Joint entropy is given as,

$$H(X, Y) = \sum_{i=1}^{M} \sum_{j=1}^{M} P(x_i y_j) \log_2 \frac{1}{P(x_i y_j)}$$

$$= P(x_1 y_1) \log_2 \frac{1}{P(x_1 y_1)} + P(x_1 y_2) \log_2 \frac{1}{P(x_1 y_2)}$$

$$+ P(x_2 y_1) \log_2 \frac{1}{P(x_2 y_1)} + P(x_2 y_2) \log_2 \frac{1}{P(x_2 y_2)}$$

$$= \frac{2}{9} \log_2 \frac{9}{2} + \frac{1}{9} \log_2 9 + \frac{2}{30} \log_2 \frac{30}{2} + \frac{18}{30} \log_2 \frac{30}{18}$$

$$= 1.5365 \text{ bits/symbol}$$

(v) To obtain conditional entropies $H(X/Y)$ and $H(Y/X)$:

$H(Y/X)$ is given as,

$$H(Y/X) = H(X,Y) - H(X)$$

$$= 1.5365 - 0.9182 = 0.6183 \text{ bits/symbol}$$

and $H(X/Y)$ is given as,

$$H(X/Y) = H(X,Y) - H(Y)$$

$$= 1.5365 - 0.8672 = 0.6692 \text{ bits/symbol}$$

(vi) To obtain mutual information $I(X;Y)$:

Mutual information is given as,

$$I(X;Y) = H(X) - H(X/Y)$$

$$= 0.9182 - 0.6692 = 0.249 \text{ bits/symbol}$$

Also $$I(X;Y) = H(Y) - H(Y/X)$$

$$= 0.8672 - 0.6183 = 0.249 \text{ bits/symbol}$$

Thus both the equations have same result.

Ex.2.12.8 *The channel transition matrix is given by,*

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

A

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Draw the channel diagram and determine the probabilities associated with outputs assuming equiprobable inputs. Also find the mutual information $I(X;Y)$ for the channel. **[Dec-2002, 10 Marks]**

Sol. : The given data is,

$$P = \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_1/x_2) & P(y_2/x_2) \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

Inputs are equiprobable. Hence probabilities of two input symbols are,

$$P(x_1) = 0.5 \quad \text{and} \quad P(x_2) = 0.5$$

(i) To draw the channel diagram :

Fig. 2.12.3 shows the channel diagram based on given channel transition matrix.

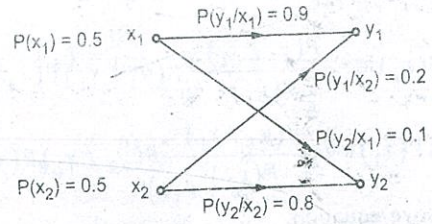

Fig 2.12.3 Channel diagram

(ii) To obtain output symbol probabilities :

Probabilities of output are given from equation 2.11.12 as,

$$\begin{bmatrix} P(y_1) \\ P(y_2) \end{bmatrix} = \left[P(x_1) \quad P(x_2)\right] \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_1/x_2) & P(y_2/x_2) \end{bmatrix}$$

Putting values in above equation

$$\begin{bmatrix} P(y_1) \\ P(y_2) \end{bmatrix} = [0.5 \quad 0.5] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} (0.5 \times 0.9) + (0.5 \times 0.2) \\ (0.5 \times 0.1) + (0.5 \times 0.8) \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}$$

Thus the probabilities of output are,

$$P(y_1) = 0.55 \quad \text{and} \quad P(y_2) = 0.45$$

(iii) To obtain mutual information :

Mutual information is given by equation 2.12.2 as,

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i y_j) \log_2 \frac{P(x_i / y_j)}{P(x_i)} \quad \text{... (2.12.40)}$$

From probability theory we know that,

$$P(x_i y_j) = P(x_i / y_j) P(y_j)$$

also $$P(x_i y_j) = P(y_j / x_i) P(x_i)$$

$$\therefore \quad P(x_i / y_j) P(y_j) = P(y_j / x_i) P(x_i)$$

$$\frac{P(x_i / y_j)}{P(x_i)} = \frac{P(y_j / x_i)}{P(y_j)}$$

Hence mutual information of equation 2.12.40 becomes,

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(y_j / x_i) P(x_i) \log_2 \frac{P(y_j / x_i)}{P(y_j)}$$

$$= P(y_1 / x_1) P(x_1) \log_2 \frac{P(y_1 / x_1)}{P(y_1)} + P(y_1 / x_2) P(x_2) \log_2 \frac{P(y_1 / x_2)}{P(y_1)}$$

$$+ P(y_2 / x_1) P(x_1) \log_2 \frac{P(y_2 / x_1)}{P(y_2)} + P(y_2 / x_2) P(x_2) \log_2 \frac{P(y_2 / x_2)}{P(y_2)}$$

Putting values in above equation,

$$I(X;Y) = (0.9)(0.5) \log_2 \frac{0.9}{0.55} + (0.2)(0.5) \log_2 \frac{0.2}{0.55}$$

$$+ (0.1)(0.5) \log_2 \frac{0.1}{0.45} + (0.8)(0.5) \log_2 \frac{0.8}{0.45}$$

$$= 0.3197 - 0.1459 - 0.1085 + 0.332 = 0.3973 \quad \text{bits/symbol.}$$

Ex.2.12.9 *A channel matrix for the ternary channel is given below.*

c: class

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 1-p \\ 0 & 1-p & p \end{bmatrix}$$

Assuming source probabilities as $P(x_1) = P$ and $P(x_2) = P(x_3)$, determine the source entropy $H(X)$ and the mutual information $I(X;Y)$. Also determine the capacity of the channel. **[May-2003, 8 Marks]**

Sol. : Given data

The channel transition matrix is given, i.e.,

$$P = \begin{bmatrix} P(y_1 / x_1) & P(y_2 / x_1) & P(y_3 / x_1) \\ P(y_1 / x_2) & P(y_2 / x_2) & P(y_3 / x_2) \\ P(y_1 / x_3) & P(y_2 / x_3) & P(y_3 / x_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 1-p \\ 0 & 1-p & p \end{bmatrix}$$

Based on above equation the channel diagram is shown below :

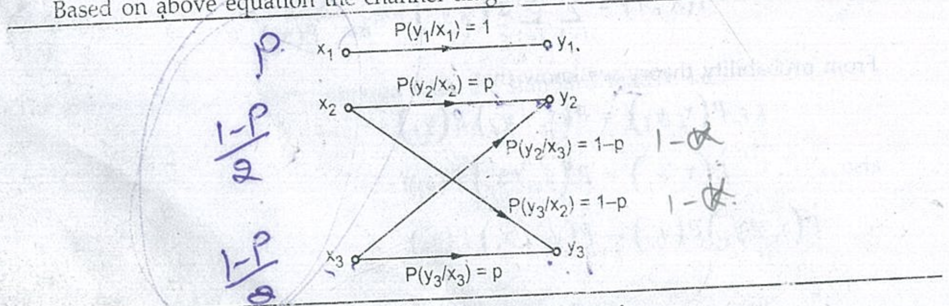

Fig. 2.12.4 Ternary channel

And $P(x_1) = P$ Hence,

$$P(x_1) + P(x_2) + P(x_3) = 1 \quad \text{Since } P(x_2) = P(x_3)$$

$$P + P(x_2) + P(x_3) = 1$$

$$\therefore \quad P(x_2) = \frac{1-P}{2}$$

$$P(x_3) = \frac{1-P}{2}$$

(i) To determine the source entropy $H(X)$:

Source entropy is given as,

$$H(X) = \sum_{k=1}^{M} P(x_i) \log_2 \frac{1}{P(x_i)}$$

$$= P \log_2 \frac{1}{P} + \left(\frac{1-P}{2}\right) \log_2 \frac{1}{\left(\frac{1-P}{2}\right)} + \left(\frac{1-P}{2}\right) \log_2 \frac{1}{\left(\frac{1-P}{2}\right)}$$

$$= P \log_2 \frac{1}{P} + (1-P) \log_2 \left(\frac{2}{1-P}\right) \quad \text{... (2.12.41)}$$

(ii) To obtain $H(X/Y)$:

We know that $H(X/Y)$ is given as,

$$H(X/Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i y_j) \log_2 \frac{1}{P(x_i / y_j)} \quad \text{... (2.12.42)}$$

Here we have to calculate $P(x_i y_j)$ and $P(x_i / y_j)$

From probability theory we know that

$$P(x_i y_j) = P(y_j / x_i) P(x_i) ;$$

$$\therefore \quad P(x_1 y_1) = P(y_1 / x_1) P(x_1) = 1 \times P = P$$

$$P(x_1 y_2) = P(y_2 / x_1) P(x_1) = 0 \times P = 0$$

$$P(x_1 y_3) = P(y_3 / x_1) P(x_1) = 0 \times P = 0$$

$$P(x_2 y_1) = P(y_1 / x_2) P(x_2) = 0 \times \frac{1-P}{2} = 0$$

$$P(x_2 y_2) = P(y_2 / x_2) P(x_2) = \alpha \times \frac{1-P}{2} = p\left(\frac{1-P}{2}\right)$$

$$P(x_2 y_3) = P(y_3 / x_2) P(x_2) = (1-p) \times \frac{1-P}{2} = (1-p)\left(\frac{1-P}{2}\right)$$

$$P(x_3 y_1) = P(y_1 / x_3) P(x_3) = 0 \times \frac{1-P}{2} = 0$$