Finite Element Method in One Dimension

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The finite element method (FEM) is a numerical analysis method for finding the solution to boundary value problems for partial differential equations (PDE). The finite element method has a variety of applications. The most popular application for this method is used for numerical modeling of physical systems in engineering and physics. Using FEM, computers can model thermodynamics, electromagnetism and fluid dynamics.

The finite element method determines approximate values of the unknowns at a certain number of point over a domain. Essentially, it divides a large domain into smaller, computable parts called finite elements. Then, these smaller parts are combined to model the entire problem.

There are many forms of finite element method. One form, called the applied element method, is used to predict the continuum and discrete behavior of structures. Each node in the division of elements is given values of normal and shear spring mechanisms. Thus, this method uses rotational elements and trigonometric properties to calculate a field.

Another form is $hp - FEM$. Like the FEM analyzed in this paper, This method solves partial differential equations based on piecewise-polynomial approximations. However, this form uses a variable size and polynomial degree. Thus, using different polynomials may yields more accurate results. Also, since their is more freedom in the construction, this form produces much faster algorithms.

For this analysis, and for simplicity, we will look at finite methods in one-dimension. Thus, this method can be modeled using the following Poisson equation.

$$-u''(x) = f(x), 0 < x < 1, u(0) = 0, u(1) = 0$$

First, construct, using the weak formulation, a function $v(x)$ such that $v(0) = 0, v(1) = 0$ to get

$$-u''v = fv$$

. Then, integrating from 0 to 1 by parts. Then,

$$\int_0^1 (-u''v)dx = -u'v\Big|_0^1 + \int_0^1 u'v'dx$$
$$= \int_0^1 (u'v')dx$$

Then, you construct a mesh. In our program, we use a uniform mesh where each polygon is a right triangle. Then, based on the mesh, a piecewise linear function,ϕ is created and a linear combination of the functions are created

$$\sum_{j=1}^{n-1} c_j \phi_j(x)$$

where the coefficients c_j are unknown. Thus,

$$\int_0^1 (u'v')dx = \int_0^1 fvdx \Longrightarrow \int_0^1 \sum_{j=1}^{n-1} c_j' \phi_j' v' = \sum_{j=1}^{n-1} c_j \int_0^1 \phi_j' v'$$

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Next choose the function $v(x)$ as ϕ_j.

$$\begin{bmatrix} g(\phi_1\phi_1) & g(\phi_1\phi_2) & g(\phi_1\phi_3) & \dots & g(\phi_1\phi_{n-1} \\ g(\phi_2\phi_1) & g(\phi_2\phi_2) & g(\phi_2\phi_3) & \dots & g(\phi_2\phi_{n-1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g(\phi_{n-1}\phi_1) & g(\phi_{n-1}\phi_2) & g(\phi_{n-1}\phi_3) & \dots & g(\phi_{n-1}\phi_{n-1}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} (h_1) \\ (h_2) \\ \vdots \\ (h_{n-1} \end{bmatrix}$$

where

$$g(\phi_i, \phi_j) = \int_0^1 dx,$$

$$h = \int_0^1 f\phi_i dx$$

Then, solve the system of equations for the coefficients and essentially find the heights of each of the nodes. Thus finding:

$$u_h(x) = \sum_i c_i\phi_i(x)$$

My FEM Code

The following code examples are using FEM in two dimensions with zero boundary conditions.

Figure 1: Function 1: $F = -2x(x-1) - 2y(y-1)$

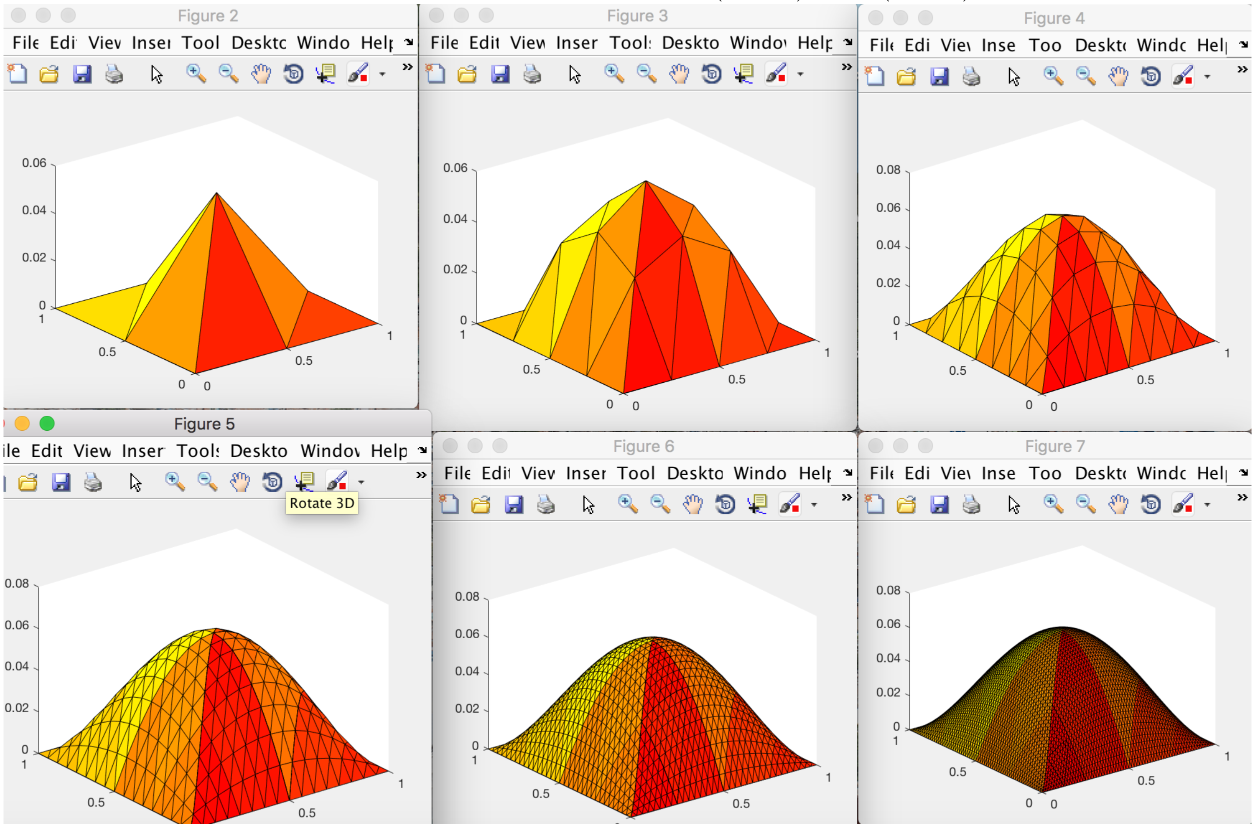


Figure 2: Function 1 Error Table

MeshLevel	L2error	reductionrate	GradientError	reductionrate2
1	0.0162805742145676	0	0.229510612252705	0
2	0.00552356292171698	0.339273225189722	0.166039991254296	0.72345234769134
3	0.00147732013352197	0.267457826489782	0.101659026127179	0.612256272475255
4	0.000380435000987196	0.257516967619082	0.0541066079807509	0.532236143134613

Figure 3: Function 2: $F = x^2(y-1.5)^3$

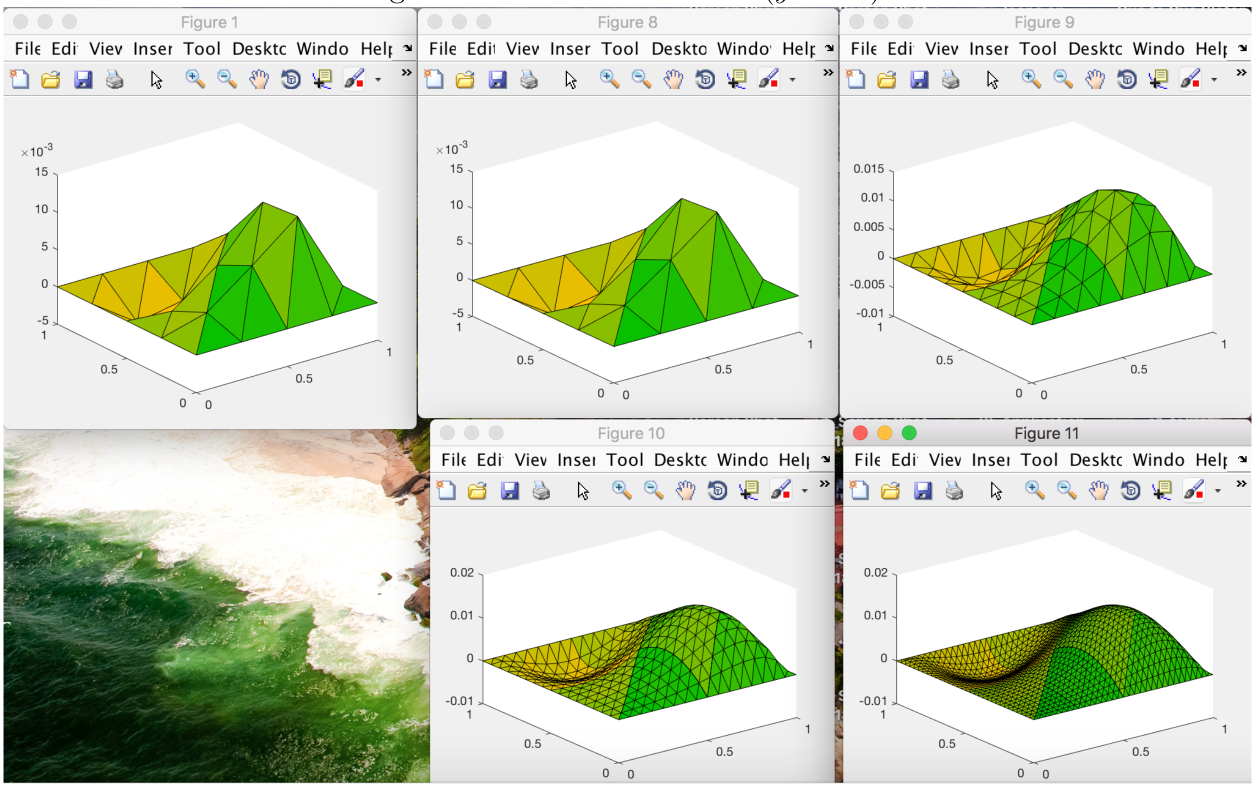


Figure 4: Function 2

MeshLevel	L2error	reductionrate	GradientError	reductionrate2
1	0.00836038905292584	0	0.00440323931508814	0
2	0.00283645615782699	0.380800615189722	0.00128832833859145	0.609516403149
3	0.000758632398905965	0.295142753156448	0.000734866502199209	0.573010935432667
4	0.000195360714903854	0.278280662619082	0.000406235921268471	0.5547582015745
5	4.92059084670305e-05	0.268483030133467	0.000220278126097272	0.5438065612596

The following functions are general FEM. These functions will include a function on boundary.

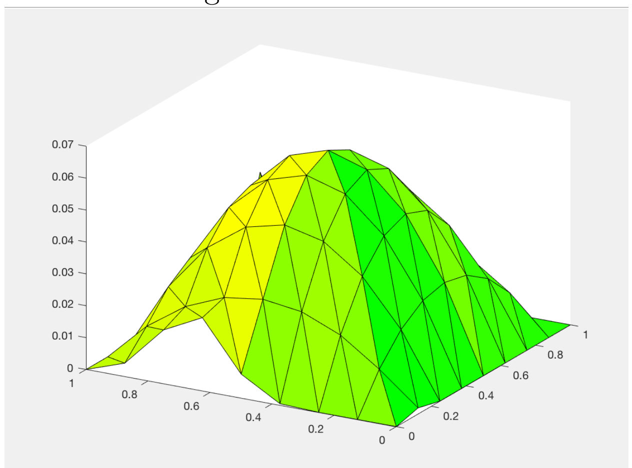
Third Function

$$F = -2x(x-1) - 2y(y-1)$$

with

$$G = x^2 * y^2$$

Figure 5: Function 3



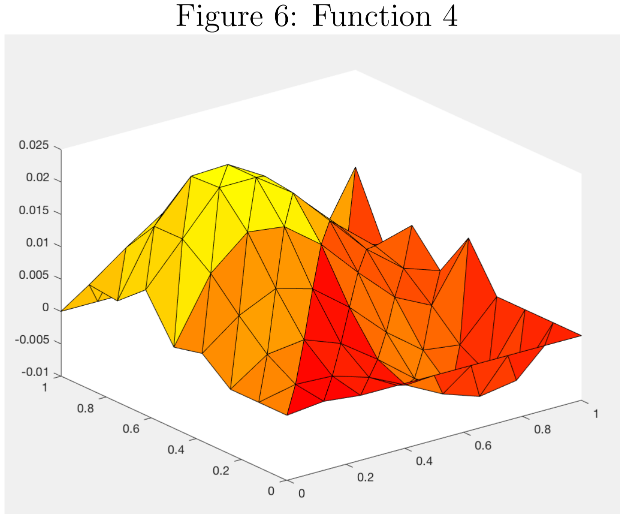
Fourth Function

$$F = x^2(x-2) + y^2(y+2)$$

with

$$G = x^2 * y^{\frac{1}{2}}$$

Figure 6: Function 4



References

[1] Li, Zhilin, and Zhonghua Qiao. *Numerical Solution of Differential Equations: Introduction to Finite Difference and Finite Element Methods*. Cambridge University Press, 2017.

[2] Strang, Gilbert; Fix, George (1973). *An Analysis of The Finite Element Method*. Prentice Hall. ISBN 0-13-032946-0.