leq n$: If $a^m \in G$ then $a^m \in H$

$\therefore H \subset G$

Case ② : $m > n$: $m = qn + r \Rightarrow a^m = a^{qn+r}$

$= (a^n)^q . a^r = e a^r$

$= a^r \in H$

$G \subset H$

$\therefore G \subset H$ & $H \subset G \Rightarrow G = H$

$o(G) = n$

$o(H) = n$

$\boxed{o(G) = o(a) = n}$

COSETS

Let H be a subgroup of G and $a \in G$, then the set

$aH = \{ah / \forall h \in H\}$

is left coset

$Ha = \{ha / \forall h \in H\}$

is Right Coset $\qquad n = \{-3, -2, -1, 0, 1, 2, 3 ---\}$

$G = (Z, +)$ ab group

$H = \{\{2n \mid n \in Z\}, +\}$ is a subgroup of G

$3 \in Z$

$H + 3 = \{ ---- -1, 1, 0, 5, 7, 9 ---\}$

$G = \cup aH$

$G \Rightarrow$ Union of all left costts of H

$G = \cup Ha$

$G \Rightarrow$ union of all right cosets of H.

Q) Find all cosets of $3Z$ in Group $(Z, +)$

Ans)

$Z = \{ \ldots -3, -2, -1, 0, 1, 2, 3 \ldots \}$

$3Z = \{ \ldots -9, -6, -3, 0, 3, 6, 9 \ldots \}$

$3Z+0 = \{ \ldots -9, -6, -3, 0, 3, 6 \ldots \}$

$3Z+1 = \{ \ldots -8, -5, -2, 1, 4, 7 \ldots \}$

$3Z+2 = \{ \ldots -7, -4, -1, 2, 5, 8 \ldots \}$

$3Z+3 = \{ \ldots -6, -3, 0, 3, 6, 9 \ldots \}$

$3Z+4 = \{ \ldots -5, -2, 1, 4, 7, 10 \ldots \}$

$3Z+(-1) = \{ \ldots -7, -4, -1, 2, 5, 8 \ldots \}$

$3Z+(-2) = \{ \ldots -8, -5, -2, 1, 4, 7 \ldots \}$

~~3z+1 = 3z+(-2) = 3z+1~~

$\therefore$ There are three distinct cosets.

$\{3Z, 3Z+1, 3Z+2\}$

Theorem

Lagrange's Theorem

If $o(G) = n$

$o(H) = m$

n / m

$\boxed{n = mK, \ K \in Z}$

$\Rightarrow$ The order of the every subgroup of a finite group is a divisor of the order of the group.

Proof: Suppose $o(G) = n$ & $o(H) = m$ where H is a subgroup of G, let the different coset of H and G be

$g_1H, g_2H \ldots g_KH$

then, $G = \bigcup_{n=1}^{K} g_iH$

$G = g_1H \cup g_2H \cup g_3H \ldots \cup g_KH$

$o(G) = o(g_1H) + o(g_2H) \ldots + o(g_KH)$

$n = o(H) + o(H) + \ldots o(H)$ (K times)

$= m + m + \ldots + m$ K times

$n = mK$

$\Rightarrow m$ is a divisor of n

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Relation of Congruence w.r.t. subgroup
If G is a group, H is subgroup of G
& $a, b \in G$ ⇒ then $a \cong b \pmod H \Leftrightarrow ab^{-1} \in H$

The relation of congruency in a group G defined as

$$a \cong b \,(\text{Mod } H) \Leftrightarrow ab^{-1} \in H$$

is an equivalency relation where H is a subgroup of G.

Proof :- ① Reflexive, Symmetric & transitive

$\boxed{a \cong a}$ $\boxed{a \cong b \Rightarrow b \cong a}$ $\boxed{a \cong b,\ b \cong c \Rightarrow a \cong c}$

① Reflexive relation :- let $a \in G$, then $a^{-1} \in G$ [Existence of inverse]
$aa^{-1} \in G \Rightarrow e \in G$. [$H$ is a subgroup then it will be a group]
$\Rightarrow a \cong a \pmod H$

② Symmetric ⇒ let $a \cong b \text{ Mod } H \Rightarrow ab^{-1} \in H$
$(ab^{-1})^{-1} \in H$ (H is a group)
$(b^{-1})^{-1} a^{-1} \in H$
$ba^{-1} \in H$
$b \cong a \pmod H$

∴ $\boxed{a \cong b \Leftrightarrow b \cong a}$

③ let $a \cong b \pmod H$ & $b \cong c \pmod H$
$ab^{-1} \in H$ and $bc^{-1} \in H$
$(ab^{-1})(bc^{-1}) \in H$ (H is a subgroup) [closure Property]
$a(b^{-1}b)c^{-1} \in H$ [By associative property in H]
$aec^{-1} \in H \Rightarrow ac^{-1} \in H$
$\Rightarrow a \cong c \pmod H$
∴ Transitive

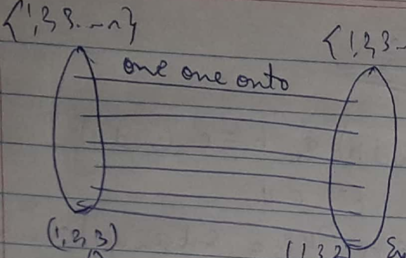
∴ Relation is an equivalence relation.

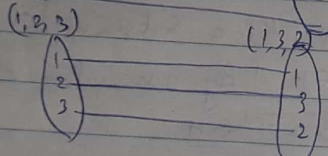
* Permutation

A one to one mapping 'σ' of the set $\{1, 2, \ldots n\}$ onto itself is called a permutation and is denoted by

$$\sigma = \begin{Bmatrix} 1 & 2 & 3 & \cdots & n \\ j_1 & j_2 & j_3 & \cdots & j_n \end{Bmatrix}$$

where $j_i = \sigma(i)$





Ex: A {1,2,3}

$$\sigma = \begin{Bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{Bmatrix}$$

* Cyclic Permutation

A permutation σ of Set A is a cyclic permutation or a cycle if there exists a finite subset $a_1, a_2, \dots a_r$ of A, such that $\sigma(a_1) = a_2, \sigma(a_2) = a_3, \sigma(a_3) = a_4 \dots \sigma(a_r) = a_1$. And it is denoted by $\sigma = [a_1, a_2, \dots a_n]$.

* Composition of two permutation

If σ and ϕ are two permutation then

$$\sigma\phi = \sigma \circ \phi$$

$$\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad \phi = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\sigma\phi = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\sigma\phi = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Q) If $\sigma = [1\ 7\ 2\ 6\ 3\ 5\ 8\ 4]$

$$\rho = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 4 & 3 & 8 & 7 & 6 & 1 \end{bmatrix}$$

①

$$\rho\sigma\rho^{-1} = (\rho(1)\ \rho(7)\ \rho(2)\ \rho(6)\ \rho(3)\ \rho(5)\ \rho(8)\ \rho(4)$$

Ⓐ

$$\rho^{-1} = \begin{bmatrix} 2 & 5 & 4 & 3 & 8 & 7 & 6 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

$$\rho\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 4 & 3 & 8 & 7 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 1 & 8 & 3 & 2 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7 & 6 & 5 & 1 & 8 & 3 & 2 & 4 \\ 6 & 7 & 8 & 2 & 1 & 4 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 1 & 8 & 3 & 2 & 4 \end{bmatrix}$$

$$\rho\sigma \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 2 & 1 & 4 & 5 & 3 \end{bmatrix}$$

$$\rho \circ \rho^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 2 & 1 & 4 & 5 & 3 \end{bmatrix} \begin{bmatrix} \cancel{2} & \cancel{5} & \cancel{4} & \cancel{3} & \cancel{8} & \cancel{7} & \cancel{6} & \cancel{1} \\ \cancel{1} & \cancel{2} & \cancel{3} & \cancel{4} & \cancel{5} & \cancel{6} & \cancel{7} & \cancel{8} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 4 & 3 & 2 & 7 & 6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{2} & \cancel{5} & \cancel{4} & \cancel{3} & \cancel{8} & \cancel{7} & \cancel{6} & \cancel{1} \\ \cancel{6} & \cancel{7} & \cancel{8} & \cancel{2} & \cancel{1} & \cancel{4} & \cancel{5} & \cancel{3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & 1 & 4 & 3 & 2 & 7 & 6 & 5 \\ 3 & 6 & 2 & 8 & 7 & 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 4 & 3 & 2 & 7 & 6 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 2 & 8 & 7 & 5 & 4 & 1 \end{bmatrix}$$

$\Rightarrow (1\ 3\ 2\ 6\ 5\ 7\ 4\ 8)$ [// cyclic]

$= [\rho(8)\ \rho(4)\ \rho(1)\ \rho(7)\ \rho(2)\ \rho(6)\ \rho(3)\ \rho(5))$

$\llcorner$ B [From 1]

$\Rightarrow$ From A & B

Both are cyclic permutations.

$\therefore$ Both are equal.

So, Hence Proved.

Disjoint cycles

$$\rho = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 4 & 3 & 8 & 7 & 6 & 1 \end{bmatrix}$$

$\Rightarrow \underbrace{(1, 2, 5, 8)}_{\text{transposition}}\ \underbrace{(3, 4)\ (6, 7)}_{\text{disjoint cycles.}}$

Even & Odd permutations.

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Q] $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}$

$\sigma = (134)(56)$

$\sigma^{-1}\rho\sigma = ?$ (disjoint cycle)

ρ → even or → odd order

$\sigma = \begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 2 & 7 & 8 & 9 \\ 3 & 4 & 1 & 6 & 5 & 7 & 8 & 9 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \end{pmatrix}$

$\sigma^{-1} = \begin{pmatrix} 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$

$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 1 & 3 & 6 & 5 & 2 & 7 & 8 \end{pmatrix}$

$\rho\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \\ \cancel{7\,8\,9\,6\,4\,5\,2\,3\,1} \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}$

$\sigma^{-1}\rho\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 1 & 3 & 6 & 5 & 2 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}$

$= \begin{pmatrix} 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \\ 8 & 9 & 5 & 2 & 6 & 3 & 1 & 4 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 5 & 2 & 6 & 3 & 1 & 4 & 7 \end{pmatrix}$

$= (184297)(356)$

∴ odd Permutation

$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}$

$= (172839)(465)$

$= (19)(13)(18)(12)(17)$
$(45)(46)$

$= 7$ (odd) transpositions | inversion.

$f = (1\ 7\ 2\ 8\ 3\ 9)(4\ 6\ 5)$
$= (19)(13)(18)(12)(17)(45)(46)$

$= LCM(6,3)$

$\boxed{O(f) = 6}$

1. Product
2. Inverse
3. Order
4. Even or Odd
5. disjoint.

A permutation is said to be even permutation if it can be expressed as product of even no. of transpositions.

A permutation is said to be odd permutation if it can be expressed as product of odd no. of transpositions.

Symmetric Group or Group of Permutations

$A = \{1,2,3\} \Rightarrow 3! = 3\times2\times1 = 6$

$\epsilon = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ $\sigma_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$ $\sigma_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

$\sigma_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ $\sigma_4 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ $\sigma_5 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

$S_3 = \{\epsilon, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$

$S_4 = \{\leftarrow 24 \rightarrow\}$

$S_5 = \{120\}$
$S_n = \{n!\}$

The group of permutation of the set $\{1,2,\ldots n\}$ is called a symmetric group of degree n and it is denoted by S_n

Q) Show that S_3 is a group.

$A = \{1, 2, 3\}$

$E_I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $\sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

then $S_3 = \{ E_I, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \}$

we have to show S_3 is group

o	E_I	σ_1	σ_2	σ_3	σ_4	σ_5
E_I	E_I	σ_1	σ_2	σ_3	σ_4	σ_5
σ_1	σ_1	E_I	σ_4	σ_5		
σ_2	σ_2					
σ_3	σ_3					
σ_4	σ_4					
σ_5	σ_5					

1) Closure Property $\Rightarrow$ As all mapping are $E_I, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$
$\therefore$ it is closed.

2) Associative Property $\Rightarrow$ Composition function is associative

3) Identity : E_I is identity

4) Existence of inverse : Symmetric.

Q) Show that S_n is a group?

Ans) ① Closure Property : let $f, g \in S_n$
$\Rightarrow f \circ g \in S_n$
$\therefore S_n$ is closed.

② Associativity : let $f, g, h \in S_n$
then we know $f \circ (g \circ h) = (f \circ g) \circ h$
(becoz composite function is associative)

③ Existence of identity $= I_n \in S_n$ is identity function i.e. $\begin{pmatrix} 1 & 2 & 3 & - & - & n \\ 1 & 2 & 3 & - & - & n \end{pmatrix}$

s.t. $f \circ I_n = f = I_n \circ f$

④ Existence of Inverse : $f \in S_n$

f is one one and onto

f^{-1} is also one & onto

s.t. $f o f^{-1} = I_n = f^{-1} o f$

$\therefore$ inverse element exist.

$\therefore$ By ①, ②, ③ & ④, then S_n is a group.

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* NORMAL SUBGROUP

If G is a group and H is a subgroup of G, then H is normal subgroup of G iff $\forall\, x \in G$ & $h \in H$ $\Rightarrow x h x^{-1} \in H$.

$$\boxed{H \trianglelefteq G \Leftrightarrow (x \in G,\ h \in H \Rightarrow x h x^{-1} \in H)}$$

Improper — $G, \{e\}$

Proper

* Single group : If G is a group and G do not have any proper normal subgroup then it is simple Group.

Ex : $G = \{1, -1, i, -i\}$ is a group.

$H = \{1, -1\}$ is a subgroup.

↓

Normal Subgroup.

$x = 1,\ h = 1$

$x^{-1} \Rightarrow 1 \quad (1)(1)(-1) = -1 \in H$

$x = i,\ h = 1$

$x^{-1} = -i \quad (i)(1)(-i) = -1 \in H$

$\boxed{x \in G,\ h \in H \Rightarrow x h x^{-1} \in H} \Rightarrow$ Normal Subgroup.

\# Some Properties of Normal Subgroup :

Theorem ① Every Subgroup of Abelian group is normal subgroup.

Proof : let H be a subgroup of abelian group G.

let $x \in G$ and $h \in H$

$x h x^{-1} = (h x) x^{-1}$ [$\because G$ is a abelian group]

$= h (x x^{-1})$ [$\because G$ is associative]

$= h \in H$

$\boxed{xhx^{-1} \in H}$

$\therefore x \in G,\ h \in H \Rightarrow xhx^{-1} \in H \Rightarrow H$ is normal subgroup.

~~Commutative Property~~

Theorem 2 : A subgroup H of a group G is a normal subgroup if and only if

$H \triangleleft G \Leftrightarrow xHx^{-1} = H \ \forall x \in G.$

$A = B$

$A \subset B \Rightarrow x \in A$ and $x \in B$

$B \subset A \Rightarrow y \in B$ and $y \in A$

Let H is normal subgroup of G

If $x \in G, h \in H \Rightarrow xhx^{-1} \in H$

$\Rightarrow xHx^{-1} \subset H$

$x \in G \Rightarrow x^{-1} \in G$ (Existence of inverse)

$x^{-1} \in G$ and $h \in H$

$(x^{-1}) h (x^{-1})^{-1} \in H$

$\Rightarrow x^{-1}hx \in H$ — (1)

$x^{-1}Hx \subset H$

$x(x^{-1}Hx)x^{-1} \subset xHx^{-1}$

$eHe \subset xHx^{-1}$

$= H \subset xHx^{-1}$ — (2)

$\Rightarrow$ By (1) & (2)

$\boxed{xHx^{-1} = H}$

$\Rightarrow$ CONVERSELY :

Let $xHx^{-1} = H$ and we will show that H is a normal subgroup.

Now, $xHx^{-1} = H \Rightarrow xHx^{-1} \subset H$

$\Rightarrow xhx^{-1} \in H \quad \forall x \in G, h \in H$

$\Rightarrow \boxed{H \triangleleft G}$

Theorem 3 $\Rightarrow$ The intersection of any two normal subgroup of a group is a normal subgroup.

Proof $\Rightarrow$ Let H_1 and H_2 be two normal subgroup of G. Then we have to show $H_1 \cap H_2$ is also normal subgroup of G.

Let $x \in G$ and $h \in H_1 \cap H_2$

Now $h \in H_1 \cap H_2$

$\Rightarrow h \in H_1$ and $h \in H_2$

Now, $x \in G$, $h \in H_1 \Rightarrow xhx^{-1} \in H_1$ [H_1 is normal subgroup]

Now, $x \in G$, $h \in H_2 \Rightarrow xhx^{-1} \in H_2$ [H_2 is normal subgroup]

$x \in G, h \in H_1 \cap H_2 \Rightarrow xhx^{-1} \in H_1 \cap H_2$,

$\Rightarrow H_1 \cap H_2$ is normal subgroup.

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H is a subgroup of G, and N is normal subgroup.

$\Rightarrow H \cap N \triangleleft H$

Proof: $H \cap N$ will be subgroup of G,

$H \cap N \subset H$

$\Rightarrow H \cap N$ will be a subgroup of H.

$x \in H$, $h \in H \cap N$,

Now, $h \in H \cap N \Rightarrow h \in H$ and $h \in N$

Now, $x \in H$, $h \in H \Rightarrow xhx^{-1} \in H$ [By closure prop.]

$x \in H \Rightarrow x \in G$ [H is a subgroup of G]

Now,

$x \in G$, $h \in N \Rightarrow \boxed{xhx^{-1} \in N}$ [N is normal subgroup of G]

$xhx^{-1} \in H \cap N \Rightarrow H \cap N \triangleleft H$

QUOTIENT GROUP

If G is a group, $H \triangleleft G$, then the set G/H of all cosets of H in G together with binary composition $Ha \cdot Hb = Hab$ is a group and it is called the quotient group of G by H.

~~Let G be a group and $H \triangleleft G$, then the set G/H~~

Q) Find the quotient group G/H and also prepare its operation table when $G\{\{1,-1,i,-i\}\ x\}$, $H = [\{1,-1\}, x]$

operation is complex multiplication

$H.1 = \{1.1 - 1.1\} \Rightarrow \{1,-1\} = H$

$H.(-1) = \{1(-1), -1(-1)\} = \{-1, 1\} = H$

$H.i = \{1.i, -1.i\} = \{i, -i\} = Hi$

$H.(-i) = \{1(-i), -1(-i)\} = \{-i, i\} = Hi$

$G/H = \{H, Hi\}$

$*$	H	Hi
H	H	Hi
Hi	Hi	H

Q) Find the quotient group G/H and also prepare its operation table when $G = (Z, +)$, $H = (4Z, +)$.

Ans) Now coset of G in H

$H+0 = \{ \dots -12, -8, -4, 0, 4, 8 \dots \}$

$H+1 = \{ \dots -11, -7, -3, 1, 5, 9 \dots \}$

$H+2 = \{ \dots -10, -6, -2, 2, 6, 10 \dots \}$

$H+3 = \{ \dots -9, -5, -1, 3, 7, 11 \dots \}$

$H+4 = H+0 = H+8 = H+12 \dots$

$H+5 = H+1 = H+9 = H+13 \dots$

$H+6 = H+2 = H+10 = \dots$

$H+7 = H+3 = H+11 = \dots$

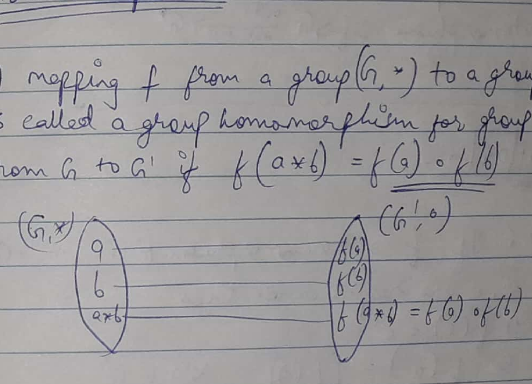
$G/H = \{H, H+1, H+2, H+3\}$

$\boxed{\begin{array}{l} Ha \cdot Hb = Hab \\ (H+a) + (H+b) = H+(a+b) \end{array}}$

	H	$H+1$	$H+2$	$H+3$
H	H	$H+1$	$H+2$	$H+3$
$H+1$	$H+1$	$H+2$	$H+3$	H
$H+2$	$H+2$	$H+3$	H	$H+1$
$H+3$	$H+3$	H	$H+1$	$H+2$

Homomorphism

A mapping f from a group $(G, *)$ to a group $(G', \circ)$ is called a group homomorphism for group morphism from G to G' if $f(a*b) = f(a) \circ f(b)$



Ex:

$(R, +)$, $(R_0, \times)$

$f: (R, +) \to (R_0, \times)$

$f(x) = 2^x$

$f(x_1 + x_2) = 2^{x_1 + x_2} = 2^{x_1} \cdot 2^{x_2}$

$= f(x_1) \cdot f(x_2)$

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Various Morphism :

1. Monomorphism : $f \to$ one one
2. Epimorphism : $f \to$ onto
3. Isomorphism : $f \to$ one-one & onto
4. Endomorphism : $f \to G \to G$
5. Automorphism : $f \Rightarrow G \xrightarrow[\text{one-one}]{\text{onto}} G$

01/05/2020

Theorem 1 : If f is homomorphism from G to G' & e, e' are respective identities

1. $f(e) = e'$
2. $f(a^{-1}) = [f(a)]^{-1} \quad \forall a \in G$

Proof (i) let $a \in G$

$\Rightarrow ae = a = ea$

$f(ae) = f(a) = f(ea)$

$f(a) f(e) = f(a) = f(e) f(a)$ [f is homomorphism]

$f(e) = e' \in G'$

(ii) let $a \in G$

$a^{-1} \in G$ [G is a group]

$aa^{-1} = e = a^{-1}a$

$f(aa^{-1}) = f(e) = f(a^{-1}a)$

$f(a) f(a^{-1}) = f(e) = f(a^{-1}) f(a)$ [f is homomorphism]

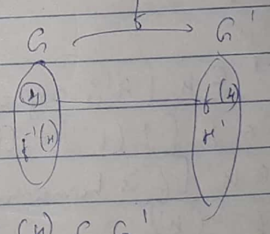
$f(a) f(a^{-1}) = e' = f(a^{-1}) f(a)$

$f(a^{-1}) = [f(a)]^{-1}$

Theorem 2 :

If f is homomorphism from group G to G' then show,

1. H is a subgroup of $G \Rightarrow f(H)$ is a subgroup of G'.
2. H' —— " —— $G' \Rightarrow f^{-1}(H')$ —— " —— G.



$f(ab) = f(a) f(b)$

$a, b \in H \Rightarrow ab^{-1} \in H$

Proof (i) $f(H) \subset G'$

$f(H) \neq \phi \Rightarrow \exists\ e' \in f(H)$ s.t. $f(e) = e'$ ($e \in H$)

$a', b' \in f(H) \Rightarrow \exists\ a, b \in H$

s.t. $f(a) = a'$, $f(b) = b'$.

$a'(b'^{-1})^{-1} = f(a)[f(b)]^{-1}$

$= f(a)\, f[b^{-1}] = f(ab^{-1})$ — ①

Now, $a, b \in H \Rightarrow ab^{-1} \in H$ [H is subgroup]

$f(a, b^{-1}) \in f(H)$

$a'(b')^{-1} \in f(H)$ [by ①]

$a', b' \in f(H) \Rightarrow a'(b')^{-1} \in f(H)$

$\therefore f(H)$ is a subgroup of G'

② $f^{-1}(H') \subset G$

$e \in G$ s.t. $f(e) = e'$

$\Rightarrow f^{-1}(H) \neq \phi$

$a, b \in f^{-1}(H')$

$f(a), f(b) \in H'$

$f(a)[f(b)]^{-1} \in H'$ (H' is subgroup)

$f(a)\, f(b^{-1}) \in H$ [$f(b^{-1}) = [f(b)]^{-1}$]

$f(ab^{-1}) \in H'$ [f is homomorphism]

$ab^{-1} \in f^{-1}(H')$

$\Rightarrow f^{-1}(H')$ is a subgroup of G.

Kernal of Homomorphism

$f : G \to G'$ (Homomorphism)

Ker f =

If f is a homomorphism of a group G into G' then the set K of all those elements of G which are mapped to the identity e' of G' is called the Kernal of the homomorphism f. It is denoted by Ker f or Ker(f).

Theorem A homomorphism f from a group G to G' is an isomorphism if and only if Ker $f = \{e\}$

Proof : f is homomorphism $\Leftrightarrow$ Ker $f = \{e\}$

① Let f is isomorphism [f is one one and onto both]

Let $a \in G$

s.t. $f(a) = e'$

$f(a) = f(e) \Rightarrow [f(e) = e']$

$a = e$ [f is one one]

Ker $f = \{e\}$

Conversely let Ker $f = \{e\}$

clearly f is onto

let $f(a) = f(b)$ $[a, b \in G]$

$f(a)[f(b)]^{-1} = f(b)[f(b)]^{-1}$

$f(a) f(b^{-1}) = e'$

$f(ab^{-1}) = e'$

$ab^{-1} \in Ker f \Rightarrow ab^{-1} = e$

f is one-one $\boxed{a = b}$

04/05/20 Recurrence Relation

$$\left.\begin{array}{l} a_r = a_{r-1} + 1 \\ a_r = 2a_{r-1} + a_{r-2} \end{array}\right\} \rightarrow \text{Differential Equations.}$$

$a_r \rightarrow$ Numeric function

$f\{0, 1, 2 \ldots\} \rightarrow R$

List : $a_i = \{3, 5, 7, 9 \ldots\}$

$a_1 = 3$
$a_2 = 5$
$a_3 = 7$

formula : $a_r = 3r + 1$

$a_0 = 1$
$a_1 = 4$
$a_2 = 7$
$a_3 = 10$

Linear recurrence relation with constant coefficient

$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \ldots = f(r)$

$$\boxed{a_r + 2a_{r-1} + 3 = (3r^2 + 1)}$$

$$\boxed{a_r = \text{Homogenous solution} + \text{Particular Solution}}$$

- Homogenous Solution [depends upon LHS side]

$C_0 a_1 + C_1 a_{r-1} + C_2 a_{r-2} + \ldots \ldots C_n a_{r-n} = f(r)$

Characteristic Equation

we substitute $a_r = A\alpha^r$, $f(r) = 0$

$\Rightarrow C_0 A\alpha^r + C_1 A\alpha^{r-1} + C_2 A\alpha^{r-2} \ldots = 0$

$A(C_0\alpha^r + C_1\alpha^{r-1} + C_2\alpha^{r-2} \ldots) = 0$

$C_0\alpha^r + C_1\alpha^{r-1}\ C_2\alpha^{r-2} \ldots = 0$

Roots will be : real, repeated real, imaginary, repeated imaginary.

This is characteristic Eqn :

① Roots are real and distinct $(\alpha_1, \alpha_2, \alpha_3 \dots)$

$$a_r = A_1 \alpha_1^r + A_2 \alpha_2^r + A_3 \alpha_3^r \dots$$

② Roots are real and repeated $(\alpha, \alpha_2, \alpha_3 \dots)$

$$a_r = (A_1 r^2 + A_2 r + A_3)\, \alpha_1^r + A_4 \alpha_4^r \dots$$

③ Roots are imaginary $(\alpha \pm i\beta)$

$$a_r = \rho^r (A_1 \cos r\theta + A_2 \sin r\theta)$$

$$\rho = \sqrt{\alpha^2 + \beta^2}, \quad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

④ Roots are imaginary and repeated $(\alpha \pm i\beta, \alpha \pm i\beta)$

$$a_r = \rho^r \left((A_1 r + A_2) \cos r\theta + (A_3 r + A_4) \sin r\theta \right)$$

$$\rho = \sqrt{\alpha^2 + \beta^2} \quad \theta = \tan^{-1}(\beta/\alpha)$$

Q1) Fibonacci Sequence is $a_r = a_{r-1} + a_{r-2}$ with boundary condition $a_0 = 0, a_1 = 1$.

Solution : $a_r - a_{r-1} - a_{r-2} = 0$

Now characteristic eqn : $a_r = A\alpha^r$

$$\alpha^r - \alpha^{r-1} - \alpha^{r-2} = 0$$

$$\alpha^{r-2}(\alpha^2 - \alpha^1 - 1) = 0$$

$$(\alpha^2 - \alpha - 1) = 0$$

$$\alpha = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\cancel{\alpha = 1/2}$, $\cancel{\beta = \sqrt{5}/2}$

$$\cancel{a_r = \rho^r (A_1 \cos r\theta + A_2 \sin r\theta)}$$

$$\cancel{\rho = \sqrt{\alpha^2 + \beta^2}, \quad \theta = \tan^{-1}(\beta/\alpha)}$$

$$a_r = A_1 \left(\frac{1 + \sqrt{5}}{2}\right)^r + A_2 \left(\frac{1 - \sqrt{5}}{2}\right)^r$$

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$a_0 = 0 \Rightarrow A_1 + A_2 \Rightarrow A_1 = -A_2$

$a_1 = 1 \Rightarrow 1 = A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right)$

$A_1 = \frac{1}{\sqrt{5}} = A_2 = \frac{-1}{\sqrt{5}}$

$\therefore a_r = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^r - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^r$

Q) $4a_r - 20a_{r-1} + 17a_{r-2} - 4a_{r-3} = 0$

Sol^n^: Characteristic eq^n^ :-

$4\alpha^{r} - 20\alpha^{r-1} + 17\alpha^{r-2} - 4\alpha^{r-3} = 0$

$\Rightarrow 4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0$

$\alpha = 4, \frac{1}{2}, \frac{1}{2}$

Homogeneous Solution is

$a_r = A_1(4)^2 + (A_2 r + A_3)\left(\frac{1}{2}\right)^r$

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Q) $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$

Sol^n^ :- Characteristic Eq^n^ :-

$a_r = A\alpha^r$, RHS $= 0$

$\Rightarrow \alpha^r + 5\alpha^{r-1} + 6\alpha^{r-2} = 0$

$\alpha^2 + 5\alpha + 6 = 0$

$\alpha^2 + 2\alpha + 3\alpha + 6 = 0$

$\boxed{\alpha = -3, -2}$

$\therefore$ H.S. $\Rightarrow \boxed{a_r = A_1(-3)^r + A_2(-2)^r}$

Let particular sol^n^ be $a_r = P_1 r^2 + P_2 r + P_3$

$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$

$(P_1 r^2 + P_2 r + P_3) + 5\left(P_1(r-1)^2 + P_2(r-1) + P_3\right) + 6\left(P_1(r-2)^2 + P_2(r-2) + P_3\right) = 3r^2$

$\Rightarrow P_1 r^2 + P_2 r + P_3 + 5\left[P_1 r^2 - 2P_1 r + P_1 + P_2 r + P_3 - P_2\right]$

$+ 6\left[P_1 r^2 - 4P_1 r + 4P_1 + P_2 r - 2P_2 + P_3\right]$

$12P_1 r^2 + r(P_2 + 5P_2 + 6P_2) - 10P_1 r - 24P_1 r + (P_3 + 5P_3 + 6P_3 \ldots)$

$\Rightarrow 12P_1 r^2 + r(12P_2 - 34P_1) + (29P_1 - 17P_2 + 12P_3) = 3r^2$

$12P_1 \cancel{r^2} = 3\cancel{r^2}$; $12P_2 - 34P_1 = 0$ & $29P_1 - 17P_2 + 12P_3 = 0$

$P_1 = 1/4$; $P_2 = 17/24$, $P_3 = 115/288$

$$\boxed{P.S. \Rightarrow a_r = \frac{1}{4} r^2 + \frac{17}{24} r + \frac{115}{288}}$$

$\boxed{T.S = HS + PS}$

Q) $a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r$

P.S :- ①

②

③ $a^2 b^r$

Ans) H.S. $\Rightarrow \alpha = -3, -2$

H.S. $\Rightarrow a_r = A_1(-2)^r + A_2(-3)^r$

P.S. $\Rightarrow$ let $a_r = P4^r$

$a_{r-1} = P4^{r-1}$, $a_{r-2} = P4^{r-2}$

$\Rightarrow$ Eq^n becomes,

$P4^r + 5P4^{r-1} + 6P4^{r-2} = 42 \cdot 4^r$

$16P + 20P + 6P = 42 \cdot 16$

$42P = 42 \cdot 16$

$\boxed{P = 16}$

$\therefore P_s = a_r = 16 \cdot 4^r$

$\boxed{TS = HS + PS}$

Q) $a_r + a_{r-1} = 3r\, 2^r$

Characteristic Eq^n :-

$\alpha^r + \alpha^{r-1} = 0$

$\alpha^{r-1}(\alpha + 1) = 0$

$\alpha = 0, -1$

H.S. $\Rightarrow A_1(0)^r + A_2(-1)^r = A_2(-1)^r$

P.S. $\Rightarrow$ let $a_r = (P_1 r + P_2)2^r$

$a_{r-1} = (P_1(r-1) + P_2)2^{r-1}$

Eq^n becomes,

$(P_1 r + P_2)2^r + [P_1(r-1) + P_2]2^{r-1} = 3r2^r$

$2(P_1 r + P_2) + (P_1 r - P_1 + P_2) = 6r$

$3P_1 r + 3P_2 - P_1 = 6r$

$3P_1 r = 6r \; ; \; 3P_2 - P_1 = 0$

$\boxed{P_1 = 2}$ $\boxed{P_2 = 2/3}$

$Ps = (2r + 2/3)2^r$

$\boxed{T.S. = Hs + Ps}$

Q] $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$

CE: $\alpha^r - 4\alpha^{r-1} + 4\alpha^{r-2} = 0$

$\alpha^2 - 4\alpha + 4 = 0$

$\alpha = 2 \,\&\, 2$

H.S. $= a_r = (A_1 r + A_2)2^r$

P.S. let $a_r = r^2(P_1 r + P_2)2^r$

$a_{r-1} = (r-1)^2(P_1(r-1) + P_2)2^{r-1}$

$a_{r-2} = (r-2)^2(P_1(r-2) + P_2)2^{r-2}$

Eq^n becomes,

$r^2(P_1 r + P_2)2^r - 4((r-1)^2(P_1(r-1) + P_2)2^{r-1} +$

$4((r-2)^2(P_1(r-2) + P_2)2^{r-2}$

$\Rightarrow r^3(P_1 \qquad) + r^2(\qquad)$

$+ r(\qquad) + 1(\qquad)$

$= \cancel{(r+1)2^r}$

$P_1 = \frac{1}{6} \,\&\, P_2 = 1$

$$P.S. = r^2\left(\frac{1}{6}r+1\right)2^r$$

$$\underline{\underline{T.S = H.S + P.S}}$$

Q) $a_r = a_{r-1} + 7$

$a_r - a_{r-1} = 7$

$CE = \alpha^r - \alpha^{r-1} = 0$

$\alpha^{r-1}(\alpha - 1) = 0$

$\alpha = 0, 1$

$H.S. = A_1(1)^r = A_1$

$P.S = a_r = P_r(1)^r = P_r$

$P_r = P(r-1) + 7$

$P_r - P(r-1) = 7$

$P_r - P_r + P = 7$

$P = 7$

$P.S. = a_r = 7r$

$T.S. = A_1 + 7r$

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Q) solve $a_r = 3a_{r-1} + 2$, $r \geq 1$
with B.C. $a_0 = 1$ using generating function.

Ans) let $A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$ ——

$$= \sum_{r=0}^{\infty} a_r z^r$$

Given eqn is $a_r = 3a_{r-1} + 2$

$$\sum_{r=1}^{\infty} a_r z^r = 3\sum_{r=1}^{\infty} a_{r-1} z^r + 2\sum_{r=1}^{\infty} z^r$$

$$\sum_{r=1}^{\infty} a_r z^r = A(z) - a_0$$

$$\sum_{r=1}^{\infty} a_{r-1} z^r = z\sum_{r=1}^{\infty} a_{r-1} z^{r-1} = zA(z)$$

$$\sum_{r=1}^{\infty} z^r = z + z^2 + z^3 + \text{——} = z(1 + z + z^2 \dots) = \frac{z}{1-z}$$

FORMULAES

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \frac{n(n-1)(n-2)}{3!} z^3$$

$$(1+z)^{-1} = 1 - z + \frac{(-1)(-1-1)}{2!} z^2 + \frac{(-1)(-1-1)(-1-2)}{3!} z^3$$

$$= 1 - z + z^2 - z^3 + \ldots$$

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots$$

$$(1-z)^{-2} = 1 + (-2)(-z) + \frac{(-2)(-2-1)}{2!}(-z)^2 + \ldots$$

$$(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \ldots$$

$$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \ldots$$

$$\Rightarrow A(z) - a_0 = 3z A(z) + \frac{2z}{1-z}$$

$$A(z)(1-3z) = 1 + \frac{2z}{1-z} = \frac{1-z+2z}{1-z} = \frac{1+z}{1-z}$$

$$A(z) = \frac{1+z}{(1-z)(1-3z)} = \frac{A}{(1-z)} + \frac{B}{(1-3z)} = \frac{-1}{(1-z)} + \frac{2}{(1-3z)}$$

$$A(z) = 2(1-3z)^{-1} - (1-z)^{-1}$$

$$\sum_{r=0}^{\infty} a_r z^r = 2(1 + 3z + 3^2z^2 + 3^3z^3 + \ldots) - (1 + z + z^2 + z^3 + \ldots)$$

$$= 2\sum_{r=0}^{\infty} 3^r z^r - \sum_{r=0}^{\infty} z^r$$

$$\Rightarrow \sum_{r=0}^{\infty} (2 \cdot 3^r - 1) z^r$$

$$\boxed{a_r = 2 \cdot 3^r - 1 \qquad r \geq 0}$$ Ans

Q) $a_r^2 - 2a_{r-1}^2 = 1$ with $a_0 = 2$

Sol^n : let $b_r = a_r^2$

eq^n becomes $b_r - 2b_{r-1} = 1$

Multiplying by z^r and summing from $r=1$ to ∞

Eqⁿ becomes,

$$\sum_{r=1}^{\infty} b_r z^r - 2\sum_{r=1}^{\infty} b_{r-1} z^r = \sum_{r=1}^{\infty} z^r$$

Let $B(z) = \sum_{r=0}^{\infty} b_r z^r = b_0 + b_1 z + b_2 z^2 + \ldots$

$$\sum_{r=1}^{\infty} b_r z^r = B(z) - b_0$$

$$\sum_{r=1}^{\infty} b_{r-1} z^r = z \sum_{r=1}^{\infty} b_{r-1} z^{r-1} = z(B(z))$$

$$\sum_{r=1}^{\infty} z^r = \frac{z}{1-z} = z(1-z)^{-1}$$

$$B(z) - b_0 - 2z(B(z)) = \frac{z}{1-z}$$

$$B(z)[1-2z] = b_0 + \frac{z}{1-z}$$

$$= 4 + \frac{z}{1-z}$$

$\boxed{a_0 = 2 \\ b_0 = 2^2 = 4}$

$$= \frac{4-3z}{1-z}$$

$$B(z) = \frac{4-3z}{(1-z)(1-2z)} = \frac{A}{(1-z)} + \frac{B}{(1-2z)}$$

$\Rightarrow A = -1$ & $B = 5$

$$B(z) = \frac{-1}{(1-z)} + \frac{5}{(1-2z)}$$

$$\sum_{r=0}^{\infty} b_r z^r = -1(1-z)^{-1} + 5(1-2z)^{-1}$$

$$= -(1+z+z^2+\ldots) + 5(1+2z+2^2z^2+2^3z^3+\ldots)$$

$$= -\sum_{r=0}^{\infty} z^r + 5\sum_{r=0}^{\infty} 2^r z^r$$

$$b_r \cancel{z^r} = (5\cdot 2^r - 1)\cancel{z^r}$$

$$b_r = 5\cdot 2^r - 1$$

$a_r = \sqrt{5\cdot 2^r - 1}$ Ans

$\boxed{b_r = a_r^2 \\ a_r = \sqrt{b_r}}$

Q) Find the generating function of 2, 4, 8, 16, 32 ____

Ans) $A(z) = a_0 + a_1 z + a_2 z^2 + \dots$

$= \sum_{r=0}^{\infty} a_r z^r$

$= 2 + 4z + 8z^2 + 16z^3 + 32z^4 + \dots$

$= 2(1 + 2z + 2^2 z^2 + 2^3 z^3 + \dots)$

$= 2(1-2z)^{-1} = \frac{2}{1-2z}$

Q) Find the generating function of $a_r = 2^r + 3^r$, $r \geq 0$.

$A(z) = \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} (2^r + 3^r) z^r$

$= \sum_{r=0} 2^r z^r + \sum_{r=0} 3^r z^r$

$= (1 + 2z + 2^2 z^2 + \dots) + (1 + 3z + 3^2 z^2 \dots)$

$= (1-2z)^{-1} + (1-3z)^{-1}$

$= \frac{1}{(1-2z)} + \frac{1}{(1-3z)} = \frac{2-5z}{(1-2z)(1-3z)}$

FUNCTIONS

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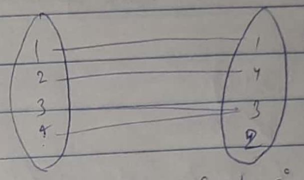
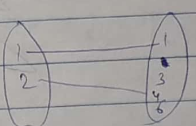
$f: A \to B$



$A = \{1, 2\}$ = domain

$B = \{1, 4, 3, 6\}$ = Co-domain

Range = $\{1, 4\}$



[Equal Functions]

Q) $A = \{1, 2\}$, $B = \{3, 6\}$

$f: A \to B$ s.t. $f(x) = x^2 + 2$ $\quad f(1) = 3$, $f(2) = 6$

$g: A \to B$ s.t. $g(x) = 3x$

$g(1) = 3$

$g(2) = 6$

Here

$f(1) = g(1)$

$f(2) = g(2) \Rightarrow f = g$

- Properties

① INTO function : $\boxed{\text{Codomain} \neq \text{Range}}$

$\{1, 2\}$ & $\{1, 3, 4, 6\}$

Range $= \{1, 4\}$ & Codomain $\{1, 3, 4, 6\}$

$\therefore$ INTO function $\Rightarrow \{1, 3, 4, 6\}$

② ONTO function : $\boxed{\text{Codomain} = \text{Range}}$

$A = \{1, 2, 3, 4\}$ $\quad f(x) = x^2$

$B = \{1, 4, 9, 16\}$

$\therefore$ Range = Codomain.

$\forall b \in B \; \exists \, a \in A$, s.t. $f(a) = b$

onto $\Rightarrow$ Surjection & one-one $\Rightarrow$ injection.

③ One-one function : $f(x) = f(y)$
$\Rightarrow x = y$

④ Many one function $\Rightarrow$ $f : I \to I$
$f(x) = x^2$
$f(-1) = 1$
$f(1) = 1$

$\therefore \{-1, 1\} \to \{1\}$

⑤ Bijective function : when f is both onto and one-one

one one $\Rightarrow f(x) = f(y) \Rightarrow x = y$

onto $\Rightarrow$ $\exists \, y \in B \; \exists \, x \in A$ s.t. $f(x) = y$.

⑥ Cardinally Equivalent Set :

$A \sim B$

$f : A \to B$

$\Rightarrow$ bijective.

Q) $f : R \to R$ s.t. $f(x) = ax+b$ where $a, b \in R$, then show that f is invertible.

Ans) f will be invertible if f is one-one and onto

one one $\Rightarrow$ $x_1, x_2 \in R$ (domain)

s.t. $f(x_1) = f(x_2)$

$ax_1 + b = ax_2 + b$

$\Rightarrow x_1 = x_2$

$\Rightarrow$ f is one-one.

onto : let $y \in R$ (Co-domain)

s.t. $\exists \frac{y-b}{a} \in R$ (domain)

$y = ax+b$

$x = \frac{y-b}{a}$

s.t. $f\left(\frac{y-b}{a}\right) = a\left(\frac{y-b}{a}\right) + b$

$= y$

$\Rightarrow$ f is onto.

$\Rightarrow$ f is invertible.

$$\boxed{f^{-1}(y) = \frac{y-b}{a}}$$

Theorem 1 :- The inverse map $f^{-1} : B \to A$ is also one-one onto.

Proof :- let $f : A \to B$ be one one onto then we have to show $f^{-1} : B \to A$ is one one onto

(1) f^{-1} is one one

let $b_1, b_2 \in B$

s.t. $f^{-1}(b_1) = f^{-1}(b_2)$ — (1)

Now, $b_1, b_2 \in B \Rightarrow \exists a_1, a_2 \in A$

s.t. $f(a_1) = b_1$, $f(a_2) = b_2$ (f is onto)

$f^{-1}(b_1) = a_1$ & $f^{-1}(b_2) = a_2$

In Eqn (1)

$a_1 = a_2$

$\Rightarrow f(a_1) = f(a_2)$ [f is function]

$\Rightarrow b_1 = b_2$

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2) f^{-1} is onto let $a \in A$

$f^{-1} : B \to A$ $\quad \exists\, b \in B$ s.t. $f(a) = b$ (f is function)

$f^{-1}(b) = a$

$\Rightarrow f^{-1}$ is onto

Theorem 2 If $f : A \to B$ & it is one-one onto then show that inverse of f is unique.

Proof: Let inverse is not unique.

Let $f : A \to B$

i.e. g, h s.t. $\quad g : B \to A$

$h : B \to A$

$\forall\, x \in B \quad g(x) = h(x)$

Now let $b \in B$ and $a_1, a_2 \in A$

$g(b) = a_1 \Rightarrow f(a_1) = b$

$h(b) = a_2 \Rightarrow f(a_2) = b$

$\therefore f(a_1) = f(a_2)$

$a_1 = a_2$ [f is one-one)

$g(b) = h(b)$

$\therefore g$ & h are equal function $\Rightarrow$

$\Rightarrow$ Inverse is always unique.

Q) Determine whether the fun^n are bijection where $f : R \to R$

① $f(x) : -3x + 4$

one one $\Rightarrow x_1, x_2 \in R$ (domain)

s.t. $f(x_1) = f(x_2)$

$-3x_1 + \cancel{4} = -3x_2 + \cancel{4}$

$\Rightarrow x_1 = x_2$

$x = \frac{y-4}{-3}$

onto

let $y \in R$ (codomain)

$\exists\, \frac{y-4}{-3} \in R$ (domain)

s.t. $f\left(\frac{y-4}{-3}\right) = -3\left(\frac{y-4}{-3}\right) + 4 = y$

$\therefore$ This is both

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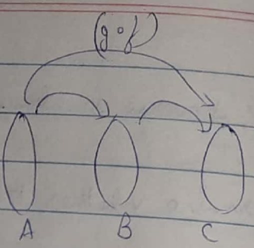
* **Composite function**



$f : A \to B$

$g : B \to C$

$g \circ f \Rightarrow A \to C$

$$g \circ (f(x)) = g[f(x)]$$

Properties:-

$f \circ g \neq g \circ f$ [it is not commutative]

$f \circ (g \circ h) = (f \circ g) \circ h$ [It is associative]

① Composite function is not ~~do~~ commutative. Here $f \circ g$ can't be defined as domain of one is not equal to codomain of another.

• $f \circ g$ will only be defined when $A = C$.

② Associative: $f \circ (g \circ h) = (f \circ g) \circ h$.

$f : C \to D$

$g : B \to C$

$h : A \to B$

$g \circ h \Rightarrow A \to C$

$f \circ (g \circ h) \Rightarrow A \to D$

$(f \circ g) \Rightarrow B \to D$

$(f \circ g) \circ h \Rightarrow A \to D.$

we will show $f \circ (g \circ h)$ and $(f \circ g) \circ h$ are equal function

Here $f \circ (g \circ h) \Rightarrow A \to D$

also $(f \circ g) \circ h \Rightarrow A \to D$

domain and codomain are same. Now we will show

$$[f \circ (g \circ h)]a = [(f \circ g) \circ h]a$$

let

$a \in A,\ b \in B,\ c \in D$ & $d \in D$

s.t. $h(a) = b$

$g(b) = c$

$f(c) = d$

$$(f \circ [g \circ h])a = [f[g \circ h]a] = f(g(h(a))) = f(g(b)) = f(c) = d$$

$$\left[(f\circ g)\circ h\right]a = f\circ g\,(h(a))$$
$$= f\circ g\,(b)$$
$$= f(g(b))$$
$$= f(c) = d.$$