DMS

Graph

- **Graph $G = (V, E)$**
  A graph written as $G = (V, E)$ consist of two components:

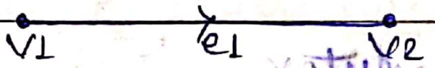
(1) The set of vertices V, also called points or nodes.

(2) The set of edges E, also called lines or arcs, connecting pair of vertices.

Each pair of vertices, connected by an edge, is called the end points or end vertices.

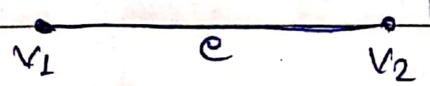
- **Directed Graph $G = (V, E)$**
  A directed graph (V, E) consist of a set of vertices V and a set of edges E that are ordered pair of elements of V.

ex: $G_1 : (\{v_1, v_2\}, \{e_1\})$



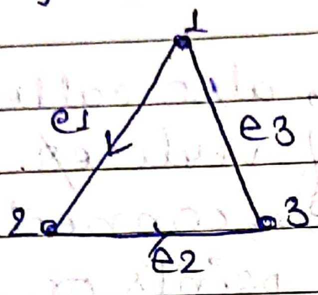
- **Undirected Graph:**
  A graph $G = (V, E)$, in which every edge is associated with an unordered pair of vertices.

ex: $G_1 = (\{v_1, v_2\}, \{e\})$



- **Mixed Graph:**
In a graph $G=(V,E)$, if some edges are directed & some are undirected then it is called Mixed graph.

ex: $G=(\{1,2,3\},\{e_1,e_2,e_3\})$



- **Isolated vertex:**
A vertex v which is not connected with any vertex of a graph G, by an edge is called Isolated vertex.

ex:

v_3 is the isolated vertex.

- **Null Graph:**
If all the vertices of a graph are isolated [i.e set $E=\phi$], the graph is called null graph.
(or a totally disconnected graph).

ex: $G=(\{v_1,v_2,v_3\},\phi)$

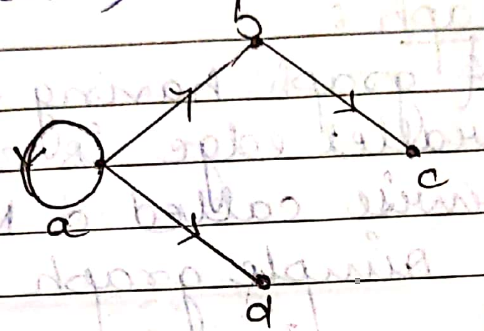
Self loop: An edge having the same vertex as both its end vertices is called self-loop.

∴ a loop is an edge of the form (a, a).

Let, V = (a, b, c, d)

E = {(a, a), (a, b), (a, d), (b, c)} set of edges.

then, G = (V, E).



The edge (a, a) is represented by a closed curve drawn at a and is called self-loop.

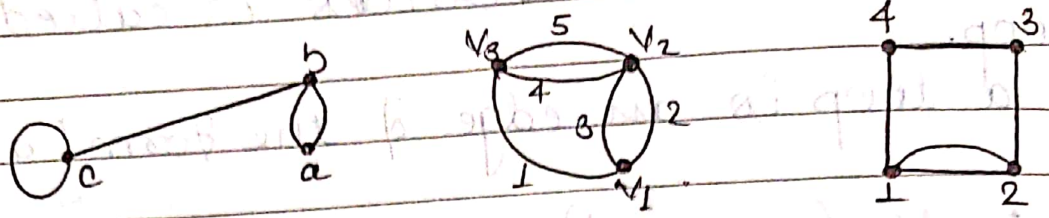
Initial & terminal vertices:

Let G = (V, E) be a graph and let e ∈ E be a directed edge associated with the ordered pair of vertices (u, v). Then the edge e is said to be initially or originating in the vertex u & terminating or ending in the vertex v. The vertices u & v are called the initial & terminal vertices respectively of the edge e. An edge e ∈ E which joins the vertex u & v is said to be incident on the vertices u & v.

Parallel edges or Multiple edges:

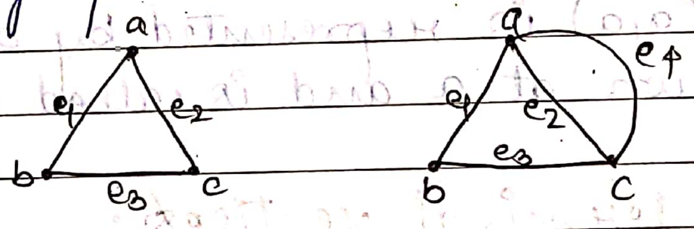
If a pair of vertices is joined b

more than one edge then these edges are called parallel edges.



Here parallel edges are a, b; 2, 3. & 4, 5; 1, 2 &

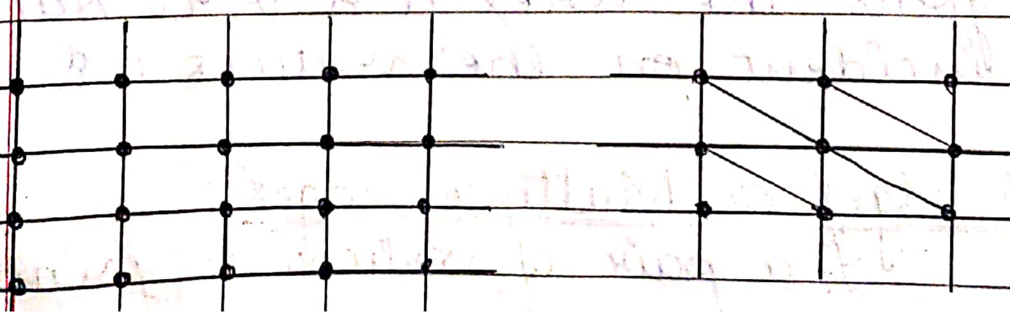
- **Simple Graph:**
A graph having neither self-loop nor parallel edge is called a simple graph otherwise called a Multigraph.
Here G_1 is simple graph & G_2 is Multigraph.



G_1 G_2

- **Finite & infinite graph:**
In a graph $G = (V, E)$, if the set V & E are both finite then it is called a finite graph otherwise a infinite graph.

ex: $V = \{V_1, V_2, V_3 \ldots\}$ $E = \{e_1, e_2, \ldots\}$, then the graph is Infinite graph.



- **Order of a graph:**
  The no. of vertices in a graph, is called its order.

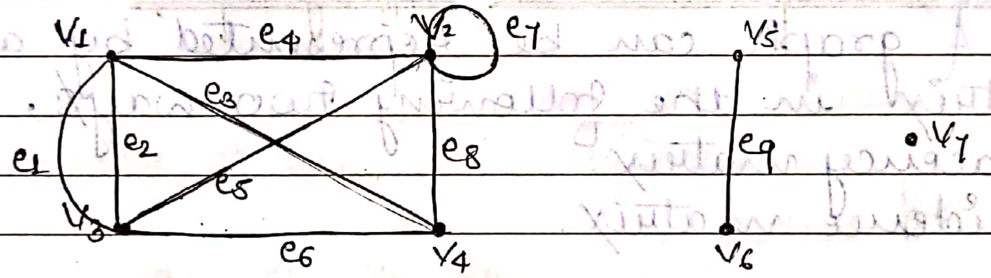
- **Size of a graph:**
  The no. of edges in a graph is called its size.

NOTE: for calculating size, a loop is treated to be of size 2.

ex: Find no. of vertices, edges, loops, isolated vertex, parallel edges in the following graph

G_1: $\cdot v_1$ $\cdot v_2$ $\cdot v_3$ $\cdot v_4$.

G_2:



Sol^n: G_1 has only 4 vertices & denoted as disconnected graph v_1, v_2, v_3, v_4.

$G_2 = (\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}, \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\})$

has i) 7 vertices ii) 9 edges

iii) one self loop, e_7 iv) one isolated vertex v_7.

v) a pair of parallel edges e_1 & e_2 with end vertices v_1 & v_3.

- **Adjacent edges & Adjacent vertices:**

Two non-parallel edges are said to be adjacent if both are incident on a common vertex.

In fig. e_1 & e_2; e_2, e_3 & e_3, e_4, e_5 are adjacent edges.

Two vertices connected by an edge are called adjacent vertices.

In fig. v_1, v_2; v_2, v_3; v_3, v_4; v_4, v_5; v_4, v_6 are adjacent vertices.

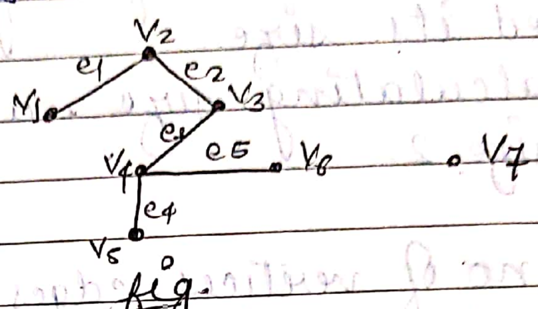


fig.

- **Matrix representation of graphs:**

A graph can be represented by a matrix in the following two ways,

(1) Adjacency matrix.

(2) Incidence matrix.

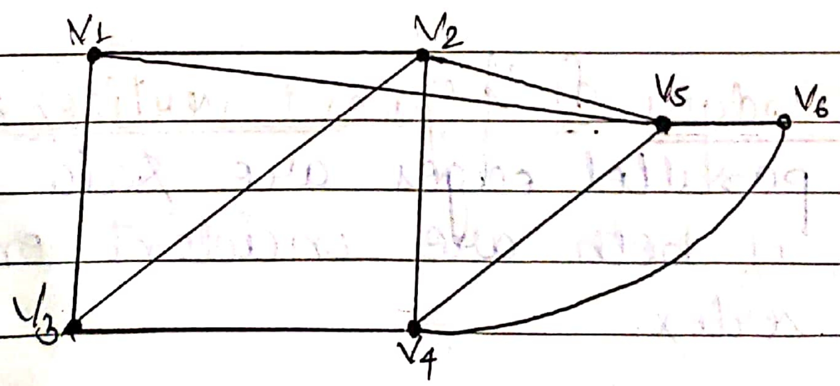
1. **Adjacency Matrix:**

Let $G = (V, E)$ be a graph with n vertic then the adjacency matrix of G, denoted by $A(G)$ is a symmetric matrix $A(G) = [a_{ij}]_n$

where, $a_{ij} = K$.

if there are K edges b/w v_i & v_j.

ex:



then adjacency matrix $A(G)$ is displayed as:

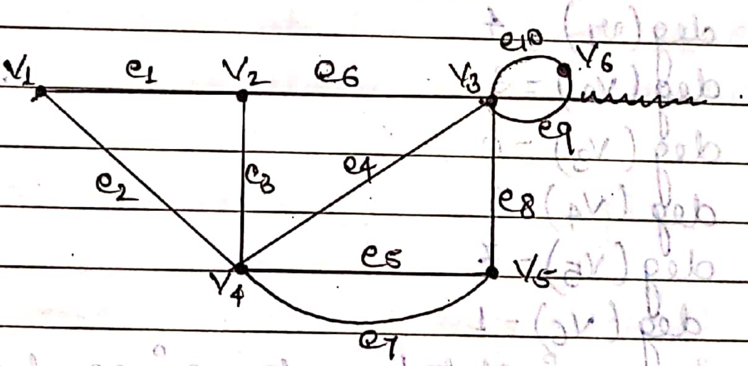
$$A(G) = \begin{array}{c|cccccc|} & V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \\ \hline V_1 & 0 & 1 & 1 & 0 & 1 & 0 \\ V_2 & 1 & 0 & 1 & 1 & 1 & 0 \\ V_3 & 1 & 1 & 0 & 1 & 0 & 0 \\ V_4 & 0 & 1 & 1 & 0 & 1 & 1 \\ V_5 & 1 & 1 & 0 & 1 & 0 & 1 \\ V_6 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

- **Incidence Matrix:**

Let G be a graph with n vertices, e edges, and no self loops. Then the incidence matrix of G, denoted by $I(G) = (a_{ij})_{n \times e}$, is an $n \times e$ matrix, where

$$a_{ij} = \begin{cases} 1, & \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } v_i \\ 0, & \text{otherwise.} \end{cases}$$

ex:



$$I(G) = \begin{array}{c|cccccccccc|} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \\ \hline V_1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ V_2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ V_3 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ V_4 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ V_5 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ V_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

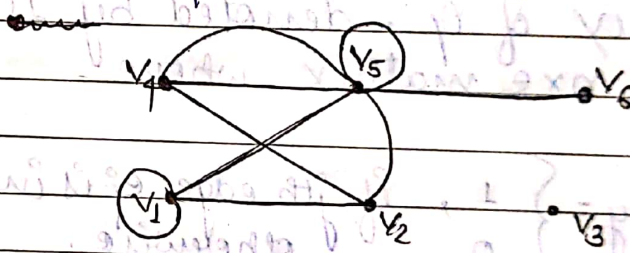
- **Degree of vertices:**

The no. of edges incident on a vertex v of a graph is called the degree of the vertex v and is denoted as deg(v).

- **Pendant vertex:**

A vertex of degree one is known as the pendant vertex.

ex: Find the degree of each vertex in the following graph. Also find if there is any isolated vertex or pendant vertex.



$deg(v_1) = 4$
$deg(v_2) = 3$
$deg(v_3) = 0$
$deg(v_4) = 3$
$deg(v_5) = 7$
$deg(v_6) = 1$

Here, v_3 is an isolated vertex since $deg(v_3) = 0$ and v_6 is a pendant vertex since $deg(v_6) = 1$.

- **Even & odd vertices:**

In a graph, if the degree of a vertex is an even integer then the vertex is called even vertex, and if the degree of a vertex is an odd integer the vertex is known as the odd vertex.

Theorem 1: The Handshaking theorem:

The sum of the degrees of all the vertices in a graph is equal to twice the no. of edges in the graph.

Proof

Each edge contributes two to the sum of degrees of vertices because an edge is incident with exactly two vertices. Therefore if $v_1, v_2 --- v_k$ are the vertices then $d(v_1) + d(v_2) + d(v_3) + -- d(v_k) = 2 \times n$ or

$$\sum_{i=1}^{k} d(V_i) = 2n$$; where n is no. of edges

Theorem 2 :- The number of vertices of odd degree in graph is always even.

Proof :- Let $G(V,E)$ be a graph. V_e and V_o be the set of vertices of even degree & odd degree i.e $V = V_e \cup V_o$

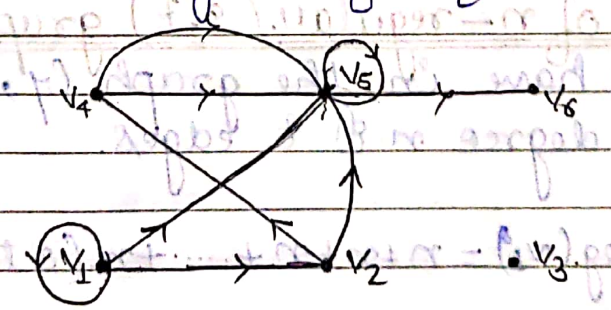
Now $\sum d(V_i) = 2n \Rightarrow \sum d(V_e) + \sum d(V_o) = 2n$.

Now $\sum d(V_e) =$ even number $= 2k \Rightarrow \sum d(V_o) = 2n - 2k = 2(n-k)$

Degree sequence of a graph:

If we find the degree of each vertex of a graph & write them in an ascending order, the sequence so obtained is known as the degree sequence of the graph.

ex: Find the degree of sequence of graph G.



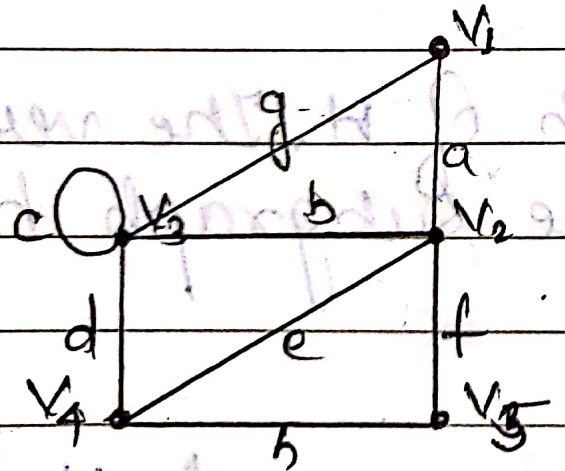
In graph G: $deg(V_1) = 4$, $deg(V_2) = 3$, $deg(V_3) = 0$
$deg(V_4) = 3$, $deg(V_5) = 7$, $deg(V_6) = 1$

The deg. of sequence: $\langle 0, 1, 3, 4, 7 \rangle$

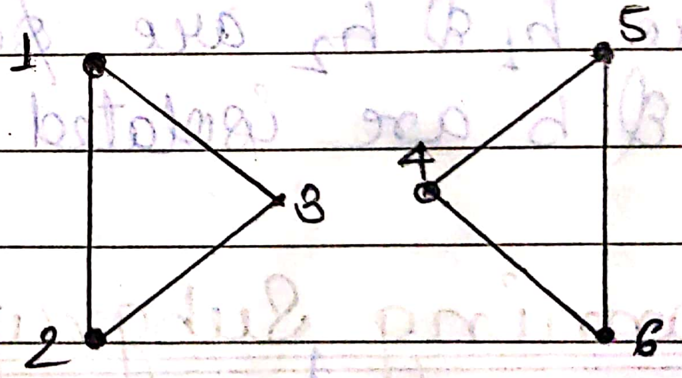
Connected & Disconnected graph:

A graph G is said to be connected if there is at least one path b/w every pair of vertices in G, otherwise it is said to be disconnected.

A disconnected graph consist of two or more connected subgraphs. Each of these subgraph is called component.



connected graph



Disconnected graph

- **Isomorphic graph:**

Two graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic to each other if there exist a bijection mapping f from V_1 to V_2.

i.e $f: V_1 \to V_2$ such that for each of the vertices v_i, v_j of V_1.

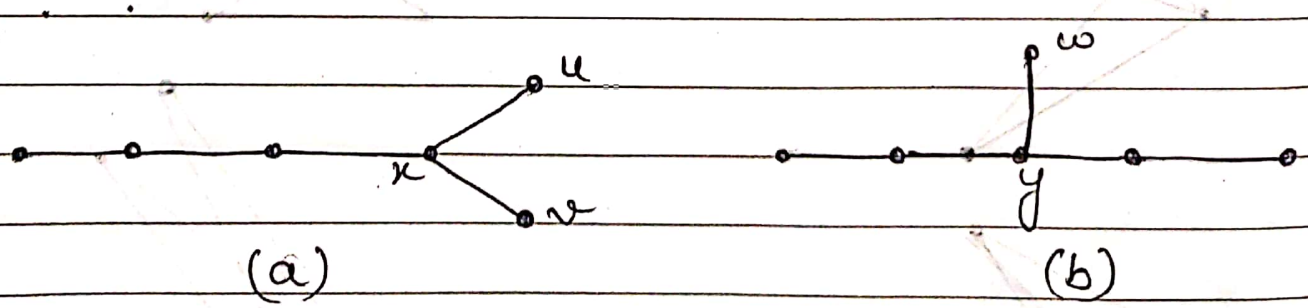
$\{v_i, v_j\} \in E_1 \Rightarrow \{f(v_i), f(v_j)\} \in E_2$.

The function f is called an isomorphism from G_1 to G_2.

NOTE: It is immediately apparent by the def^n of isomorphism that two isomorphic graphs must have:

(1) Same no. of vertices.
(2) Same no. of edges.
(3) An equal no. of vertices with a given degree (i.e same degree sequence).

However, these cond^n are by no means sufficient. For instance, the two graphs given below satisfy all 3 cond^n, yet they are not isomorphic.

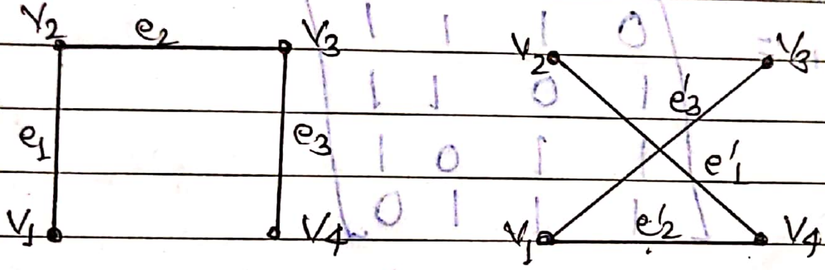


Two graphs that are not isomorphic.

- ## Self — complementary graph :

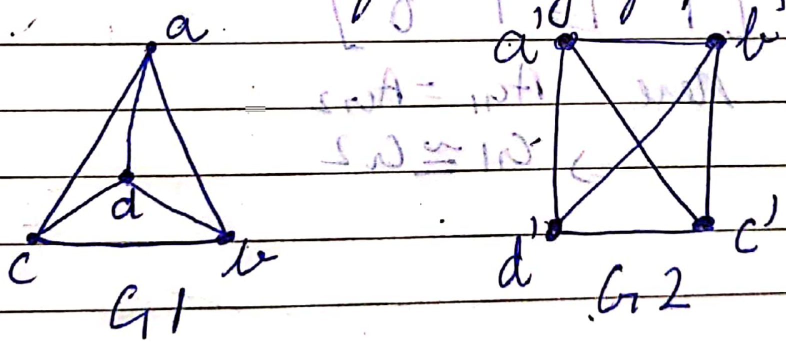
A simple graph $G=(V,E)$ is a self complementary graph if the graph G and its complementary graph $\overline{G}$ ~~is~~ are isomorphic.

ex: The following graph G is a self-complementary graph & $\overline{G}$ is the complementary graph of G.



G $\qquad\qquad\qquad\qquad$ $\overline{G}$

Q Show that following graphs are isomorphic.



sol^n 1) G1 and G2 have same no. of vertices i.e 4

2) G1 and G2 have same no of edges i.e 6

3) ~~All the vertice~~ Degree sequence of G1 and G2 are same i.e $\{3,3,3,3\}$

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Let us define a mapping

$$f: V1 \to V2$$
s.t $f(a) = a'$
$f(b) = b'$
$f(c) = c'$,
$f(d) = d'$

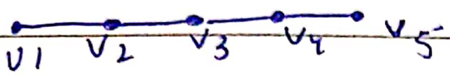
Now adjacency matrix of G1 and G2 is given by

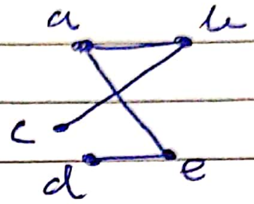
$$A_{G1} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\cancel{AG}.\ A_{G2} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

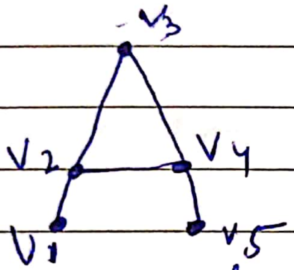
~~AG~~ Here $A_{G1} = A_{G2}$
$\Rightarrow G1 \cong G2$

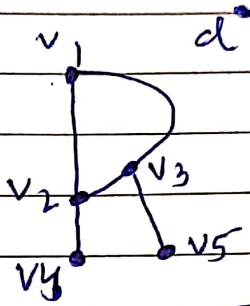
Determine whether the following graphs are isomorphic or not

a) 

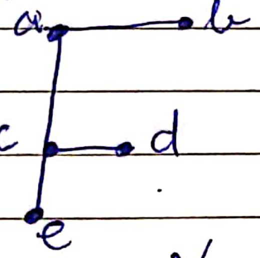


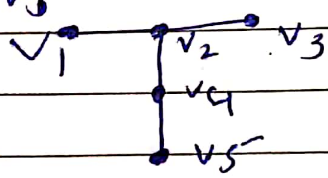
(iso)

b) 

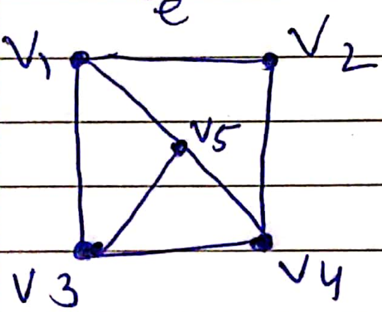


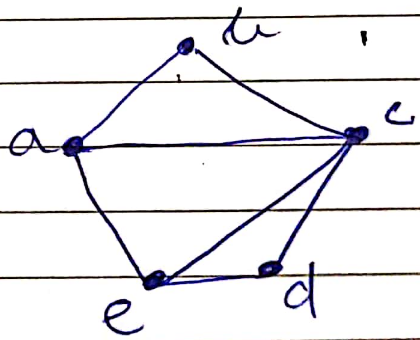
(iso)

c) 



(iso)

d 



(non iso)