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Graph Theory - II

* **Path, Circuits and Cycles:-**

Let u and v be two vertices in a graph G. A path from u to v in G is an alternating sequence of vertices & edges of G having the form

$$u = v_1, e_1, v_2, e_2, \ldots\ldots, v_{n-1}, e_{n-1}, v_n, e_n, v_{n+1} = v,$$

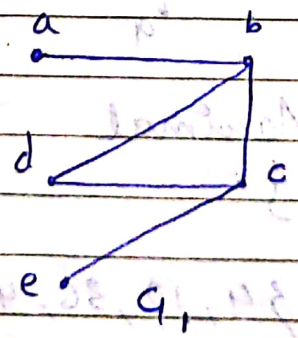
beginning with vertex u called initial vertex and ending with vertex v called the terminal vertex.

If the graph G is directed, the path is called a directed path.

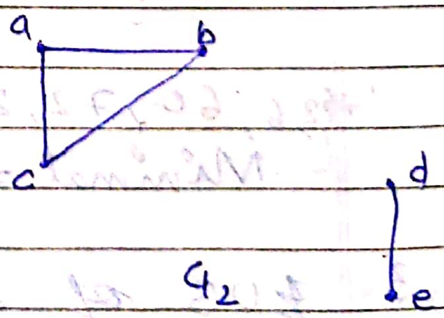
* **Connected graph:-**

Let G be a graph. A vertex u is said to be connected to a vertex v if there is a u-v path in G.

A graph G is called a connected graph if for any two vertices u, v of G, there is a u-v path in G, otherwise it is called disconnected graph.



(connected graph)



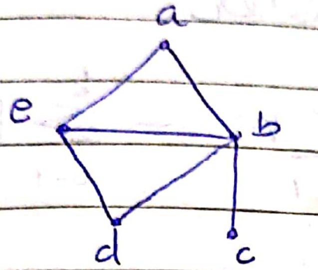
(Disconnected graph)

* Component of a graph:- (Maximal connected sub graph of G):-

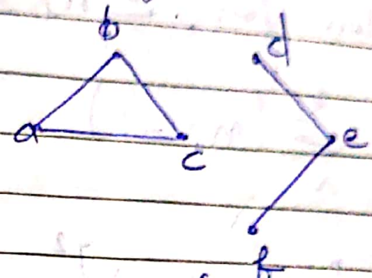
A subgraph H of graph G is called a component of G if

i) any two vertices of H are connected in H.

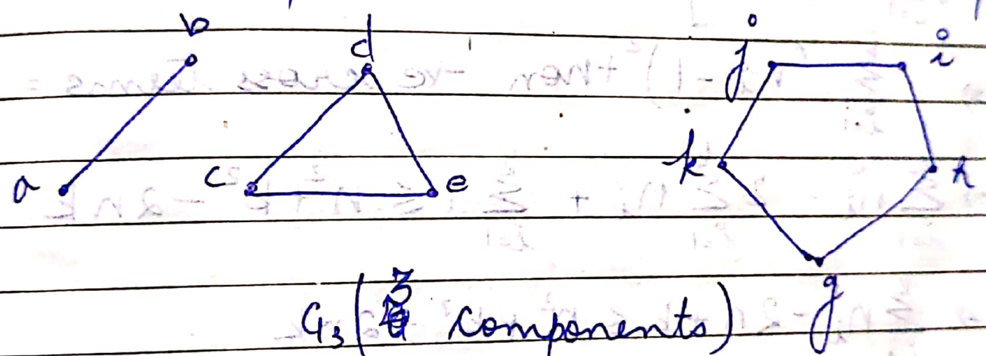
ii) H is not properly contained in any connected subgraph of G.



G_1 (1 component)



G_2 (2 components)



G_3 (3 components)

Theorem:- A simple graph with n vertices & k components cannot have more than $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof:- Let n_i be the no.'s of vertices in i^{th} component, $1 \le i \le k$. Then $\sum_{i=1}^{k} n_i = n$ ——(1)

A component with n_i vertices will have maximum no. of edges when it is complete. The no. of edges in a complete graph K_{n_i} is

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$$\frac{1}{2} n_i(n_i - 1) \quad \text{——— (2)}$$

Hence the maximum no. of edges is

$$\frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1) = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right] \quad [\text{using (i)}] \quad \text{——— (3)}$$

Now for $\sum_{i=1}^{k} n_i^2$, consider

$$\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$$

$$\Rightarrow \left[\sum_{i=1}^{k}(n_i-1)\right]^2 = (n-k)^2 = n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 + \text{non -ve cross terms} = n^2+k^2-2n$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \leq n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2n + k \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k \quad \text{——— (4)}$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$$

Substituting (4) in (3), we get,

$$\frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1) \leq \frac{1}{2}\left[n^2 - (k-1)(2n-k) - n\right]$$

$$= \frac{1}{2}\left(n^2 - 2nk + k^2 + n - k\right)$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

Hence Proved.

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Theorem:- Show that a simple graph G with n vertices is connected if it has more than $\frac{1}{2}(n-1)(n-2)$ edges.

Proof:- A simple graph is connected, if it has only one component.
Let the graph is not connected and has two components. By theorem, the maximum no. of edges is

$\frac{1}{2}(n-2)(n-2+1) = \frac{1}{2}(n-1)(n-2)$

So if the no. of edges is more than $\frac{1}{2}(n-1)(n-2)$ the graph will get connected. Hence the result.

* Eulerian Path & Circuits :-

Euler Path :- A path is a connected graph G is called Euler path if it includes every edge exactly once. Since the path contains every edge exactly once, it is also called Euler trail.

Euler Circuit :- An euler path that is a circuit is called Euler ~~(or Eulerian) graph.~~ circuit i.e a closed Euler path is Euler circuit.

* Hamiltonian Path & Circuit :-

Hamiltonian Path :-
A path is a connected graph G is called Hamiltonian ~~cycle~~ path if it ~~contains~~ includes every vertex exactly once.

Hamiltonian cycle :-
A cycle in a connected graph G

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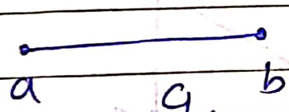
is called Hamiltonian cycle if it contains each vertex of G exactly once except the starting & ending vertex, which are same (edges of cycle are distinct).

A graph G which has a Hamiltonian cycle is called a Hamiltonian graph.

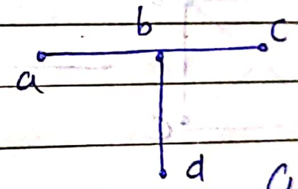
Q. Give an example of connected graph that has:-

i) Neither an Euler circuit Nor a Hamiltonian cycle.
ii) A Euler circuit but no Harmiltonian Cycle
iii) A Hamiltonian cycle but no Euler circuit
iv) Both " " and " "

Ans. i)

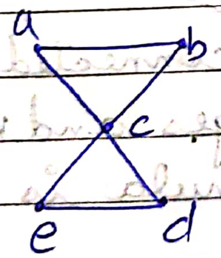


G_1

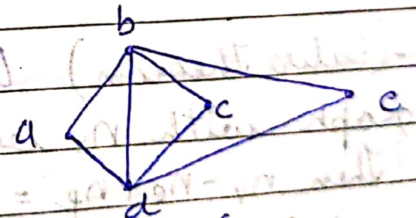


G_2

ii)



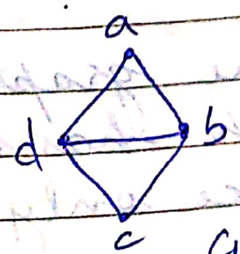
G_1



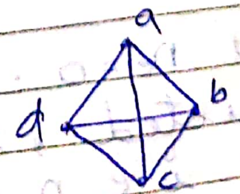
G_2

Euler circuit : c, a, b, c, d, e, c in G_1
and : a, b, d, c, b, e, d, a in G_2

iii)



G_1



G_2

Hamiltonian cycle : a, b, c, d

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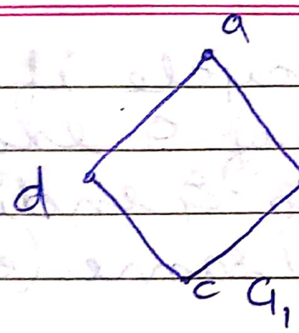
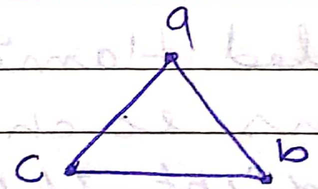
iii)





In G_1 Euler Circuit : a, b, c, d, a
Hamiltonian cycle : a, b, c, d, a

In G_2 Euler circuit and Hamiltonian cycle are a, b, c, a.