Introduction to Vectors

What are Vectors?

Vectors can be thought of as ordered lists or tuples. They are used to store values from a data set.

A vector can be represented as follows:

- $\boldsymbol{v}$
- $\vec{v}$
- $\bar{v}$

Now, you will learn about vectors with the help of an example. The following tables contain information about the height and weight of five people.

Person	Height (cm)	Weight (kg)
1	155	85
2	170	85
3	150	80
4	165	80
5	155	70

We can represent the height of all five people using the vector $\vec{h} = (155 \quad 170 \quad 150 \quad 165 \quad 155)$.

We can also represent the height and weight of the first person using the vector $\vec{v}_1 = (155 \quad 85)$.

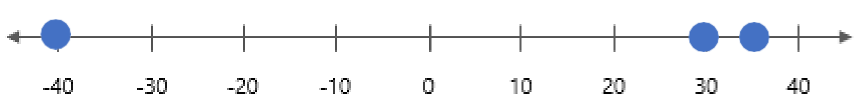
Similarly, we can create vectors for all five people.

1-D Coordinate Space: $\mathbb{R}^1$

The one-dimensional coordinate system is defined using a single line called the number line. Each point on this line represents a real number and vice versa. All points on this line are measured with respect to the origin.

Consider the following situation. A rod is being heated, and its temperature is being recorded after every hour. The following table shows the temperatures after the first, second and third hour.

Time (hours)	Temperature (°C)
1	-40
2	35
3	30



Here, the temperature of the rod is a data point. Every data point (real number) corresponds to a unique point on the number line and vice versa.

2-D Coordinate Space: $\mathbb{R}^2$

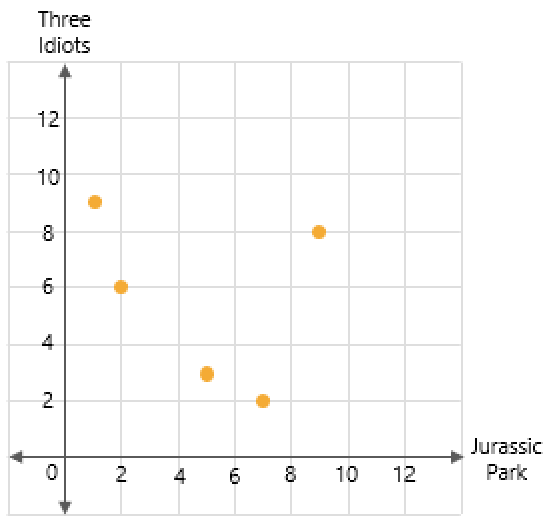
Some important points related to the 2-D coordinate space are as follows:

- The 2-D coordinate space is defined using two perpendicular lines called the x-axis and the y-axis.
- The point of intersection of the axes is called the origin, and all the positions are measured relative to the origin.
- Any point in the two-dimensional space is represented using an ordered pair of the form (a, b), where a is the x coordinate and b is the y coordinate.

Consider the following table that shows the ratings given by five users to two movies.

User	***Jurassic Park*** **(Adventure)**	***Three Idiots*** **(Comedy)**
Anmol	7	2
Babita	2	6
Chester	5	3
Disha	9	8
Euclid	1	9

Consider the rating given by Anmol. His ratings can be represented by the ordered pair $(7, 2)$, where the first and the second coordinates refer to the ratings given to *Jurassic Park* and *Three Idiots*, respectively. We can represent the ratings given by all five users in a similar manner and plot them on the two-dimensional plane as shown below.



3-D Coordinate Space: $\mathbb{R}^3$

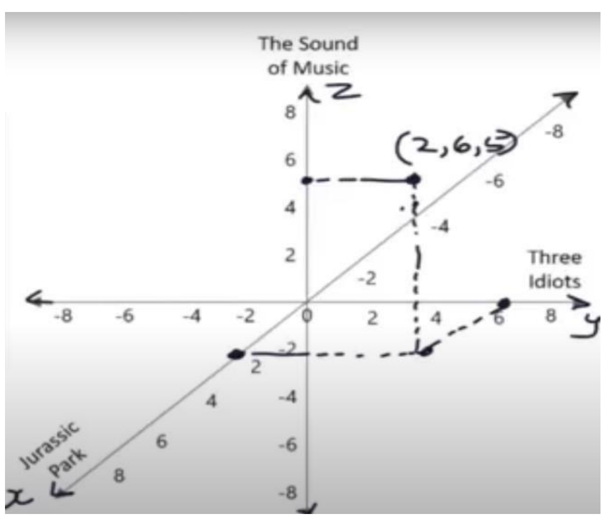
Some important points related to the 3-D coordinate space are as follows:

- The three-dimensional coordinate space is defined using three perpendicular lines called the x-axis, the y-axis and the z-axis.
- Any point in the three-dimensional space is represented using an ordered triplet of the form (a, b, c) where a is the x coordinate, b is the y coordinate and c is the z coordinate.
- The point of intersection of the axes is called the origin, and all the positions are measured relative to the origin.

Consider the following table. The table shows the ratings given by five users to three movies.

User	*Jurassic Park* (Adventure)	*Three Idiots* (Comedy)	*The Sound of Music* (Romance)
Anmol	7	2	6
Babita	2	6	5
Chester	5	3	9
Disha	9	8	6
Euclid	1	9	5

Consider the rating given by Babita. Her ratings can be represented by the ordered triplet $(2, 6, 5)$, where the first, second and third coordinates refer to the ratings given to *Jurassic Park*, Three Idiots and *The Sound of Music*, respectively. We can represent the ratings given by all the five users in a similar manner and plot them on the three-dimensional plane. The rating given by Babita is as shown below.



n-Dimensional Coordinate Space: $\mathbb{R}^n$

Each point in the n-dimensional space is represented by a tuple containing n coordinates.

Consider the following table that shows the ratings given by five users to five movies.

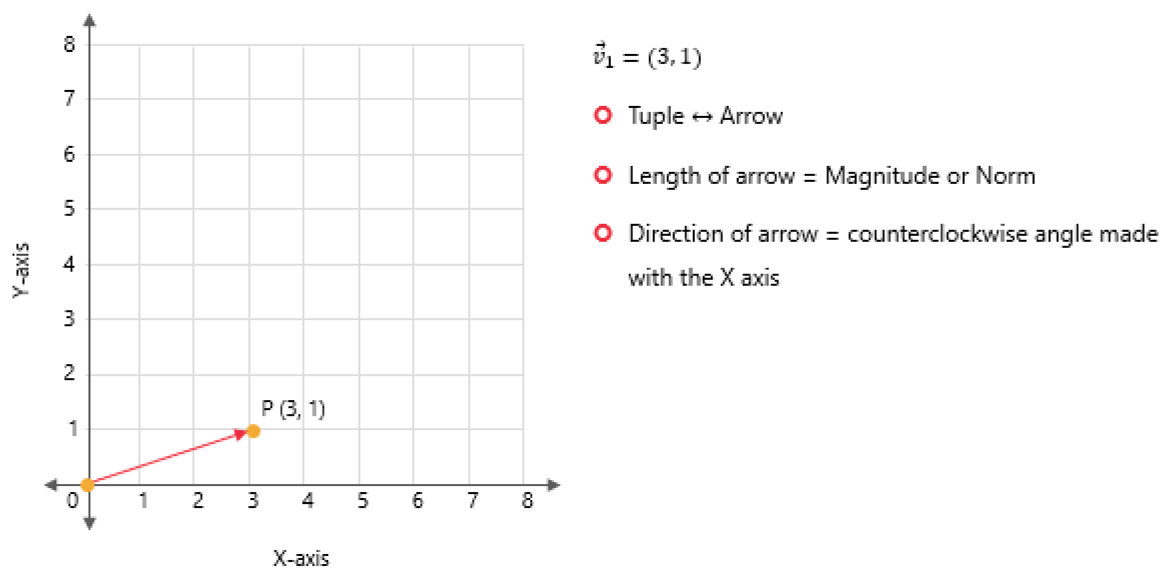
User	***Jurassic Park* (Adventure)**	***Three Idiots* (Comedy)**	***The Sound of Music* (Romance)**	***Dracula* (Horror)**	***Macbeth* (Tragedy)**
Anmol	7	9	8	6	2
Babita	2	7	9	10	6
Chester	5	4	3	7	3
Disha	9	2	5	10	8
Euclid	1	6	3	8	9

Consider the rating given by Babita. Her ratings can be represented by the ordered tuple $(2, 7, 9, 10, 6)$, where the first, second, third, fourth and fifth coordinates refer to the ratings given to *Jurassic Park*, *Three Idiots*, *The Sound of Music*, *Dracula* and *Macbeth*, respectively. We can represent the ratings given by all the five users in a similar manner; however, it will be difficult to visualise them.

Plotting Vectors

Plotting a vector means representing a vector by an arrow in space. The arrow connects a point by a line from the origin. The arrowhead gives the sense of direction of the vector.

The vector $(3, 1)$ can be plotted as shown in the image given below.

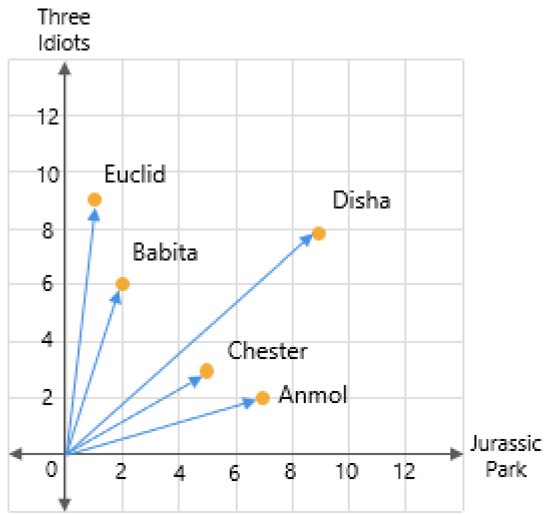


Plotting vectors makes it easy for us to understand data.

Consider the following ratings given by five users for two movies and the corresponding plot.

User	Jurassic Park (Adventure)	Three Idiots (Comedy)
Anmol	7	2
Babita	2	6
Chester	5	3
Disha	9	8
Euclid	1	9

User movie ratings



By looking at the graph, we can easily interpret that Anmol and Chester like *Jurassic Park* more than *Three Idiots*, and Babita and Euclid like *Three Idiots* more than *Jurassic Park*.

If we were to handle a larger set of users, this visualisation would help us understand user preferences efficiently.

Magnitude of a Vector

Some important points related to the magnitude of a vector are as follows:

- Consider a vector $\vec{u} = (u_1, u_2)$. The magnitude is denoted by $||\vec{u}||$
- $||\vec{u}|| = \sqrt{u_1^2 + u_2^2}$
- In the n-dimensional space, $||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$
- Consider a vector $\vec{u} = (4, 3)$. Then, $||\vec{u}|| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$

Unit Vector

Some important points related to the unit vector are as follows:

- Its magnitude is unity.
- It indicates the direction of a vector.
- The unit vector in the direction of u is given by

$$\hat{u} = \frac{\vec{u}}{||\vec{u}||}$$

Some standard unit vectors in the two-dimensional and three-dimensional space are shown below.

STANDARD UNIT VECTORS

$$\hat{\imath} = \begin{bmatrix}1\\0\end{bmatrix} \quad \hat{\jmath} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\hat{\imath} = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \hat{\jmath} = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \hat{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Zero Vector

Some important points related to the zero vector are as follows:

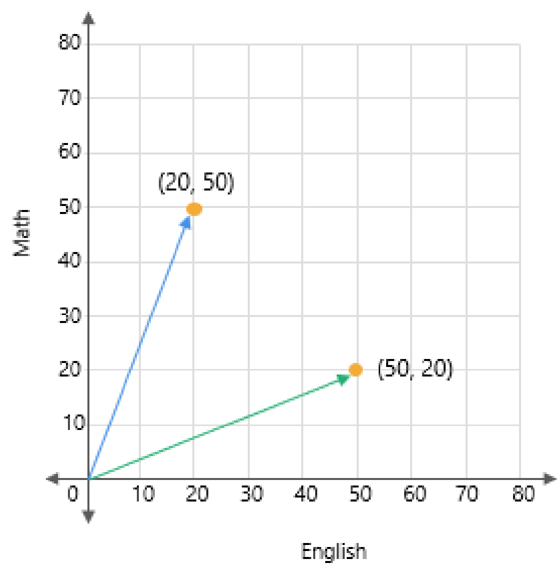
- The magnitude of a zero vector is 0.
- The zero vector can be denoted by $\vec{0}$ or **0** or $\bar{0}$.
- Each dimension has a unique zero vector:
    - 2 dimensions, $\vec{0} = (0,0)$
    - 3 dimensions, $\vec{0} = (0,0,0)$
    - In n dimensions, $\vec{0} = (0,0,\ldots,0) - n$ components

Addition of Vectors

Consider the marks scored by two students Sakshi and Manas in English and Mathematics.

Subject	Sakshi	Manas
English	20	50
Math	50	20

- $\vec{s} = \begin{bmatrix} 20 \\ 50 \end{bmatrix}$
- $\vec{m} = \begin{bmatrix} 50 \\ 20 \end{bmatrix}$



The subject-wise marks scored by Manas and Sakshi can be represented by the vectors $\vec{m}$ and $\vec{s}$, respectively.

If we want to know the total of the subject-wise marks scored by Sakshi and Manas we can simply add the two vectors $\vec{m}$ and $\vec{s}$. The vectors $\vec{m}$ and $\vec{s}$ can be added as shown below.

$$\vec{m} + \vec{s} = [50 \quad 20] + [20 \quad 50] = [(50+20) \quad (20+50)] = [70 \quad 70]$$

The following points need to be kept in mind while adding two vectors:

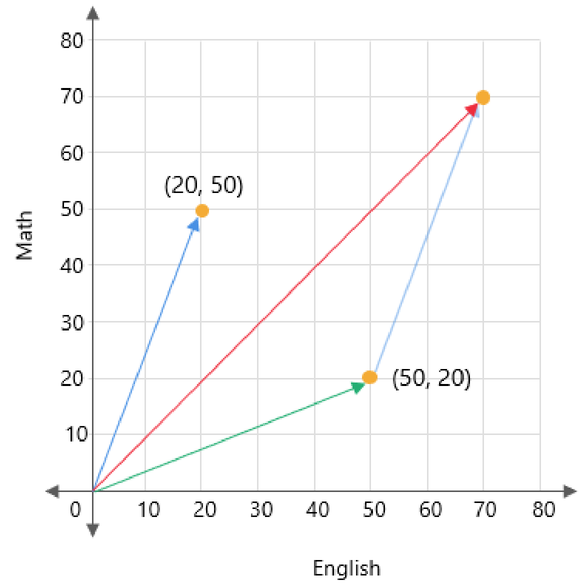
- Generally, when we add two vectors, we add the corresponding components as shown above.
- Both vectors should have the same dimension.
- Addition results in a third vector of the same dimension.
- The order of addition does not matter, i.e., $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Visualising the Addition of Vectors

Consider the following example where marks of two students Sakshi and Manas in two subjects English and Math are as given in the table.

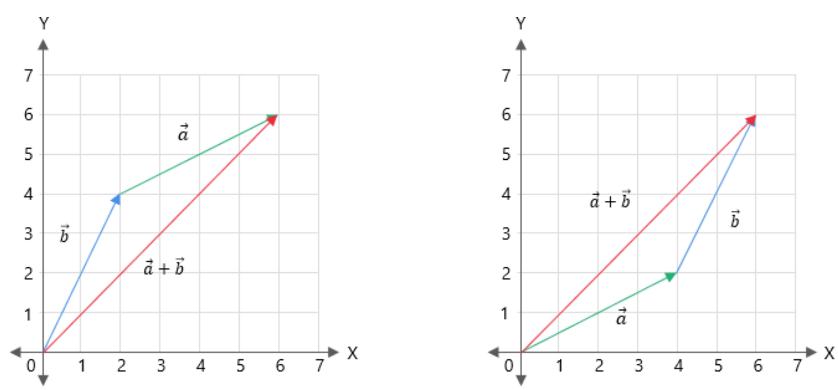
Subject	Sakshi	Manas
English	20	50
Math	50	20

- $\vec{s} = \begin{bmatrix} 20 \\ 50 \end{bmatrix}$
- $\vec{m} = \begin{bmatrix} 50 \\ 20 \end{bmatrix}$
- The subject-wise totals is $\vec{s} + \vec{m} = \begin{bmatrix} 70 \\ 70 \end{bmatrix}$



The two vectors $\vec{m}$ and $\vec{s}$ can also be added as shown in the graph given above. The tail of the vector $\vec{s}$ is placed on the head of the second vector, and the resultant is obtained by joining the tail of the vector $\vec{m}$ and the head of the vector $\vec{s}$.

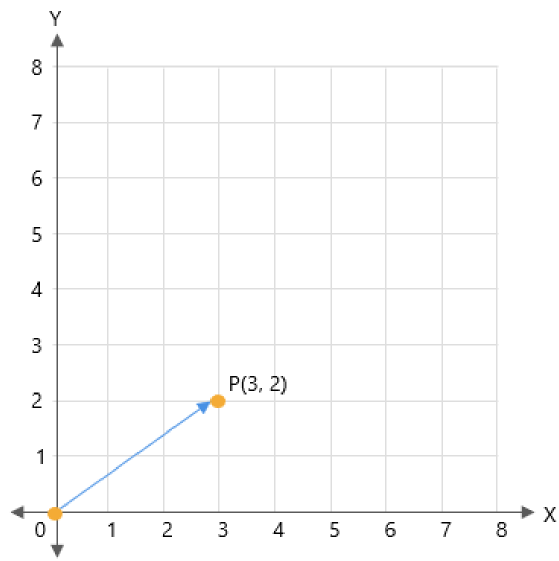
The following figure shows the two ways to add vectors $\vec{a}$ and $\vec{b}$



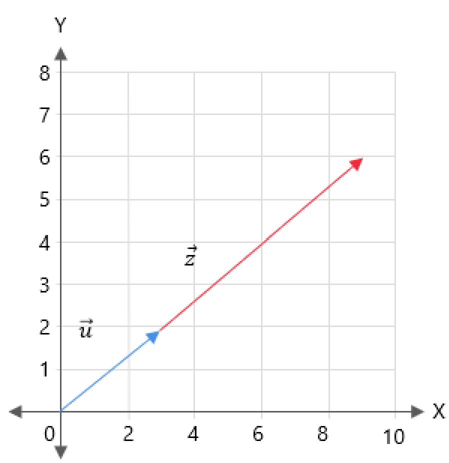
Scalar Multiplication

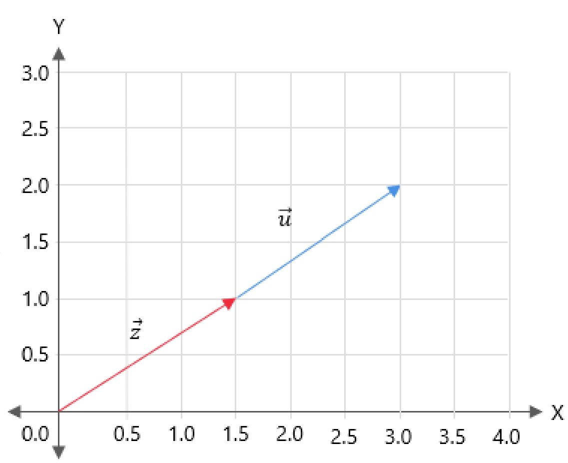
When we multiply a vector by a scalar, we simply multiply each of the components of the vector by that scalar as shown in the example given below.

- Multiply a vector with a scalar
- Consider $\vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- Then, $2\vec{a} = \begin{bmatrix} 2 \times 3 \\ 2 \times 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$



Multiplying a vector by a scalar can either stretch or contract the vector as shown below.

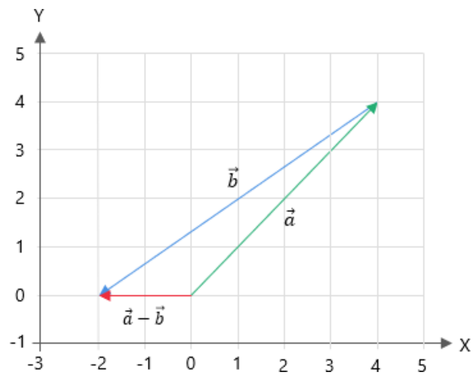




Vector Subtraction

The vector subtraction of two vectors $\vec{a}$ and $\vec{b}$ can be thought of as the addition of $\vec{a}$ and $-\vec{b}$

- $\vec{u} = \vec{a} - \vec{b} = \vec{u} = \vec{a} + (-\vec{b})$
- Consider $\vec{a} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$
- Then, $\vec{a} - \vec{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} + \begin{bmatrix} -6 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$



Dot Product or Scalar Product

Some important points related to the dot product are as follows:

- The dot product or the scalar product is a product of two vectors that gives a scalar.
- This product can be used to find the angle between two vectors.
- This product also helps to find the magnitude of a vector, i.e., $||\vec{a}\,|| = \sqrt{\vec{a}\,.\,\vec{a}}$
- If $\vec{a} = [1 \quad 2]$ and $\vec{b} = [6 \quad 15]$, then $\vec{a}\,.\,\vec{b} = (1 \times 6) + (2 \times 15) = 36$.

The following table shows the ratings given by 10 people to five movies. Suppose we are interested in the ratings given by Euclid and want to know whose ratings are like the ones given by him. We consider Anmol and Howard. So, we find the dot product between the rating vectors as shown below.

User	Jurassic Park (Adventure)	Three Idiots (Comedy)	The Sound of Music (Romance)	Dracula (Horror)	Macbeth (Tragedy)
Anmol	7	9	8	6	2
Babita	2	7	9	10	6
Chester	5	4	3	7	3
Disha	9	2	5	10	8
Euclid	1	6	3	8	9
Farhaan	7	4	7	6	5
Gayatri	8	5	9	3	8
Howard	1	6	5	9	8
Indira	3	6	7	6	9
Jasmine	3	2	2	4	7

- User movie ratings

$\vec{a} = \begin{pmatrix} 7 \\ 9 \\ 8 \\ 6 \\ 2 \end{pmatrix}$ $\vec{e} = \begin{pmatrix} 1 \\ 6 \\ 3 \\ 8 \\ 9 \end{pmatrix}$ $\vec{h} = \begin{pmatrix} 1 \\ 6 \\ 5 \\ 9 \\ 8 \end{pmatrix}$

$\vec{a}.\vec{e} = (7)(1)+(9)(6)+8(3)+6(8)+2(9)$
$= \underline{151}$

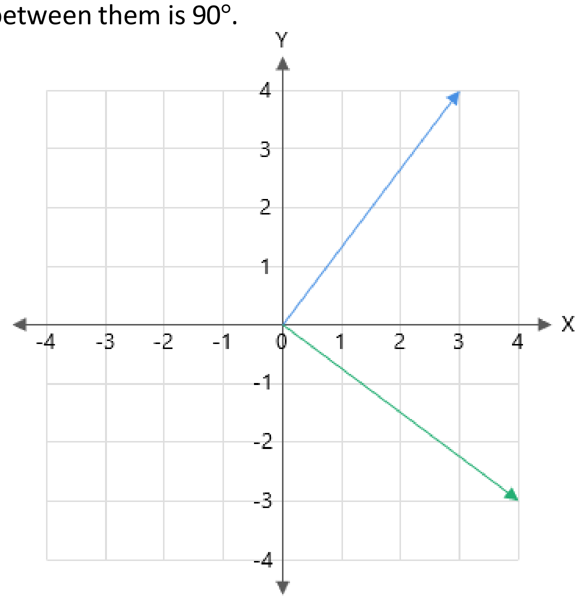
$\vec{h}.\vec{e} = \underline{196}$

$\vec{a}\,.\,\vec{e}$ is less than $\vec{h}.\vec{e}$, which implies that the ratings given by Howard are closer to the ratings given by Euclid than those given by Anmol.

Orthogonal Vectors

Two vectors are said to be orthogonal if the angle between them is 90°.

- The dot product of two orthogonal vectors is zero
- Consider $\vec{a} = (3,4)$ and $\vec{b} = (4,-3)$, then, $\vec{a}.\vec{b} = 3 \times 4 + 4 \times -3 = 0$



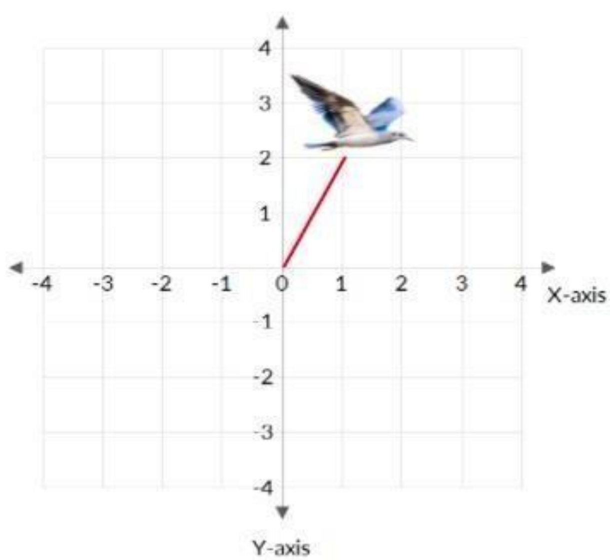
NORMS

Some important points related to norms are as follows:

- A norm is a function that helps in finding the distance between two vectors.
- It can also be used to calculate the magnitude of a vector.
- The norm of a vector $\vec{u}$ is denoted by $||\vec{u}||$.

L^2 or Euclidean Norm

Suppose a bird starts flying from $(0,0)$ to $(1,2)$. We want to calculate the distance travelled by this bird, which is given by the L^2 norm of $(1,2)$.



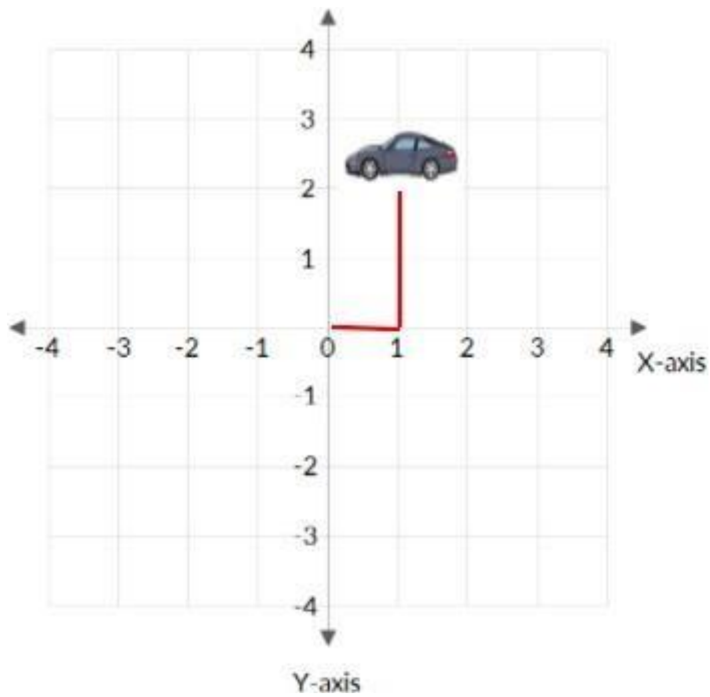
Distance travelled by the bird $= \sqrt{12 + 22} = \sqrt{5} = ||1,2||^2$

Generally, the L^2 norm of a vector gives us its distance from the origin and

$$||(x_1 + x_2 + x_n)||^2 = \sqrt{||(x_1{}^2 + x_2{}^2 + x_n{}^2)||}$$

L^1 or Taxicab Norm or Manhattan Distance

Suppose a car starts moving from $(0,0)$ to $(1,2)$. Unlike the bird, the car can only travel along the horizontal and vertical road networks. We want to calculate the distance travelled by this car, which is given by the L^1 norm of $(1,2)$.



Distance travelled by the car = $|1| + |2| = 3 = ||1, 2||^1$

In general, the L^1 norm of a vector gives its distance from the origin and

$$||(x_1 + x_2 + x_n)||^1 = |x_1| + |x_2| + \cdots + |x_n|$$

The following table shows the ratings given by 10 people to five movies. Suppose we are interested in the ratings given by Euclid and want to know whose ratings are like the ones given by him. We consider Anmol and Howard. So, we find the norm of the difference of the rating vectors as shown below.

User	Jurassic Park (Adventure)	Three Idiots (Comedy)	The Sound of Music (Romance)	Dracula (Horror)	Macbeth (Tragedy)
Anmol	7	9	8	6	2
Babita	2	7	9	10	6
Chester	5	4	3	7	3
Disha	9	2	5	10	8
Euclid	1	6	3	8	9
Farhaan	7	4	7	6	5
Gayatri	8	5	9	3	8
Howard	1	6	5	9	8
Indira	3	6	7	6	9
Jasmine	3	2	2	4	7

$$\vec{a} = \begin{pmatrix} 7 \\ 9 \\ 8 \\ 6 \\ 2 \end{pmatrix} \quad \vec{e} = \begin{pmatrix} 1 \\ 6 \\ 3 \\ 8 \\ 9 \end{pmatrix} \quad \vec{h} = \begin{pmatrix} 1 \\ 6 \\ 5 \\ 9 \\ 8 \end{pmatrix}$$

$$\|\vec{a} - \vec{e}\| = 11.09$$

$$\|\vec{h} - \vec{e}\| = 2.45$$

User movie ratings

Since $\|\vec{a} - \vec{e}\|$ greater than $\|\vec{h} - \vec{e}\|$, it implies that the ratings given by Howard are closer to the ratings given by Euclid than the ones given by Anmol.

Linear Combinations of Vectors

Linear Combinations

The following table shows the numbers of shirts and pants produced by factory A and factory B per day. The numbers of shirts and pants produced in a month depend on the number of days for which the factories are functional.

Factory	Shirts (per day)	Pants (per day)
A	500	300
B	100	100

Factory production

How many shirts and pants will be produced in a month if factory A is open for 20 days and factory B is open for 25 days?

$20 \times 500 + 25 \times 100$

$20 \times 300 + 25 \times 100$

We can use vectors to store the number of shirts and pants produced by factories A and B. To get the total numbers of shirts and pants produced in a month, we multiply the vectors by the number of days for which the factories are functional and then add them as shown below.

$$20[500 \quad 300] + 25[100 \quad 100] = [3500 \quad 8500]$$

$20[500 \quad 300] + 25[100 \quad 100]$ is a linear combination of the vectors $[500 \quad 300]$ and $[100 \quad 100]$.

When we scale multiple vectors and add them together, we get a linear combination of those vectors.

The linear combinations of $\vec{u} = [1 \quad 0]$ and $\vec{v} = [0 \quad 1]$ are as shown below.

$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Linear combinations:

$$2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

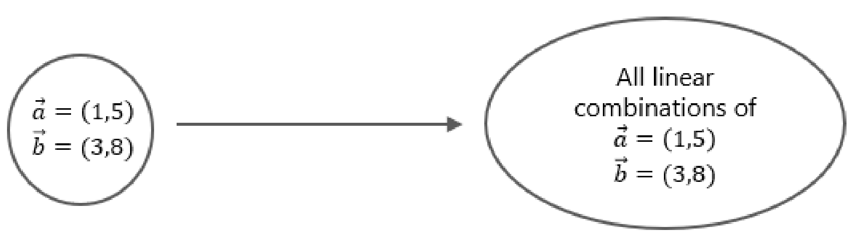
$$1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(-2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

It can easily be checked that every vector in the two-dimensional space can be written as a linear combination of $\vec{u} = [1 \quad 0]$ and $\vec{v} = [0 \quad 1]$.

Span

- A set of all vectors created by all linear combination of a set of vectors



Span of $\vec{a} = (1,5)$ and $\vec{b} = (3,8)$

If $3[1 \quad 5] + 2[3 \quad 8] = [9 \quad 31]$, then we say $[9 \quad 31]$ belongs to the span of $[1 \quad 5]$ and $[3 \quad 8]$ because it is a linear combination of $[1 \quad 5]$ and $[3 \quad 8]$.

- For a single vector, span can be a point or a line.
- For a linear combination of 2 vectors, the span can be a point, a line or a plane.
- For a linear combination of 3 vectors, the span can be a point, a line, plane or the entire 3D space.

Linear Independence and Linear Dependence

Some important points related to linear independence and linear dependence are as follows:

- Consider the vectors $\vec{u} = [1 \quad 1]$ and $\vec{v} = [2 \quad 2]$. We know that $[2 \quad 2] = 2[1 \quad 1]$ or $[2 \quad 2]$ is a linear combination of $[1 \quad 1]$. We say that $\vec{u} = [1 \quad 1]$ and $\vec{v} = [2 \quad 2]$ are linearly dependent.
- Consider the vectors $\vec{u} = [1 \quad 1]$ and $\vec{v} = [0 \quad 2]$. It can be verified that $[0 \quad 2]$ is not a linear combination of $[1 \quad 1]$. We say that $\vec{u} = [1 \quad 1]$ and $\vec{v} = [2 \quad 2]$ are linearly independent.

- Linearly Independent Set: If no single vector can be obtained by a linear combination of any of the other vectors present in the set.
- Linearly Dependent Set: If even one vector can be obtained as a linear combination of any of the other vectors.

Basis

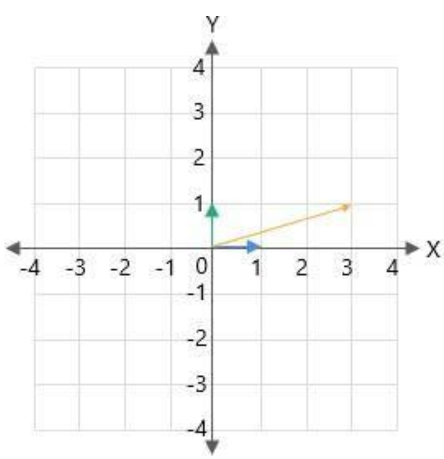
Some important points related to basis are as follows:

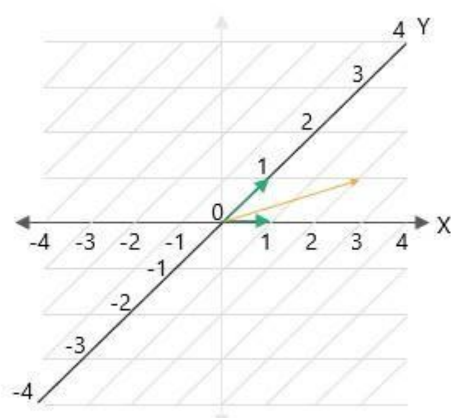
- Basis is a set of linearly independent vectors whose span is equal to the vector space.
    - For example, $\vec{u} = [1 \quad 0]$ and $\vec{v} = [0 \quad 1]$ are linearly independent vectors whose span is the two-dimensional space. Therefore, we say that $\{[1 \quad 0], [0 \quad 1]\}$ is a basis of the two- dimensional space.
    - For example, $[1 \quad 5]$ and $[3 \quad 8]$ are linearly independent vectors whose space is the two- dimensional space. Therefore, we say that $\{[1 \quad 5], [3 \quad 8]\}$ is a basis of the two-dimensional space.

- $\{[1 \quad 0], [0 \quad 1]\}$ is called the standard basis of the two-dimensional space.
- $\{[1 \quad 0 \quad 0], [0 \quad 1 \quad 0], [0 \quad 0 \quad 1]\}$ is called the standard basis of the three-dimensional space.

Change of Basis

The following figure shows the change that occurs when we change the basis from $\vec{u}\to = [1\ 0\ ]$ and $v\to =$ $[0\ 1\ ]$ to $a\to = [1\ 0\ ]$ and $\vec{b}\to = [1\ 1\ ]$.





Orthogonal and Orthonormal Bases

Some important points related to orthogonal and orthonormal bases are as follows:

- When the basis vectors are perpendicular to each other, they form orthogonal vectors.
    - $\{(2,0),(0,2)\}$ is an orthogonal basis.
- When the basis vectors are perpendicular to each other and the magnitude of each vector is 1, they form an orthonormal basis.
    - $\{(1,0),(0,1)\}$ is an orthogonal basis.

Matrices

Introduction to Matrices

Matrices are used to represent data sets. We can, for instance, use a matrix to represent this table, which shows the height and weight of five people, as shown below.

Person	Height (cm)	Weight (Kg)
1	155	85
2	170	85
3	150	80
4	165	80
5	155	70

We can represent this data as a matrix.

$$M = \begin{bmatrix} 155 & 85 \\ 170 & 85 \\ 150 & 80 \\ 165 & 80 \\ 155 & 70 \end{bmatrix} \longrightarrow \text{Row}$$

$\downarrow$ Column

Essentially, a matrix is a rectangular array of numbers, which is used to store values from tabular data.

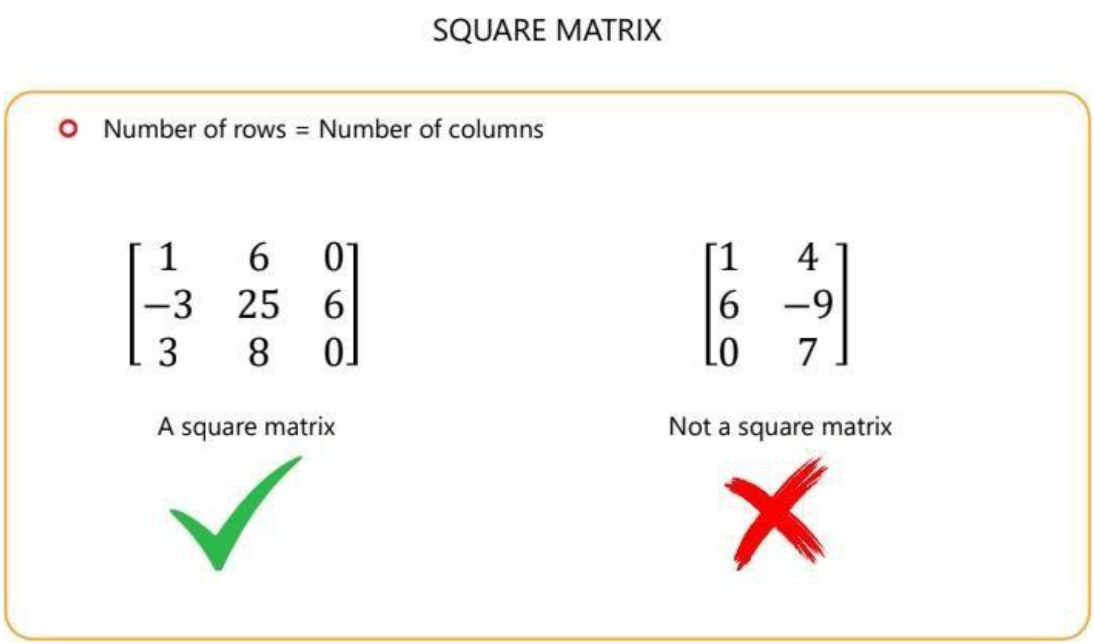
The components of matrices are listed below.

- Elements
- Rows
- Columns
- Dimensions

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{(2\times3)}$$

The matrix shown above has 2 rows and 3 columns. The first column is (1 2) and the first row is (1 2 3). The entries in a matrix are known as its elements. Since the matrix above has 2 rows and 3 columns, we say that its dimensions are 2*3.

Here is a list of standard matrices and their properties.



DIAGONAL MATRIX

- It is a square matrix.
- Entries outside main diagonal are zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

A diagonal matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 7 \end{bmatrix}$$

Not a diagonal matrix

SCALAR MATRIX

- It is a diagonal matrix.
- Entries along the diagonal are equal.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

A scalar matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

Not a scalar matrix

IDENTITY MATRIX AND ZERO MATRIX

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

An identity matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A zero matrix

Operation on Matrices

Matrix Addition

The tables below show the number of hammers, pliers and screwdrivers manufactured by factories A, B and C for the year 2006 and 2007. We can represent this data using matrices, as shown, and the sum of the matrices gives the total production for the years 2006 and 2007.

2006

Factory Code	Hammers	Pliers	Screw Drivers
A	200	250	0
B	150	160	60
C	270	300	100

$$\begin{bmatrix} 200 & 250 & 0 \\ 150 & 160 & 60 \\ 270 & 300 & 100 \end{bmatrix} + \begin{bmatrix} 250 & 300 & 50 \\ 160 & 110 & 70 \\ 200 & 240 & 100 \end{bmatrix}$$

2007

Factory Code	Hammers	Pliers	Screw Drivers
A	250	300	50
B	160	110	70
C	200	240	100

The matrices can be added as shown below.

$$\begin{bmatrix} 200 & 250 & 0 \\ 150 & 174 & 66 \\ 273 & 300 & 102 \end{bmatrix} + \begin{bmatrix} 250 & 375 & 59 \\ 100 & 174 & 78 \\ 200 & 300 & 88 \end{bmatrix} = \begin{bmatrix} 200+250 & 250+375 & 0+59 \\ 150+100 & 174+174 & 66+78 \\ 273+200 & 300+300 & 102+88 \end{bmatrix} = \begin{bmatrix} 450 & 625 & 59 \\ 250 & 348 & 144 \\ 473 & 600 & 190 \end{bmatrix}$$

Matrix addition is essentially adding the elements of matrices component wise. Two matrices can be added only if their dimensions are the same, and their sum is again a matrix of the same dimensions.

Scalar Multiplication of Matrices

The number of masks, gloves and sanitisers manufactured by factories A and B is shown below. The factories are planning to double their production. How can this change be reflected in a matrix?

Well, to reflect the change, you can simply multiply the matrix by the scalar 2, as shown below.

Product	Factory A	Factory B
Masks	200	250
Gloves	150	174
Sanitizers	273	300

$$\begin{bmatrix} 200 & 250 \\ 150 & 174 \\ 273 & 300 \end{bmatrix} \times 2 = \begin{bmatrix} 200 \times 2 & 250 \times 2 \\ 150 \times 2 & 174 \times 2 \\ 273 \times 2 & 300 \times 2 \end{bmatrix} = \begin{bmatrix} 400 & 500 \\ 300 & 348 \\ 546 & 600 \end{bmatrix}$$

Scalar multiplication involves multiplying each element of a matrix by a scalar, and it yields a matrix of the same dimensions.

Matrix Subtraction

Consider that these tables represent the tuition fees to be paid by three students (in units) and the advance amounts they have paid already. The amount that remains to be paid by each student can be calculated easily as the difference between two matrices.

Student Name	Science	Math
Sheetal	200	250
Chaya	150	174
Nirmal	273	300

Student Name	Science	Math
Sheetal	100	250
Chaya	0	130
Nirmal	273	250

Here is how you can calculate the difference of the matrices representing the data above.

$$\begin{matrix} 200 & 250 \\ 150 & 174 \\ 273 & 300 \end{matrix} - \begin{matrix} 100 & 250 \\ 0 & 130 \\ 273 & 250 \end{matrix} = \begin{matrix} 200-100 & 250-250 \\ 150-0 & 174-130 \\ 273-273 & 300-250 \end{matrix} = \begin{matrix} 100 & 0 \\ 150 & 44 \\ 0 & 50 \end{matrix}$$

Transpose of a Matrix

The transpose of a matrix, which is another matrix, is obtained by interchanging its rows and columns.

If the order of a matrix is $m \times n$, then the order of its transpose is $n \times m$. The concept of transpose is depicted below.

$$\underset{(3 \times 2)}{A = \begin{bmatrix} 1 & 6 \\ 3 & 25 \\ 3 & 8 \end{bmatrix}} \longrightarrow \underset{(2 \times 3)}{A^T = \begin{bmatrix} 1 & 3 & 3 \\ 6 & 25 & 8 \end{bmatrix}}$$

Matrix Multiplication

Two matrices can be multiplied only if the number of columns in the first matrix equals the number of rows in the second. The concept of matrix multiplication is depicted below.

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 5 \\ 7 & 7 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$(2 \times 3) \qquad (3 \times 2) \qquad (2 \times 2)$$

A $(m \times n)$ matrix can be multiplied by a $(n \times p)$ matrix to give a $(m \times p)$ matrix

Matrix–Vector Multiplication

A matrix and a vector can be multiplied as shown below.

Consider $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{bmatrix}_{(3\times 2)}$ and $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{(2\times 1)}$

Then $A\vec{v} = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Another way to think about this is:

$$A\vec{v} = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 14 \end{bmatrix}$$

Matrix–Matrix Multiplication

Two matrices can be multiplied using matrix–vector multiplication, as shown below.

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{bmatrix}_{(3\times 2)} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}_{(2\times 2)}$$

$$AB = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} & 1 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 14 & 9 \end{bmatrix}$$

We can also multiply matrices using the usual multiplication, as shown below.

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{bmatrix}_{(3\times 2)} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}_{(2\times 2)}$$

$$AB = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2\times 2 + (-1)\times 3 & 2\times 1 + (-1)\times 2 \\ 3\times 2 + 0\times 3 & 3\times 1 + 0\times 2 \\ 1\times 2 + 4\times 3 & 1\times 1 + 4\times 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 14 & 9 \end{bmatrix}$$

We can easily verify that multiplying any matrix with the identity matrix returns the same matrix.

Multiplication of matrices is not a commutative operation. You can consider this example for a better understanding.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 13 & 15 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 10 & 14 \end{bmatrix}$$

$$AB \neq BA$$

Determinants

A determinant is a number associated with a square matrix. It tells us the important properties of these matrices. The determinant can be calculated as shown below.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A| = ad - bc$.

For $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $|A| = 2 \times 2 - 1 \times 3 = 1$

Inverse

The inverse of a matrix A is A^{-1}, such that $AA^{-1} = I$ or $A^{-1}A = I$. A and A^{-1} are inverses of each other. The concept of inverse is depicted below.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

For $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $A^{-1} = \frac{1}{1} \begin{bmatrix} 0 & -3 \\ -1 & 2 \end{bmatrix}$

Systems of Linear Equations

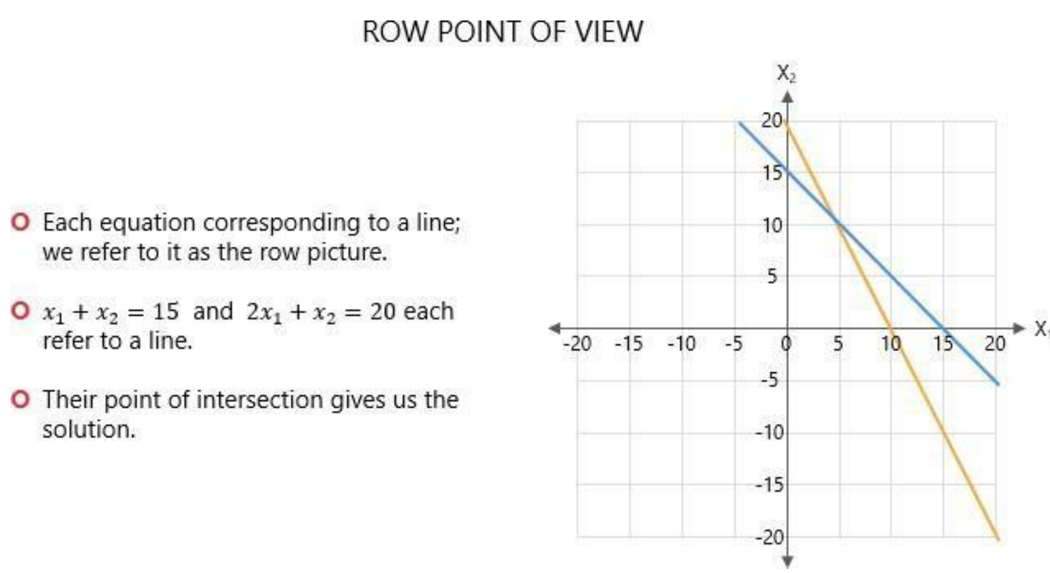
Suppose x_1 is the cost of an apple and x_2 is the cost of an orange.

2 x_1 + 3 x_2 will give us the total cost of 2 apples and 3 oranges, and 2 x_1 + x_2 represents the cost of 2 apples and 1 orange. Now, suppose the cost of 2 apples and 3 oranges is ₹40 and the cost of 2 apples and 1 orange is ₹20. We can represent this information using equations as shown below.

$$2x_1 + 3x_2 = 40$$

$$2x_1 + x_2 = 20$$

Such a collection of linear equations is called a system of linear equations.



COLUMN POINT OF VIEW

$$x_1 + x_2 = 15$$
$$2x_1 + x_2 = 20$$

- The equations can be rewritten as: $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix}$
- The column vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are scaled by $x_1$and x_2, respectively, to give us $\begin{bmatrix} 15 \\ 20 \end{bmatrix}$

Solution for a System of Linear Equations

The solution for a system of equations is a set of values of the variables that satisfy the system.

Consider this example:

- You have the equations $x_1 + x_2 = 15$ and $2x_1 + x_2 = 20$.
- Find the values of x_1 and x_2 that make both the equations true simultaneously, that is, the solution for this system of equations.
- Essentially, you are looking for $(x_1 \quad x_2)$ or $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Every system can be represented using matrices, as shown below.

Consider the following system:

$x_1 + x_2 = 15$

$2x_1 + x_2 = 20$

We can write it as: $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix}$, that is $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix}$.

This is of the form $A\vec{x} = \vec{b}$.

Here,

$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 15 \\ 20 \end{bmatrix}$

To find a solution for this system, you need to find the vector $\vec{x}$. You can find $\vec{x}$ as shown below.

Consider $A\vec{x} = \vec{b}$

$\Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{b}$ (multiplying both sides by A^{-1})

$\Rightarrow I\vec{x} = A^{-1}\vec{b}$ $(A^{-1}A = I)$

$\Rightarrow \vec{x} = A^{-1}\vec{b}$ $(I\vec{x} = \vec{x})$

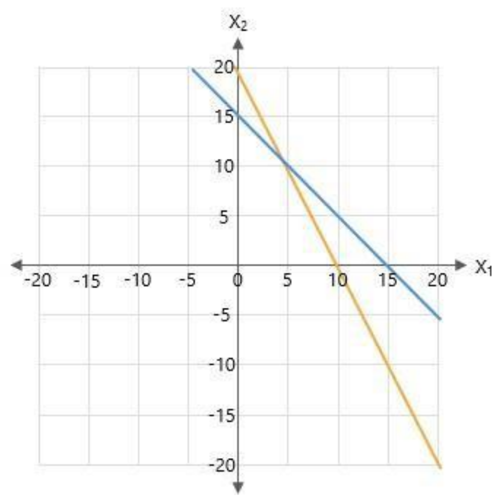
Given a system of equations, you can have:

- A unique solution,
- No solution and
- Infinitely many solutions.

You can consider these examples for a better understanding.

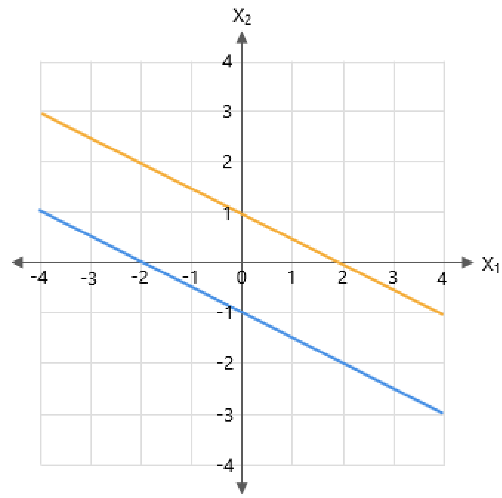
CASE I: UNIQUE SOLUTION

$x_1 + x_2 = 15$
$2x_1 + x_2 = 20$



CASE II: NO SOLUTION

$$x_1 + 2x_2 = 2$$
$$x_1 + 2x_2 = -2$$



CASE III: INFINITELY MANY SOLUTIONS

$$x_1 + 2x_2 = 2$$
$$3x_1 + 6x_2 = 6$$