GROUP THEORY

Maths

15/April/20

Commutative Property : $a * b = b * a$

Associative Property : $a * (b * c) = (a * b) * c$.

Closure Property : $a * b \in S$
$a \in S$ & $b \in S$.

Extensive of Identity : $\boxed{a * E = a}$
$E \rightarrow$ Identity Element.
Addition Identity = 0 & Multiplicative = 1

Inverse Element : $\boxed{\begin{array}{l} 1 \rightarrow -1 \\ 2 \rightarrow 2 \end{array}}$ $\therefore$ $\boxed{a * a^{-1} = E}$

Addition Inverse : $1 \rightarrow -1$
Multiplication Inverse : $2 \rightarrow 1/2$

Algebraic Structure :

1] SEMI-GROUP :
$(G, *)$ is a semi group
if 1) Closure Property : $a, b \in G \Rightarrow a * b \in G$
2) Associativity : $a * (b * c) = (a * b) * c$

2] MonoID : $(G, *) \rightarrow$ semigroup
① Closure Property
② Associativity
③ Ex~~tenis~~istence of Identity Element.

Q) Group
(a) Closure
(b) Associativity
(c) Existence of Identity
(d) Existence of Inverse

4 Q) Abelian Group
(a) Closure
(b) Associativity
(c) Existence of Identity
(d) Existence of Inverse
(e) Commutative Property

NOTE: Matrix Multiplication is not commutative property

Q) Show that $G = \{-1, 1, i, -i\}$ is abelian group for matrix multiplication.

Finite Group :
Infinite Group :

	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	+1
-i	-i	i	+1	-1

(a) Satisfies closure Property
(b) Associativity Property.
(c) 1 is identity
(d) Inverse element exists
(e) Commutative Property

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Q) $Z_5 = \{0, 1, 2, 3, 4\}$ is abelian group

$$a +_5 b = \begin{cases} a+b & ; a+b < 5 \\ a+b-5 & ; a+b \geq 5 \end{cases}$$

$$3 +_5 4 = 7 - 5 = 2$$

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	1
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(a) Closure Property is satisfied.
(b) Associative Property is satisfied.
(c) 0 is the identity element
(d) It is inverse
(e) $a+b = b+a$ ∴ It is Commutative Property.

Q) Show that $a * b = \frac{ab}{2}$, $\forall a, b \in Q^+$ is an abelian group?

Ans) (1) closure property : let $a, b \in Q^+$
$$a * b = \frac{ab}{2} \in Q^+$$

(2) Associativity :
$$a, b, c \in Q^+$$
$$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{abc}{4}$$
$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{abc}{4}$$

(3) Existence of Identity :
$$e = 2 \in Q^+$$
s.t. $a * 2 = \frac{a\cancel{2}}{\cancel{2}} = a$

$a * e = a$
$\frac{\cancel{a}e}{2} = \cancel{a}$
$\boxed{e = 2}$

(4) Existence of Inverse :
$\forall a \in Q^+$
$\exists \frac{4}{a} \in Q^+$
s.t. $a \times \frac{4}{4} = \frac{a \, 4/2}{2} = a$

$a * b = e$
$\frac{ab}{2} = 2$
$b = 4/a$

(5) Commutativity
$a * b = \frac{ab}{2}$ & $b * a = \frac{ba}{2}$

Q) S is a set of real no. except -1. Then show that $(S, \circ)$ is a group where $\circ$ is operation defined as
$a \circ b = a + b + ab$, $a, b \in S$.
$1 \circ 2 = 1 + 2 + 1.2 = 5$
$5 \circ 10 = 5 + 10 + 5.10 = 65$

Ans ① Closure : $a, b \in S \; \{a, b \neq -1\}$

$a \circ b \neq -1$

Let $a \circ b = -1 \Rightarrow a + b + ab = -1$

$(1 + a) + b(1 + a) = 0$

$(1 + a) + (b + 1) = 0$

$a = -1 \;\&\; b = -1$

contradiction

$\therefore \; a \circ b \neq -1$

② $a \circ b = a + b + ab$

$a \circ (b \circ c) = a \circ (b + c + bc)$
$= a + b + c + bc + ab + ac + abc$

$(a \circ b) \circ c = (a + b + ab) \circ c$
$= a + b + c + ab + c(a + b + ab)$
$= a + b + c + ab + bc + ac + abc$

$\therefore$ it is satisfied.

③ $(a \circ e) = a$

$\cancel{a} + e + ae = \cancel{a}$

$e(1 + a) = 0$

$\boxed{e = 0}$

existence of identity $e = 0 \in Q^+$ is identity element

$a \circ 0 = a + 0 + a.0 = a$

④ Existence of Inverse : $a \circ b = e = 0$

$a + b + ab = 0$

$b(1 + a) = 0$

$b = \dfrac{-a}{1 + a}$

$\forall \; a \in S$

$\exists \; \dfrac{-a}{1 + a} \in S$

is inverse element.

$a \circ \left(\dfrac{-a}{1 + a}\right) = a + \left(\dfrac{-a}{1 + a}\right) + a\left(\dfrac{-a}{1 + a}\right)$

$= \dfrac{a - a^2 - a - a^2}{1 + a} = 0$

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Ex: 5

Q) $\{f_1, f_2, \ldots f_6\}$ $(g \circ f)$ $\boxed{f \circ g \neq g \circ f}$

$f_1 = z$; $f_2 = \dfrac{1}{z}$; $f_3 = 1 - z$; $f_4 = \dfrac{z}{z - 1}$

$f_5 = \dfrac{1}{1 - z}$; $f_6 = \dfrac{z - 1}{z}$

$(f_1 \circ f_2)_z = f_1[f_2(z)]$
$= f_1\left(\dfrac{1}{z}\right) = z$

$f_3 \circ f_4 = f_3[f_4(z)]$
$=$

	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_6	f_1	f_5	f_4	f_2
f_4	f_4	f_5	f_6	f_1	f_2	f_3
f_5	f_5	f_4	f_2	f_3	f_6	f_1
f_6	f_6	f_3	f_4	f_2	f_1	f_5

① Closure
② Associative
③ Identity : f_1
④ Inverse : exists
⑤ It is not commutative.

$\therefore$ It is not an abelian group, but it is a group.

Ex: 6) $G = \{(a, b), a, b \in R, a \neq 0\}$ is non abelian group for $\circ$ defined as $(a, b) \circ (c, d) = (ac, bc + d)$

Ans) ① $(a, b), (c, d) \in G$

$a, b, c, d \in R \; (a, b, c, d \neq 0)$

$(ac, bc + d) \in R$

$(ac, bc + d) \in G, \; (ac, bc + d \neq 0)$

$\therefore$ It is Closure.

② $(a, b) \circ [(c, d) \circ (e, f)] = [(a, b) \circ (c, d)] \circ (e, f)$

$\Rightarrow (a, b) \circ [ce, de + f] = [ac, bc + d] \circ (e, f)$

$\Rightarrow [ace, bce + de + f] = (ace, bce + de + f)$

$\therefore$ It is associative.

③ $ae = a$

$(a, b) \circ (e_1, e_2) = (a, b)$

$ae_1 = a \qquad be_1 + e_2 = b$

$\boxed{e_1 = 1} \qquad e_2 = b - b$

$\boxed{e_2 = 0}$

$\therefore (1, 0) = (e_1, e_2)$

$\exists \; (1, 0) \in G$

s.t. $(a, b) \circ (1, 0) = (a, b.1 + 0)$

$(1, 0) \circ (a, b) = (a, b)$

4. Existence of inverse : $(a,b)\circ(c,d) = (1,0)$

$ac, bc+d = 1, 0$

$ac = 1, \quad bc + d = 0$

$c = \frac{1}{a}, \quad \frac{b}{a} + d = 0$

$d \Rightarrow -b/a$

$\forall (a,b) \in G \Rightarrow \exists \left(\frac{1}{a}, \frac{-b}{a}\right) \in G.$

$(a,b)\circ\left(\frac{1}{a}, \frac{-b}{a}\right) = a\cdot\frac{1}{a}, \; b\times\frac{1}{a} - \frac{b}{a} \Rightarrow (1,0)$

Becoz composite fun^n is not commutative, therefore it is a non-abelian group.

Properties :

Property ① Identity element is always unique.

Proof :

let G be a group

let e_1, e_2 be two identity element

let $a \in G$

If e_1 is identity $\Rightarrow ae_1 = a$

If e_2 is identity $\Rightarrow ae_2 = a$

$\Rightarrow ae_1 = ae_2 \Rightarrow e_1 = e_2$

$\therefore$ Identity is always unique.

Property ② Inverse element of an element is always unique.

proof

Let G be a group & let $a \in G$

Also let $b, c \in G$

such that $a^{-1} = b, \; a^{-1} = c.$

i.e; a have two inverse element

Now $a^{-1} = b \Rightarrow ab = e = ba$

$a^{-1} = c \Rightarrow ac = e = ca$

$(ba = e) \times c$

$c\,ba = ce = c$

$c\,ab = c$ [associative Property $ab = ba$]

$eb = c$ [associative $eb = be$]

$be = c$

$\boxed{b = c}$

$\therefore$ The inverse element of an element is always unique.

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① Identity is unique.

② Inverse of each element is unique.

Property ③ If G is group & $a, b \in G$,

(i) $(a^{-1})^{-1} = a$

(ii) $(ab)^{-1} = b^{-1}a^{-1}.$

(i) $a \in G$ and G is group

$\Rightarrow a^{-1} \in G$ [Existence of inverse]

Such that $aa^{-1} = e = a^{-1}a$

$a^{-1}a = e = aa^{-1}$

$\boxed{(a^{-1})^{-1} = a}$

(ii) $(ab)^{-1} = b^{-1}a^{-1}$

$\Rightarrow (ab)(b^{-1}a^{-1}) = e$

$\Rightarrow (b^{-1}a^{-1})(ab) = e$

$a \in G, \; b \in G \Rightarrow a^{-1}, b^{-1} \in G$

a, a^{-1}, b, b^{-1}, ab all are elements of G.

$(ab).(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$ [Associative Law]

$= a(e)a^{-1}$ [Existence of Inverse]

$= aa^{-1}$ [$a \times a^{-1} = e$]

$= e$ (Identity)

$(b^{-1}a^{-1}).(ab) = b^{-1}(a^{-1}a)b$

$= b^{-1}eb$

$= b^{-1}b$

$= e$

$\Rightarrow (ab)(b^{-1}a^{-1}) = e = (b^{-1}a^{-1})(ab)$

Property 4 : Cancellation Laws
- Left Cancellation law
- Right Cancellation law

$\Rightarrow a, b, c \in G$ and G is group.

then (i) $ab = ac \Rightarrow b = c$ [Left cancellation law]

(ii) $ba = ca \Rightarrow b = c$ [Right cancellation law]

Proof : $a, b, c \in G$

& G is group. $\Rightarrow a^{-1} \in G$

$ab = ac$

$a^{-1}(ab) = a^{-1}(ac)$ [associative]

$(a^{-1}a)b = (a^{-1}a)c$ [associative]

$eb = ec$ [inverse]

$b = c$ [Identity]

(ii) $ba = ca$

$(ba)a^{-1} = (ca)a^{-1}$ [Associative]

$b(aa^{-1}) = c(aa^{-1})$ [Associative]

$be = ce$ [inverse]

$b = c$ [Identity]

definitions :- order of group and order of element.

$(G, *)$

$\rightarrow O(G) \rightarrow$ order of group

$G = \{1, -1, i, -i\} \Rightarrow \boxed{e = 1}$

n is the order of a $\Rightarrow \boxed{a^n = e}$

$O(G) = 4$

$o(1) = (1)^1 = 1$

$o(-1) = (-1)^2 = 1 \Rightarrow e = o(-1) = 2$

$o(i) = (i)^4 = 1 \Rightarrow o(i) = 4$

$o(-i) = (-i)^4 = 1 \Rightarrow o(-i) = 4$

(i) Identity Ka order will always be 1.

(ii) $\boxed{o(\text{element}) \le O(\text{Group})}$

(iii) Order of $a = O(a) = n \Rightarrow a^n = e$.

also $a^m = e$

then m is a multiple of n.

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Subgroup

If $(G, *)$ is a group and $H \subseteq G$. Then H is a subgroup of G if

(i) H is closed for composition $*$ $a, b \in H \Rightarrow a * b \in H$.

(ii) H is a group for induced composition.

(i) Identity of G & H are same

(ii) inverse of each element s

(iii) if $O(a) \Rightarrow H$ is G and $O(a) \Rightarrow H$

Types of Subgroup → Proper Sub Group
→ Im Proper Sub Group or Trivial Subgroup

$(G, *)$ is a group

(i) G itself

(ii) $\{e\}$

- Proper Sub Group

$G = \{Z, +\}$

$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$e = 0$,

$H = \{2n, n \in I^+\}$

$n = \{0, 2, 4, 6, 8, \ldots\}$

$\boxed{H \subset G}$

(i) Closure

(ii) Identity Element $= 0$

(iii) Inverse

(iv) Associative

$\therefore \boxed{H \subseteq G}$

So H is a proper Sub Group.

A non void subset H of group $(G, *)$ is a subgroup if and only if $a, b \in H \Rightarrow ab^{-1} \in H$

H is a subgroup of $G \Leftrightarrow (a, b^{-1} \in H \Rightarrow ab^{-1} \in H)$

Solⁿ :- let H is a subgroup of $(G, *)$

$\Rightarrow H$ is a group.

Now, $a, b \in H \Rightarrow b^{-1} \in H$ [existence of Inverse]

$a \in H, b^{-1} \in H \Rightarrow ab^{-1} \in H$ [closure Property]

$\Rightarrow a, b \in H \Rightarrow ab^{-1} \in H$

let condition is true,

$\Rightarrow a, b \in H \Rightarrow ab^{-1} \in H$

Now, we show H is a subgroup of G.

$\Rightarrow H$ is a group itself.

$a \in H, a \in H \Rightarrow aa^{-1} \in H$ [By condition]

$e \in H$ [$\therefore$ identity exist in H]

$e \in H, a \in H \Rightarrow ea^{-1} \in H$ [By condition]

$\therefore a^{-1} \in H$ [Existence of Inverse]

Let $a, b \in H \Rightarrow b \in H$

$b^{-1} \in H$

$\Rightarrow a, b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H$

$\Rightarrow ab \in H$ (closure Property)

And it is always associative as $H \subset G$.

$\Rightarrow H$ is a group itself.

Theorem:
The intersection of any two subgroup of G is again a subgroup.

Solⁿ: Let $(G,*)$ be a group & H_1, H_2 be its two subgroup.

we have to show $H_1 \cap H_2$ is also a subgroup of G.

(i) $a, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in (H_1 \cap H_2)$

(ii) $H_1 \cap H_2$ is non void.

(ii) $\Rightarrow e \in H_1, e \in H_2$ [Where e is identity of G, H_1, H_2]

$\Rightarrow e \in H_1 \cap H_2$

$\Rightarrow H_1 \cap H_2 \neq 0$

(i) $\Rightarrow a, b \in H_1 \cap H_2 \Rightarrow a, b \in H_1$ and $a, b \in H_2$.

$\Rightarrow ab^{-1} \in H_1$ and $ab^{-1} H_2$ (Condition)

$\Rightarrow ab^{-1} \in H_1 \cap H_2$

$\therefore H_1 \cap H_2$ is a subgroup of G.

Theorem
The union of any two subgroup of G is not necessarily subgroup.

Proof: Let $G = \{Z, +\}$ is a group.

$H_1 = \{2n, n \in Z\}$

$H_2 = \{3n, n \in Z\}$

H_1 & H_2 are two subgroup of G.

$H_1 \cup H_2 = \{-6, -4, -3, 2, 0, 2, 3, 4, 6\}$

$2, 3 \in H_1 \cup H_2$

$2 + 3 = 5 \notin H_1 \cup H_2 \Rightarrow$ it does not satisfy closure property. Here it is not a subgroup of G.

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Subgroup $\Leftrightarrow a, b \in H \Rightarrow ab^{-1} \in H$

G

$H \subset G$

subgroup

(i) $H \neq \phi$

(ii) H itself is a group

closure, asso, e, a^{-1}

If $a, b \in H \Rightarrow ab^{-1} \in H$

Proper

Improper | trivial G {e}

Theorem
The union of two subgroup is a subgroup if one is contained in another.

Proof: Let $(G, *)$ be a group and H_1 & H_2 be two subgroup

we have to show,

$H_1 \cup H_2$ is a subgroup $\Leftrightarrow H_1 \subset H_2$ or $H_2 \subset H_1$.

H_1, H_2

(i) Let $H_1 \subset H_2$ or $H_2 \subset H_1$

If $H_1 \subset H_2 \Rightarrow H_1 \cup H_2 = H_2$ & H_2 is subgroup.

$\Rightarrow H_1 \cup H_2$ is also a subgroup

$H_2 \subset H_1 \Rightarrow H_1 \cup H_2 = H_1$ & H_1 is subgroup

$\Rightarrow H_1 \cup H_2$ is also a subgroup.

(ii) Let $H_1 \cup H_2$ is a subgroup

$\Rightarrow H_1 \subset H_2$ or $H_2 \subset H_1$.

Let $H_1 \not\subset H_2 \Rightarrow \exists a \in H_1$ such that $a \notin H_2$.

Let $b \in H_2$

$a \in H_1, b \in H_2 \Rightarrow a, b \in H_1 \cup H_2$ [subgroup $= H_1 \cup H_2 \Rightarrow H_1 \cup H_2$ is a group $\Rightarrow$ satisfies closure]

$\Rightarrow ab \in H_1 \cup H_2$

~~$b \in H_2 \Rightarrow b^{-1} \in H_2$ [Existence of Identity]~~

~~$ab \in H_1 \cup H_2$, $b^{-1} \in H_1 \cup H_2$~~

$\Rightarrow ab \in H_1$ or $ab \in H_2$

If $ab \in H_2, b \in H_2 \Rightarrow (ab)b^{-1} \in H_2 \Rightarrow a(bb^{-1}) \in H_2$

$\Rightarrow ae \in H_2 \Rightarrow a \in H_2$

$\Rightarrow$ a contradiction $\Rightarrow ab \notin H_2$

$\Rightarrow ab \in H_1, a \in H_1 \Rightarrow a^{-1}(ab) \in H_1 \Rightarrow (a^{-1}a)b \in H_1$

$\Rightarrow eb \in H_1 \Rightarrow b \in H_1$.

$\Rightarrow H_2 \subset H_1$

We have show that G is a group and center of group G is given by $Z(G) = \{x \in G \mid gx = xg \ \forall g \in G\}$ is a subgroup.

$\boxed{eg = e = ge}$

Proof: let $x_1, x_2 \in Z(G)$

$gx_1 = x_1 g$, $gx_2 = x_2 g$

$\exists e \in G$ such that $eg = ge \ \forall g \in G$

$\Rightarrow e \in Z(G)$

$\Rightarrow Z(G) \neq \phi$

$gx_2 = x_2 g \Rightarrow x_2^{-1}(gx_2)x_2^{-1} = x_2^{-1}(x_2 g)x_2^{-1}$

$= x_2^{-1} g (x_2 x_2^{-1}) = (x_2^{-1} x_2)(g x_2^{-1})$

$= x_2^{-1} g e = e g x_2^{-1}$

$= x_2^{-1} g = g x_2^{-1}$

$\Rightarrow x_2^{-1} \in Z(G)$

$(x_2 x_2^{-1})g = x_2(x_2^{-1}g) = x_1(g x_2^{-1}) = (x_1 g)x_2^{-1}$

$= g(x_1)x_2^{-1} = g(x_1 x_2^{-1})$

$x_1 x_2^{-1} \in Z(G)$

$Z(G)$ is a subgroup.

Q) subgroup $= \{a, b \in H \Rightarrow ab^{-1} \in H\}$

$Z(G) = \{x \in G \mid gx = xg \ \forall g \in G\}$

$G = \{a + b\sqrt{2} \mid a, b \in Q\}$ is a group w.r.t. addition

(i) associative (✓) (iii) identity element $e = 0$

(ii) Closure (✓) (iv) $a^{-1} \Rightarrow (-a - b\sqrt{2})$

—x— x — x — x — x —

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Q) If $a, b \in G$, then we have to show $ax = b$, $ya = b$ {have unique solution}

proof: $a \in G \Rightarrow a^{-1} \in G$ [Existence of ~~identity~~ Inverse]

$a^{-1} \in G, b \in G \Rightarrow a^{-1}b \in G$ [closure Property]

$ax = a(a^{-1}b) = (aa^{-1})b = eb = b$

$x \to a^{-1}b$ is solution of eqn $ax = b$.

If it is not unique, $\Rightarrow$ let x_1, x_2 are soln of $ax = b$

$ax_1 = b$, $ax_2 = b$

$ax_1 = ax_2$

$x_1 = x_2$ (left cancellation law)

$\therefore$ soln is unique.

(ii) $a^{-1} \in G, b \in G$

$\Rightarrow ba^{-1} \in G$ [closure Property]

Now, $\boxed{ya = b}$

$(ba^{-1})a = b(a^{-1}a) = b(e) = b$

$\Rightarrow ba^{-1}$ is soln of eqn $ya = b$

Now, we show that this is unique.

let y_1, y_2 be soln of $ya = b$

$y_1 a = b$, $y_2 a = b$

$y_1 a = y_2 a$

$y_1 = y_2$ [Right cancellation law]

$\Rightarrow$ soln will be unique.

* Cyclic Group: a group which can be generated through a single element.

ex: $G = \{1, -1, i, -i\}$ {Multiplication Composition}

$(i)^4 = 1$

$(i)^2 = -1$

$(i)^1 = i$

$(i)^3 = -i$ {i is generator}

$\therefore$ G is a cyclic group.

Ex:-2 $(Z^*, +)$ {Addition Composition} {1 is generator}

* A group G is a cyclic group if there exists a element $a \in G$ such that $G = [a]$ i.e; every element can be written as power of a(x) when composition of addition, we add no. of add (+)

—x— x — x — x —

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Properties of a cyclic group

(i) Every cyclic Group is abelian

(ii) If 'a' is a generator of a cyclic group G, then a^{-1} is also its generator.

3) The order of finite cyclic group is equal to order of its generator.

$G = \{1, -1, i, -i\}$ $\qquad o(G) = o(a)$

$o(G) = 4$

$o(i) = 4$

$i^4 = i^2 \times i^2$

$= -1 \times -1$

$= 1 = e$

Proof :

$G = [a] \qquad a \in G \quad o(a) = n \Rightarrow a^n = e$

H is a sub group of G whose order is n.

$o(G) = m$

Case ① : $m \leq n$: If $a^m \in G$ then $a^m \in H$

$\therefore H \subset G$

Case ② : $m > n$: $m = qn + r \Rightarrow a^m = a^{qn+r}$

$= (a^n)^q . a^r = e a^r$

$= a^r \in H$

$G \subset H$

$\therefore G \subset H \; \& \; H \subset G \Rightarrow G = H$

$o(G) = n$

$o(H) = n$

$\boxed{o(G) = o(a) = n}$

COSETS

Let H be a subgroup of G and $a \in G$, then the set

$aH = \{ah \mid \forall\, h \in H\}$

is left coset

$Ha = \{ha \mid \forall\, h \in H\}$

is Right Coset

$G = (Z, +)$ ab group $\qquad n = \{-3, -2, -1, 0, 1, 2, 3 ---\}$

$H = \{\{2n \mid n \in Z\}, +\}$ is a subgroup of G

$3 \in Z$

$H + 3 = \{ ---- -1, 1, 3, 5, 7, 9 ---- \}$

$G = \cup\, aH$ $\qquad G = \cup\, Ha$

$G \Rightarrow$ Union of all left costs of H $\qquad G \Rightarrow$ union of all right cosets of H.

Q) Find all cosets of 3Z in Group (Z, +)

Ans)

$Z = \{ ---- -3, -2, -1, 0, 1, 2, 3 ---\}$

$3Z = \{ ---- -9, -6, -3, 0, 3, 6, 9 ---\}$

$3Z + 0 = \{ ---- -9, -6, -3, 0, 3, 6 ---\}$

$3Z + 1 = \{ ---- -8, -5, -2, 1, 4, 7 ---\}$

$3Z + 2 = \{ ---- -7, -4, -1, 2, 5, 8 ---\}$

$3Z + 3 = \{ ---- -6, -3, 0, 3, 6, 9 ---\}$

$3Z + 4 = \{ ---- -5, -2, 1, 4, 7, 10 ---\}$

$3Z + (-1) = \{ ---- -7, -4, -1, 2, 5, 8 ---\}$

$3Z + (-2) = \{ ---- -8, -5, -2, 1, 4, 7 ---\}$

~~3Z+2 = 3Z+(-2) = 3Z+1~~

$\therefore$ There are three distinct cosets.

$\{3Z, 3Z+1, 3Z+2\}$

Theorem

Lagrange's Theorem

If $o(G) = n$

$o(H) = m$

n / m

$\boxed{n = mK, \; K \in Z}$

$\Rightarrow$ The order of the every subgroup of a finite group is a divisor of the order of the group.

Proof : Suppose $o(G) = n$ & $o(H) = m$

where H is a subgroup of G,

Let the different coset of H and G be

$g_1H, g_2H ---- g_kH$

then, $G = \bigcup_{i=1}^{k} g_iH$

$G = g_1H \cup g_2H \cup g_3H ---- \cup g_kH$

$o(G) = o(g_1H) + o(g_2H) ---- + o(g_kH)$

$n = o(H) + o(H) + ---- o(H)$ (K times)

$= m + m + ---- + m$ K times

$n = mK$

$\Rightarrow m$ is a divisor of n

27/04/2020

Relation of Congruence w.r.t. subgroup

If G is a group, H is subgroup of G & $a, b \in G \Rightarrow$ then $a \cong b \pmod H \Leftrightarrow ab^{-1} \in H$

The relation of congruency in a group G defined as

$$a \cong b \,(\text{Mod } H) \Leftrightarrow ab^{-1} \in H$$

is an equivalency relation where H is a subgroup of G.

Proof :- ① Reflexive, Symmetric & transitive

$\boxed{a \cong a}$ $\boxed{a \cong b \ \& \Rightarrow b \cong a}$ $\boxed{a \cong b,\ b \cong c \Rightarrow a \cong c}$

① Reflexive relation :- let $a \in G$, then $a^{-1} \in G$ [Existence of inverse]

$aa^{-1} \in G \Rightarrow e \in G$. [H is a subgroup then it will be a group]

$\Rightarrow a \cong a \pmod H$

② Symmetric $\Rightarrow$ let $a \cong b \bmod H \Rightarrow ab^{-1} \in H$

$(ab^{-1})^{-1} \in H$ (H is a group)

$(b^{-1})^{-1} a^{-1} \in H$

$ba^{-1} \in H$

$b \cong a \pmod H$

$\therefore \boxed{a \cong b \Leftrightarrow b \cong a}$

③ Let $a \cong b \pmod H$ & $b \cong c \pmod H$

$ab^{-1} \in H$ and $bc^{-1} \in H$

$(ab^{-1})(bc^{-1}) \in H$ (H is a subgroup) [closure Property]

$a(b^{-1}b)c^{-1} \in H$ [By associative property in H]

$aec^{-1} \in H \Rightarrow ac^{-1} \in H$

$\Rightarrow a \cong c \pmod H$

$\therefore$ Transitive

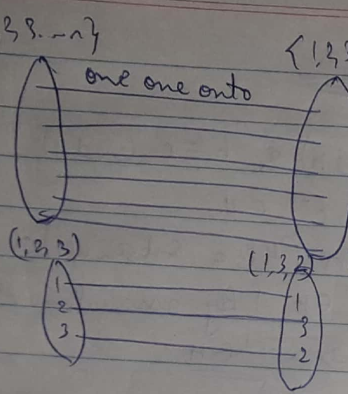
$\therefore$ Relation is an equivalence relation.

* Permutation

A one to one mapping 'σ' of the set $\{1, 2, \text{----} n\}$ onto itself is called a permutation and is denoted by

$$\sigma = \begin{Bmatrix} 1 & 2 & 3 & \cdots & n \\ j_1 & j_2 & j_3 & \cdots & j_n \end{Bmatrix}$$

where $j_i = \sigma(i)$



Ex:- $A \{1,2,3\}$

$$\sigma = \begin{Bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{Bmatrix}$$

* Cyclic Permutation

A permutation σ of set A is a cyclic permutation or a cycle if there exists a finite subset $a_1, a_2, \text{----} a_r$ of A, such that $\sigma(a_1) = a_2$, $\sigma(a_2) = a_3$, $\sigma(a_3) = a_4 \text{---} \sigma(a_r) = a_1$. And it is denoted by $\sigma = [a_1, a_2, \text{---} a_n]$.

* Composition of two permutation

If σ and ϕ are two permutation then

$\sigma\phi = \sigma \circ \phi$

$$\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad \phi = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\sigma\phi = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{1\ 3\ 2} \\ 3\ 2\ 1 \end{bmatrix} \begin{bmatrix} 1\ 2\ 3 \\ \cancel{1\ 3\ 2} \end{bmatrix}$$

$$\sigma\phi = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Q) If $\sigma = [1\ 7\ 2\ 6\ 3\ 5\ 8\ 4]$

$$\rho = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 4 & 3 & 8 & 7 & 6 & 1 \end{bmatrix} \quad ①$$

$$\rho\sigma\rho^{-1} = (\rho(1)\ \rho(7)\ \rho(2)\ \rho(6)\ \rho(3)\ \rho(5)\ \rho(8)\ \rho(4)) \quad (A)$$

$$\rho^{-1} = \begin{bmatrix} 2 & 5 & 4 & 3 & 8 & 7 & 6 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

$$\rho\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 4 & 3 & 8 & 7 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 1 & 8 & 3 & 2 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7 & 6 & 5 & 1 & 8 & 9 & 2 & 4 \\ 6 & 7 & 8 & 2 & 1 & 4 & 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 1 & 8 & 3 & 2 & 4 \end{bmatrix}$$

$$\rho\sigma \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 2 & 1 & 4 & 5 & 3 \end{bmatrix}$$

$$\rho \circ \rho^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 2 & 1 & 4 & 5 & 3 \end{bmatrix}\begin{bmatrix} \cancel{2} & \cancel{5} & \cancel{4} & \cancel{3} & \cancel{8} & \cancel{7} & \cancel{6} & \cancel{1} \\ \cancel{1} & \cancel{2} & \cancel{3} & \cancel{4} & \cancel{5} & \cancel{6} & \cancel{7} & \cancel{8} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 4 & 3 & 2 & 7 & 6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{2} & \cancel{5} & \cancel{4} & \cancel{3} & \cancel{8} & \cancel{7} & \cancel{6} & \cancel{1} \\ \cancel{6} & \cancel{7} & \cancel{8} & \cancel{2} & \cancel{1} & \cancel{4} & \cancel{5} & \cancel{3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & 1 & 4 & 3 & 2 & 7 & 6 & 5 \\ 3 & 6 & 2 & 8 & 7 & 5 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 4 & 3 & 2 & 7 & 6 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 2 & 8 & 7 & 5 & 4 & 1 \end{bmatrix}$$

$\Rightarrow (1\ 3\ 2\ 6\ 5\ 7\ 4\ 8)$ (// cyclic)

$= [\rho(8)\ \rho(4)\ \rho(1)\ \rho(7)\ \rho(2)\ \rho(6)\ \rho(3)\ \rho(5)]$ — (B) [From (1)]

$\Rightarrow$ From (A) & (B)

Both are cyclic permutations.

$\therefore$ Both are equal.

So, Hence Proved.

Disjoint cycles

$$\rho = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 4 & 3 & 8 & 7 & 6 & 1 \end{bmatrix}$$

$\Rightarrow (1, 2, 5, 8)\ (3, 4)\ (6, 7)$ disjoint cycles.

transposition

Even & odd permutations.

28/04/2020

Q) $\rho = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{bmatrix}$

$\sigma = (1\,3\,4)(5\,6)$

$\sigma^{-1}\rho\sigma = ?$ (disjoint cycle)

ρ → even or odd order

$$\sigma = \begin{bmatrix} 1 & 3 & 4 & 5 & 6 & 2 & 7 & 8 & 9 \\ 3 & 4 & 1 & 6 & 5 & 7 & 8 & 9 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \end{bmatrix}$$

$$\sigma^{-1} = \begin{bmatrix} 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

$$\sigma^{-1} = \begin{Bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 1 & 3 & 6 & 5 & 2 & 7 & 8 \end{Bmatrix}$$

$$\rho\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}$$

$$\sigma^{-1}\rho\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 1 & 3 & 6 & 5 & 2 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \\ 8 & 9 & 5 & 2 & 6 & 3 & 1 & 4 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 5 & 2 & 6 & 3 & 1 & 4 & 7 \end{pmatrix}$$

$= (1\,8\,4\,2\,9\,7)(3\,5\,6)$

$\therefore$ odd permutation

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}$$

$= (1\,7\,2\,8\,3\,9)\ (4\,6\,5)$

$= (1\,9)(1\,3)(1\,8)(1\,2)(1\,7)(4\,5)(4\,6)$

$= 7$ (odd) transpositions | inversion.

$f = (1\ 7\ 2\ 8\ 3\ 9)(4\ 6\ 5)$
$= (19)(13)(18)(12)(17)(45)(46)$

$= LCM(6, 3)$

$\boxed{O(f) = 6}$

1. Product
2. Inverse
3. order
4. Even or Odd
5. disjoint.

A permutation is said to be even permutation if it can be expressed as ~~of~~ product of even no. of transpositions.

A permutation is said to be odd permutation if it can be expressed as product of odd no. of transpositions.

Symmetric Group or Group of Permutations

$A = \{1, 2, 3\} \Rightarrow 3! = 3\times2\times1 = 6$

$\xi_I = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ $\sigma_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$ $\sigma_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

$\sigma_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ $\sigma_4 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ $\sigma_5 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

$S_3 = \{\xi_I, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$

$S_4 = \{\longleftarrow 24 \longrightarrow\}$

$S_5 = \{120\}$
$S_n = \{n!\}$

The group of permutation ~~set~~ of the set $\{1, 2, \ldots n\}$ is called a symmetric group of degree n and it is denoted by S_n

Q) Show that S_3 is a group.

$A = \{1, 2, 3\}$

$\xi_I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$\sigma_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ $\sigma_5 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ $\sigma_4 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

then $S_3 = \{\xi_I, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$

we have to show S_3 is group

o	ξ_I	σ_1	σ_2	σ_3	σ_4	σ_5
ξ_I	ξ_I	σ_1	σ_2	σ_3	σ_4	σ_5
σ_1	σ_1	ξ_I	σ_4	σ_5		
σ_2	σ_2					
σ_3	σ_3					
σ_4	σ_4					
σ_5	σ_5					

1) Closure Property $\Rightarrow$ As all mapping are $\xi_I, \sigma_1, \sigma_2, \ldots$
$\therefore$ it is closed.

2) Associative Property $\Rightarrow$ Composition function is associative

3) Identity : ξ_I is identity

4) Existence of inverse : Symmetric

Q) Show that S_n is a group?

Ans) ① Closure Property : let $f, g \in S_n$
$\Rightarrow f\circ g \in S_n$
$\therefore S_n$ is closed.

② Associativity : let $f, g, h \in S_n$
then we know $f\circ(g\circ h) = (f\circ g)\circ h$
(becoz composite function is associative)

③ Existence of identity = $I_n \in S_n$ is identity function i.e. $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$

s.t. $f\circ I_n = f = I_n\circ f$

(4) Existence of Inverse : $f \in S_n$

f is one one and onto

f^{-1} is also one & onto

s.t. $f \circ f^{-1} = I_n = f^{-1} \circ f$

$\therefore$ inverse element exist.

$\therefore$ By (1), (2), (3) & (4), then S_n is a group.

--x--------x--------x--------x--------x

29/04/2020

NORMAL SUBGROUP

If G is a group and H is a subgroup of G, then H is normal subgroup of G iff $\forall x \in G$ & $h \in H$ $\Rightarrow x h x^{-1} \in H$.

$$\boxed{H \trianglelefteq G \Leftrightarrow (x \in G, h \in H \Rightarrow x h x^{-1} \in H)}$$

Improper — $G, \{e\}$

Proper

Simple group : If G is a group and G do not have any proper normal subgroup then it is simple Group.

Ex : $G = \{1, -1, i, -i\}$ is a group.

$H = \{1, -1\}$ is a subgroup.

$\downarrow$

Normal Subgroup.

$x = 1$, $h = -1$

$x^{-1} \Rightarrow 1$ $(1)(1)(-1) = -1 \in H$

$x = i$, $h = 1$

$x^{-1} = -i$ $(i)(1)(-i) = -1 \in H$

$\boxed{x \in G, h \in H \Rightarrow x h x^{-1} \in H} \Rightarrow$ Normal Subgroup.

Some Properties of Normal Subgroup :

Theorem (1) Every Subgroup of Abelian group is normal subgroup.

Proof : Let H be a subgroup of abelian group G.

Let $x \in G$ and $h \in H$

$x h x^{-1} = (hx) x^{-1}$ {$\because G$ is a abelian group}

$= h (x x^{-1})$ [$\because G$ is associative]

$= h \in H$

$\boxed{x h x^{-1} \in H}$

$\therefore x \in G, h \in H \Rightarrow x h x^{-1} \in H \Rightarrow H$ is normal subgroup.

~~Commutative Property~~

Theorem 2 : A subgroup H of a group G is a normal subgroup if and only if

$H \trianglelefteq G \Leftrightarrow x H x^{-1} = H$ $\forall x \in G$.

$A = B$

$A \subset B \Rightarrow x \in A$ and $x \in B$

$B \subset A \Rightarrow y \in B$ and $y \in A$

Let H is normal subgroup of G

If $x \in G, h \in H \Rightarrow x h x^{-1} \in H$

$\Rightarrow x H x^{-1} \subset H$

$x \in G \Rightarrow x^{-1} \in G$ (Existence of inverse)

$x^{-1} \in G$ and $h \in H$

$(x^{-1}) h (x^{-1})^{-1} \in H$

$\Rightarrow x^{-1} h x \in H$ —— (1)

$x^{-1} H x \subseteq H$

~~$x(x^{-1}) H$~~ $x (x^{-1} H x) x^{-1} \subset x H x^{-1}$

$e H e \subset x H x^{-1}$

$= H \subset x H x^{-1}$ —— (2)

$\Rightarrow$ By (1) & (2)

$\boxed{x H x^{-1} = H}$

$\Rightarrow$ CONVERSELY :

Let $x H x^{-1} = H$ and we will show that H is a normal subgroup.

Now, $x H x^{-1} = H \Rightarrow x H x^{-1} \subset H$

$\Rightarrow x h x^{-1} \in H$ $\forall x \in G, h \in H$

$\Rightarrow \boxed{H \trianglelefteq G}$

Theorem 3 $\Rightarrow$ The intersection of any two normal subgroup of a group is a normal subgroup.

Proof $\Rightarrow$ Let H_1 and H_2 be two normal subgroup of G.

Then we have to show $H_1 \cap H_2$ is also normal subgroup of G.

Let $x \in G$ and $h \in H_1 \cap H_2$

Now $h \in H_1 \cap H_2$

$\Rightarrow h \in H_1$ and $h \in H_2$

Now, $x \in G$, $h \in H_1 \Rightarrow xhx^{-1} \in H_1$ [H_1 is normal subgroup]

Now, $x \in G$, $h \in H_2 \Rightarrow xhx^{-1} \in H_2$ [H_2 is normal subgroup]

$x \in G, h \in H_1 \cap H_2 \Rightarrow xhx^{-1} \in H_1 \cap H_2$,

$\Rightarrow H_1 \cap H_2$ is normal subgroup.

30/04/2020

\# H is a subgroup of G, and N is normal subgroup.

$\Rightarrow H \cap N \triangleleft H$

Proof: $H \cap N$ will be subgroup of G,

$H \cap N \subset H$

$\Rightarrow H \cap N$ will be a subgroup of H.

$x \in H$, $h \in H \cap N$,

Now, $h \in H \cap N \Rightarrow h \in H$ and $h \in N$

Now, $x \in H, h \in H \Rightarrow xhx^{-1} \in H$ [By closure prop.]

$x \in H \Rightarrow x \in G$ [H is a subgroup of G]

Now,

$x \in G$, $h \in N \Rightarrow \boxed{xhx^{-1} \in N}$ [N is normal subgroup of G]

$xhx^{-1} \in H \cap N \Rightarrow \underline{H \cap N \triangleleft H}$

QUOTIENT GROUP

If G is a group, $H \triangleleft G$, then the set G/H of all cosets of H in G together with binary composition $Ha . Hb = Hab$ is a group and it is called the quotient group of G by H.

~~Let G be a group and $H \triangleleft G$, then the set G/H~~

Q) Find the quotient group G/H and also prepare its operation table when $G\{\{1,-1,i,-i\}\times\}$, $H = [\{1,-1\}, \times]$

operation is complex multiplication

$H.1 = \{1.1 - 1.1\} \Rightarrow \{1,-1\} = H$

$H.(-1) = \{1(-1), -1(-1)\} = \{-1, 1\} = H$

$H.i = \{1.i, -1.i\} = \{i, -i\} = Hi$

$H.(-i) = \{1(-i), -1(-i)\} = \{-i, i\} = Hi$

$G/H = \{H, Hi\}$

	H	Hi
H	H	Hi
Hi	Hi	H

Q) Find the quotient group G/H and also prepare its operation table when $G = (Z, +)$, $H = (4Z, +)$.

Ans) Now coset of G in H

$H+0 = \{ ---- -12, -8, -4, 0, 4, 8 -\}$

$H+1 = \{ ---- -11, -7, -3, 1, 5, 9 --\}$

$H+2 = \{ ---- -10, -6, -2, 2, 6, 10 --\}$

$H+3 = \{ ---- -9, -5, -1, 3, 7, 11 -\}$

$H+4 = H+0 = H+8 = H+12$

$H+5 = H+1 = H+9 = H+13$

$H+6 = H+2 = H+10 =$

$H+7 = H+3 = H+11 =$

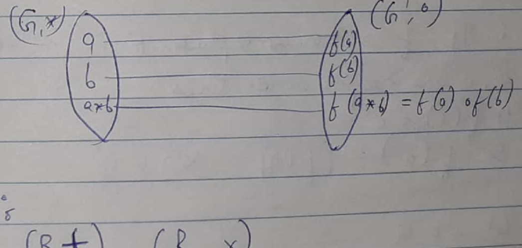
$G/H = \{\underline{H, H+1, H+2, H+3}\}$

$\boxed{\begin{array}{l} Ha \cdot Hb = Hab \\ (H+a) + (H+b) = H + (a+b) \end{array}}$

	H	$H+1$	$H+2$	$H+3$
H	H	$H+1$	$H+2$	$H+3$
$H+1$	$H+1$	$H+2$	$H+3$	H
$H+2$	$H+2$	$H+3$	H	$H+1$
$H+3$	$H+3$	H	$H+1$	$H+2$

\# Homomorphism

A mapping f from a group $(G, *)$ to a group $(G', \circ)$ is called a group homomorphism for group morphism from G to G' if $f(a*b) = f(a) \circ f(b)$



Ex:

$(R, +)$, $(R_0, \times)$

$f: (R, +) \rightarrow R_0, \times)$

$f(x) = 2^x$

$f(x_1 + x_2) = 2^{x_1 + x_2} = 2^{x_1} . 2^{x_2}$

$= f(x_1) . f(x_2)$

Various Morphism :

① Monomorphism : $f \to$ one one
② Epimorphism : $f \to$ onto
③ Isomorphism : $f \to$ one-one & onto
④ Endomorphism : $f \to G \to G$
⑤ Automorphism : $f \to G \xrightarrow[\text{one-one}]{\text{onto}} G'$

———x ———x ———x

01/05/2020

Theorem 1 : If f is homomorphism from G to G' & e, e' are respective identities

① $f(e) = e'$
② $f(a^{-1}) = [f(a)]^{-1} \quad \forall a \in G$

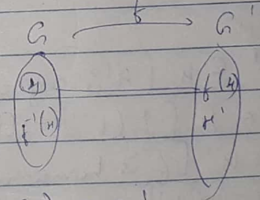
Proof ① Let $a \in G$

$\Rightarrow ae = a = ea$
$f(ae) = f(a) = f(ea)$
$f(a)f(e) = f(a) = f(e)f(a)$ [f is homomorphism]
$f(e) = e' \in G'$

(ii) Let $a \in G$
$a^{-1} \in G$ [G is a group]
$aa^{-1} = e = a^{-1}a$
$f(aa^{-1}) = f(e) = f(a^{-1}a)$
$f(a)f(a^{-1}) = f(e) = f(a^{-1})f(a)$ [f is homomorphism]
$f(a)f(a^{-1}) = e' = f(a^{-1})f(a)$
$f(a^{-1}) = [f(a)]^{-1}$

Theorem 2 :

If f is homomorphism from group G to G' then show,
① H is a subgroup of $G \Rightarrow f(H)$ is a subgroup of G'.
② H' —— 〃 —— $G' \Rightarrow f^{-1}(H')$ —— 〃 —— G.



$f(ab) = f(a)f(b)$
$a, b \in H \Rightarrow ab^{-1} \in H$

Proof ① $f(H) \subset G'$
$f(H) \neq \phi \Rightarrow \exists\, e' \in f(H)$ s.t. $f(e) = e'$ $(e \in H)$
$a', b' \in f(H) \Rightarrow \exists\, a, b \in H$
s.t. $f(a) = a'$, $f(b) = b'$.

$a'(b')^{-1} = f(a)[f(b)]^{-1}$
$= f(a) f[b^{-1}] = f(ab^{-1})$ — ①

Now, $a, b \in H \Rightarrow ab^{-1} \in H$ [H is subgroup]
$f(ab^{-1}) \in f(H)$
$a'(b')^{-1} \in f(H)$ [by ①]
$a', b' \in f(H) \Rightarrow a'(b')^{-1} \in f(H)$
$\therefore f(H)$ is a subgroup of G'

② $f^{-1}(H') \subset G$
$e \in G$ s.t. $f(e) = e'$
$\Rightarrow f^{-1}(H') \neq \phi$
$a, b \in f^{-1}(H')$
$f(a), f(b) \in H'$
$f(a)[f(b)]^{-1} \in H'$ (H' is subgroup)
$f(a) f(b^{-1}) \in H'$ [$f(b^{-1}) = [f(b)]^{-1}$]
$f(ab^{-1}) \in H'$ [f is homomorphism]
$ab^{-1} \in f^{-1}(H')$
$\Rightarrow f^{-1}(H')$ is a subgroup of G.

Kernel of Homomorphism

$f : G \to G'$ (Homomorphism)
Ker f =

If f is a homomorphism of a group G into G' then the set K of all those elements of G which are mapped to the identity e' of G' is called the kernel of the homomorphism f. It is denoted by Ker f or Ker(f).

Theorem A homomorphism f from a group G to G' is an isomorphism if and only if Ker $f = \{e\}$

Proof : f is homomorphism $\Leftrightarrow$ Ker $f = \{e\}$
① Let f is isomorphism [f is one one and onto both]
Let $a \in G$
s.t. $f(a) = e'$
$f(a) = f(e) \Rightarrow [f(e) = e']$
$a = e$ [f is one one]
Ker $f = \{e\}$
Conversely let Ker $f = \{e\}$

clearly f is onto

let $f(a) = f(b)$ $[a, b \in G]$

$f(a)[f(b)]^{-1} = f(b)[f(b)]^{-1}$

$f(a) f(b^{-1}) = e'$

$f(ab^{-1}) = e'$

$ab^{-1} \in \text{Ker} f \Rightarrow ab^{-1} = e$

f is one-one $\quad \boxed{a = b}$

04/05/20 **Recurrence Relation**

$\left.\begin{cases} a_r = a_{r-1} + 1 \\ a_r = 2a_{r-1} + a_{r-2} \end{cases}\right\} \rightarrow$ Differential Equations.

$a_r \rightarrow$ Numeric function

$f\{0, 1, 2 \ldots\} \rightarrow R$

Let : $a_i = \{3, 5, 7, 9 \ldots\}$

formula : $a_r = 3r + 1$

$a_0 = 1$
$a_1 = 4$
$a_2 = 7$
$a_3 = 10$

$a_1 = 3$
$a_2 = 5$
$a_3 = 7$

Linear recurrence relation with constant coefficient

$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \ldots = f(r)$

$$\boxed{a_r + 2a_{r-1} + 3 = (3r^2 + 1)}$$

$$\boxed{a_r = \text{Homogenous Solution} + \text{Particular Solution}}$$

- Homogeneous Solution [depends upon LHS side]

$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \ldots C_n a_{r-n} = f(r)$

Characteristic Equation

we substitute $a_r = A\alpha^r$, $f(r) = 0$

$\Rightarrow C_0 A\alpha^r + C_1 A\alpha^{r-1} + C_2 A\alpha^{r-2} \ldots = 0$

$A(C_0\alpha^r + C_1\alpha^{r-1} + C_2\alpha^{r-2} \ldots) = 0$

$C_0\alpha^r + C_1\alpha^{r-1}\ C_2\alpha^{r-2} \ldots = 0$

Roots will be : real, repeated real, imaginary, repeated imaginary.

This is characteristic Eqn :

① Roots are real and distinct $(\alpha_1, \alpha_2, \alpha_3 \ldots)$

$a_r = A_1\alpha_1^r + A_2\alpha_2^r + A_3\alpha_3^r \ldots$

② Roots are real and repeated $(\alpha_1, \alpha_2, \alpha_3 \ldots)$

$a_r = (A_1 r^2 + A_2 r + A_3)\alpha_1^r + A_4\alpha_4^r \ldots$

③ Roots are imaginary $(\alpha \pm i\beta)$

$a_r = \rho^r (A_1 \cos r\theta + A_2 \sin r\theta)$

$\rho = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

④ Roots are imaginary and repeated $(\alpha \pm i\beta, \alpha \pm i\beta)$

$a_r = \rho^r ((A_1 r + A_2)\cos r\theta + (A_3 r + A_4)\sin r\theta)$

$\rho = \sqrt{\alpha^2 + \beta^2}$ $\quad \theta = \tan^{-1}(\beta/\alpha)$

Q1) Fibonacci Sequence is $a_r = a_{r-1} + a_{r-2}$ with boundary condition $a_0 = 0, a_1 = 1$.

Solution : $a_r - a_{r-1} - a_{r-2} = 0$

Now characteristic eqn : $a_r = A\alpha^r$

$\alpha^r - \alpha^{r-1} - \alpha^{r-2} = 0$

$\alpha^{r-2}(\alpha^2 - \alpha^1 - 1) = 0$

$(\alpha^2 - \alpha - 1) = 0$

$\alpha = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

~~$\alpha = 1/2$, $\beta = \sqrt{5}/2$~~

~~$a_r = \rho^r (A_1 \cos r\theta + A_2 \sin r\theta)$~~

~~$\rho = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}(\beta/\alpha)$~~

$a_r = A_1\left(\frac{1 + \sqrt{5}}{2}\right)^r + A_2\left(\frac{1 - \sqrt{5}}{2}\right)^r$

$a_0 = 0 \Rightarrow A_1 + A_2 \Rightarrow A_1 = -A_2$

$a_1 = 1 \Rightarrow 1 = A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right)$

$A_1 = \frac{1}{\sqrt{5}} = A_2 = \frac{-1}{\sqrt{5}}$

$\therefore a_r = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^r - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^r$

Q] $4a_r - 20a_{r-1} + 17a_{r-2} - 4a_{r-3} = 0$

Solⁿ: Characteristic eqⁿ:
$4\alpha^{r} - 20\alpha^{r-1} + 17\alpha^{r-2} - 4\alpha^{r-3} = 0$

$\Rightarrow 4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0$

$\alpha = 4, \frac{1}{2}, \frac{1}{2}$

Homogeneous Solution is

$\boxed{a_r = A_1(4)^r + (A_2 r + A_3)(1/2)^r}$

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Q] $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$

Solⁿ: Characteristic Eqⁿ:
$a_r = A\alpha^r$, RHS $= 0$

$\Rightarrow \alpha^r + 5\alpha^{r-1} + 6\alpha^{r-2} = 0$

$\alpha^2 + 5\alpha + 6 = 0$

$\alpha^2 + 2\alpha + 3\alpha + 6 = 0$

$\boxed{\alpha = -3, -2}$

$\therefore$ H.S. $\Rightarrow \boxed{a_r = A_1(-3)^r + A_2(-2)^r}$

let particular solⁿ be $a_r = P_1 r^2 + P_2 r + P_3$

$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$

$(P_1 r^2 + P_2 r + P_3) + 5(P_1(r-1)^2 + P_2(r-1) + P_3) + 6(P_1(r-2)^2 + P_2(r-2) + P_3) = 3r^2$

$\Rightarrow P_1 r^2 + P_2 r + P_3 + 5[P_1 r^2 - 2P_1 r + P_1 + P_2 r + P_3 - P_2]$

$+ 6[P_1 r^2 - 4P_1 r + 4P_1 + P_2 r - 2P_2 + P_3]$

$12P_1 r^2 + r(P_2 + 5P_2 + 6P_2) - 10P_1 r - 24P_1 r + P_3 + 5P_3 + 6P_3 \ldots$

$\Rightarrow 12P_1 r^2 + r(12P_2 - 34P_1) + (29P_1 - 17P_2 + 12P_3) = 3r^2$

$12P_1 r^2 = 3r^2$; $12P_2 - 34P_1 = 0$ & $29P_1 - 17P_2 + 12P_3 = 0$

$P_1 = 1/4$; $P_2 = 17/24$, $P_3 = 115/288$

$\boxed{\text{P.S.} \Rightarrow a_r = \frac{1}{4}r^2 + \frac{17}{24}r + \frac{115}{288}}$

$\boxed{\text{T.S} = \text{HS} + \text{PS}}$

Q] $a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r$

P.S:- ①
②
③ $a^2 b^r$

Ans: H.S. $\Rightarrow \alpha = -3, -2$

H.S. $\Rightarrow a_r = A_1(-2)^r + A_2(-3)^r$

P.S. $\Rightarrow$ let $a_r = P4^r$

$a_{r-1} = P4^{r-1}$, $a_{r-2} = P4^{r-2}$

$\Rightarrow$ Eqⁿ becomes,

$P4^r + 5P4^{r-1} + 6P4^{r-2} = 42 \cdot 4^r$

$16P + 20P + 6P = 42 \cdot 16$

$42P = 42 \cdot 16$

$\boxed{P = 16}$

$\therefore P_s = a_r = 16 \cdot 4^r$

$\boxed{\text{TS} = \text{HS} + \text{PS}}$

Q] $a_r + a_{r-1} = 3r\,2^r$

Characteristic Eqⁿ:

$\alpha^r + \alpha^{r-1} = 0$

$\alpha^{r-1}(\alpha + 1) = 0$

$\alpha = 0, -1$

H.S. $\Rightarrow A_1(0)^r + A_2(-1)^r = A_2(-1)^r$

P.S. $\Rightarrow$ let $a_r = (P_1 r + P_2)2^r$

$a_{r-1} = (P_1(r-1) + P_2)2^{r-1}$

Eq^n becomes,

$(P_1 r + P_2)2^r + [P_1(r-1) + P_2]2^{r-1} = 3r2^r$

$2(P_1 r + P_2) + (P_1 r - P_1 + P_2) = 6r$

$3P_1 r + 3P_2 - P_1 = 6r$

$3P_1 r = 6r$; $3P_2 - P_1 = 0$

$\boxed{P_1 = 2}$ $\boxed{P_2 = 2/3}$

$PS = (2r + 2/3)2^r$

$\boxed{T.S. = HS + PS}$

Q) $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$

CE : $\alpha^r - 4\alpha^{r-1} + 4\alpha^{r-2} = 0$

$\alpha^2 - 4\alpha + 4 = 0$

$\alpha = 2 \& 2$

H.S. $= a_r = (A_1 r + A_2)2^r$

P.S. let $a_r = r^2(P_1 r + P_2)2^r$

$a_{r-1} = (r-1)^2 (P_1(r-1) + P_2)2^{r-1}$

$a_{r-2} = (r-2)^2 (P_1(r-2) + P_2)2^{r-2}$

Eq^n becomes,

$r^2(P_1 r + P_2)2^r - 4((r-1)^2(P_1(r-1) + P_2)2^{r-1} +$

$4((r-2)^2(P_1(r-2) + P_2)2^{r-2}$

$\Rightarrow r^3(P_1 \quad\quad) + r^2(\quad\quad)$

$+ r(\quad\quad) + 1(\quad\quad)$

$= \cancel{(\quad)}$

$P_1 = \frac{1}{6}$ & $P_2 = 1$

P.S. $= r^2\left(\frac{1}{6}r + 1\right)2^r$

T.S = HS + PS

Q) $a_r = a_{r-1} + 7$

$a_r - a_{r-1} = 7$

CE $= \alpha^r - \alpha^{r-1} = 0$

$\alpha^{r-1}(\alpha - 1) = 0$

$\alpha = 0, 1$

H.S. $= A_1(1)^r = A_1$

PS $= a_r = Pr(1)^r = Pr$

$Pr = P(r-1) + 7$

$Pr - P(r-1) = 7$

$Pr - Pr + P = 7$

$\boxed{P = 7}$

P.S. $= a_r = 7r$

T.S. $= A_1 + 7r$

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Q) solve $a_r = 3a_{r-1} + 2$, $r \geq 1$

with B.C. $a_0 = 1$ using generating function.

Ans) let $A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 \ldots$

$= \sum_{r=0}^{\infty} a_r z^r$

Given eq^n is $a_r = 3a_{r-1} + 2$

$\sum_{r=1}^{\infty} a_r z^r = 3\sum_{r=1}^{\infty} a_{r-1} z^r + 2\sum_{r=1}^{\infty} z^r$

$\sum_{r=1}^{\infty} a_r z^r = A(z) - a_0$

$\sum_{r=1}^{\infty} a_{r-1} z^r = z\sum_{r=1}^{\infty} a_{r-1} z^{r-1} = zA(z)$

$\sum_{r=1}^{\infty} z^r = z + z^2 + z^3 + \ldots = z(1 + z + z^2 \ldots) = \frac{z}{1-z}$

FORMULAES

$$\boxed{(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \frac{n(n-1)(n-2)}{3!} z^3}$$

$$(1+z)^{-1} = 1 - z + \frac{(-1)(-1-1)}{2!} z^2 + \frac{(-1)(-1-1)(-1-2)}{3!} z^3$$

$$= 1 - z + z^2 - z^3 + \dots$$

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \dots$$

$$(1-z)^{-2} = 1 + (-2)(-z) + \frac{(-2)(-2-1)}{2!}(-z)^2 + \dots$$

$$(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \dots$$

$$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \dots$$

$$\Rightarrow A(z) - a_0 = 3z A(z) + \frac{2z}{1-z}$$

$$A(z)(1-3z) = 1 + \frac{2z}{1-z} = \frac{1-z+2z}{1-z} = \frac{1+z}{1-z}$$

$$A(z) = \frac{1+z}{(1-z)(1-3z)} = \frac{A}{(1-z)} + \frac{B}{(1-3z)} = \frac{1}{(1-z)} + \frac{2}{(1-3z)}$$

$$A(z) = 2(1-3z)^{-1} - (1-z)^{-1}$$

$$\sum_{r=0}^{\infty} a_r z^r = 2(1 + 3z + 3^2z^2 + 3^3z^3 + \dots) - (1 + z + z^2 + z^3 + \dots)$$

$$= 2\sum_{r=0}^{\infty} 3^r z^r - \sum_{r=0}^{\infty} z^r$$

$$\Rightarrow \sum_{r=0}^{\infty} (2.3^r - 1) z^r$$

$$\boxed{a_r = 2.3^r - 1 \quad r \geq 0}$$ Ans

Q) $a_r^2 - 2a_{r-1}^2 = 1$ with $a_0 = 2$

Sol^n: let $b_r = a_r^2$

eq^n becomes $b_r - 2b_{r-1} = 1$

Multiplying by z^r and summing from $r=1$ to ∞

Eq^n becomes,

$$\sum_{r=1}^{\infty} b_r z^r - 2\sum_{r=1}^{\infty} b_{r-1} z^r = \sum_{r=1}^{\infty} z^r$$

let $B(z) = \sum_{r=0}^{\infty} b_r z^r = b_0 + b_1 z + b_2 z^2 + \dots$

$$\sum_{r=1}^{\infty} b_r z^r = B(z) - b_0$$

$$\sum_{r=1}^{\infty} b_{r-1} z^r = z \sum_{r=1}^{\infty} b_{r-1} z^{r-1} = z(B(z))$$

$$\sum_{r=1}^{\infty} z^r = \frac{z}{1-z} = z(1-z)^{-1}$$

$$B(z) - b_0 - 2z(B(z)) = \frac{z}{1-z}$$

$$B(z)[1-2z] = b_0 + \frac{z}{1-z}$$

$$= 4 + \frac{z}{1-z}$$

$$\boxed{\begin{aligned} a_0 &= 2 \\ b_0 &= 2^2 = 4 \end{aligned}}$$

$$= \frac{4-3z}{1-z}$$

$$B(z) = \frac{4-3z}{(1-z)(1-2z)} = \frac{A}{(1-z)} + \frac{B}{(1-2z)}$$

$$\Rightarrow A = -1 \;\&\; B = 5$$

$$B(z) = \frac{-1}{(1-z)} + \frac{5}{(1-2z)}$$

$$\sum_{r=0}^{\infty} b_r z^r = -1(1-z)^{-1} + 5(1-2z)^{-1}$$

$$= -(1 + z + z^2 + \dots) + 5(1 + 2z + 2^2z^2 + 2^3z^3 + \dots)$$

$$= -\sum_{r=0}^{\infty} z^r + 5\sum_{r=0}^{\infty} 2^r z^r$$

$$b_r \cancel{z^r} = (5.2^r - 1)\cancel{z^r}$$

$$b_r = 5.2^r - 1$$

$$a_r = \sqrt{5.2^r - 1}$$ Ans

$$\boxed{\begin{aligned} b_r &= a_r^2 \\ a_r &= \sqrt{b_r} \end{aligned}}$$

Q) Find the generating function of 2, 4, 8, 16, 32 ___

Ans) $A(z) = a_0 + a_1 z + a_2 z^2 + \dots$

$= \sum_{r=0}^{\infty} a_r z^r$

$= 2 + 4z + 8z^2 + 16z^3 + 32z^4 + \dots$

$= 2(1 + 2z + 2^2 z^2 + 2^3 z^3 + \dots)$

$= 2(1-2z)^{-1} = \frac{2}{1-2z}$

Q) Find the generating function of $a_r = 2^r + 3^r$, $r \geq 0$.

$A(z) = \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} (2^r + 3^r) z^r$

$= \sum_{r=0}^{\infty} 2^r z^r + \sum_{r=0}^{\infty} 3^r z^r$

$= (1 + 2z + 2^2 z^2 + \dots) + (1 + 3z + 3^2 z^2 + \dots)$

$= (1-2z)^{-1} + (1-3z)^{-1}$

$= \frac{1}{1+2z} + \frac{1}{1-3z} = \frac{2-5z}{(1-2z)(1-3z)}$

FUNCTIONS

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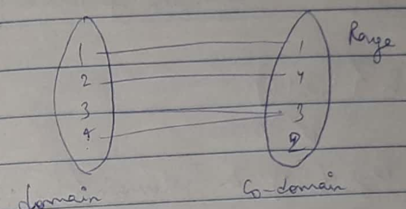
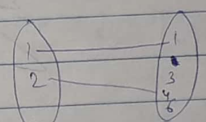
$f : A \to B$



$A = \{1, 2\}$ = domain

$B = \{1, 4, 3, 6\}$ = Co-domain

Range $= \{1, 4\}$



[Equal Functions]

Q) $A = \{1, 2\}$, $B = \{3, 6\}$

$f : A \to B$ s.t. $f(x) = x^2 + 2$ $\quad f(1) = 3$, $f(2) = 6$

$g : A \to B$ s.t. $g(x) = 3x$

$g(1) = 3$

$g(2) = 6$

Here

$f(1) = g(1)$

$f(2) = g(2) \Rightarrow f = g$

• Properties

① INTO function : $\boxed{\text{Codomain} \neq \text{Range}}$

$\{1, 2\}$ & $\{1, 3, 4, 6\}$

Range $= \{1, 4\}$ & Codomain $\{1, 3, 4, 6\}$

$\therefore$ INTO function $\Rightarrow \{1, 3, 4, 6\}$

② ONTO function : $\boxed{\text{Codomain} = \text{Range}}$

$A = \{1, 2, 3, 4\}$ $\quad f(x) = x^2$

$B = \{1, 4, 9, 16\}$

$\therefore$ Range = Codomain.

$\forall b \in B \; \exists a \in A$, s.t. $f(a) = b$

onto ⇒ Surjection & one-one ⇒ injection

③ One-one function : $f(x) = f(y) \Rightarrow x = y$

④ Many one function ⇒ $f : I \to I$

$f(x) = x^2$

$f(-1) = 1$

$f(1) = 1$

$\therefore \{-1, 1\} \to \{1\}$

⑤ Bijective function : when f is both onto and one-one

one one ⇒ $f(x) = f(y) \Rightarrow x = y$

onto ⇒ $\exists y \in B \; \exists x \in A$ s.t. $f(x) = y$

⑥ Cardinally Equivalent Set :

$A \sim B$

$f : A \to B$ ⇒ bijective

Q) $f: R \to R$ s.t. $f(x) = ax+b$ where $a, b \in R$, then show that f is invertible.

Ans) f will be invertible if f is one-one and onto

one one $\Rightarrow$ $x_1, x_2 \in R$ (domain)
s.t. $f(x_1) = f(x_2)$
$ax_1 + b = ax_2 + b$
$\Rightarrow x_1 = x_2$
$\Rightarrow$ f is one-one.

onto : let $y \in R$ (Co-domain)
s.t. $\exists\ \frac{y-b}{a} \in R$ (domain)

$y = ax+b$
$x = \frac{y-b}{a}$

s.t. $f\left(\frac{y-b}{a}\right) = a\left(\frac{y-b}{a}\right)+b$
$= y$

$\Rightarrow$ f is onto.
$\Rightarrow$ f is invertible

$\boxed{f^{-1}(y) = \frac{y-b}{a}}$

Theorem 1: The inverse map $f^{-1}: B \to A$ is also one-one onto.

Proof: let $f: A \to B$ be one one onto then we have to show $f^{-1}: B \to A$ is one one onto

(1) f^{-1} is one one
let $b_1, b_2 \in B$
s.t. $f^{-1}(b_1) = f^{-1}(b_2)$ — (1)
Now, $b_1, b_2 \in B \Rightarrow \exists\ a_1, a_2 \in A$
s.t. $f(a_1) = b_1$, $f(a_2) = b_2$ (f is onto)

$f^{-1}(b_1) = a_1$ & $f^{-1}(b_2) = a_2$

In Eqn (1)
$a_1 = a_2$
$\Rightarrow f(a_1) = f(a_2)$ [f is function]
$\Rightarrow b_1 = b_2$

2) f^{-1} is onto let $a \in A$
$\exists\ b \in B$ s.t. $f(a) = b$ (f is function)
$f^{-1}: B \to A$ $\quad f^{-1}(b) = a$
$\Rightarrow f^{-1}$ is onto

Theorem 2 If $f: A \to B$ & it is one-one onto then show that inverse of f is unique.

Proof: let inverse is not unique.
let $f: A \to B$
i.e. g, h s.t. $g: B \to A$
$h: B \to A$
$\forall\ x \in B$ $g(x) = h(x)$
Now let $b \in B$ and $a_1, a_2 \in A$
$g(b) = a_1 \Rightarrow f(a_1) = b$
$h(b) = a_2 \Rightarrow f(a_2) = b$
$\therefore f(a_1) = f(a_2)$
$a_1 = a_2$ [f is one-one]
$g(b) = h(b)$
$\therefore$ g & h are equal function

$\Rightarrow$ Inverse is always unique.

Q) Determine whether the funⁿ are bijection where $f: R \to R$

(1) $f(x) := -3x+4$
one one $\Rightarrow$ $x_1, x_2 \in R$ (domain)
s.t. $f(x_1) = f(x_2)$
$-3x_1 + \cancel{4} = -3x_2 + \cancel{4}$
$\Rightarrow x_1 = x_2$

$x = \frac{y-4}{-3}$

onto
let $y \in R$ (codomain)
$\exists\ \frac{y-4}{-3} \in R$ (domain)
s.t. $f\left(\frac{y-4}{-3}\right) = -3\left(\frac{y-4}{-3}\right) + 4 = y$

$\therefore$ This is both

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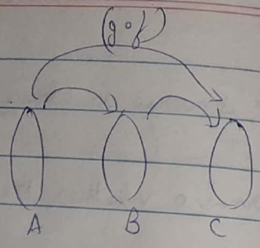
* **Composite function**



$f : A \to B$

$g : B \to C$

$g \circ f \Rightarrow A \to C$

$$g \circ (f(a)) = g[f(a)]$$

Properties:-

$f \circ g \neq g \circ f$ [it is not commutative]

$f \circ (g \circ h) = (f \circ g) \circ h$ [It is associative]

① Composite function is not ~~do~~ commutative.
Here $f \circ g$ cannot be defined as domain of one is not equal to codomain of another.

∘ $f \circ g$ will only be defined when $A = C$.

② Associative: $f \circ (g \circ h) = (f \circ g) \circ h$.

$f : C \to D$

$g : B \to C$

h : $A \to B$

$g \circ h \Rightarrow A \to C$

$f \circ (g \circ h) \Rightarrow A \to D$

$(f \circ g) \Rightarrow B \to D$

$(f \circ g) \circ h \Rightarrow A \to D.$

We will show $f \circ (g \circ h)$ and $(f \circ g) \circ h$ are equal function

Here $f \circ (g \circ h) \Rightarrow A \to D$
also $(f \circ g) \circ h \Rightarrow A \to D$

domain and codomain are same. Now we will show $[f \circ (g \circ h)]a = [(f \circ g) \circ h]a$

Let
$a \in A,\ b \in B,\ c \in D$ & $d \in D$

s.t. $h(a) = b$

$g(b) = c$

$f(c) = d$

$$\begin{aligned}(f \circ [g \circ h])a &= [f[g \circ h]a] \\ &= f(g(h(a))) \\ &= f(g(b)) \\ &= f(c) \\ &= d\end{aligned}$$

$$\begin{aligned}[(f \circ g) \circ h]a &= f \circ g(h(a)) \\ &= f \circ g(b) \\ &= f(g\,b) \\ &= f(c) = d.\end{aligned}$$