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Lab

1) $E[f(X)] = \int_{\infty} dx\, f(x)\, P_X(x)$

a) $E[f(W) + g(U)]$

$= \int_{\infty} dx \left(f(W) + g(U)\right) P_X(x)$

$= \int_{\infty} dx\, f(W)\, P_X(x) + \int_{\infty} dx\, g(U)\, P_X(x)$

$= E[f(W)] + E[g(U)]$

b) $E[X + a] = \int_{\infty} dx\, (X + a)\, P_X(x)$

$= \int_{\infty} dx\, x\, P_X(x) + a \underbrace{\cancel{\int_{\infty} dx\, P_X(x)}}_{1}$

$= E[X] + a$

(Since the sum of all probabilities is 1.)

c) $E[aX] = \int_{\infty} dx\,(aX)\, P_X(x) = a\int_{\infty} dx\, x\, P_X(x)$

$= aE[X]$

d) $Var(aX) = E[(aX - E[aX])^2]$

$= E[(aX - aE[X])^2]$ (Using c)

$= E[a^2(X - E[X])^2]$

$= a^2 E[(X - E[X])^2]$ (Again, using c)

$= a^2\, Var(X)$

2) Linearity : $E[af(X) + bg(Y)]$

$= aE[f(X)] + bE[g(Y)]$

Let $Z = X + Y$

$Var(Z) = E[(Z - E[Z])^2]$

$= E[Z^2 - 2Z\,E[Z] + (E[Z])^2]$

$= E[(X+Y)^2 - 2(X+Y)\,E[X+Y] + (E[X+Y])^2]$

$$= E\left[X^2 + 2XY + Y^2 - 2(X+Y)(E[X] + E[Y]) + (E[X] + E[Y])^2 \right]$$

$$= E\left[\left(X^2 - 2XE[X] + E[X]^2\right) + \left(Y^2 - 2YE[Y] + E[Y]^2\right) + 2XY - 2XE[Y] - 2YE[X] + 2E[X]E[Y] \right]$$

$$= E\left[(X - E[X])^2 \right] + E\left[(Y - E[Y])^2 \right] + 2E[XY] - 2E[X]E[Y] - \cancel{2E[Y]E[X]} + \cancel{2E[X]E[Y]}$$

$$= Var(X) + Var(Y) + 2C_{XY}$$

where $C_{XY} = E[XY] - E[X]E[Y]$