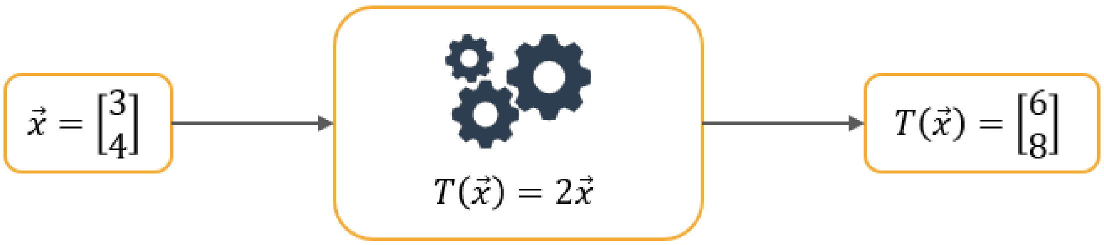
Linear Transformations

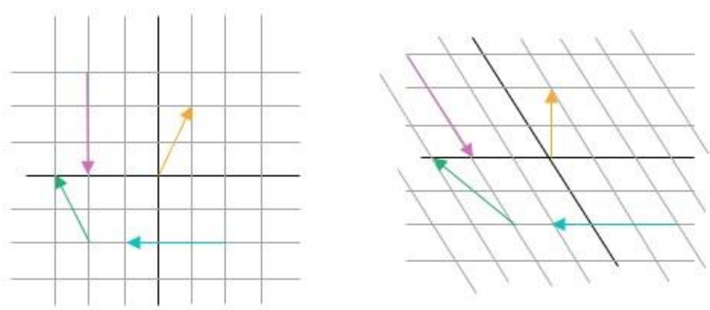
Transformations

A transformation is a function that takes a vector as an input and produces an output vector



Linear Transformations

- For a transformation to be linear:
  - A line should remain a line after transformation.
  - Origin should remain fixed.
  - Grid lines should stay parallel and equidistant.



To verify that a transformation is linear we need to verify the following properties.

- To maintain linearity, transformation should obey the following properties:
  - $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
  - $T(c\vec{u}) = c\,T(\vec{u})$

Every matrix can be used to define a linear transformation.

$$\begin{bmatrix}1\\2\\1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 2 & 3\\4 & 5 & 1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix} \longrightarrow \begin{bmatrix}8\\15\end{bmatrix}$$

$$\vec{x} \qquad\qquad T(\vec{x}) = A\vec{x} \qquad\qquad \vec{b}$$

The transformations defined using matrices are always linear.

- Matrices as a transformation follow both the properties:
  - $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
  - $T(c\vec{u}) = c\,T(\vec{u})$
- Every linear transformation can be expressed as a matrix.

Standard Transformations

The following standard transformations can be defined using matrices.

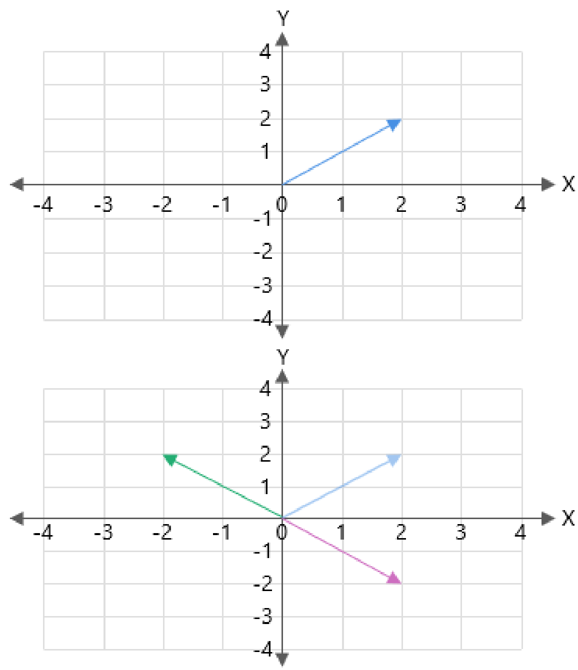
- **Reflection**

About Y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -x_1 \\ y_1 \end{bmatrix}$$

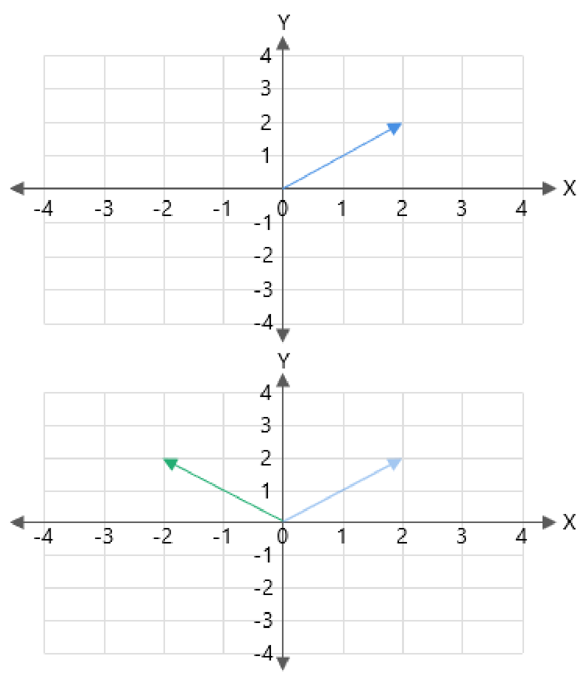
About X-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ -y_1 \end{bmatrix}$$



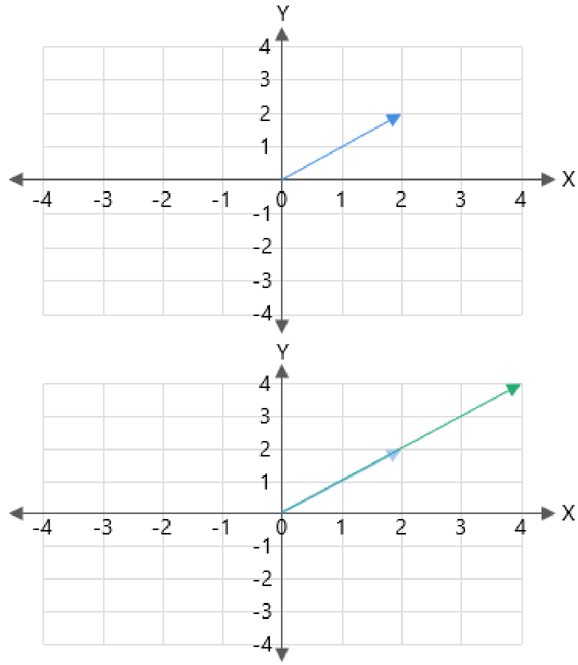
- **Rotation**

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -x_1 \\ y_1 \end{bmatrix}$$



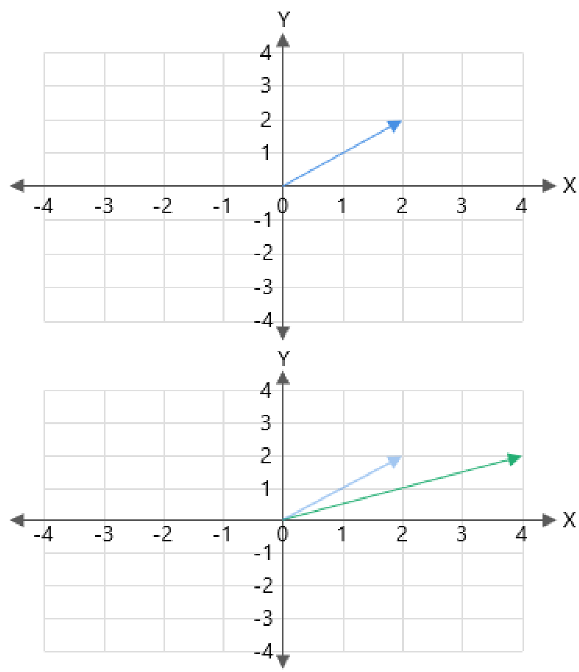
- Dilation

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2y_1 \end{bmatrix}$$



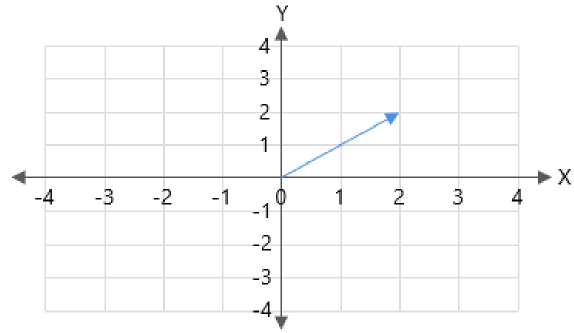
- Shear

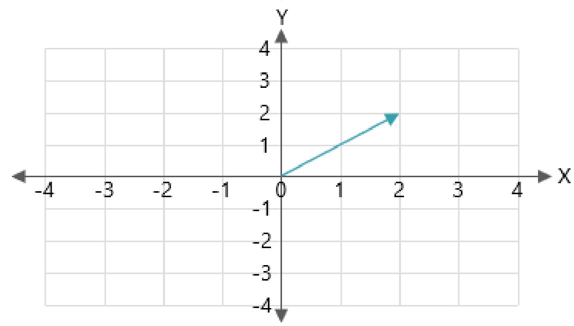
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ y_1 \end{bmatrix}$$



- Identity

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}x_1\\y_1\end{bmatrix}$$





Reversing a Transformation

To reverse a transformation, we need to follow the following steps. Note that not all transformations can be reversed.

- Transformation done by matrix A can be reversed
  - Say, $\vec{x}$ is transformed by transformation T, i.e., $T\vec{x}$
  - To get the original $\vec{x}$, we can multiply by T^{-1}, i.e., $T^{-1}T\vec{x}$
- Not all transformations can be reversed
  - Transformation by a singular/non-invertible matrix cannot be reversed

For example,

$$\vec{u}=\begin{bmatrix}3\\-2\end{bmatrix}, T=\begin{bmatrix}3 & 2\\1 & 2\end{bmatrix}$$

Transforming, vector $\vec{u}$ to $\vec{u'}$:

$\vec{u'} = T\vec{u} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

Getting back the original transformation:

$T^{-1}\vec{u'} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 12 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Coordinates of a vector in a changed basis

Consider vector $\vec{a} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$ in $\mathbb{R}^2$. Components of $\vec{a}$ depend on the basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We know it can be written as $\vec{a} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix}$. As a matrix, the standard basis vectors in $\mathbb{R}^2$ can be written as $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If basis vector changes, the components of vector $\vec{a}$ will also change.

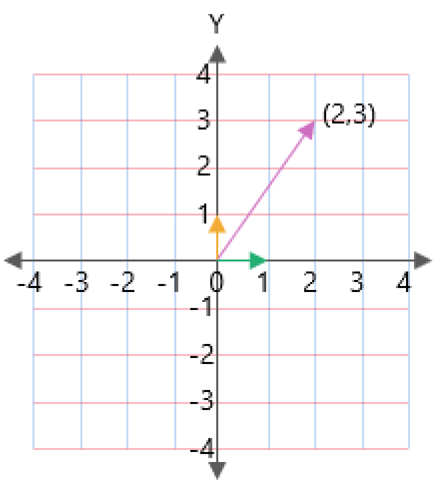
To find the updated components of vector $\vec{a}$ in the new basis, we use the change of basis matrix. Consider the basis is $B_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$?

In B_1, $\vec{a} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

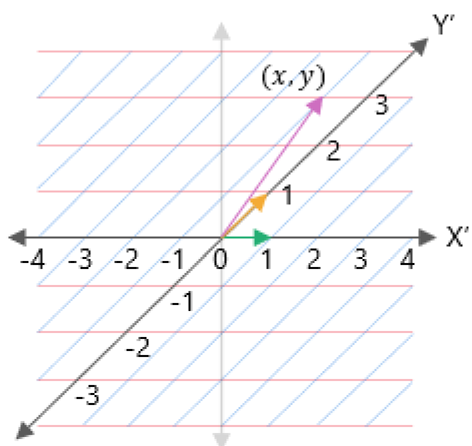
In B_2, $\vec{a'} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

So, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$,

that is, $\begin{bmatrix} x \\ y \end{bmatrix} = B_2^{-1}\vec{a} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$



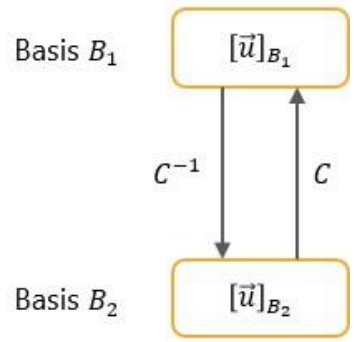
B_1



B_2

To summarize, we can say that to find the coordinates of a vector in a changed basis B_2 we follow the following steps.

- $[\vec{u}]_{B_2} = {B_2}^{-1} B_1 [\vec{u}]_{B_1}$
- Similarly, $[\vec{u}]_{B_1} = {B_1}^{-1} B_2 [\vec{u}]_{B_2}$, where $C = {B_1}^{-1} B_2$
- C is the change of basis matrix
- $C^{-1} = ({B_1}^{-1} B_2)^{-1} = {B_2}^{-1} B_1$
- Rewriting: $[\vec{u}]_{B_1} = C[\vec{u}]_{B_2}$ and $[\vec{u}]_{B_2} = C^{-1}[\vec{u}]_{B_1}$

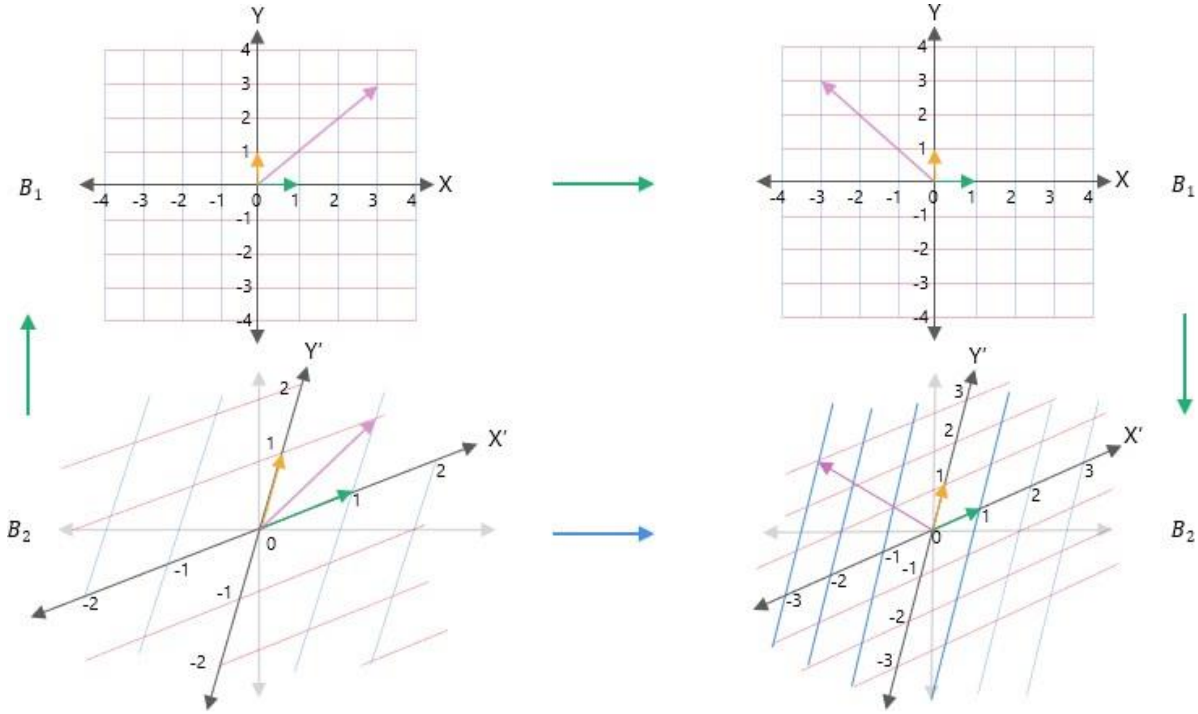


Transformations in a changed basis

If we need to perform transformations in a changed basis cannot use the matrices which we used in the case of standard basis. To get the new matrix we need to follow the following steps.

- First, convert vector $[\vec{u}]_{B_2}$ to $[\vec{u}]_{B_1}$ by multiplying with C and get the vector in the standard basis.
- Apply the transformation matrix T to the resultant vector $[\vec{u}]_{B_1}$; this gives us the transformed vector $[v]_{B_1}$ in the standard basis.
- Then, convert the transformed vector $[v]_{B_1}$ to the vector in basis B_2, i.e., $[v]_{B_2}$ by multiplying $[v]_{B_1}$ with C^{-1}.

The following figure shows the reflection of the vectors $\begin{bmatrix}1\\1\end{bmatrix}$ in the basis $\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$.



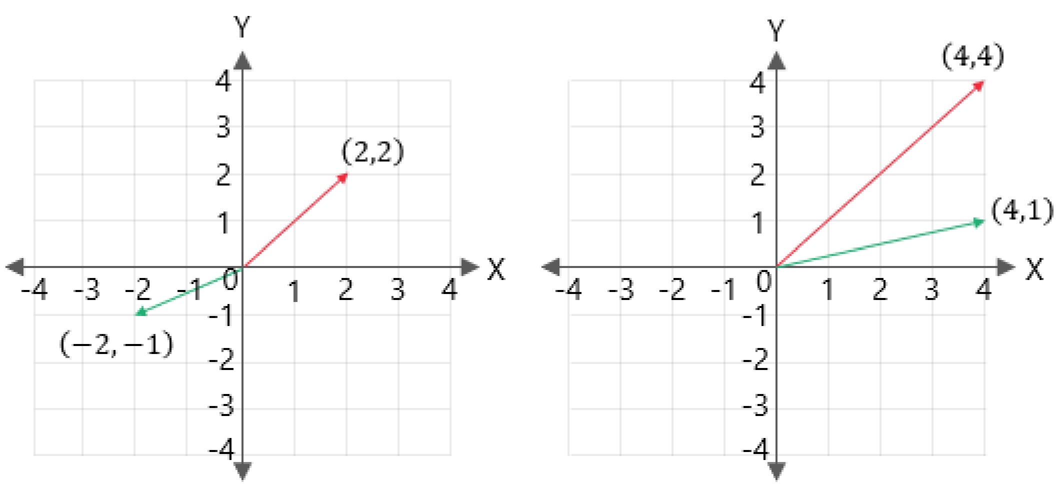
Eigenvalues and Eigenvectors

Eigenvectors and Eigenvalues

- An eigenvector is a non-zero vector $\vec{x}$ such that $T(\vec{x}) = \lambda\vec{x}$ for some real number λ. Basically, T only scales $\vec{x}$ and does not change its direction, as λ is a scalar.
- λ is called the eigenvalue.
- Each eigenvalue has a corresponding eigenvector.
- A matrix can have multiple eigenvector and eigenvalue pairs.

Consider the following example.

For the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, consider the transformation $T(\vec{x}) = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$.



The direction of the vector $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ does not change; in fact, the vector has only been scaled, whereas if we consider the vector $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$, its direction has changed.

Finding Eigenvectors and Eigenvalues

Suppose $\vec{x}$ is an eigenvector of the transformation $T(\vec{x}) = A\vec{x}$ corresponding to eigenvalue λ.

Then, $A\vec{x} = \lambda\vec{x}$

$\Rightarrow\ AI\vec{x} = \lambda I\vec{x}$ (multiplying both sides by I)

$\Rightarrow\ A\vec{x} = \lambda I\vec{x}$ (since $AI = A$)

$\Rightarrow\ A\vec{x} - \lambda I\vec{x} = 0$

$\Rightarrow (A - \lambda I)\vec{x} = 0$

Either $\vec{x} = \vec{0}$ (trivial solution) or $(A - \lambda I) = 0$

Solve for λ (eigenvalues) by solving $\det(A - \lambda I) = 0$

Consider the following examples where we need to find eigenvectors and eigenvalues.

Given matrix $A = \begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix}$, let's compute the eigenvalues of matrix A.

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 3-\lambda & -5 \\ -6 & 4-\lambda \end{bmatrix}\right) = 0$$

$$(3-\lambda)(4-\lambda) - (-5)(-6) = 0$$

$$12 - 4\lambda - 3\lambda + \lambda^2 - 30 = 0$$

$$12 - 7\lambda + \lambda^2 - 30 = 0$$

$$\lambda^2 - 7\lambda - 18 = 0$$

$$\lambda^2 - 9\lambda + 2\lambda - 18 = 0$$

$$\lambda(\lambda - 9) + 2(\lambda - 9) = 0$$

$$(\lambda + 2)(\lambda - 9) = 0$$

$$\lambda = -2 \quad , \quad \lambda = 9$$

Given matrix $A = \begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix}$ and its eigenvalues, let's compute the eigenvectors of matrix A.

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\lambda = 9$, $\lambda = -2$

$\underline{\lambda = 9}$

$A\vec{x} = \lambda \vec{x}$

$\begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix} \vec{x} = 9\vec{x}$

$\Rightarrow \begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 9 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow 3x_1 - 5x_2 = 9x_1$

$\Rightarrow -6x_1 + 4x_2 = 9x_2$

$\Rightarrow \begin{cases} -6x_1 - 5x_2 = 0 \\ -6x_1 - 5x_2 = 0 \end{cases}$ $\quad x_2 = 1$, $x_1 = -\frac{5}{6}$

$6x_1 + 5x_2 = 0$ $\quad \vec{x} = \begin{bmatrix} -\frac{5}{6} \\ 1 \end{bmatrix}$

$x_1 = -\frac{5}{6} x_2$

$\lambda = -2$

$A\vec{x} = \lambda \vec{x}$

$\begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow 3x_1 - 5x_2 = -2x_1$

$\Rightarrow -6x_1 + 4x_2 = -2x_2$

$\Rightarrow 5x_1 - 5x_2 = 0$

$-6x_1 + 6x_2 = 0$

$5x_1 = 5x_2$ $\quad -6x_1 = -6x_2$

$x_1 = x_2$ $\quad x_1 = x_2$

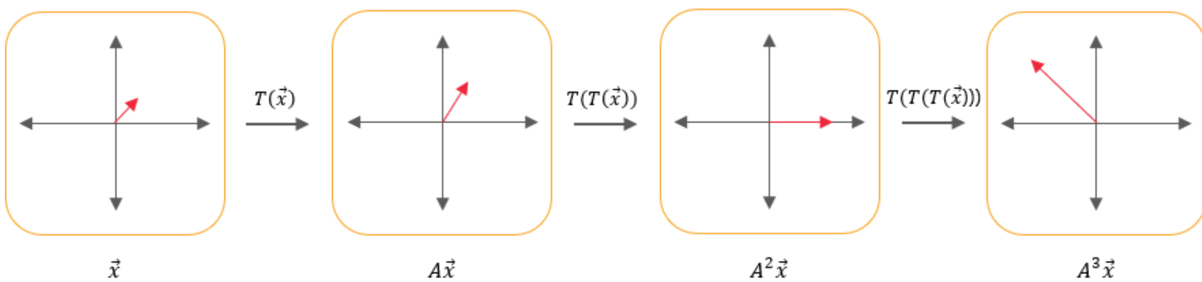
$x_2 = 1$, $x_1 = 1$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigendecomposition

Need for Eigendecomposition

Consider the vector $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the transformation $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.



$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix}$ $\quad A^3 = \begin{bmatrix} 3 & -6 \\ 6 & -3 \end{bmatrix}$

Suppose we had to apply the transformation on the vector 100 times then we would need to calculate A^{100} , which is a tedious process. If the matrix was a diagonal matrix, it would be easier to calculated higher powers.

If $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$

If the matrix A is a diagonal matrix, then we can easily find the resultant vector after transforming it multiple times. Diagonalization helps us to relate a matrix to a diagonal matrix.

Finding Eigendecomposition

To find the Eigendecomposition of a matrix we use the following steps.

- $\det(A - \lambda I) = 0$ (Get eigenvalues and eigenvectors)
- $A = CDC^{-1}$ (C is the change of basis matrix having eigenvectors as its columns and D is the diagonal matrix with eigenvalues as diagonal elements)

If we have the Eigendecomposition of a matrix we can apply the transformation in the following manner.

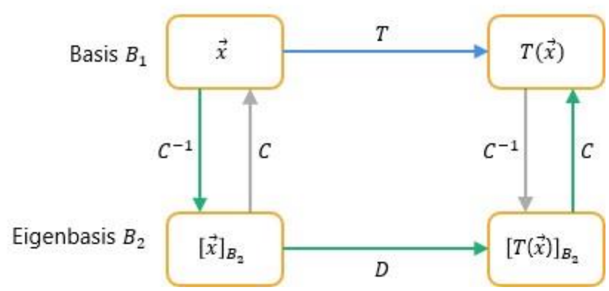
We take an input vector $\vec{x}$ and convert it to $[\vec{x}]_{B_2}$ by multiplying by C^{-1}.

We then apply transformation D (diagonal matrix) to $[\vec{x}]_{B_2}$ that is $D[\vec{x}]_{B_2}$ and get $[T(\vec{x})]_{B_2}$.

We then get the transformed vector back to the original basis that is $T(\vec{x})$ by multiplying the result with C.

$T(\vec{x}) = CDC^{-1}(\vec{x})$

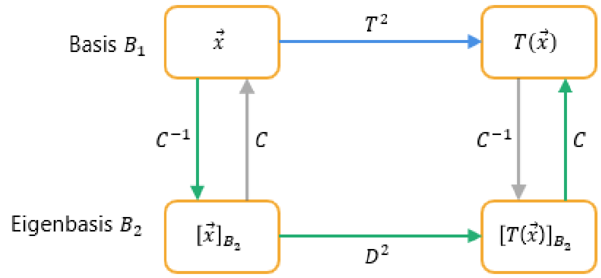
This implies that $T = CDC^{-1}$.



If we have the Eigendecomposition of a matrix and this matrix is used to define a transformation, then applying this transformation multiple times becomes easy because we basically need to calculate powers of a diagonal matrix.

If we need to do this transformation twice:

$$\begin{aligned} T^2 &= CDC^{-1}CDC^{-1} \\ &= CDIDC^{-1} \\ &= CDDC^{-1} \\ &= CD^2C^{-1} \end{aligned}$$



And for n times: $T^n = CD^nC^{-1}$