CSCI 335 Software Design and Analysis III

The Disjoint Set Class

Equivalence Relations

- **Relation R** on a set S: for every pair (a,b) with a, b in S, a R b is either T or F. If a R b is T => a is related to b.
  - Examples:

    Relation < on the set {0,1,2,3,4,5,6}

    ....Many more....
- **Equivalence** relation R:
  - a R a, for all a (reflexive)
  - a R b => b R a (symmetric)
  - a R b and b R c => a R c (transitive)
- Examples (which ones of the following are equivalence relations?):
  - =
  - ≤
  - Set S is set of cities, a R b iff there is a direct two-way road from a to b

Dynamic Equivalence Problem

- Given set S={a1,a2,a3,a4,a5,a6}
- Given equivalence relation ~
  - a1~a2, a2~a3, a5~a6
  - ai ~ ai for i=1,..,6 (from reflexivity)
  - Symmetry => a2~a1, a3~a2, a6~a5
  - Transitivity => a1 ~ a3, a3 ~ a1
- Equivalence class of element *a* in S:

  all elements related to *a*
- {a1,a2,a3},{a4},{a5,a6} are the <u>equivalence classes</u> in above example
- S is partitioned into **disjoint** equivalence classes.
  - Any equivalent relation R on S, partitions S into a disjoint set of equivalence classes.

Union-Find

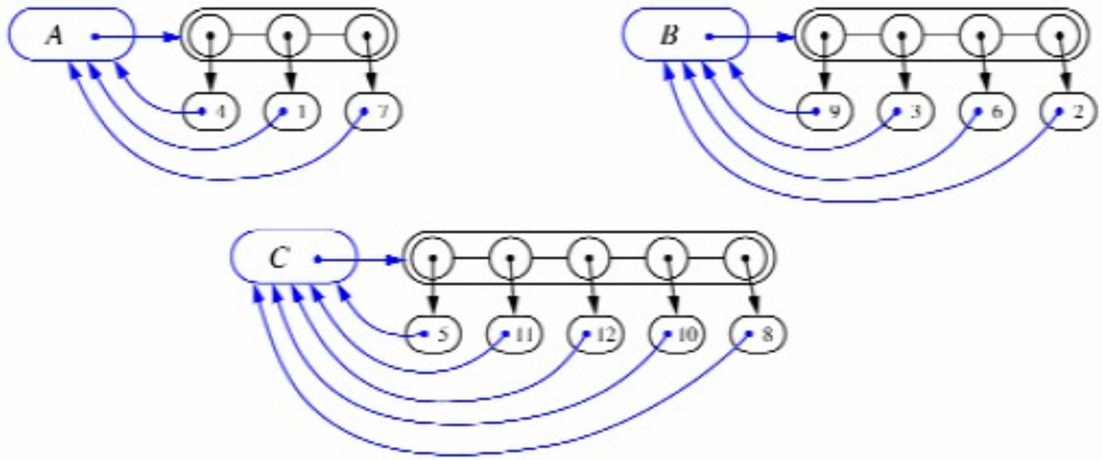
- Find:
  - given an element find the equivalence class it is in
- Union:
  - Add the relationship a~b:
    - Perform a Find on a and b. If already related nothing to do.
    - Else Unite the equivalence classes of a and b into a new class.
- **Dynamic:** the equivalence classes can change with time, i.e. Unions are applied at any time.
- **On-line:** a union or find "arrives" at each instance of time and needs to be processed.
- **Off-line:** the sequence of unions and finds is given, and you need to calculate final sets (not dynamic).

Union-Find

- Assume set S = {0,1,2,…,N-1} (can be hashed)
- Solutions?
- O(1) find?
  - Union ?

List-based Implementation

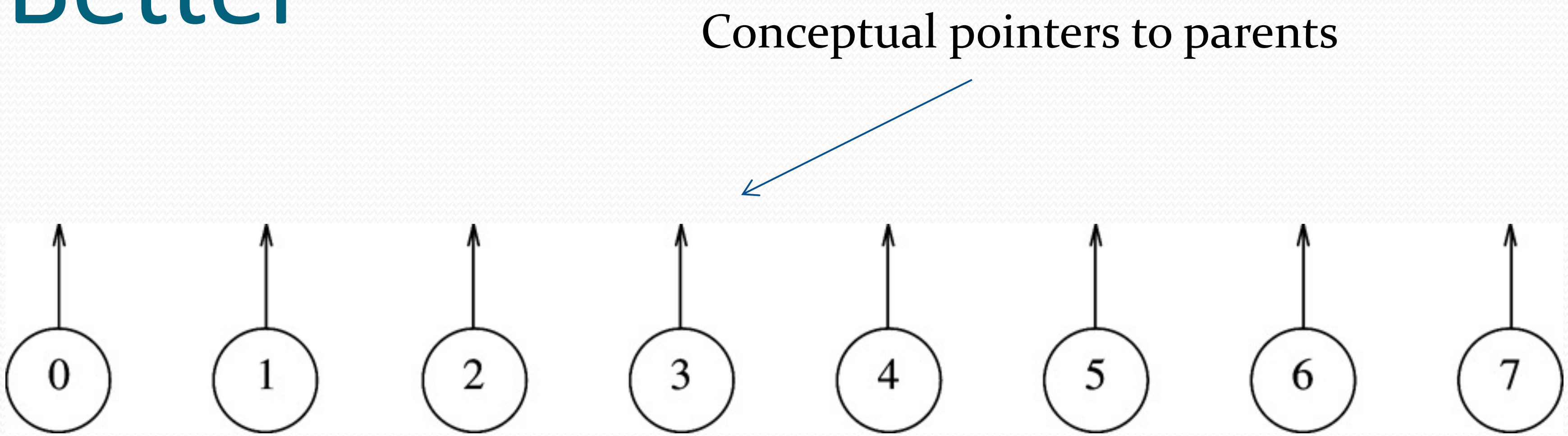
- Each set is stored in a sequence represented with a linked-list
- Each node should store an object containing the element and a reference to the set name



Analysis of List-based Representation

- When doing a union, always move elements from the smaller set to the larger set
  - Each time an element is moved it goes to a set of size at least double its old set
  - Thus, an element can be moved at most O(log n) times
- Total time needed to do n unions and finds is O(n log n).

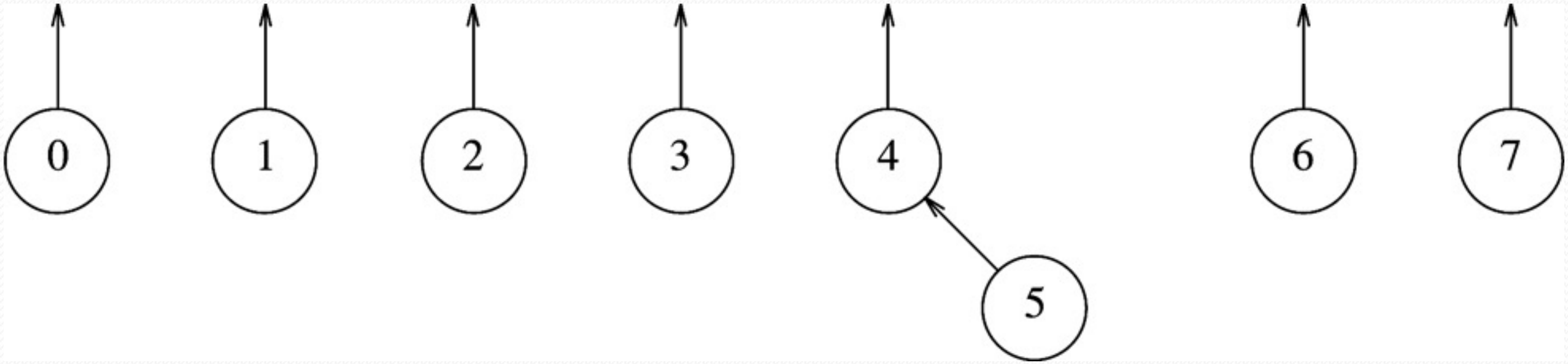
Better



Each equivalence class is represented by a tree.
The name of the class is the root (i.e. name of the representative).
Initially each element is one class containing itself (reflexivity).
Can be implemented by array

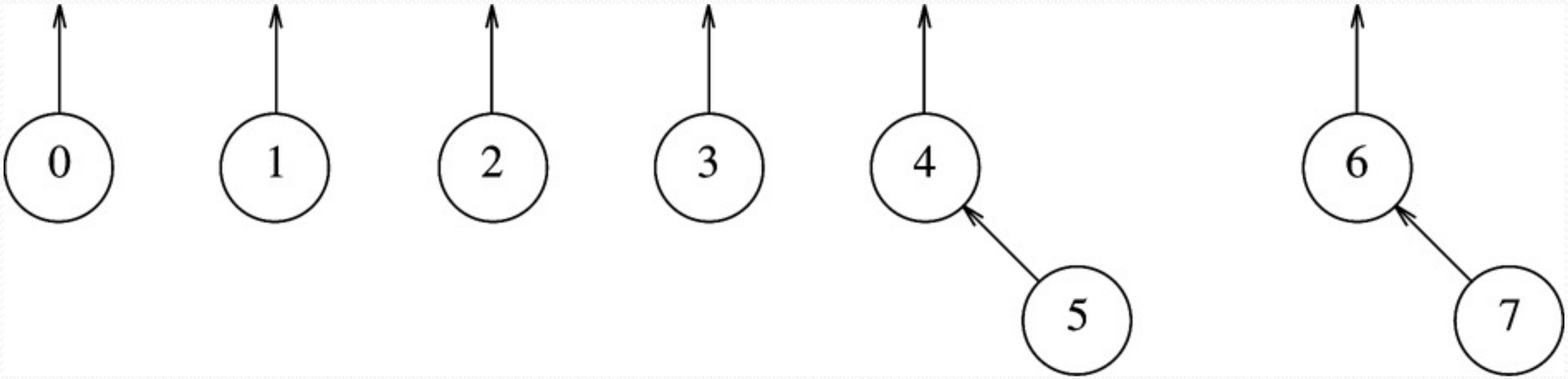
Better

After Union(4,5)



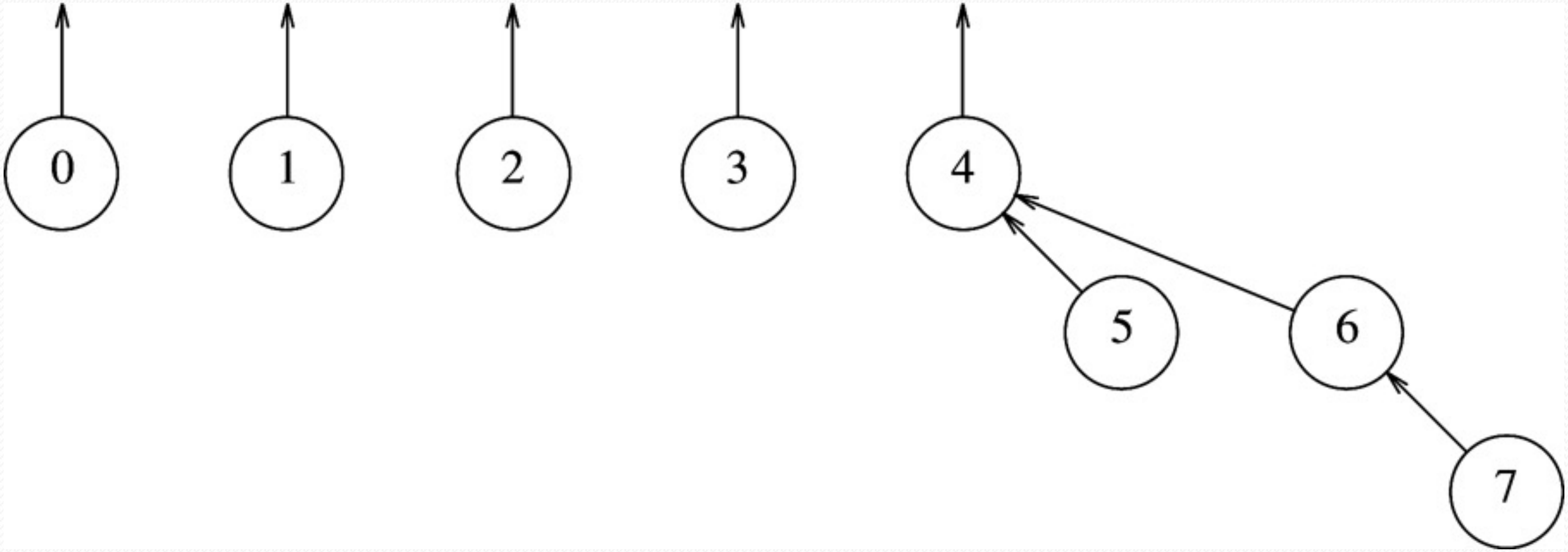
Better

After Union(6,7)



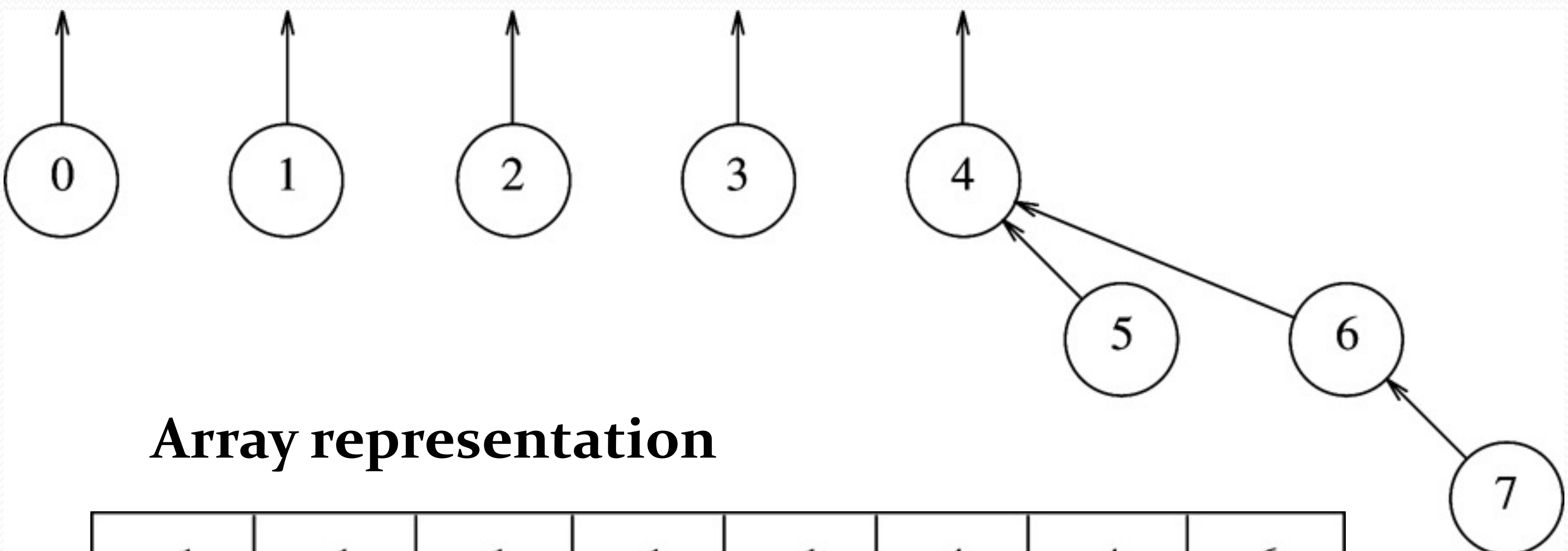
Better

After Union(4,6)



Better

After Union(4,6)



Array representation

–1	–1	–1	–1	–1	4	4	6
0	1	2	3	4	5	6	7

Implementation

```
class DisjSets
{
  public:
    explicit DisjSets( int numElements );

    int find( int x ) const;
    int find( int x );
    void unionSets( int root1, int root2 );

  private:
    vector<int> s;
};
```

Analysis

- Union's cost?

Analysis

- Union's cost?
- Union: Fast O(1)

Analysis

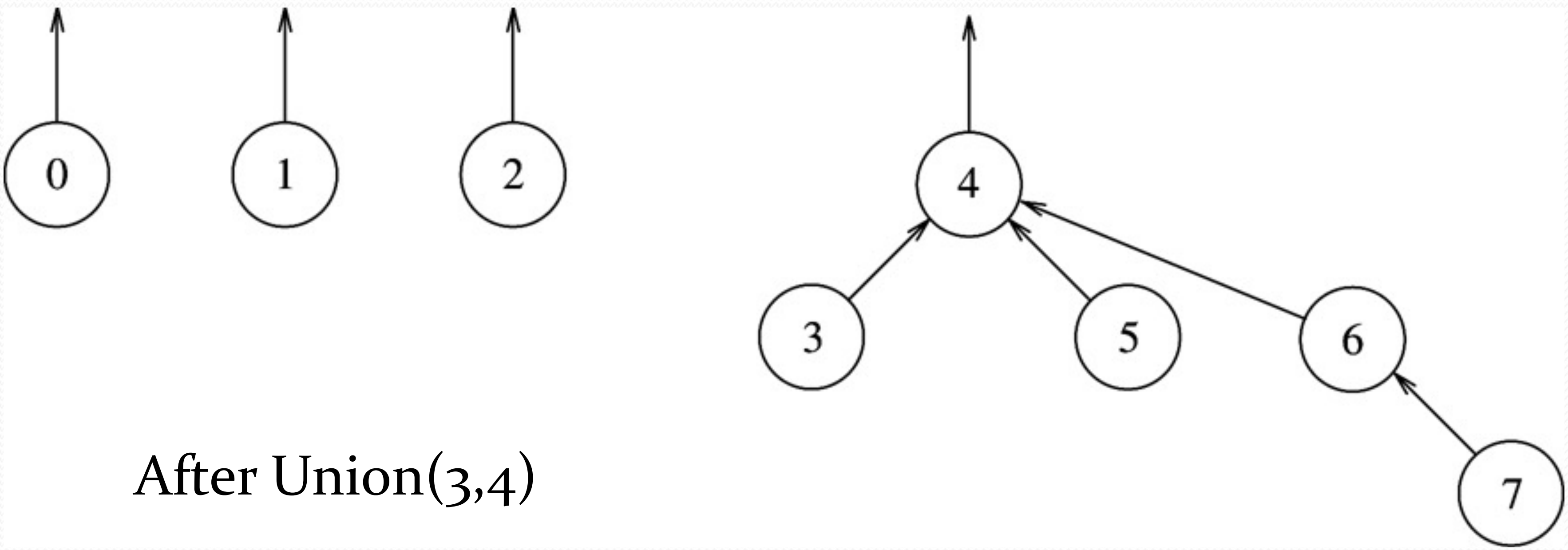
- Union's cost?
- Union: Fast O(1)
- Find's cost?

Analysis

- Union's cost?
- Union: Fast O(1)
- Find's cost ?
- Find: Worst-case depth of deeper node O(N)
  => O(MN) for M mixed find/union operations
  - Can we improve this?
- Average running time: depends on what is *average:*
  - *Θ(M) or Θ(MN) or Θ(MlogN) for M unions*

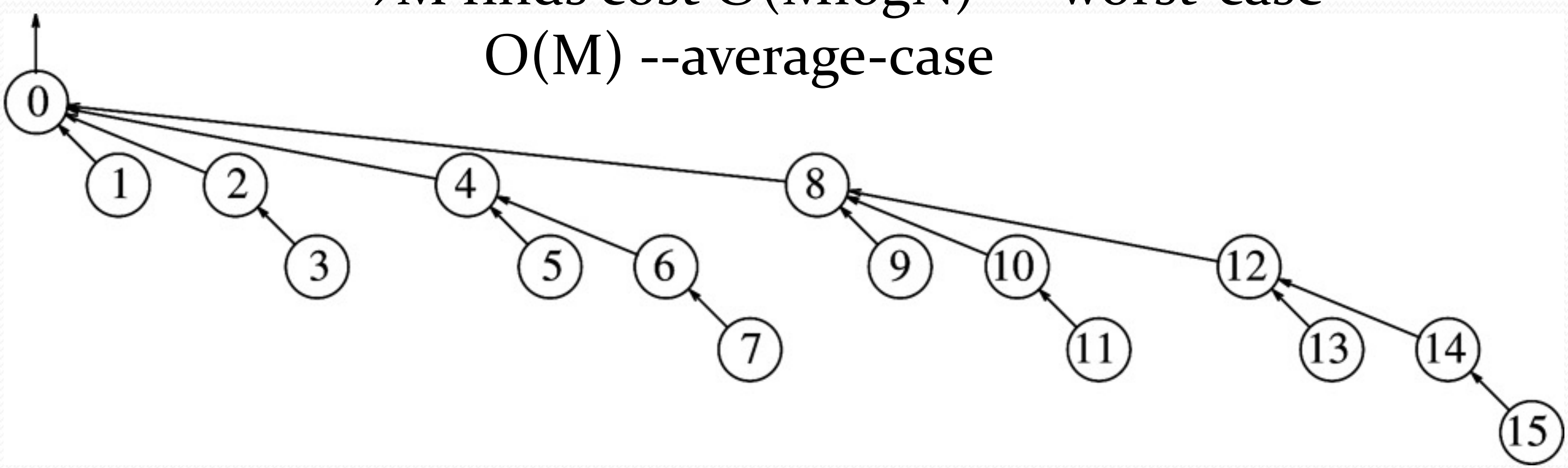
Smart Union Algorithms

- Union by size: Smaller tree becomes subtree of larger



Smart Union Algorithms

Depth of any node can't be more than logN
⇒Find costs O(logN) --worst-case
⇒M finds cost O(MlogN) --worst-case
O(M) --average-case

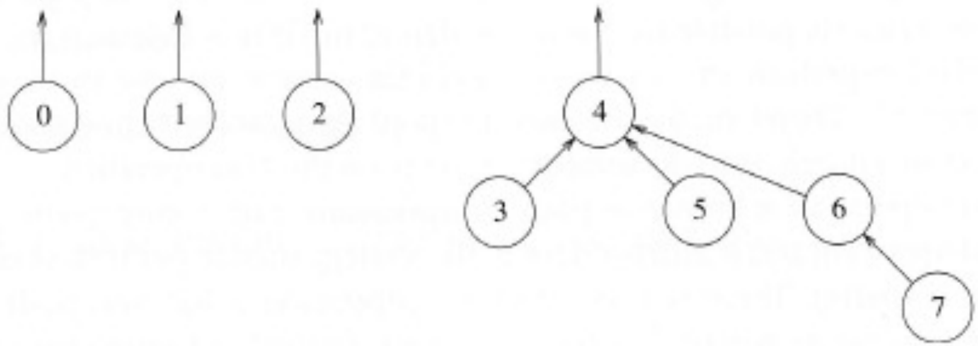


Worst-case for union when N=16 (binomial tree)
[result after 16 unions]
[union of equal size trees]

Smart Union Algorithms

- **Union-by-height**: shortest tree under tallest
- Height is increased by one **iff** when two trees of equal height are combined.
- Again O(logN) find

Smart Union Algorithms



0	1	2	3	4	5	6	7
–1	–1	–1	4	–5	4	4	6

Union-by-size

0	1	2	3	4	5	6	7
–1	–1	–1	4	–3	4	4	6

Union-by-height

Smart Union Algorithms

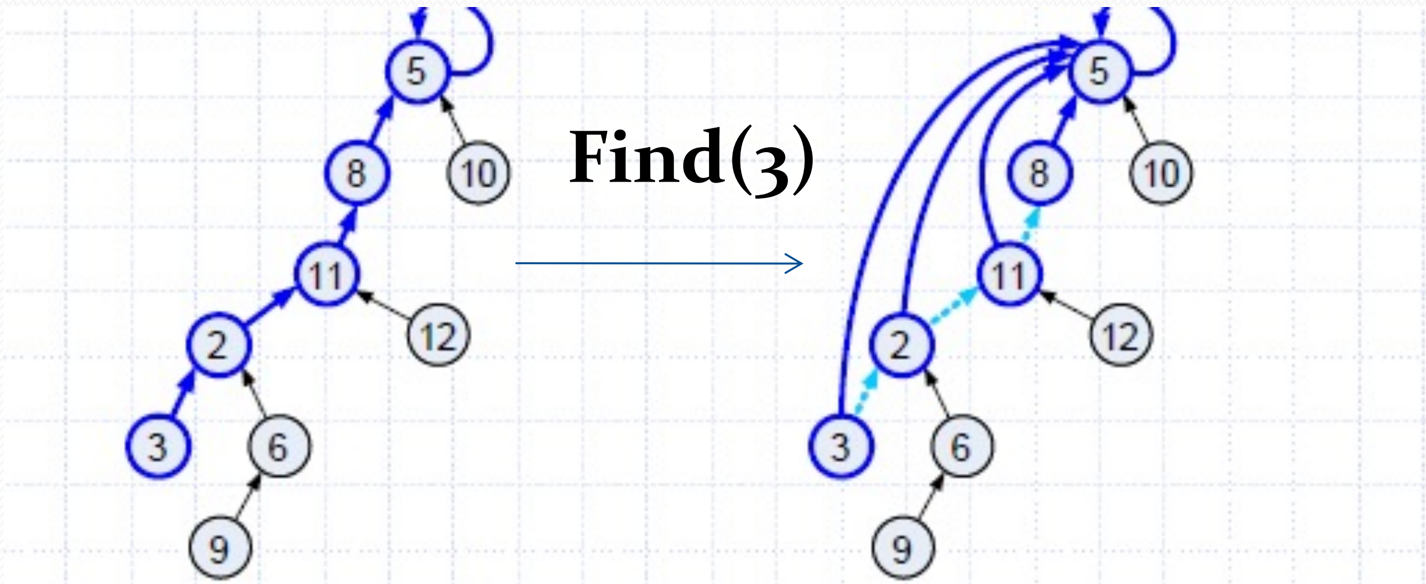
```
/**
 * Union two disjoint sets.
 * For simplicity, we assume root1 and root2 are distinct
 * and represent set names.
 * root1 is the root of set 1.
 * root2 is the root of set 2.
 */
void DisjSets::unionSets( int root1, int root2 )
{
    if( s[ root2 ] < s[ root1 ] )  // root2 is deeper
        s[ root1 ] = root2;        // Make root2 new root
    else
    {
        if( s[ root1 ] == s[ root2 ] )
            s[ root1 ]--;          // Update height if same
        s[ root2 ] = root1;        // Make root1 new root
    }
}
```

Path Compression

- Improve on the O(MlogN) worst-case find() cost for M finds
- Ideas?

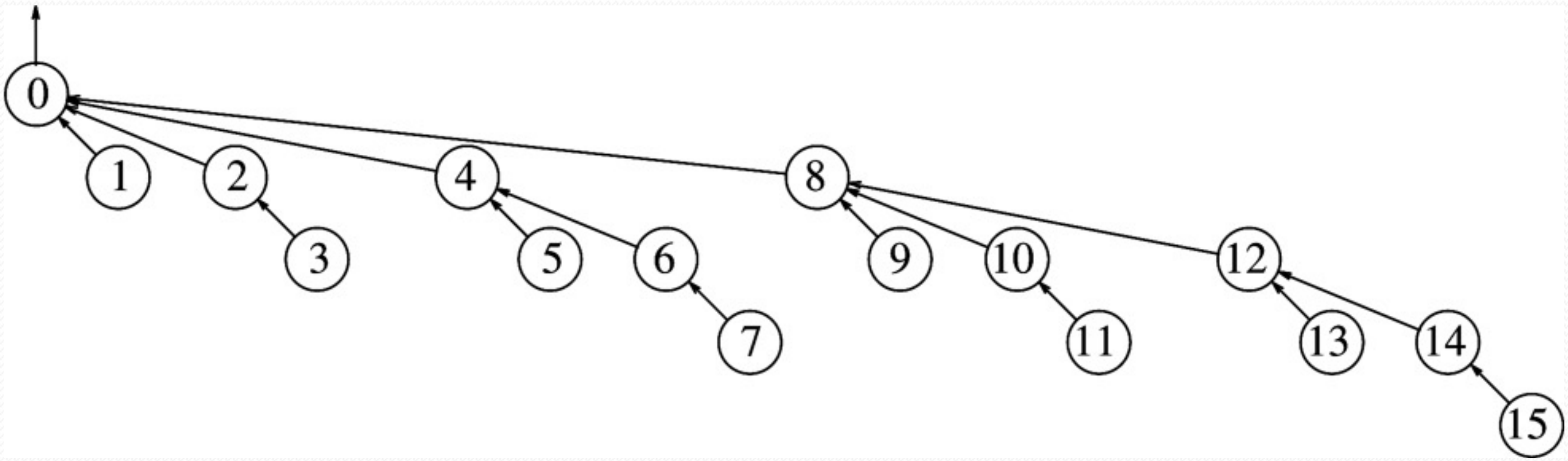
Path Compression

- Improve on the O(MlogN) worst-case find() cost for M finds
- It can appear frequently
- **Path compression:** During find(x) make all nodes on the path from x to the root to be children of the root.

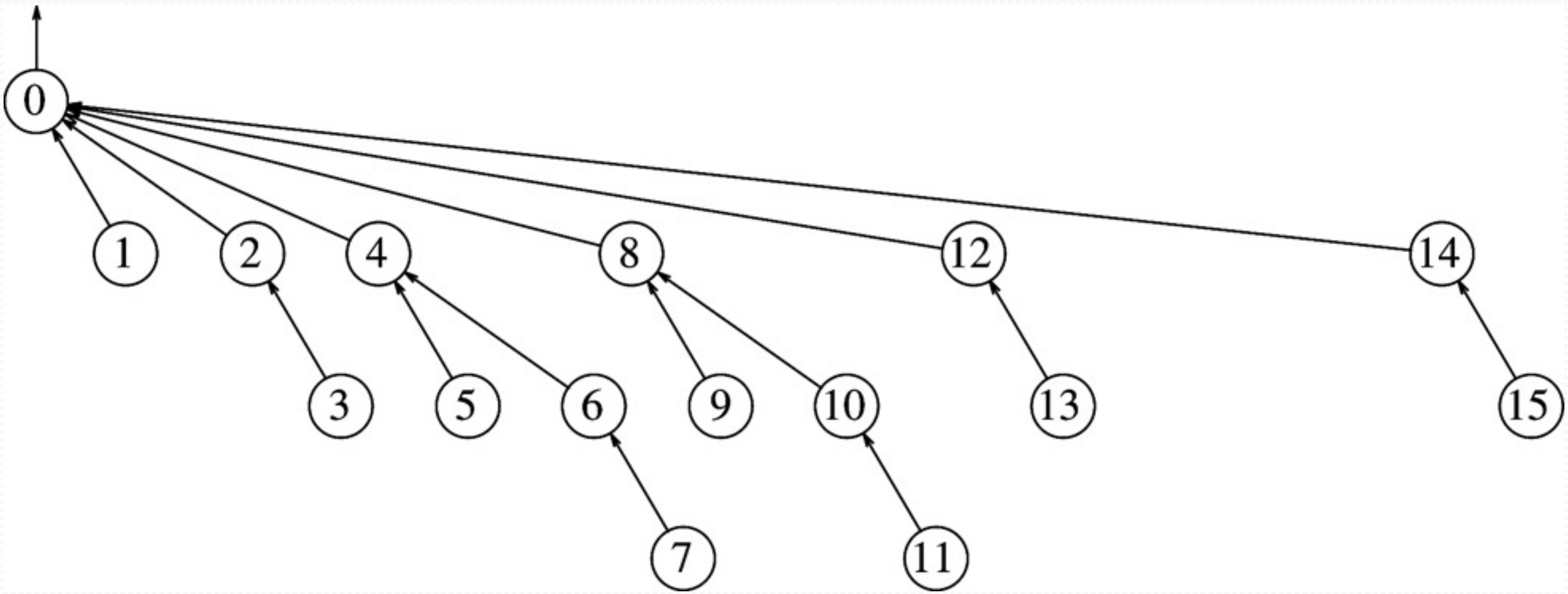


Path Compression

Apply find (14) in this set



Path Compression



Path Compression

```
/**
 * Perform a find with path compression.
 * Error checks omitted again for simplicity.
 * Return the set containing x.
 */
int DisjSets::find( int x )
{
    if( s[ x ] < 0 )
        return x;
    else
        return s[ x ] = find( s[ x ] );
}
```

Union-by-rank

- **Union-by-rank:** after compression the height of tree may not be accurate.
- It is now called **rank** => it is an upper bound on the actual height.

Path Compression

(A) Union-by-size + Path Compression

or

(B) Union-by-rank + Path Compression

Average case: Not sure whether Path Compression helps

Worst case: Path Compression helps a lot

Union-by-rank is preferred because it requires fewer updates on heights

Worst-Case Analysis of Union-Find

- Definition: **log*N** is number of times logarithm of N needs to be applied until it gets to ≤ 1.
- For example: log*65536=4, because ($65536=2^{16}$)

  log log log log 65536 = 1
- Example: log*2^{20} = ?
- Example: log*2^{65536} = ? (2^{65536} has 20,000 digits in decimal form)
- log*N grows very slowly as N becomes bigger !

Analysis of Union-Find

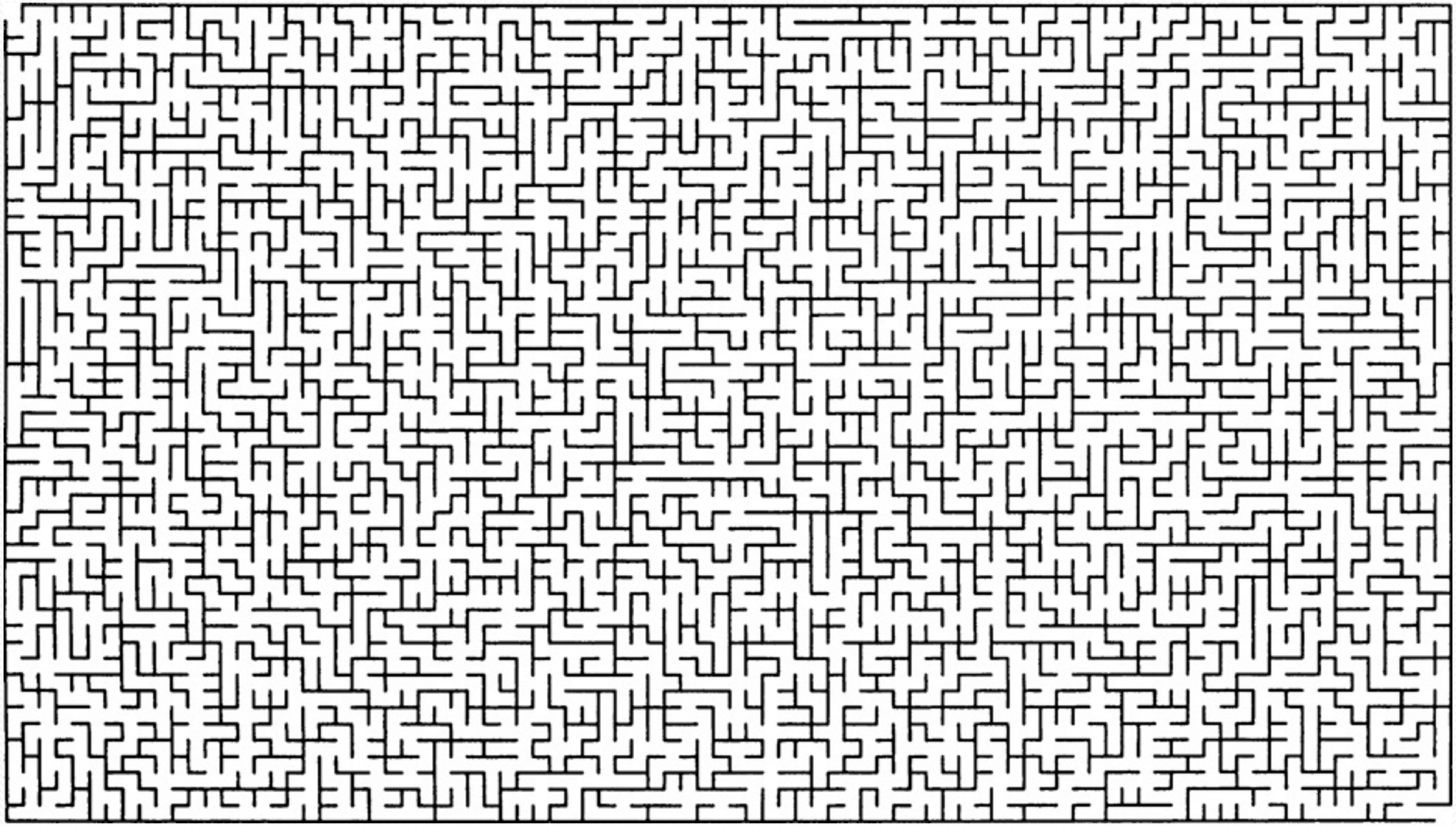
Any sequence of $M=\Omega(N)$ union/find operations takes a total of $O(M \log^* N)$ running time.

Model:

Union/Finds in any order

Union-by-rank **with** path compression

A fun application

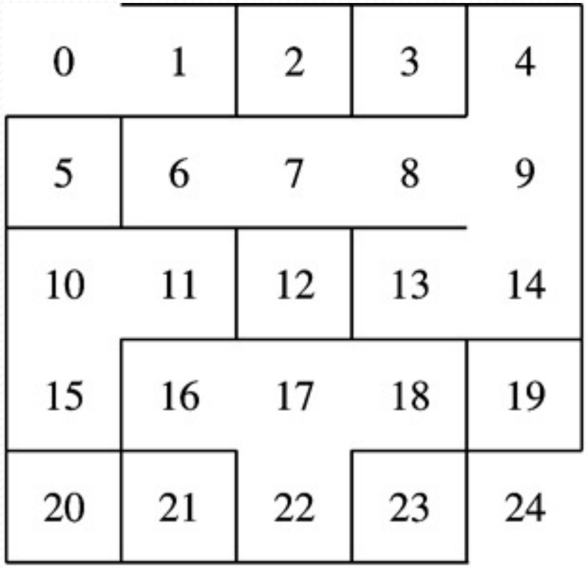


A fun application

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24

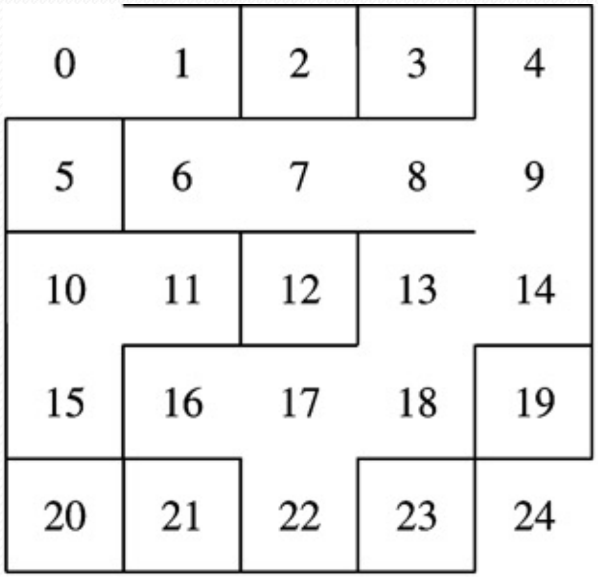
{0} {1} {2} {3} {4} {5} {6} {7} {8} {9} {10} {11} {12} {13} {14} {15} {16} {17} {18} {19} {20} {21} {22} {23} {24}

A fun application



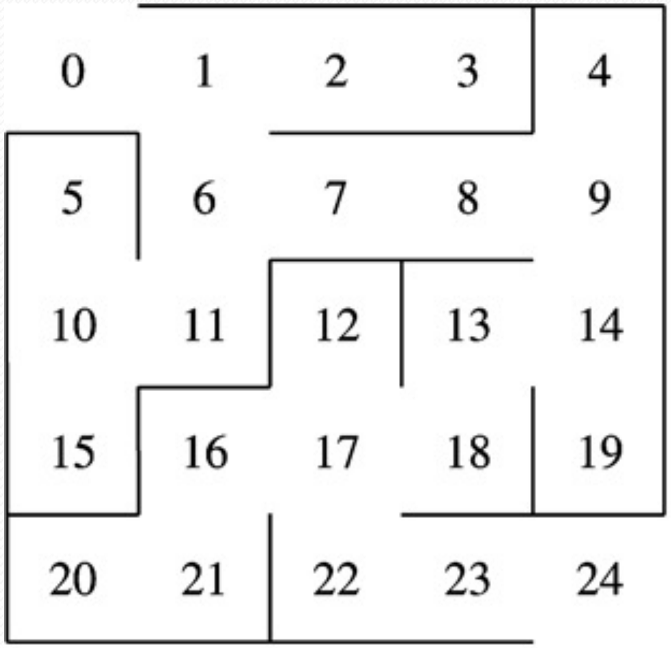
{0, 1} {2} {3} {4, 6, 7, 8, 9, 13, 14} {5} {10, 11, 15} {12} {16, 17, 18, 22} {19} {20} {21} {23} {24}

A fun application



{0, 1} {2} {3} {4, 6, 7, 8, 9, 13, 14, 16, 17, 18, 22} {5} {10, 11, 15} {12} {19} {20} {21} {23} {24}

A fun application



{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24}