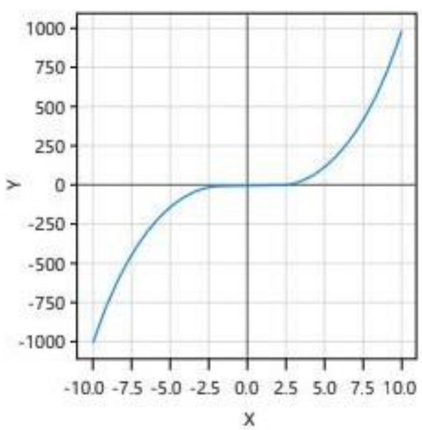
Multivariable Calculus

Introduction to Multivariable Functions

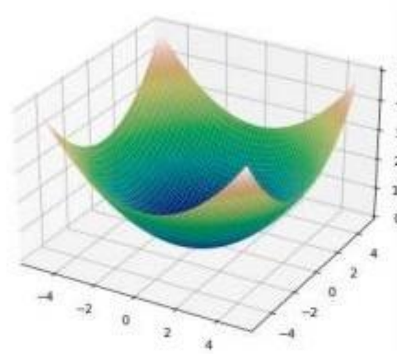
- Function of single variable
- Maps one variable to another

$$y = f(x) = x^3$$



- Function of two or more variables
- Maps a tuple of variables to another variable

$$z = f(x, y) = x^2 + y^2$$



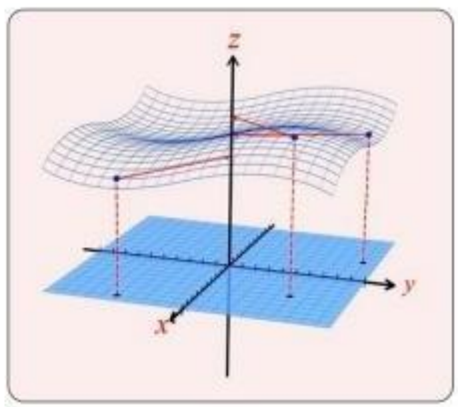
The following are examples of multivariable functions in the real world:

- The elevation on a mountain at a point is determined by two coordinates.
- A person’s BMI is determined by their height and weight.
- Occupancy in a hospital depends upon the season, cost and probability of flu, among other factors.
- The profit earned by a store depends upon the price of each of its items.
- The standard of living depends upon a person’s income, expenses, number of dependents, etc.

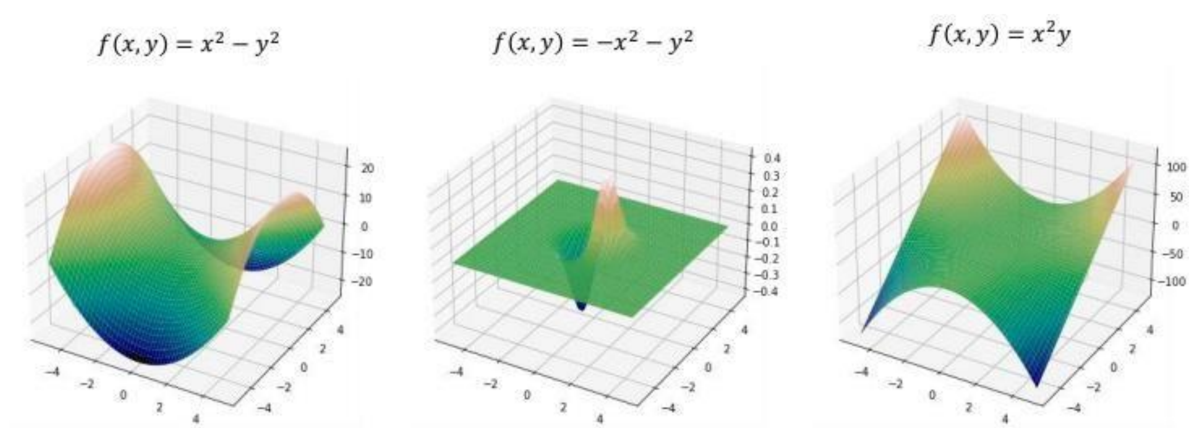
- Temperature may depend on the season, location and the time of the day.

Functions of Two Variables

- A multivariable function takes more than one parameter: $y = f(x, y)$.
- Each ordered pair (x, y) is the domain of the function.
- Each (x, y) is mapped to a real number $z,$ which is the range of the function.
- Each point on the graph of the function is given by an ordered triplet, (x, y, z).
- The graph of a function with two variables is a surface.



The figure below shows some examples of multivariable functions.



Functions of More Than Two Variables

- Consider the function $f(x, y, z) = x^2 + 2xy - z$
  - $x,\ y$ and z represent a point in space and are mapped to a fourth quantity.
- Consider the function $f(x, y, t) = (2x - y)\sin t$
  - x and y can represent a point on a plane, t can represent time, and all three are mapped to a fourth quantity.

Clearly, it is not possible to visualise higher dimensions.

Partial Derivatives

- Let us consider the example of an electricity bill as a function of temperature, i.e., $e(t)$:

$$\frac{de}{dt} = \frac{Change\ in\ electricity\ bill}{Change\ in\ temperature}$$

- The slope of the tangent gives the instantaneous rate of change of the bill as a function of temperature.
- Let's say the electricity bill depends upon two variables: temperature, t, and the size of the house, s, i.e., $e(t, s)$.
- The electricity bill (e) is a function of two variables: temperature (t) and the size of the house (s), i.e., $e(t, s)$.
- Each of these rates of change will be the partial derivative of the electricity bill, e.
- The function e has two partial derivatives: $\frac{\partial e(t,s)}{\partial t}$ and $\frac{\partial e(t,s)}{\partial s}$.

- Keeping the size of the house fixed, $\frac{\partial e(t,s)}{\partial t} = \frac{Change\ in\ electricity\ bill\ (e)}{Change\ in\ temperature\ (t)}$.
- Keeping temperature fixed, $\frac{\partial e(t,s)}{\partial s} = \frac{Change\ in\ electricity\ bill\ (e)}{Change\ in\ size\ of\ the\ house\ (s)}$.

For example, $f(x, y) = 2x^3 + 3y^2$;

$$\frac{\partial f}{\partial x} = 6x^2$$

$$\frac{\partial f}{\partial y} = 6y$$

Gradients

- Gradients are the first order partial derivatives of a function
- Consider $f(x, y) = 2xy^2$
- Partial derivatives:

$$\frac{\partial f}{\partial x} = 2y^2$$

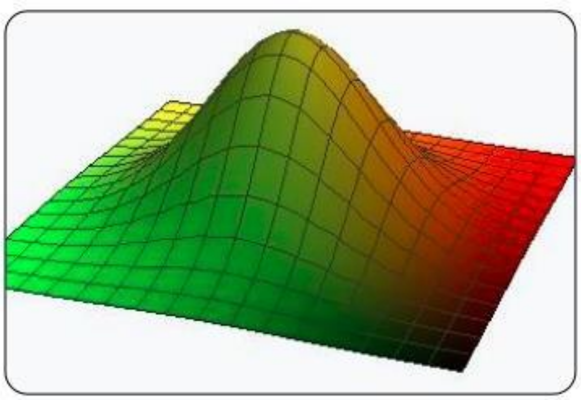
$$\frac{\partial f}{\partial y} = 4xy$$

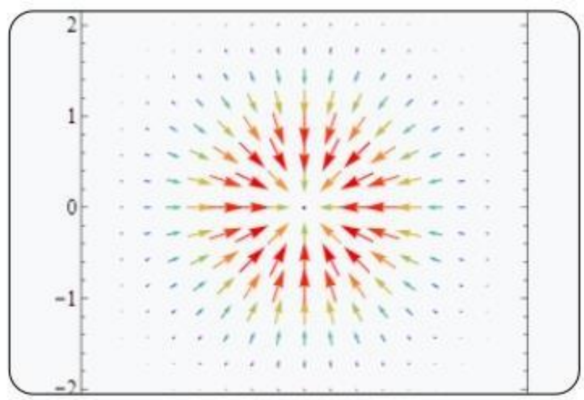
- The gradient of $f(x, y)$ is a vector of its partial derivatives

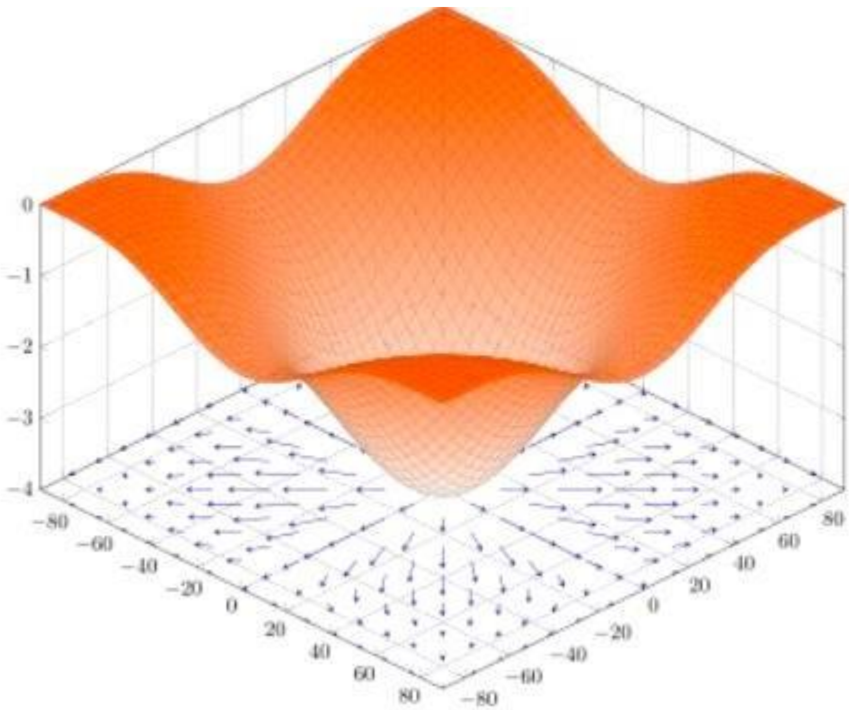
$$\nabla f(x, y) = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}] = [2y^2, 4xy]$$

- For specific values of x and y, the gradient will give a vector pointing in the direction of the steepest slope of this function

The value of gradient at a point gives us the direction of the steepest ascent at that point. The size of the arrows in the diagrams below help us judge how steep the slope is. The longer the arrows, the steeper the slope.



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