Assignment1-STA510

2024-09-22

Problem_1

a. Find the probability that the sum of three dice is either 9 or 10.

The total number of possible outcomes is:

$$6 \times 6 \times 6 = 216$$

Sum of 9

Count combinations of three numbers sum up to 9:

- (1, 2, 6) -> permute in 6 ways:(1,2,6), (1,6,2), (2,1,6), (2,6,1), (6,1,2), (6,2,1)
- (1, 3, 5) -> permute in 6 ways:(1,3,5), (1,5,3), (3,1,5), (3,5,1), (5,1,3), (5,3,1)
- (1, 4, 4) -> permute in 3 ways:(1,4,4), (4,1,4), (4,4,1)
- (2, 2, 5) -> permute in 3 ways:(2,2,5), (2,5,2), (5,2,2)
- (2, 3, 4) -> permute in 6 ways:(2,3,4), (2,4,3), (3,2,4), (3,4,2), (4,2,3),(4,3,2)
- (3, 3, 3) -> only 1 permutation

$$6 + 6 + 3 + 3 + 6 + 1 = 25$$

Sum of 10

Count combinations of three numbers sum up to 10:

- (1, 3, 6)-> permute in 6 ways: (1,3,6), (1,6,3), (3,1,6), (3,6,1), (6,1,3), (6,3,1)
- (1, 4, 5)-> permute in 6 ways: (1,4,5), (1,5,4), (4,1,5), (4,5,1), (5,1,4), (5,4,1)
- (2, 2, 6)-> permute in 3 ways: (2,2,6), (2,6,2), (6,2,2)
- (2, 3, 5)-> permute in 6 ways: (2,3,5), (2,5,3), (3,5,2), (3,2,5), (5,3,2), (5,2,3)
- (2, 4, 4)-> permute in 3 ways: (2,4,4), (4,2,4), (4,4,2)
- (3, 3, 4)-> permute in 3 ways: (3,3,4), (3,4,3), (4,3,3)

$$6 + 6 + 3 + 6 + 3 + 3 = 27$$

Probabilities

$$P(\text{sum} = 9) = \frac{25}{216} \approx 0.1157$$

$$P(\text{sum} = 10) = \frac{27}{216} \approx 0.125$$

Thus, the sum of 10 is slightly more likely than the sum of 9.

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b.

At least one six in 4 throws

The probability of rolling a six on one die is $p = \frac{1}{6}$, and the probability of not rolling a six on one die is $P(\text{not a six}) = \frac{5}{6}$. Thus, the probability of getting at least one six in 4 throws is:

$$P(\text{at least one six}) = 1 - \left(\frac{5}{6}\right)^4 \approx 0.5177$$

At least two sixes in 24 throws

Probability of getting and not getiing two sixes on one throw of two dice:

$$P(\text{two sixes}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

$$P(\text{not two sixes}) = 1 - \frac{1}{36} = \frac{35}{36}$$

$$P(\text{no two sixes in 24 throws}) = \left(\frac{35}{36}\right)^{24}$$

Probability of getting at least one time two sixes:

$$P(B) = 1 - P(\text{no two sixes in 24 throws})$$

$$P(B) = 1 - \left(\frac{35}{36}\right)^{24} = 1 - 0.5086 = 0.4914$$

Thus, the sum of at least 1 six in 4 throws is slightly more likely than the at least 2 sixes in 24 throws.

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c.

At least one 6 in one dice in 6 throws:

Probability of not getiing a 6 on one throw and in all 6 throws:

$$P(\text{no } 6) = \frac{5}{6}$$

$$P(\text{no 6 in 6 throws}) = \left(\frac{5}{6}\right)^6$$

Probability of getting at least one 6:

$$P(C) = 1 - P(\text{no 6 in 6 throws})$$

$$P(C) = 1 - \left(\frac{5}{6}\right)^6 = 1 - 0.3348 = 0.6652$$

At least two 6 in one dice in 12 throws:

$$P(X \geq k) = 1 - P(X < k)$$

$$P(D) = 1 - [P(X = 0) + P(X = 1)]$$

$$P(D) = 1 - \left[C(12, 0) \times \left(\frac{5}{6}\right)^{12} \times \left(\frac{1}{6}\right)^0 + C(12, 1) \times \left(\frac{5}{6}\right)^{11} \times \left(\frac{1}{6}\right)^1\right]$$

$$P(D) = 1 - [0.1121 + 0.3363] = 1 - 0.4484 = 0.5516$$

At least three 6 in one dice in 18 throws:

$$P(E) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$P(E) = 1 - \left[C(18, 0) \times \left(\frac{5}{6}\right)^{18} \times \left(\frac{1}{6}\right)^0 + C(18, 1) \times \left(\frac{5}{6}\right)^{17} \times \left(\frac{1}{6}\right)^1 + C(18, 2) \times \left(\frac{5}{6}\right)^{16} \times \left(\frac{1}{6}\right)^2\right]$$

$$P(E) = 1 - [0.0374 + 0.1345 + 0.2240] = 1 - 0.3959 = 0.6041$$

Therefore,the most likely event is at least one 6 when throwing a die 6 times.

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d. Probability of Needing Exactly 8 Throwa to Get 3 Sixes

This is a problem modeled using the **negative binomial distribution**:

$$P(X = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

- $r = 3$ (we want 3 sixes),
- $p = \frac{1}{6}$,

- $x = 8$ (number of throws).

Using the formula:

$$P(X = 8) = \binom{7}{2} \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^5 \approx 0.0390$$

Thus, the probability of needing exactly 8 throws to get 3 sixes is approximately **3.90%**.

Problem_2

Based on the cumulative distribution function (CDF) for a random variable as:

$$F_a(t) = e^{-e^{-(t-a)}}, \quad t \in \mathbb{R}.$$

a. Verify that this is a proper cumulative density function (CDF)

- $F_a(t)$ is non-decreasing as its derivative is non-negative.
- $\lim_{t\to-\infty} F_a(t) = 0$ and $\lim_{t\to\infty} F_a(t) = 1$.
- $F_a(t)$ is continuous.

The function is non-decreasing because it is a composition of monotonic (increasing) functions.

Limits:

- As $t \to -\infty$, $e^{-(t-a)} \to \infty$, and hence $F_a(t) \to 0$.
- As $t \to \infty$, $e^{-(t-a)} \to 0$, and hence $F_a(t) \to 1$.

b. Proof

$$E(T) = a + \gamma$$

where:

$$\gamma = -\int_0^\infty \log(x)e^{-x}dx \approx 0.5772$$

is Euler's constant.

$$F_a(t) = e^{-e^{-(t-a)}}, \quad t \in \mathbb{R}$$

The probability density function (PDF) $f_a(t)$ is the derivative of the CDF:

$$f_a(t) = \frac{d}{dt}F_a(t)$$

$$f_a(t) = e^{-e^{-(t-a)}} \cdot \left(e^{-(t-a)}\right)$$

$$f_a(t) = e^{-(t-a)} e^{-e^{-(t-a)}} = e^{-(t-a)} \cdot F_a(t)$$

The mean of a random variable T is defined as:

$$E(T) = \int_{-\infty}^{\infty} t f_a(t)\, dt$$

$$E(T) = \int_{-\infty}^{\infty} t \cdot e^{-(t-a)} e^{-e^{-(t-a)}}\, dt$$

Use the hint given in the problem:

$$s = t - a \quad \text{so that} \quad t = s + a$$

$$E(T) = \int_{-\infty}^{\infty} (s+a) e^{-s} e^{-e^{-s}}\, ds$$

$$E(T) = \int_{-\infty}^{\infty} s e^{-s} e^{-e^{-s}}\, ds + a \int_{-\infty}^{\infty} e^{-s} e^{-e^{-s}}\, ds$$

The second integral can be simplified. Recognize that it is the integral over the PDF(he integral of a PDF over its entire domain must equal 1):

$$\int_{-\infty}^{\infty} e^{-s} e^{-e^{-s}}\, ds = 1$$

Then:

$$E(T) = a + \int_{-\infty}^{\infty} s e^{-s} e^{-e^{-s}}\, ds$$

The first integral involves $s e^{-s} e^{-e^{-s}}$. To handle this, we make the substitution $x = e^{-s}$, which simplifies the integral:

$$ds = -\frac{dx}{x}, \quad s = -\log(x)$$

$$\int_{-\infty}^{\infty} s e^{-s} e^{-e^{-s}}\, ds = \int_{0}^{\infty} (-\log(x)) e^{-x}\, dx$$

$$\int_{0}^{\infty} \log(x) e^{-x}\, dx = -\gamma$$

$$E(T) = a + \gamma$$

where $\gamma \approx 0.5772$ is Euler's constant.

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c. Calculate probabilities

For this part, we will use the CDF $F_a(t) = e^{-e^{-(t-a)}}$ with $a = 5$.

Calculate $P(T \leq 4)$**:**

$$P(T \leq 4) = F_a(4) = e^{-e^{-(4-5)}} = e^{-e^{1}} = e^{-2.71828} \approx 0.06599.$$

Calculate $P(T > 9)$**:**

$$P(T > 9) = 1 - F_a(9) = 1 - e^{-e^{-(9-5)}} = 1 - e^{-e^{-4}} = 1 - e^{-0.01832} \approx 1 - 0.98185 = 0.01815.$$

Calculate $P(5 < T \leq 6)$**:**

$$P(5 < T \leq 6) = F_a(6) - F_a(5).$$

$$F_a(6) = e^{-e^{-(6-5)}} = e^{-e^{-1}} = e^{-0.36788} \approx 0.6922,$$
$$F_a(5) = e^{-e^{-(5-5)}} = e^{-e^{0}} = e^{-1} \approx 0.3679.$$

Therefore:

$$P(5 < T \leq 6) \approx 0.6922 - 0.3679 = 0.3243.$$

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d. Show that the inverse cumulative distribution function

$$U = e^{-e^{-(t-a)}}.$$

Taking the natural logarithm on both sides:

$$\log(U) = -e^{-(t-a)}.$$

$$-\log(U) = e^{-(t-a)}.$$

Now take the natural logarithm again:

$$\log(-\log(U)) = -(t-a).$$

$$t = a - \log(-\log(U)).$$

This is the inverse CDF:

$$F_a^{-1}(U) = a - \log(-\log(U)).$$

Problem_3

a. Maximum Likelihood

Likelihood Function: Given n independent observations $x_1, x_2, \ldots, x_n$ from the distribution, the likelihood function $L(\lambda)$ is the product of the individual probability density functions:

$$L(\lambda) = \prod_{i=1}^{n} f_\lambda(x_i) = \prod_{i=1}^{n} \frac{1}{2\lambda} \exp\left(-\frac{|x_i|}{\lambda}\right)$$

Log-Likelihood: To simplify the maximization, we take the log of the likelihood function, which gives us the log-likelihood function:

$$\log L(\lambda) = \sum_{i=1}^{n} \log\left(\frac{1}{2\lambda} \exp\left(-\frac{|x_i|}{\lambda}\right)\right)$$

Breaking this down:

$$\log L(\lambda) = \sum_{i=1}^{n} \left(-\log(2\lambda) - \frac{|x_i|}{\lambda}\right)$$

$$\log L(\lambda) = -n\log(2\lambda) - \frac{1}{\lambda}\sum_{i=1}^{n} |x_i|$$

Maximizing the Log-Likelihood: To find the MLE, we take the derivative of $\log L(\lambda)$ with respect to λ and set it equal to 0:

$$\frac{d}{d\lambda} \log L(\lambda) = -\frac{n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n} |x_i|$$

Setting the derivative equal to 0:

$$-\frac{n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n} |x_i| = 0$$

Multiply through by λ^2 to eliminate the denominator:

$$-n\lambda + \sum_{i=1}^{n} |x_i| = 0$$

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} |x_i|$$

This shows that the MLE of λ is indeed $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} |x_i|$.

Is $\hat{\lambda}$ an Unbiased Estimator?

An estimator $\hat{\lambda}$ is said to be unbiased if its expected value is equal to the true parameter λ:

$$E(\hat{\lambda}) = \lambda$$

Expected Value of $\hat{\lambda}$:

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} |x_i|$$

Since the observations x_i are independent and identically distributed (i.i.d.), the expectation of $\hat{\lambda}$ is:

$$E(\hat{\lambda}) = E\left(\frac{1}{n} \sum_{i=1}^{n} |x_i|\right)$$

$$E(\hat{\lambda}) = \frac{1}{n} \sum_{i=1}^{n} E(|x_i|)$$

Since each $E(|x_i|) = \lambda$ (given in the problem), we get:

$$E(\hat{\lambda}) = \frac{1}{n} \sum_{i=1}^{n} \lambda = \lambda$$

Thus, $E(\hat{\lambda}) = \lambda$, meaning that $\hat{\lambda}$ is an **unbiased estimator** of λ.