If X & Y are independent,

$$P_Z(z) = \int_{\infty} dx \; P_X(x)\, P_Y(z-x)$$

Now suppose X & Y are Gaussian random variables.

Then, $$P_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\}$$

$$P_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left\{-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right\}$$

$$\therefore P_Z(z) = \int_{-\infty}^{+\infty} dx \left(\frac{1}{2\pi}\right) \frac{1}{\sigma_x \sigma_y} \exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\}$$

$$\times \exp\left\{-\frac{(z-x-\mu_y)^2}{\;}\right\}$$

$$\left(\qquad 2\sigma_y^2 \right)$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{+\infty} dx \, \exp\left\{ -\frac{1}{2\sigma_x^2}\left(x^2 - 2x\mu_x + \mu_x^2\right) - \frac{1}{2\sigma_y^2}\left(x^2 + 2x(\mu_y - z) + (\mu_y - z)^2\right) \right\}$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \int_{\infty} dx \, \exp\left\{ \overbrace{\left(-\frac{1}{2\sigma_x^2} - \frac{1}{2\sigma_y^2}\right)}^{A} x^2 + \underbrace{\left(\frac{\mu_x}{\sigma_x^2} - \frac{\mu_y - z}{\sigma_y^2}\right)}_{B} x + \underbrace{\left(\frac{-\mu_x^2}{2\sigma_x^2} - \frac{(\mu_y - z)^2}{2\sigma_y^2}\right)}_{C} \right\}$$

$$Ax^2 + Bx + C$$

$$= A\left(x^2 + \frac{B}{A}x + \frac{C}{A}\right)$$

$$= A\left[(x)^2 + 2(x)\left(\frac{B}{2A}\right) + \left(\frac{B}{2A}\right)^2 - \left(\frac{B}{2A}\right)^2 + \frac{C}{A}\right]$$

$$= A\left(x + \frac{B}{2A}\right)^2 + \left(C - \frac{B^2}{4A}\right)$$

So,

$$P_z(z) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{ C - \frac{B^2}{4A} \right\}$$

$$\times \int_{\infty} dx \ \exp\left\{ A\left(x + \frac{B}{2A}\right)^2 \right\}$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \sqrt{\frac{\pi}{-A}} \ \exp\left\{ C - \frac{B^2}{4A} \right\}$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \sqrt{\frac{\pi}{\frac{1}{2\sigma_x^2} + \frac{1}{2\sigma_y^2}}} \ \exp\left\{ -\frac{\mu_x^2}{2\sigma_x^2} - \frac{(\mu_y - z)^2}{2\sigma_y^2} - \frac{\left(\frac{\mu_x}{\sigma_x^2} - \frac{\mu_y - z}{\sigma_y^2}\right)^2}{4\left(-\frac{1}{2\sigma_x^2} - \frac{1}{2\sigma_y^2}\right)} \right\}$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \sqrt{\frac{2\pi\sigma_x^2\sigma_y^2}{\sigma_x^2 + \sigma_y^2}} \ \exp\left\{ -\frac{\mu_x^2}{2\sigma_x^2} - \frac{(\mu_y - z)^2}{2\sigma_y^2} \right.$$

$$\left. + \left(\frac{1}{\ \ }\right)\left(\frac{\mu_x^2}{\ \ } + \frac{(\mu_y - z)^2}{\ \ } - \mu_x(\mu_y - z) \right) \right\}$$

$$\left(\sigma_x^2+\sigma_y^2\right)\left(2\sigma_x^2 \quad 2\sigma_y^2 \qquad\qquad \right)\Bigg)$$

which can be again cast into the form :

$$P_z(z) = \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} \exp\left(\alpha z^2 + \beta z + \gamma\right)$$

$$= \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} \exp\left\{\alpha\left(z+\frac{\beta}{2\alpha}\right)^2\right\}$$

$$\times \exp\left\{\gamma - \frac{\beta^2}{4\alpha}\right\}$$

→ <u>gaussian</u>

$$\therefore -\frac{(z-\mu_z)^2}{2\sigma_z^2} = \alpha\left(z+\frac{\beta}{2\alpha}\right)^2$$

So, $-\dfrac{1}{2\sigma_z^2} = \alpha \Rightarrow \sigma_z^2 = -\dfrac{1}{2\alpha}$

& $-\mu_z = \dfrac{\beta}{2\alpha} \Rightarrow \mu_z = -\dfrac{\beta}{2\alpha}$

<u>Using convolution theorem :-</u>

Th^{m}: If $W(z) = (f * g)(z)$

$$= \int_{\infty} dx \, f(x)\, g(z-x)$$

Then, $\tilde{w}(f) = \tilde{f}(f)\,\tilde{g}(f)$

Here, $P_Z(z) = \int_{\infty} dx \; P_X(x)\, P_Y(z-x)$

So, $\tilde{P}_Z(f) = \tilde{P}_X(f)\,\tilde{P}_Y(f)$

$$\tilde{P}_X(f) = \int_{\infty} dx \, P_X(x)\, e^{-2\pi i f x}$$

$$= \int_{\infty} dx \left[\frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} \right\} \right] e^{-2\pi i f x}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{\infty} dx \, \exp\left\{ -\frac{1}{2\sigma_x^2}\left(x^2 + \mu_x^2 - 2x\mu_x\right) - 2\pi i f x \right\}$$

$$= \frac{1}{\quad} \int dx \, \exp\left\{ \left(-\frac{1}{2\sigma_x^2}\right) x^2 + \left(\frac{\mu_x}{\sigma_x^2} + \frac{\pi i f}{\sigma_x^2}\right) \right.$$

$$\sqrt{2\pi}\,\sigma_x \int_{\infty} \qquad \qquad + \left(-\frac{\mu_x^2}{2\sigma_x^2}\right)\Bigg\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{\infty} dx \, \exp\left[\left(-\frac{1}{2\sigma_x^2}\right) \left\{ x^2 - 2(\mu_x + i\pi f)x + \mu_x^2 \right\} \right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{\infty} dx \, \exp\left[\left(-\frac{1}{2\sigma_x^2}\right) \left\{ \left(x - (\mu_x + i\pi f)\right)^2 - (\mu_x + i\pi f)^2 + \mu_x^2 \right\} \right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{\infty} dx \, \exp\left[\left(-\frac{1}{2\sigma_x^2}\right) \left\{ x - (\mu_x + i\pi f) \right\}^2 - \frac{-2i\pi f \mu_x + \pi^2 f^2}{2\sigma_x^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left\{ \frac{2i\pi f \mu_x - \pi^2 f^2}{2\sigma_x^2} \right\} \sqrt{\frac{\pi}{1/2\sigma_x^2}}$$

$$= \exp\left\{ \frac{2i\pi f \mu_x - \pi^2 f^2}{\quad} \right\}$$

$$= \exp\left\{\frac{\qquad}{2\sigma_x^2}\right\}$$

Illy, $$\tilde{P}_y(f) = \exp\left\{\frac{2i\pi f\mu_y - \pi^2 f^2}{2\sigma_y^2}\right\}$$

So, $$\tilde{P}_Z(f) = \exp\left\{\frac{2i\pi f\mu_x - \pi^2 f^2}{2\sigma_x^2} + \frac{2i\pi f\mu_y - \pi^2 f^2}{2\sigma_y^2}\right\}$$

Now, $$P_Z(z) = \int_{\infty} df\, \tilde{P}_Z(f)\, e^{2i\pi f z}$$

$$= \int_{\infty} df\, \exp\left\{\left(-\frac{\pi^2}{2\sigma_x^2} - \frac{\pi^2}{2\sigma_y^2}\right)f^2 + \left(\frac{i\pi\mu_x}{\sigma_x^2} + \frac{i\pi\mu_y}{\sigma_y^2} + 2i\pi z\right)f\right\}$$

$$= \int_{\infty} df\, \exp\left[\left(-\frac{\pi^2}{2\sigma_x^2} - \frac{\pi^2}{2\sigma_y^2}\right)(f-\Lambda)^2 + \Omega\right]$$

ω

$$= e^{\Omega} \sqrt{\frac{\cancel{\pi}}{\frac{\pi^{\cancel{2}}}{2\sigma_x^2} + \frac{\pi^{\cancel{2}}}{2\sigma_y^2}}}$$

$$= e^{\Omega} \sqrt{\frac{2\sigma_x^2 \sigma_y^2}{\pi(\sigma_x^2 + \sigma_y^2)}}$$

$\longrightarrow$ <u>gaussian.</u>