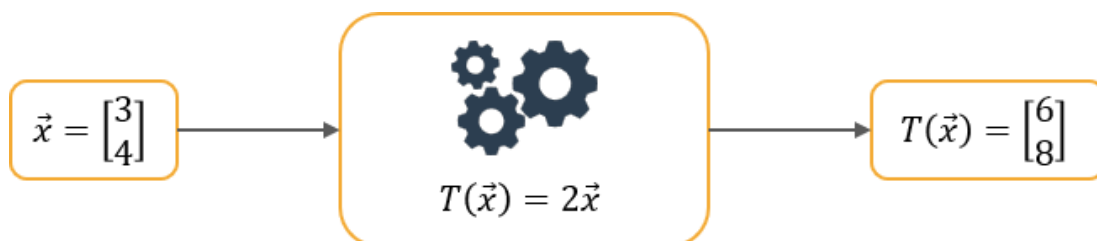


Linear Transformations

Transformations

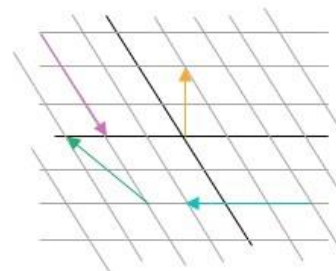
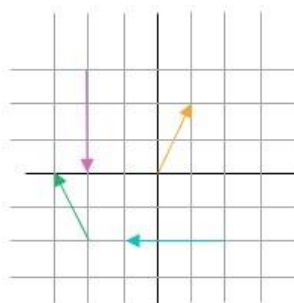
A transformation is a function that takes a vector as an input and produces an output vector



Linear Transformations

○ For a transformation to be linear:

- A line should remain a line after transformation.
- Origin should remain fixed.
- Grid lines should stay parallel and equidistant.

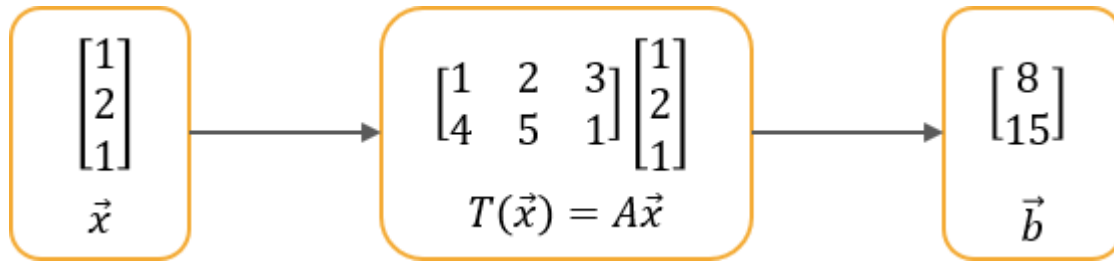


To verify that a transformation is linear we need to verify the following properties.

○ To maintain linearity, transformation should obey the following properties:

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- $T(c\vec{u}) = c T(\vec{u})$

Every matrix can be used to define a linear transformation.



The transformations defined using matrices are always linear.

○ Matrices as a transformation follow both the properties:

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- $T(c\vec{u}) = c T(\vec{u})$

○ Every linear transformation can be expressed as a matrix.

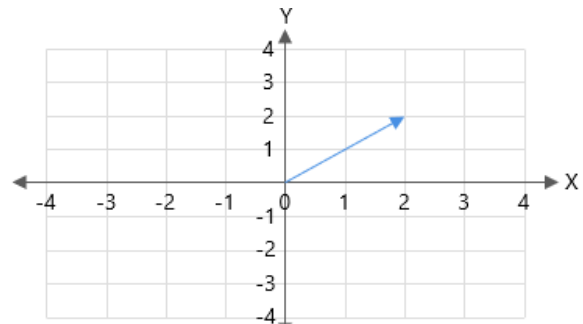
Standard Transformations

The following standard transformations can be defined using matrices.

○ Reflection

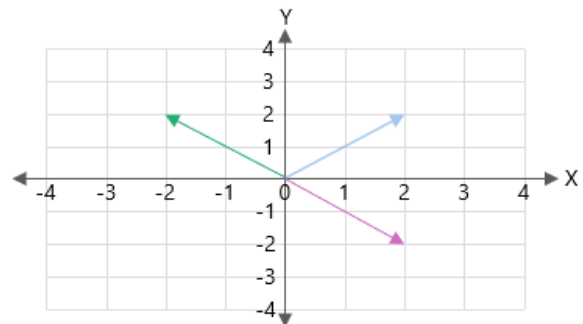
About Y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -x_1 \\ y_1 \end{bmatrix}$$



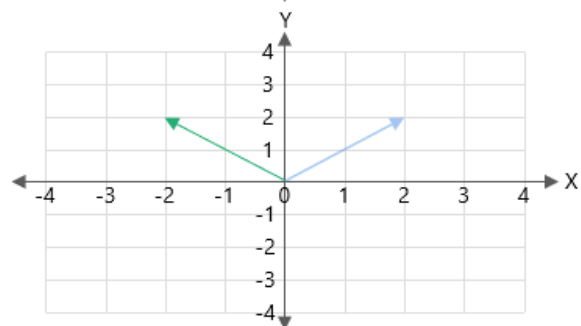
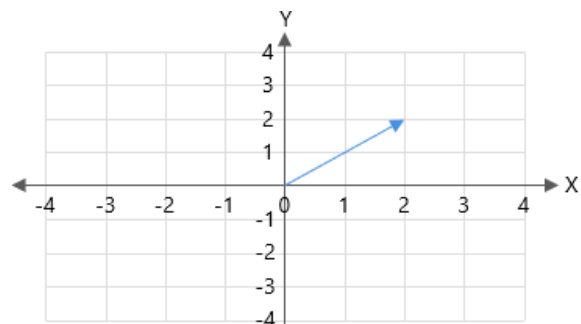
About X-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ -y_1 \end{bmatrix}$$



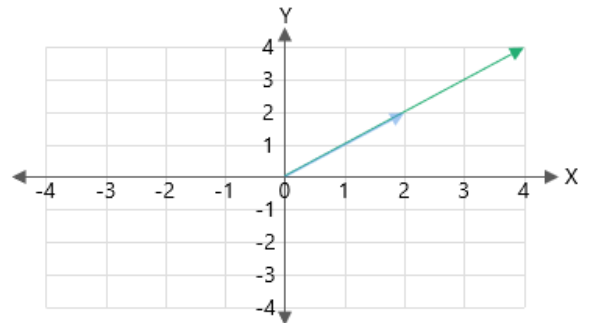
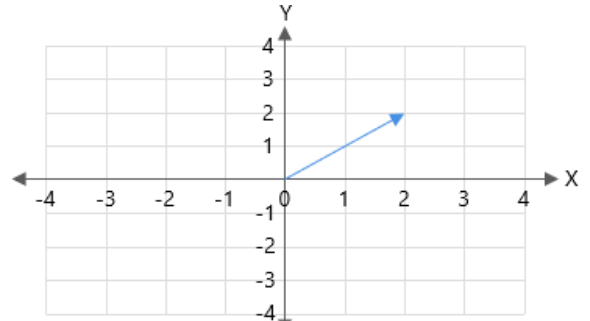
○ Rotation

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix}$$



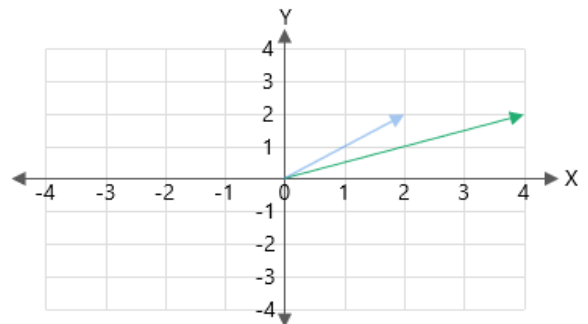
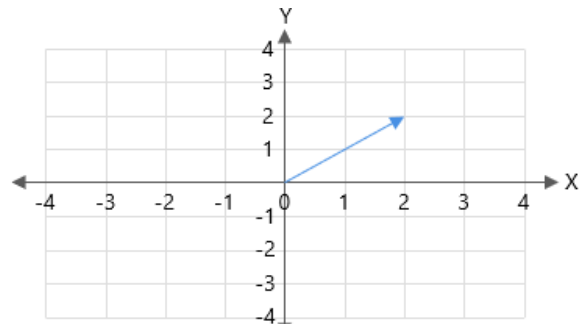
○ Dilation

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2y_1 \end{bmatrix}$$



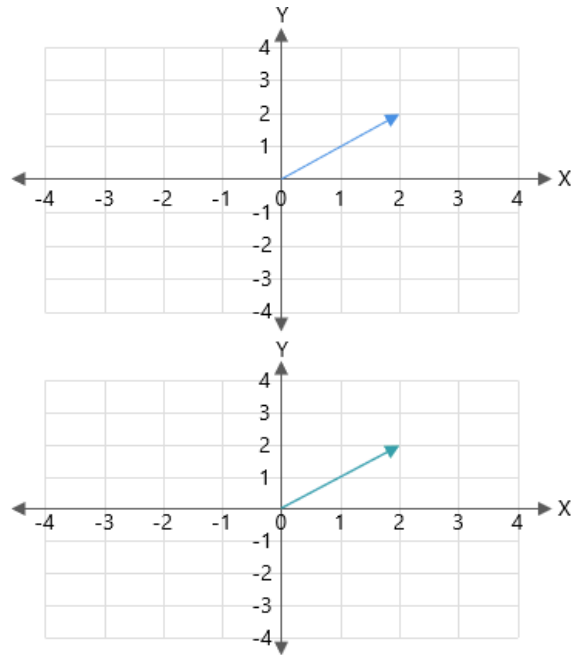
○ Shear

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ y_1 \end{bmatrix}$$



○ Identity

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$



Reversing a Transformation

To reverse a transformation, we need to follow the following steps. Note that not all transformations can be reversed.

- Transformation done by matrix A can be reversed
 - Say, \vec{x} is transformed by transformation T , i.e., $T\vec{x}$
 - To get the original \vec{x} , we can multiply by T^{-1} , i.e., $T^{-1}T\vec{x}$
- Not all transformations can be reversed
 - Transformation by a singular/non-invertible matrix cannot be reversed

For example,

$$\vec{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, T = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

Transforming, vector \vec{u} to \vec{u}' :

$$\vec{u}' = T\vec{u} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Getting back the original transformation:

$$T^{-1}\vec{u}' = \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 12 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Coordinates of a vector in a changed basis

Consider vector $\vec{a} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$ in \mathbb{R}^2 . Components of \vec{a} depend on the basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We know it can be written as $\vec{a} = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 9 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$. As a matrix, the standard basis vectors in

\mathbb{R}^2 can be written as $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If basis vector changes, the components of vector \vec{a} will also change.

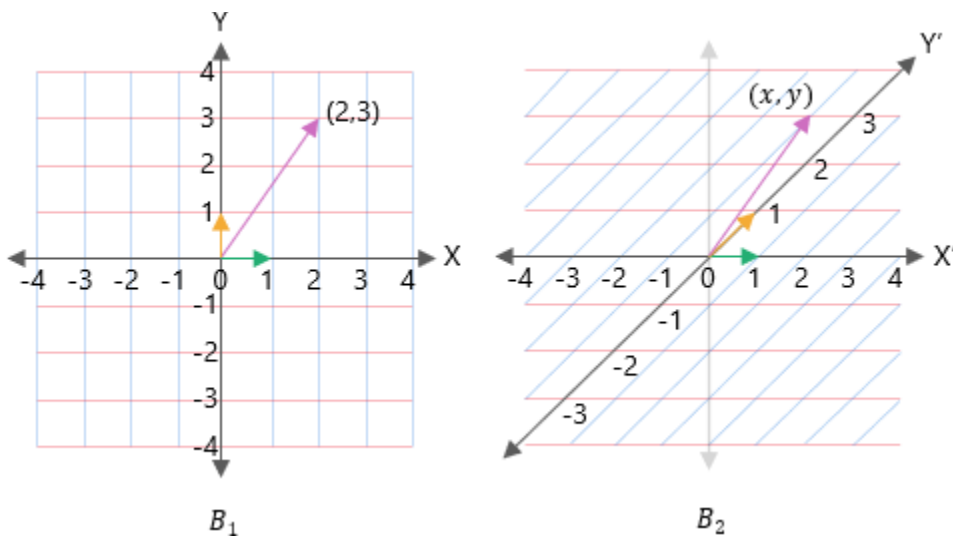
To find the updated components of vector \vec{a} in the new basis, we use the change of basis matrix. Consider the basis is $B_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$?

$$\text{In } B_1, \vec{a} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\text{In } B_2, \vec{a}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

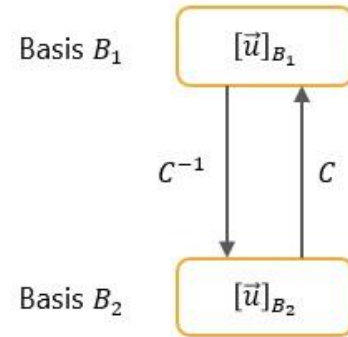
$$\text{So, } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

$$\text{that is, } \begin{bmatrix} x \\ y \end{bmatrix} = B_2^{-1} \vec{a} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$



To summarize, we can say that to find the coordinates of a vector in a changed basis B_2 we follow the following steps.

- $[\vec{u}]_{B_2} = B_2^{-1} B_1 [\vec{u}]_{B_1}$
- Similarly, $[\vec{u}]_{B_1} = B_1^{-1} B_2 [\vec{u}]_{B_2}$, where $C = B_1^{-1} B_2$
- C is the change of basis matrix
- $C^{-1} = (B_1^{-1} B_2)^{-1} = B_2^{-1} B_1$
- Rewriting: $[\vec{u}]_{B_1} = C [\vec{u}]_{B_2}$ and $[\vec{u}]_{B_2} = C^{-1} [\vec{u}]_{B_1}$

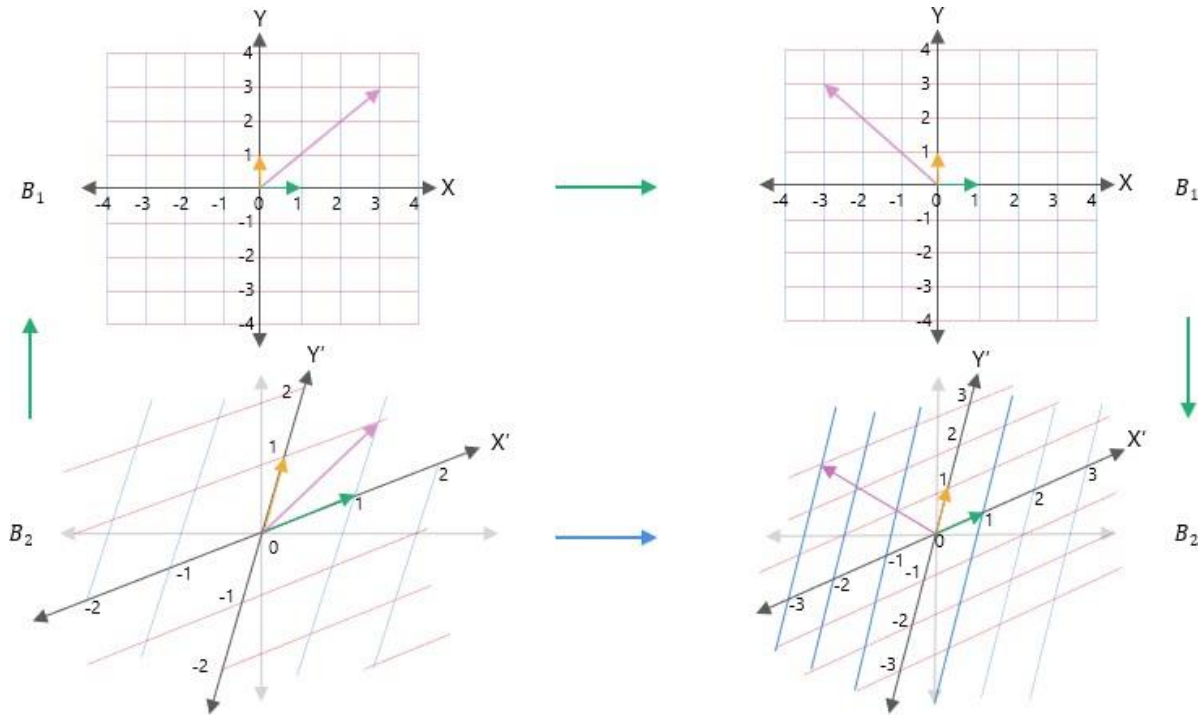


Transformations in a changed basis

If we need to perform transformations in a changed basis cannot use the matrices which we used in the case of standard basis. To get the new matrix we need to follow the following steps.

- First, convert vector $[\vec{u}]_{B_2}$ to $[\vec{u}]_{B_1}$ by multiplying with C and get the vector in the standard basis.
- Apply the transformation matrix T to the resultant vector $[\vec{u}]_{B_1}$; this gives us the transformed vector $[\vec{v}]_{B_1}$ in the standard basis.
- Then, convert the transformed vector $[\vec{v}]_{B_1}$ to the vector in basis B_2 , i.e., $[\vec{v}]_{B_2}$ by multiplying $[\vec{v}]_{B_1}$ with C^{-1} .

The following figure shows the reflection of the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the basis $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.



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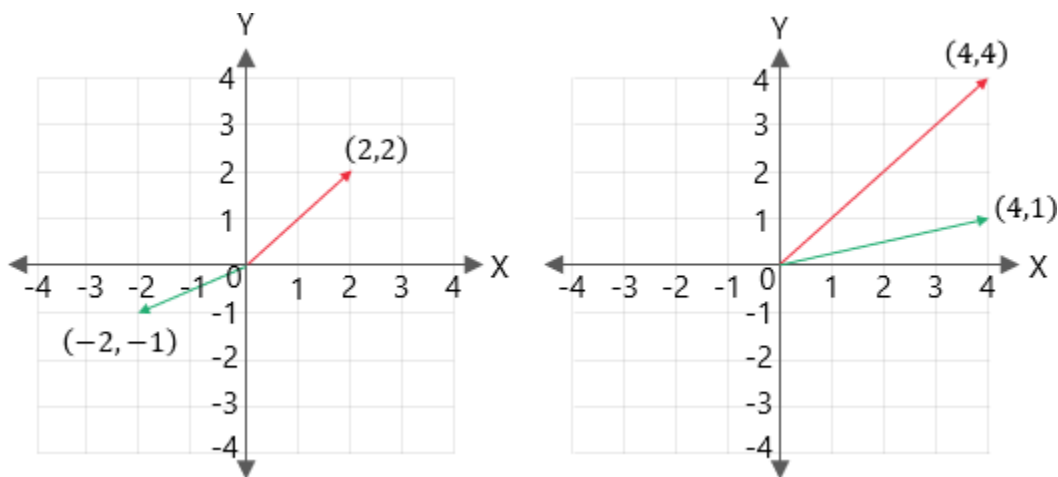
Eigenvalues and Eigenvectors

Eigenvectors and Eigenvalues

- An eigenvector is a non-zero vector \vec{x} such that $T(\vec{x}) = \lambda\vec{x}$ for some real number λ . Basically, T only scales \vec{x} and does not change its direction, as λ is a scalar.
- λ is called the eigenvalue.
- Each eigenvalue has a corresponding eigenvector.
- A matrix can have multiple eigenvector and eigenvalue pairs.

Consider the following example.

For the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, consider the transformation $T(\vec{x}) = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$.



The direction of the vector $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ does not change; in fact, the vector has only been scaled, whereas if we consider the vector $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$, its direction has changed.

Finding Eigenvectors and Eigenvalues

Suppose \vec{x} is an eigenvector of the transformation $T(\vec{x}) = A\vec{x}$ corresponding to eigenvalue λ .

Then, $A\vec{x} = \lambda\vec{x}$

$$\Rightarrow AI\vec{x} = \lambda I\vec{x} \quad (\text{multiplying both sides by } I)$$

$$\Rightarrow A\vec{x} = \lambda I\vec{x} \quad (\text{since } AI = A)$$

$$\Rightarrow A\vec{x} - \lambda I\vec{x} = 0$$

$$\Rightarrow (A - \lambda I)\vec{x} = 0$$

Either $\vec{x} = \vec{0}$ (trivial solution) or $(A - \lambda I) = 0$

Solve for λ (eigenvalues) by solving $\det(A - \lambda I) = 0$

Consider the following examples where we need to find eigenvectors and eigenvalues.

Given matrix $A = \begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix}$, let's compute the eigenvalues of matrix A .

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 0 \\ \det\left(\begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= 0 \\ \det\left(\begin{bmatrix} 3-\lambda & -5 \\ -6 & 4-\lambda \end{bmatrix}\right) &= 0 \\ (3-\lambda)(4-\lambda) - (-5)(-6) &= 0 \\ 12 - 4\lambda - 3\lambda + \lambda^2 - 30 &= 0 \\ 12 - 7\lambda + \lambda^2 - 30 &= 0 \\ \lambda^2 - 7\lambda - 18 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda^2 - 9\lambda + 2\lambda - 18 &= 0 \\ \lambda(\lambda - 9) + 2(\lambda - 9) &= 0 \\ (\lambda + 2)(\lambda - 9) &= 0 \\ \lambda = -2, \lambda = 9 \end{aligned}$$

Given matrix $A = \begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix}$ and its eigenvalues, let's compute the eigenvectors of matrix A .

$$\begin{aligned} \lambda = 9, \lambda = -2 \quad \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \lambda = -2 \quad A\vec{x} = \lambda\vec{x} \\ \underline{\lambda = 9} \quad A\vec{x} &= \lambda\vec{x} \quad \begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 9 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 9 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 3 & -5 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \Rightarrow 3x_1 - 5x_2 &= 9x_1 \quad \Rightarrow 3x_1 - 5x_2 = -2x_1 \\ \Rightarrow -6x_1 + 4x_2 &= 9x_2 \quad \Rightarrow -6x_1 + 4x_2 = -2x_2 \\ \Rightarrow \begin{cases} -6x_1 - 5x_2 = 0 & x_2 = 1 \\ -6x_1 - 5x_2 = 0 & x_1 = -\frac{5}{6} \end{cases} \quad \Rightarrow \begin{cases} 5x_1 - 5x_2 = 0 \\ -6x_1 + 6x_2 = 0 \end{cases} \\ 6x_1 + 5x_2 = 0 \quad \vec{x} &= \begin{bmatrix} -\frac{5}{6} \\ 1 \end{bmatrix} \quad 5x_1 = 5x_2 \quad -6x_1 = -6x_2 \\ x_1 &= -\frac{5}{6}x_2 \quad x_1 = x_2 \quad x_2 = 1, x_1 = 1 \\ \vec{x} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

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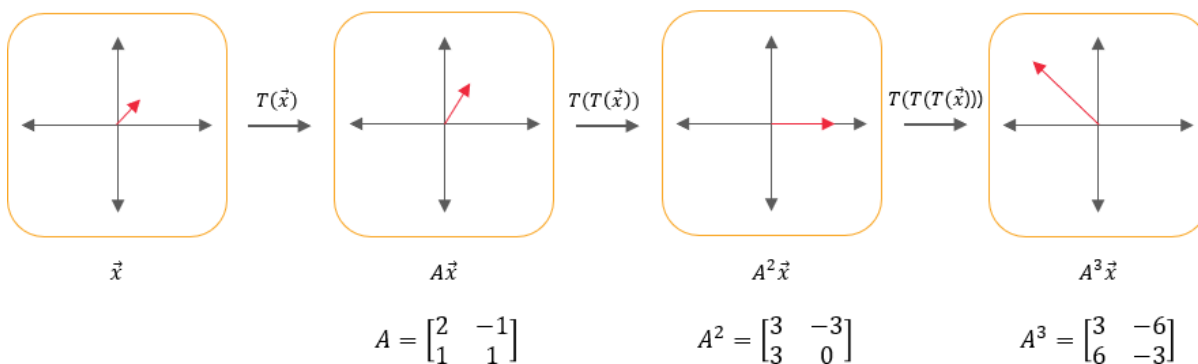
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Eigendecomposition

Need for Eigendecomposition

Consider the vector $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the transformation $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.



Suppose we had to apply the transformation on the vector 100 times then we would need to calculate A^{100} , which is a tedious process. If the matrix was a diagonal matrix, it would be easier to calculate higher powers.

$$\text{If } A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

If the matrix A is a diagonal matrix, then we can easily find the resultant vector after transforming it multiple times. Diagonalization helps us to relate a matrix to a diagonal matrix.

Finding Eigendecomposition

To find the Eigendecomposition of a matrix we use the following steps.

- $\det(A - \lambda I) = 0$ (Get eigenvalues and eigenvectors)
- $A = CDC^{-1}$ (C is the change of basis matrix having eigenvectors as its columns and D is the diagonal matrix with eigenvalues as diagonal elements)

If we have the Eigendecomposition of a matrix we can apply the transformation in the following manner.

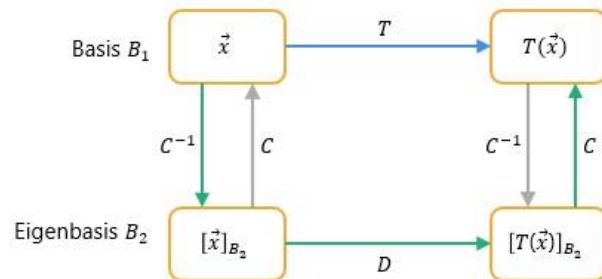
We take an input vector \vec{x} and convert it to $[\vec{x}]_{B_2}$ by **multiplying** by C^{-1} .

We then **apply transformation** D (diagonal matrix) to $[\vec{x}]_{B_2}$ that is $D[\vec{x}]_{B_2}$ and get $[T(\vec{x})]_{B_2}$.

We then get the transformed vector back to the original basis that is $T(\vec{x})$ by **multiplying** the result with C .

$$T(\vec{x}) = CDC^{-1}(\vec{x})$$

This implies that $T = CDC^{-1}$.

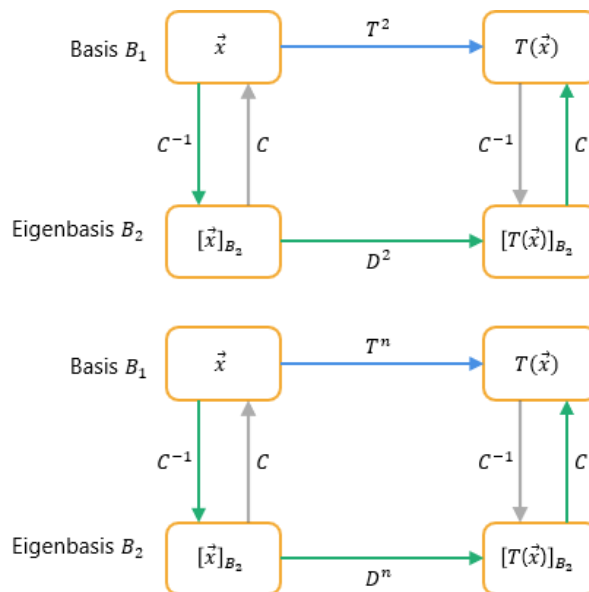


If we have the Eigendecomposition of a matrix and this matrix is used to define a transformation, then applying this transformation multiple times becomes easy because we basically need to calculate powers of a diagonal matrix.

If we need to do this transformation twice:

$$\begin{aligned} T^2 &= CDC^{-1}CDC^{-1} \\ &= CDIDC^{-1} \\ &= CDDC^{-1} \\ &= CD^2C^{-1} \end{aligned}$$

And for n times: $T^n = CD^nC^{-1}$



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