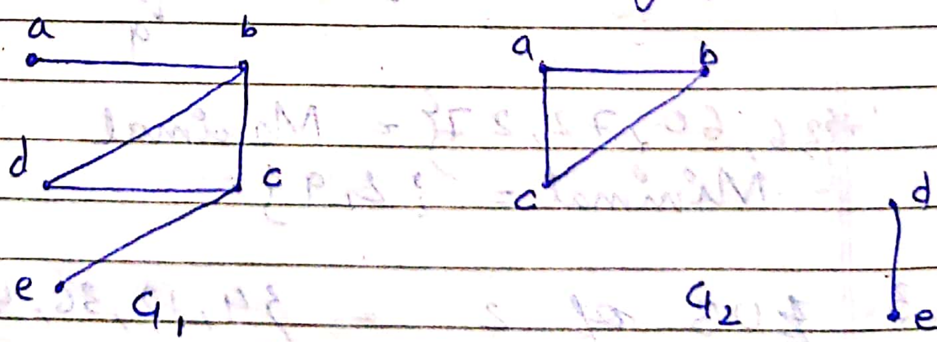
18/05/16

Graph Theory - II

* **Path, Circuits and Cycles:-**
Let u and v be two vertices in a graph G. A path from u to v in G is an alternating sequence of vertices & edges of G having the form

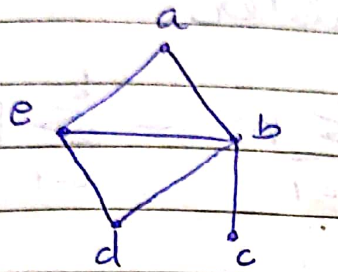
$u = v_1, e_1, v_2, e_2, \ldots\ldots, v_{n-1}, e_{n-1}, v_n, e_n, v_{n+1} = v$,
beginning with vertex u called initial vertex and ending with vertex v called the terminal vertex.
If the graph G is directed, the path is called a directed path.

* **Connected graph:-**
Let G be a graph. A vertex u is said to be connected to a vertex v if there is a u-v path in G.
A graph G is called a connected graph if for any two vertices u, v of G, there is a u-v path in G, otherwise it is called disconnected graph.

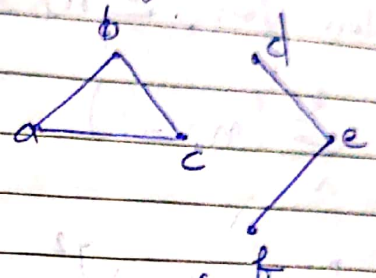


(connected graph) (Disconnected graph)

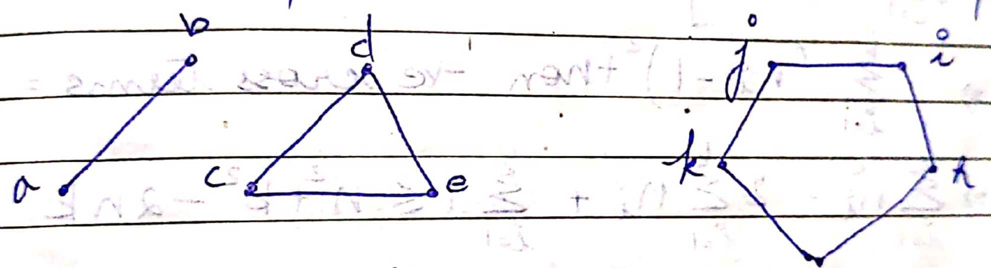
* Component of a graph:
(Maximal connected sub graph of G):-
A subgraph H of graph G is called a component of G if
i) any two vertices of H are connected in H.
ii) H is not properly contained in any connected subgraph of G.



G_1 (1 component)



G_2 (2 components)



G_3 (3 components)

Theorem:- A simple graph with n vertices & k components cannot have more than $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof:- Let n_i be the no.'s of vertices in i^{th} component, $1 \le i \le k$. Then $\sum_{i=1}^{k} n_i = n$ —(1)

A component with n_i vertices will have maximum no. of edges when it is complete. The no. of edges in a complete graph K_{n_i} is

$$\frac{1}{2} n_i(n_i-1) \qquad \text{--- (2)}$$

Hence the maximum no. of edges is

$$\frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1) = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right] \quad [\text{using } (i)] \qquad \text{--- (3)}$$

Now for $\sum_{i=1}^{k} n_i^2$, consider

$$\sum_{i=1}^{k}(n_i-1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n-k$$

$$\Rightarrow \left[\sum_{i=1}^{k}(n_i-1)\right]^2 = (n-k)^2 = n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 + \text{non -ve cross terms} = n^2+k^2-2n$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \le n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2n + k \le n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \le n^2+k^2-2nk+2n-k \qquad \text{--- (4)}$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \le n^2-(k-1)(2n-k)$$

Substituting (4) in (3), we get,

$$\frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1) \le \frac{1}{2}\left[n^2-(k-1)(2n-k)-n\right]$$

$$= \frac{1}{2}\left(n^2-2nk+k^2+n-k\right)$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

Hence Proved.

Theorem :- Show that a simple graph G with n vertices is connected if it has more than $\frac{1}{2}(n-1)(n-2)$ edges.

Proof :- A simple graph is connected, if it has only one component.
Let the graph is not connected and has two components. By theorem, the maximum no. of edges is
$\frac{1}{2}(n-2)(n-2+1) = \frac{1}{2}(n-1)(n-2)$

So if the no. of edges is more than $\frac{1}{2}(n-1)(n-2)$ the graph will get connected. Hence the result.

* Eulerian Path & Circuits :-

Euler Path:- A path is a connected graph G is called Euler path if it includes every edge exactly once. Since the path contains every edge exactly once, it is also called Euler trail.

Euler Circuit:- An euler path that is a circuit is called Euler ~~(or Eulerian)~~ ~~graph.~~ circuit i.e a closed Euler path is Euler circuit.

* Hamiltonian Path & Circuit:-

Hamiltonian Path :-
A path is a connected graph G is call Hamiltonian ~~cycle~~ path if it ~~contains~~ includes every vertex exactly once.

Hamiltonian cycle :-
A cycle in a connected graph G

is called Hamiltonian cycle if it contains each vertex of G exactly once except the starting & ending vertex, which are same (edges of cycle are distinct).

A graph G which has a Hamiltonian cycle is called a Hamiltonian graph.

Q. Give an example of connected graph that has:-

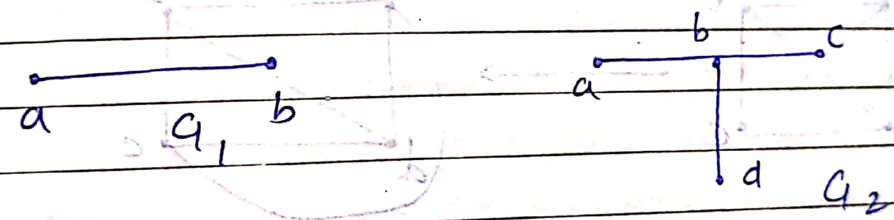
i) Neither an Euler circuit Nor a Hamiltonian cycle.

ii) A Euler circuit but no Harmiltonian Cycle

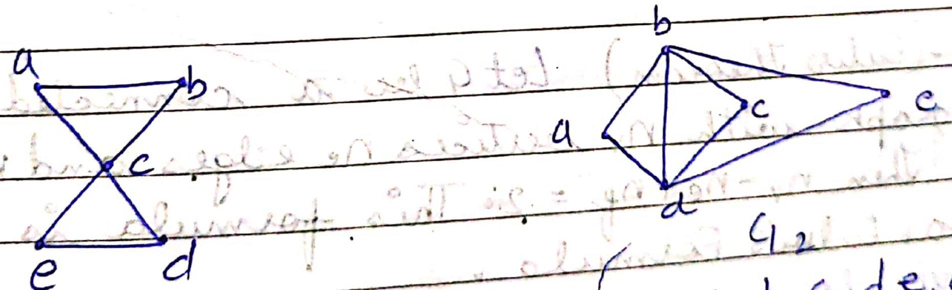
iii) A Hamiltonian cycle but no Euler circuit

iv) Both " " and " "

Ans. i)



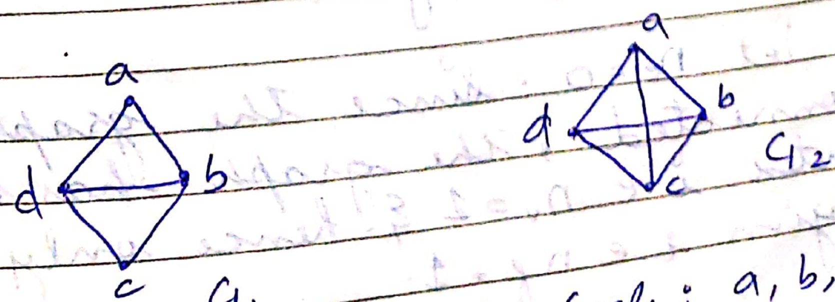
ii)



G_1

Euler circuit : c, a, b, c, d, e, c in G_1
and : a, b, d, c, b, e, d, a in G_2

iii)



Hamiltonian cycle : a, b, c, d

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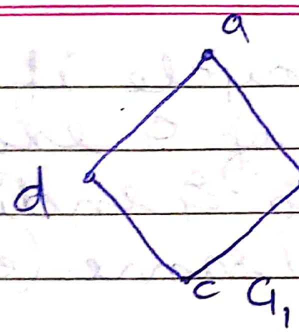
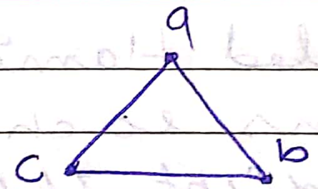
iii)



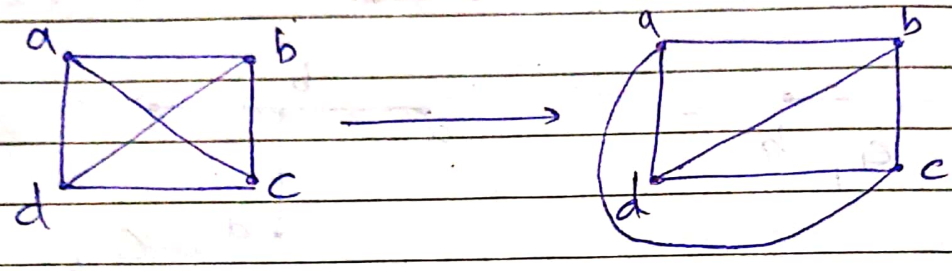


In G_1 Euler circuit : a, b, c, d, a
Hamiltonian cycle : a, b, c, d, a

In G_2 Euler circuit and Hamiltonian cycle are a, b, c, a.

* Planar graphs:-

A graph is called planar if it can be drawn in a plane such that no two edges intersect except at their common end vertices, if any.

a) 

Theorem:- (Euler Theorem) Let G be a connected planar graph with n_v vertices n_e edges and n_f faces. Then $n_v - n_e + n_f = 2$. This formula is known as Euler's Formula.

Proof:- We prove the theorem by induction on n_e (no. of edges)

Basis Step: let $n_e = 0$. Since the graph is connected, the graph has only one vertex i.e $n_v = 1$ & hence only one region i.e $n_f = 1$.

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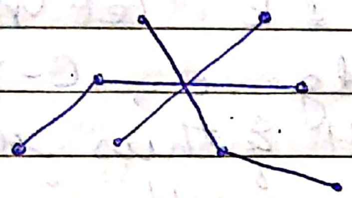
Then $n_v - n_e + n_f = 1 - 0 + 1 = 2$

Hence, the result is true.

Inductive Hypothesis :- let $n_e = k$, k be a +ve integer. Assume that $n_v - n_e + n_f = 2$ for any connected planar graph with $n_e = k$

Inductive Step :- let G be a connected planar graph with $n_e = k+1$ edges & $n_v = t+1$ vertices



Suppose G has no cycles, than G has no interior region & has only exterior region $\therefore$ $n_f = 1$, we now show that G contains a vertex of degree 1. choose vertex v in G

if $\deg(v) = 1$, we are done

if $\deg(v) \neq 1 \Rightarrow \deg(v) > 1$. let v_1 be the vertex adjacent to v. Because G has no cycles, v_1 is diff. from v. If $\deg(v_2) \neq 1$, we find a vertex v_3 adjacent to v_1 diff from v, v_1 & v_2. Since G has a finite no. of vertices, it follows there is a vertex u of degree 1. we now delete this vertex u from the graph G & form a new connected planar graph H with k edges & t vertices.

By the inductive hypothesis, for a graph H with $n_e = k$ edges

$$n_v - n_e + n_f = 2 \text{ in } H$$
$$\Rightarrow t - k + n_f = 2$$
$$\Rightarrow (t+1) - (k+1) + n_f = 2$$
$$\Rightarrow n_v - n_e + n_f = 2 \text{ in } G$$

Now, suppose that G has a cycle C. Let e be an edge in C. Construct a new graph H from G by deleting the edge e in C. i.e $H = G - \{e\}$. H is still connected planar graph with $n_v = t+1$, $n_e = k$. Let $n_f = m$ in G. After deleting edge e, $C - \{e\}$ is not a cycle in H & thus will not form a boundary in H. $\therefore n_f = m-1$ in H

By inductive hypothesis for H
$$n_v - n_e + n_f = 2 \text{ in } H$$
$$\Rightarrow (t+1) - k + (m-1) = 2$$
$$\Rightarrow (t+1) - (k+1) + m = 2$$
$$\Rightarrow n_v - n_e + n_f = 2 \text{ in } G$$

Hence the result follows from induction

Theorem :- If a connected graph G is Eulerian then every vertex of G has even degree.

Proof :- Let G be an Eulerian graph. Then G has an Euler circuit which begins & ends at u (say) and is of the form
$$u = v_1, e_1, v_2, e_2, \ldots, v_{i-1}, e_{i-1}, v_i, \ldots, v_n, e_n, v_{n+1} = u$$
when we travel along the circuit,

then each time, we visit a vertex v_i, we use two edges e_{i-1} & e_i (one in & one out). This is also true for starting vertex u because we also ends at u. Since an Euler circuit crosses every edge once, each occurrence of v in circuit represents a contribution of 2 to its degree.

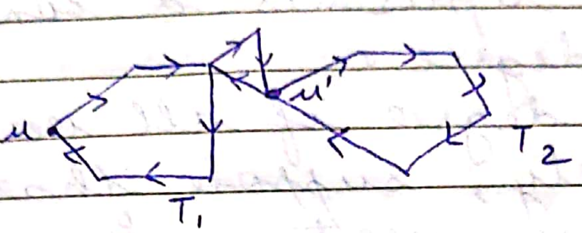
Thus, degree of all vertices is even. Conversely, suppose that G is connected and every vertex is even. We construct a circuit T_1 at any edge e beginning with u. T_1 by adding an edge after the other.

If T_1 is not closed at any step i.e the end vertex $v \neq u$, then only an odd no. of the edges incident on v appear in T_1. But v is of even degree hence we can extend T_1 by another edge incident on v. Thus, we can continue to extend T_1 until T_1 returns to its initial vertex u. If T_1 includes all the edges of G then T_1 is an Euler circuit.

Suppose T_1 does not include all edges of G. Consider the graph H obtained by deleting all edges of T_1 from G. H may not be connected but each vertex of H has even degree since each vertex in T_1 contains even no. of edges incident on it.

Since G is connected, there is

an edge e' of H which has an endpoint u' in T_1. We construct a trail T_2 in H beginning at u' & using e'. Since all vertices of H are of even degree, we continue to extent T_2 until T_2 returns to u'.



Clearly T_2 can be put in T_1 to form a larger closed trail (circuit) in G. We continue this process until all the edges of G are used. We finally obtain an Euler circuit in G. So G is Eulerian graph.

Non-Planar graph:-

A graph which cannot be drawn without intersecting its edges is called non-planar graph.

Regions OF A graph:-

Every planar representation of a graph divides the plane into diff. region called phases of the graph.

Property OF a region:-

- A region is known by set of edges & vertices constructing its boundary
- Region is not define in non-planar graph.

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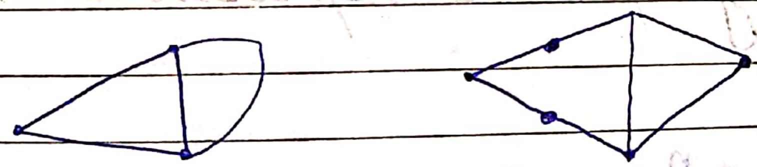
• Region is a property of specific plane representation of a graph.

Degree OF Region :-
Let R be a region in a planar representation of graph. the degree of region r is represented by deg(R) is the no. of edges traversed in a shortest closed path about the boundary of R.

Homeomorphic Graph :-
2 graphs are said to be homeomorphic graph if one can be obtained from the other by creation of edges in series (i.e insertion of vertices of degree 2) the following is eg. of homeomorphic graph :-

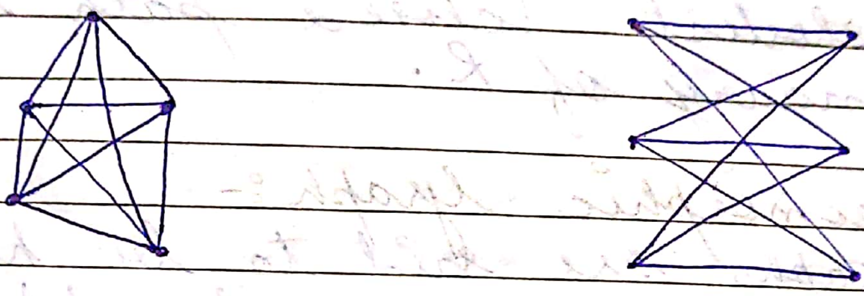


Kuratowski's Graph :-
complete Bipartite graph
The complete graph K_5 & $K_{3,3}$ are non-planar graph & these are known as kuratowski's graph 1st & 2nd graph resp. These 2 graphs are very imp. because these are used to find whether given graph is planar or not by using the property that if 2 graphs are homeomorphic than they are simultaneously planar or

non-planar.

Kuratowski's Theorem :-
A graph is non-planar if & only if it contains a sub graph i.e homeomorphic to K_5 or $K_{3,3}$

K_5 $K_{3,3}$



Q. Suppose that a connected planar graph has 30 vertices each of degree 3. In how many regions the plane is divided by planar representation of this graph.

$n_v - n_e + n_f = 2$

$30 - 3$

$\Sigma \deg v_i = 2e$

$30 \times 3 = 2e$

$e = 45$

$30 - 45 + n_f = 2$

$\boxed{n_f = 17}$

Q. Suppose that connected planar graph has 30 edges & its planner rep divides the plane into 20 region how many vertices this graph has.

$$n_v - 30 + 20 = 2$$

$$\boxed{n_v = 12}$$

Q. Show that complete bipartite graph $K_{3,3}$ is not planar graph.

Ans. We shall find the soln to this problem by method of contradiction.
Let $K_{3,3}$ be a planar graph
By Eulers theorem :-

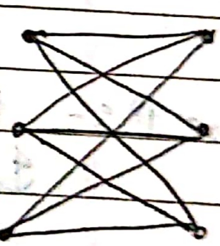
$$n_v - n_e + n_f = 2$$

here $n_v = 6$

$n_e = 9$

$$6 - 9 + n_f = 2$$

$$\boxed{n_f = 5}$$



here $K_{3,3}$ consists cycle of length 4
∴ the total no. of appearances of edges in boundary of 5 faces is $\geq$ to 5·4 but in planar rep. a edge may appear in atmost 2 diff faces.
Thus the total no. of appearances of 9 edges ≤ 18. Hence, a contradiction
∴ $K_{3,3}$ is Non planar graph.

Q. Let G be a connected simple planar graph with $n_v \geq 3$ & n_e edges than prove that $n_e \leq 3n_v - 6$.

Ans. Consider a planar representation of a graph :-

Case I :- When $n_v = 3$

$n_e \leq 3 \cdot 3 - 6$

$\boxed{9 - 6 = 3}$

When $n_v = 3 \rightarrow$ G is a simple graph $\therefore$ max^n no. of edges $= 3$, $n_e \leq 3$.

Hence $3n_v - 6 = 3 \cdot 3 - 6 = 3$

Hence Theorem is true.

Case II :- $n_v > 3$.

If G does not contain any cycle than

$n_e = n_v - 1$

$$3n_v - 6 = (n_v - 1) + (n_v - 2) + (n_v - 3)$$

$$3n_v - 6 = n_e + (n_v - 2) + (n_v - 3)$$

$$\boxed{3n_v - 6 \geq n_e}$$

Further we suppose that $n_v > 3$ & G consists of a cycle with min no. of 3 edges. $\therefore$ the no. of edges in the boundary of a face is >= 3 & hence we obtain

$$\boxed{3n_f < 2n_e}$$

than by Eulers theorem :-

$$n_v - n_e + n_f = 2$$

$$n_f = 2 - n_v + n_e$$

$$3(2 - n_v + n_e) \leq 2n_e$$

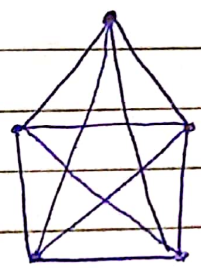
$$6 - 3n_v + 3n_e \leq 2n_e$$

$$-n_e \geq 6 - 3n_v$$

$$\boxed{3n_v - 6 \geq n_e}$$

Q. Show that graph K_5 is not a planar graph.

Ans. Here K_5 is a simple graph



Here $n_v = 5$ $\therefore$ max no. of edges =

$$\frac{n_v(n_v - 1)}{2}$$

$$= \frac{5(5-1)}{2}$$

$$= 10$$

We suppose that K_5 is planar

$$\Rightarrow n_e \leq 3n_v - 6$$

$$10 \leq 3 \cdot 5 - 6$$

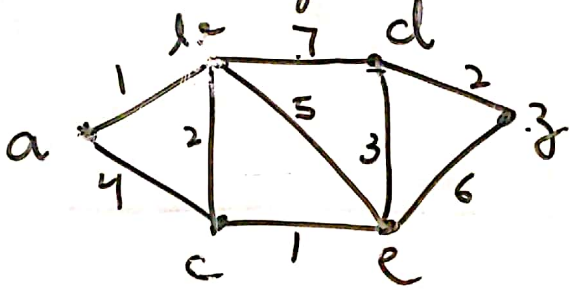
$$10 \leq 15 - 6$$

$$10 \leq 9$$

Not possible

$\Rightarrow K_5$ is Non planar

Dijkstra's algo [For shortest path]



Select a

$L(b) = 1$; $P(a, b)$
$L(c) = 4$; $P(a, c)$
$L(d) = \infty$; -
$L(e) = \infty$; -
$L(z) = \infty$; -

Select b

$L(c) = Min[L(c), L(b) + w(b, c)] = Min(4, 3) = 3$; $P(a, b, c)$
$L(d) = Min[L(d), L(b) + w(b, d)] = Min(\infty, 8) = 8$; $P(a, b, d)$
$L(e) = Min[L(e), L(b) + w(b, e)] = Min(\infty, 6) = 6$; $P(a, b, e)$
$L(z) = Min[L(z), L(b) + w(b, z)] = Min(\infty, \infty) = \infty$; -

Select c

$L(d) = Min[L(d), L(c) + w(c, d)] = Min(8, \infty) = 8$; $P(a, b, d)$
$L(e) = Min[L(e), L(c) + w(c, e)] = Min(6, 4) = 4$; $P(a, b, c, e)$
$L(z) = Min[L(z), L(c) + w(c, z)] = Min(\infty, \infty) = \infty$; -

Select e

$L(d) = Min[L(d), L(e) + w(d, e)] = Min(8, 7) = 7$; $P(a, b, c, e, d)$
$L(z) = min[L(z), L(e) + w(z, e)] = Min(\infty, 10) = 10$; $P(a, b, c, e, z)$

Select d.

$L(z) = Min[L(z), L(d) + w(d, z)]$
$= min(10, 7+2) = 9$

$P(a, b, c, e, d, z)$

Coloring of graph

§ 7.34. The coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

§ 7.35. Chromatic Number. *Definition :* **[Raj. B.E. III, 03]**

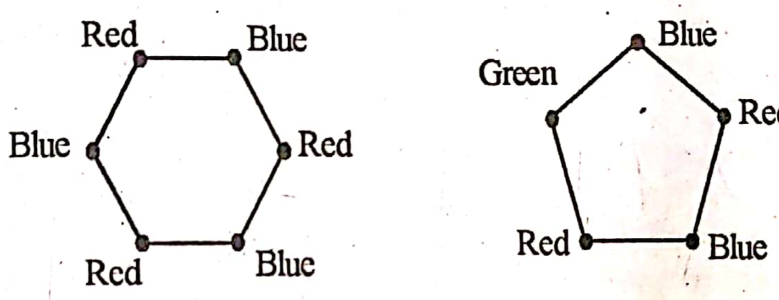
The least number of colors required for proper coloring of a graph G is called its Chromatic Number.

Notation : The chromatic number of a graph G is denoted by $\chi(G)$.

If $\chi(G) = k$, then the graph is called ***k-Chromatic.***

From the definition of chromatic number it is clear that

1. Chromatic number of null graph is 1.
2. Chromatic number of complete graph K_n of n vertices is n *i.e.* $\chi(k_n) = n$.
3. Chromatic number of a graph having one or more edges is alteast two.
4. Chromatic number of a graph having a triangle is at least three.
5. If a graph is a circuit with n vertices then
   (i) It is 2-chromatic if n is *even*. (ii) It is 3-chromatic if n is *odd.*



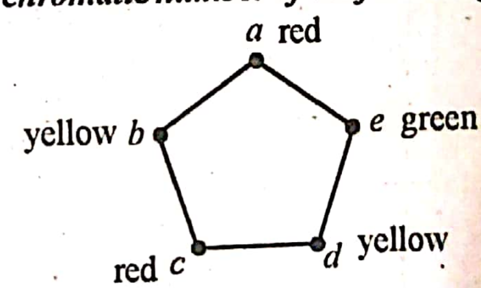
(a) Circuit with ***even*** number of vertices (b) Circuit with ***odd*** number of vertices

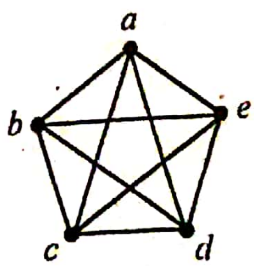
Fig. 7.61

6. If a graph is a circuit and it is given that its chromatic number is even then it has even number of vertices (edges). If a circuit graph is 3-chromatic then number of vertices in the graph is odd.

Illustrative Examples

Ex.1. ***Determine the chromatic number of the following graph C_5 :***

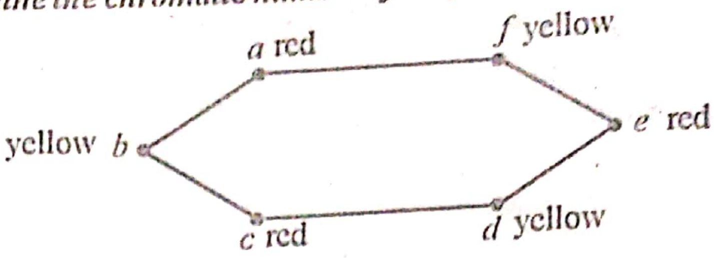




Sol. Start coloring from the vertex a and assign it red. Then we assign b yellow, e red, d yellow. Since the vertex e is adjacent to a and d, so we assign it a different new color say green. Thus we need 3 colors.

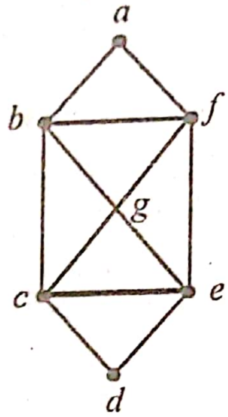
Therefore chromatic number of C_5 is 3. **Ans.**

Ex.2. ***Determine the chromatic number of the following graph :***

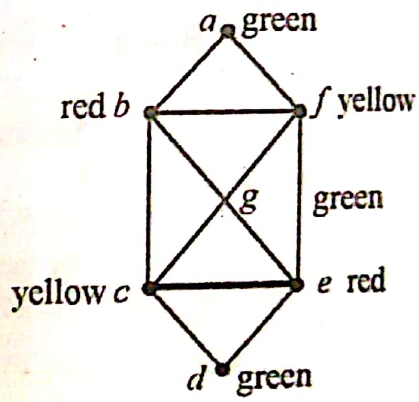


Sol. Start coloring from the vertex a and assign it the color (say red). Then we assign b yellow, c red, d yellow, e red and f yellow. Thus we need only two colors. Therefore chromatic number of C_6 is 2. **Ans.**

Ex. 3. ***Find the chromatic number of the following graph :***

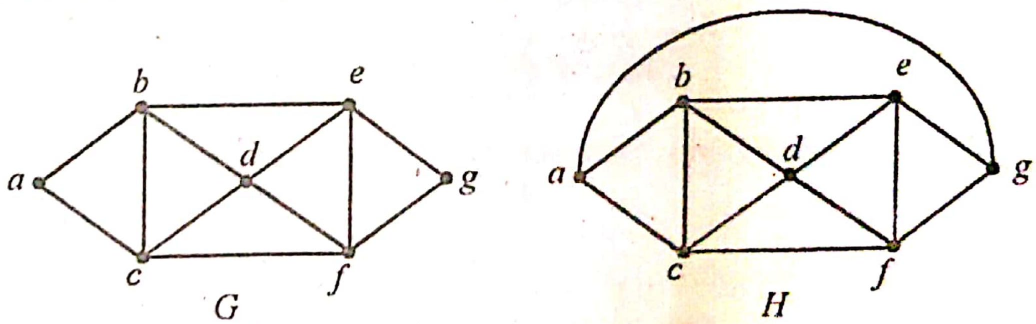


Sol. Suppose we assign green to vertex a, red to vertex b and yellow to vertex f. Then the vertex g can be colored green as it is adjacent to vertices b and f. Further the vertex c can be colored yellow, as it is adjacent to vertices b and g.



Simiarly e can be colored red and d can be colored green. This completes the coloring of the given graph with exactly 3 colors.

Ex.4. ***What are the chromatic numbers of the graphs G and H as shown in following figure.***

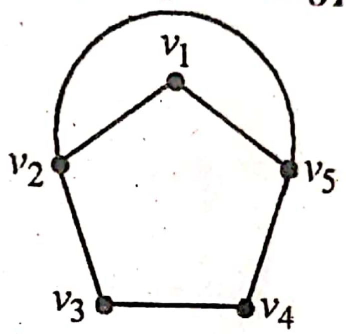


Sol. The chromatic number of G is 3 (see above Ex. 3).

Consider the graph H. Assign the colors as for G except for the vertex g, since g is adjacent to a, e, f. So we need a fourth color.

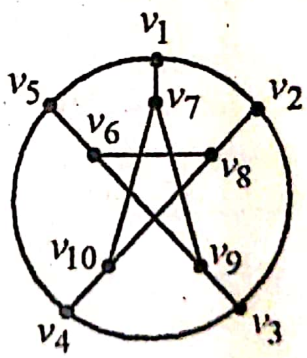
Hence chromatic number of H is 4.

Ex.5. ***Find the chromatic number of the following planar graph:***



Sol. First, assign a color (say red) to v_1. Then v_2 and v_5 can be assigned colours (say blue and green) respectively. The vertex v_3 can be assigned either red or green and v_5 blue or red. Thus the chromatic number of the given graph is 3. **Ans.**

Ex.6. ***Find the chromatic number of the following graph :***



Sol. Start assigning a color (say red) to the vertex v_1. Then the vertices v_2, v_5 and v_7 can be assigned another color (say blue). Next vertices v_4, v_6, v_8 and v_9 can be assigned again red color. Vertices v_{10}, v_3 and v_{10} can neither be assigned red nor blue colors. Therefore, assign v_3

Thus chromatic number of the given graph is 3. **Ans.**

and v_{10} a third new colour (say green).

Ex. 7. ***If G is a connected planar graph with v vertices, e edges, r regions with $r \geq 3$, then $e \leq 3v - 6$.***

... $2e = \sum \deg(r)$, because each edge occurs on the boundary of a region exactly