Substituting $A \cap B = D$ ...

Permutations

§ 2.6. Permutation. *Definition :*

An arrangement of a finite set of n objects in a given particular order is called ...*rmutation of the objects (taken all at a time).*

Any arrangement of any $r < n$ of these objects in a given order is called a r-permutation a permutation of the n objects taken r at a time.

Notation : We shall use $P(n,r)$ or ${}^nP_r$ to represent the number of permutations of r objects ...t of n objects.

$$P(n,n) = \frac{\quad}{(n-n)!} = \frac{\quad}{0!}$$

For example, if $n = 4$, then the number of permutations

$$P(4,4) = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4 \times 3 \times 2 \times 1 = 24$$

§ 2.7. Permutations with Repeatitions :

Theorem 2.3. *The number of permutations of n objects of which n_1 are alike, n_2 are* ...*ke, ..., n_r are alike is :* $\dfrac{n!}{n_1!, n_2!, \ldots. n_r!}$)

Illustrative Examples

Ex. 1. *There are 8 chairs in a room. In how many ways 5 students can sit on them ?*

Sol. $\because n = 8$, and $r = 5$

$\therefore$ Number of permutation $P(n,r) = P(8,5) = \dfrac{8!}{(8-5)!} = \dfrac{8!}{3!}$

or, number of ways of sitting $= \dfrac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$

$= 6{,}720$ Ans.

$f(x) = g(x)$

$\Rightarrow f = g$

Ex. 2. *In how many ways can three examinations be scheduled within a five-day, so that no two examinations are scheduled on the same day ?*

Sol. Here the examination (Papers) and the days are different, therefore the required of ways to conduct the examination

$$= 5 \times 4 \times 3 = 60$$

Ex. 3. *How many words can be formed from the letters of the word 'KRISHAN'*

Sol. ∵ The number of different letters is 7.

∴ By the formula, permutations $P(n, n) = n!$

Required number of words $= 7!$

$$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040 \text{ words.}$$

Ex. 4. *Out of 7 rooms, 4 rooms are assigned to 4 Programmers as offices and remaining 3 rooms for Computer terminals. Find the number of assignments that can made.*

Sol. Let the programmers be different but Computer terminals are identical. Therefore the required number of assignments

$$= 7 \times 6 \times 5 \times 4 = 840$$

Ans

Ex. 5. *How many words can be formed from the letters of the word 'POSTMAN' when :*

(*a*) Each word starts with P

(*b*) Each word starts with P and ends with N.

(*c*) Three letters MAN always occur simultaneously.

(*d*) Three letters MAN never occur

(*e*) Two vowels are kept at their present places.

(*f*) The letter T always appear in the middle

Sol. (*a*) When each word starts with P, then the place of P is fixed, therefore the words will be formed out of the remaining 6 letters.

The required permutations $= 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 720$ words. Ans

(*b*) When each word starts with P and ends with N, then the places of P and N are fixed therefore the words will be formed out of the remaining 5 letters viz O,S,T,M,A.

The required permutations $= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ words. Ans

(*c*) Three letters MAN always occur simultaneously, therefore MAN can be treated as a single object. As such out of remaining 5 letters, $5\,!$ words can be formed but 3 letters of MAN can also be permuted in $3!$ ways.

Therefore the total number of permutations =

$5! \times 3! = 5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1 = 720$ words. Ans.

(*d*) ∵ Total permutations $= 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$

and the number of words having MAN simultaneously $= 720$

∴ Number of words in which MAN never occur simultaneously

$= 5{,}040 - 720 = 4{,}320$ Ans.

(*e*) Since two vowels are kept at their present places, therefore (7 − 2) = 5 letters are to be permuted.

∴ Permutations $= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ words. **Ans.**

(*f*) Since the letter *T* always appear in the middle, therefore remaining 6 letters are to be permuted.

∴ Permutations $= 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ words. **Ans.**

Ex. 6. ***From the word INDEPENDENCE, how many permutations :***

(*a*) ***can be formed ?***

(*b*) ***which start and end with E ?***

(*c*) ***in which both D occur simultaneously ?***

(*d*) ***in which I and P do not occur simultaneously ?***

Sol. (*a*) Here $n = 12$ Let E = p; N = q; D = r

Here p or (E) = 4; q or (N) = 3; r or (D) = 2;

and I, P, C = 1 each

$$\therefore \text{Total permutations} = \frac{n!}{p!q!r!} = \frac{12!}{4!3!2!} = 16,63,200$$ **Ans.**

(*b*) Out of 4 E, two E are assigned the first and last positions, then

$n = 10$; p or (E) = 2; q or (N) = 3; r or (D) = 2;

and I, P, C = 1 each

$$\therefore \text{Total permutations} = \frac{10!}{2!3!2!} = 75,600$$ **Ans.**

(*c*) When both D occur simultaneously (treating D as one word), then

$n = 11$; p or (E) = 4; q or (N) = 3

and D, I, P, C = 1 each

$$\therefore \text{Total permutations} = \frac{11!}{3!4!} = 2,77,200$$ **Ans.**

(*d*) When I and P occur simultaneously

Number of words containing I and P simultaneously (i.e., treating I and P as a word)

∵ $n = 11$, P or (E) = 4; q or (N) = 3; r or (D) = 2

and I, or, P, C = 1 each

But I and P can be arranged simultaneously in 2 ! ways,

$$\therefore \text{No of permutations} = \frac{11!}{4!3!2!} \times 2! = 2,77,200$$

∴ Number of words which do not contain I and P simultaneously

= 16,63,200−2,77,200 = 13,86,000. **Ans.**

Ex. 7. ***Out of the digits 1, 2, 3, 4, 3, 2, 1, how many numbers :***

(*a*) ***can be constructed ?***

(*b*) ***can be constructed by taking first four digits?***

(*c*) ***odd digits will be at odd places ?***

Sol. (*a*) Here $n=7$; p or $(1)=2$; q or $(2)=2$; r or $(3)=2$

$\therefore$ Total number of permutations $= \dfrac{n!}{p!q!r!} = \dfrac{7!}{2!2!2!} = 630$

(*b*) Numbers which can be constructed with first four digits ($n=4$) are

$$4! = 24$$

(*c*) Placing odd digits at odd places :

$\because$ Four odd digits (1, 3, 3, 1) in which two are same and the other two are also same can be arranged at four odd places and three even digits (2, 4, 2) in which two are same and one is different can be arranged at three even places .

$\therefore$ Total number of permutations $= \dfrac{4!}{2!2!} \times \dfrac{3!}{2!} = 6 \times 3 = 18$

Combinations

§ 2.8. Combination. ***Definition*** : *A group of some or all objects out of finite number of objects irrespective of their positions is called a combination.*

Remark. The permutation of the combinations is meaningless.

For example, if we have pens of three colours say Red (R), Blue (B) and Green (G), out of these three colours, we can arrange three (all) or two pens as follows :

Combinations of 2 colours (pens)

(*i*) R , B

(*ii*) R , G

(*iii*) B , G

Combinations of 3 colours (pens)

(*i*) R, B, G

The number of combinations of some (r) or all (n) out of n colours denoted by $C(n, r)$ or $\binom{n}{r}$ are given by the following formula :

Number of combinations of r objects out of n different objects :

$$^nC_r = C(n,r) = \frac{n!}{r!(n-r)!} \quad C(n,r) = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

§ 2.9. Relation between Permutation and Combination :

$\because$ $\quad P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$ $\quad$ (*i*)

and $\quad C(n,r) = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$ $\quad$ (*ii*)

By (*i*) and (*ii*) ,

(1) $\quad C(n,r) = \dfrac{P(n,r)}{r!}$

(2) ***The number of combinations of r objects taking p objects at a time out of n different objects*** $= C(n-p, r-p)$

(3) ***The number of combinations of r objects when p objects are never selected out of n different objects*** $= C(n-p, r)$

...umber of combinations of 1, 2, 3, 4 .... and n at a time out of n different objects

$$= C(n,1)+C(n,2)+C(n,3)+\ldots\ldots+C(n,n) = 2^n - 1.$$

Illustrative Examples

Ex. 1. *Out of 10 flowers, in how many ways 6 flowers can be selected for the worship of ...e God?*

Sol. Total objects $(n) = 10$ flowers; selected objects $(r) = 4$ flowers

$\therefore$ Total combinations $= C(10,4) = \dfrac{10!}{(10-4)!4!} = \dfrac{10!}{6!4!}$

$$= \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{6\times5\times4\times3\times2\times1\times4\times3\times2\times1} = 210 \text{ ways.}$$ **Ans.**

Ex. 2. *Out of 7 men and 4 women, how many committees of 6 members be formed ...aving atleast two women?*

Sol. The committees can be constituted in following three ways having atleast three ...omen :

(*i*) 2 women and 4 men

(*ii*) 3 women and 3 men

(*iii*) 4 women and 2 men

$\therefore$ The number of committees in the above three cases are

$$= [C(4,2)\times C(7,4)] + [C(4,3)\times C(7,3)] + [C(4,4)\times C(7,2)]$$

$$= \left(\frac{4\times3}{2\times1}\times\frac{7\times6\times5\times4}{4\times3\times2\times1}\right) + \left(\frac{4}{1}\times\frac{7\times6\times5}{3\times2\times1}\right) + \left(\frac{1}{1}\times\frac{7\times6}{2\times1}\right)$$

$$= 210 + 140 + 21 = 371$$

Ans.

Ex. 3. *A trader wants to distribute gift packets of 4 items out of 10 different items. In ...w many ways the trader can select items out of these items?*

Sol. Total objects $(n) = 10$ items

and number of selected objects at a time $(r) = 4$ items

$\therefore$ Total number of combinations $= C(10,4) = \dfrac{10!}{(10-4)!\,4!} = \dfrac{10!}{6!4!}$

$$= \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{6\times 5\times 4\times 3\times 2\times 1\times 4\times 3\times 2\times 1} = 21$$

Ex. 4. *Out of 15 hockey players, in how many ways a team be formed, if:*

(a) *a team has 11 players.*

(b) *all the teams have the same captain*

(c) *two players are not included in any of the team.*

Sol. Total players $n = 15$ and players in the team $r = 11$

$\therefore$ (a) Number of teams $= C(15,11) = \dfrac{15}{(15-11)!(11)!} = \dfrac{15!}{(4)!(11)!}$

$$= \frac{15\times 14\times 13\times 12}{4\times 3\times 2\times 1} = 1365 \text{ ways}$$

(b) Since all the teams are having the same captain, so $n = 15 - 1 = 14$

The number of players to be selected $11 - 1 = 10$

$\therefore$ Number of teams $= C(14,10) = \dfrac{14!}{(14-10)!(10)!} = \dfrac{14!}{(4)!(10)!}$

$$= \frac{14\times 13\times 12\times 11}{4\times 3\times 2\times 1} = 1{,}001 \text{ ways}$$

(c) Since two players are not to be included in any of the team, so 11 players are to be selected out of the remaining $15 - 2 = 13$ players.

$\therefore$ Number of teams $= C(13,11) = \dfrac{(13)!}{(13-11)!\,(11)!} = \dfrac{(13)!}{(2)!(11)!}$

$$= \frac{13\times 12}{2\times 1} = 78$$

Ans

Ex. 5. ***There are 6 white, 4 blue and 5 green balls in a bag. In how many ways the balls are drawn ?***

(a) ***any four balls***

(b) ***three white balls***

(c) ***two white and one blue balls***

(d) ***one ball of each colour.***

Sol. Total balls in the bag $(n) = 6 + 4 + 5 = 15$

(a) Any 4 balls can be taken out in $C(15, 4)$ ways

$\therefore$ No of combinations $= \dfrac{15!}{(15-4)!\,(4)!} = \dfrac{15!}{(11)!(4)!}$

$$= \frac{15\times 14\times 13\times 12}{4\times 3\times 2\times 1} = 1{,}365 \text{ ways.}$$

Ans.

(b) Out of 6 white balls, 3 white balls can be drawn in $C(6,3)$ ways.

$\therefore$ No of combinations $= \dfrac{6!}{(6-3)!(3)!} = \dfrac{6!}{3!3!} = \dfrac{6\times 5\times 4}{3\times 2\times 1} = 20$

Ans.

(c) Out of 6 white and 4 blue balls, 2 white and 1 blue balls respectively can be drawn in $C(6,2)\times C(4,1)$ ways.

$\therefore$ No of combinations $= \dfrac{6!}{(6-2)!(2)!}\times\dfrac{4!}{(4-1)!(1)!} = \dfrac{6\times5}{2\times1}\times\dfrac{4}{1} = 60$ Ans.

(d) 1 ball of each colour can be drawn in $C(6,1)\times C(4,1)\times C(5,1)$ ways.

$\therefore$ No of combinations $= \dfrac{6!}{(6-1)!(1)!}\times\dfrac{4!}{(4-1)!(1)!}\times\dfrac{5!}{(5-1)!(1)!}$

$$=\frac{6}{1}\times\frac{4}{1}\times\frac{5}{1}=120$$

Ans.

Ex. 6. ***Find the number of different out comes when 3 dice are rolled.***

Sol. Here throwing 3 dice is equivalent to the selections of 3 digits out of 6 digits 1, 2, 3, 4, 5 and 6 where repeation is allowed.

Therefore the required number of different results

$$= C(6+3-1, 3) = C(8, 3) = 56$$

Ans.

Ex. 7. ***Find the number of ways to seat 5 boys in a row of 12 chairs.***

Sol. The given problem is equivalent to selection of 12 objects out of 6 different objects, where every student is of different types and 7 (empty chairs) of the same type of objects.

Therefore by the formula, the number of arrangements $= \dfrac{12!}{7!}$ Ans.

Aliter : First of all, we arrange 5 students in a row by 5 ! methods. Then the remaining 7 empty chairs are distributed arbitrary either to 2 students or at both the extremities. Thus this problem of distribution is equivalent to keep 7 balls of the same colour in 6 boxes.

Therefore number of methods = $5!\times C(6+7-1,7)$

$$= 5!\times\frac{12!}{7!\,5!}=\frac{12!}{7!}$$

Ans.

Ex. 8 . ***A person has 6 friends. In how many ways he can invite one or more for the lunch?***

Sol. Out of 6 friends, 1,2,3,4,5 and 6 may be invited for the lunch. Therefore the number of combinations of 1 or more friends

$$= C(6,1)+C(6,2)+C(6,3)+C(6,4)+C(6,5)+C(6,6)$$

$$= 6+15+20+15+6+1=63$$

Ans:

Using formula (2^n-1), no of combinations $= (2^6-1)$

$$= 64-1=63.$$

Ans.