Derivatives

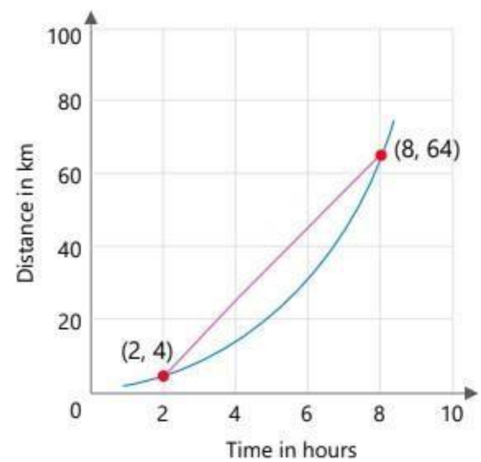
Introduction to Derivatives

- Consider the example of a cyclist travelling along the curve $y = x^2$. Suppose we want to find the average speed of the cyclist from the 2^{nd} to the 8^{th} hour.

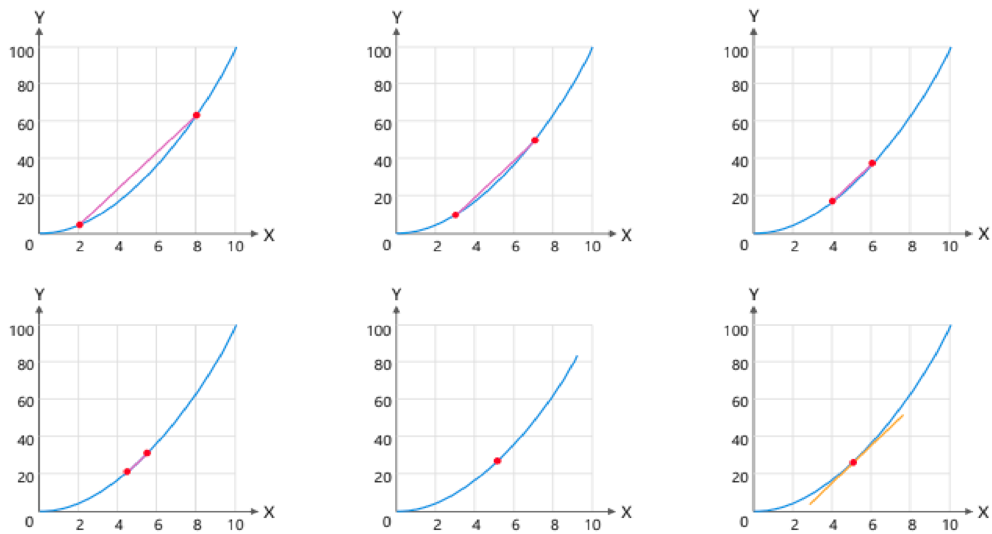
- Consider the distance traveled by a cyclist in km (y) and time in hours (x) related by $y = x^2$
- Average rate of change = Slope of secant between two points
- Between hours 2 and 8:

$$\text{Slope} = \frac{64-4}{8-2} = \frac{60}{6} = 10\ km/hr$$

- Average speed of the cyclist from 2^{nd} to 8^{th} hour is 10 km/hr



What if we want to find the speed at exactly 5 hours? We need two points to find the rate of change. Consider the second point extremely (infinitesimally) close to 5 – it can be 4.9999 or 5.0001, or even closer. Plotting a secant using these two points point will give the tangent to this curve at (5,25). The slope of this tangent will give the instantaneous rate of change at that point – also referred to as its **derivative**.



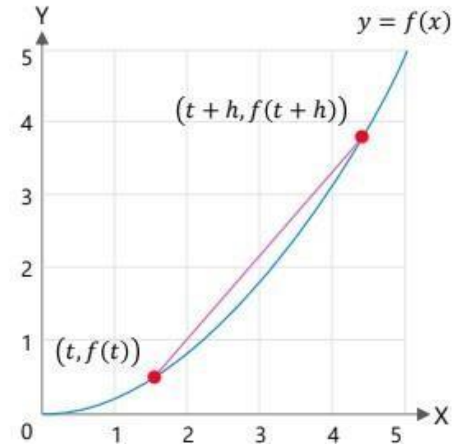
- The average rate of change = Slope $(m) = \frac{\text{Change in distance}}{\text{Change in time}}$, that is, $m = \frac{\Delta f(t)}{\Delta t} = \frac{f(t_2)-f(t_1)}{t_2-t_1}$.
- For the instantaneous rate of change, the difference between t_2 and t_1, i.e., Δt, should be infinitesimally small. The value of the slope at the point is the derivative of f.

DERIVATIVES

- $m = \frac{\Delta f(t)}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$
- $m = \frac{f(t+h) - f(t)}{t+h-t} = \frac{f(t+h) - f(t)}{h}$
- h should be infinitesimally small, approaching 0; hence, we use limits

$$\frac{df(t)}{dt} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

- This expression gives us the change in $f(t)$, given an infinitesimally small change in t

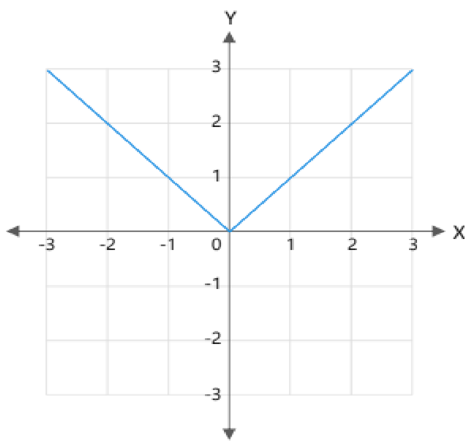


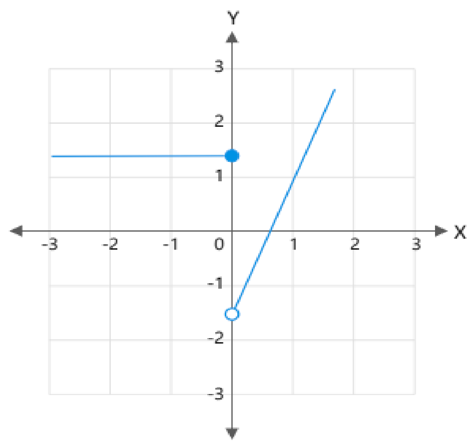
- The derivative of a function $f(x)$ with respect to x is given by $\frac{df(x)}{dx}$ or $f'(x)$.
- It indicates how the function f changes for a very small change in x, i.e., dx.

Conditions of Differentiability

- A function is differentiable at a point if:
  - It is continuous at that point,
  - The tangent at that point is not vertical and
  - There is no sudden change in direction at that point.

The functions below are not differentiable at $x = 0$.



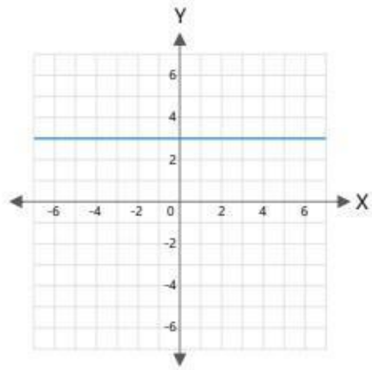


Methods of Differentiation

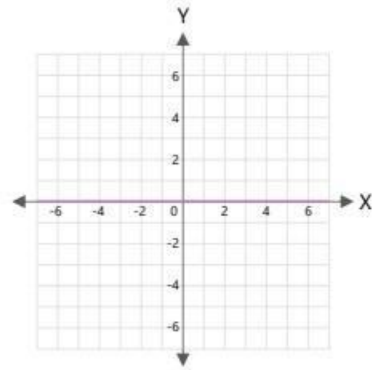
DERIVATIVE OF A CONSTANT

- If $f(x) = 3$, then $\frac{df(x)}{dx} = \frac{d}{dx}(3) = 0$
- Derivative of a constant is 0
- Generally, for a constant c,

  if $f(x) = c$, then $\frac{df(x)}{dx} = \frac{d}{dx}(c) = 0$



$f(x) = 3$



$f'(x) = 0$

POWER RULE

- If $f(x) = \frac{1}{x} = x^{-1}$,

  then $\frac{df(x)}{dx} = \frac{d}{dx}(x^{-1})$

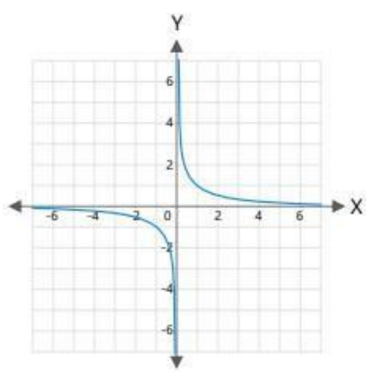
  $= -1(x^{-1-1})$

  $= -x^{-2}$

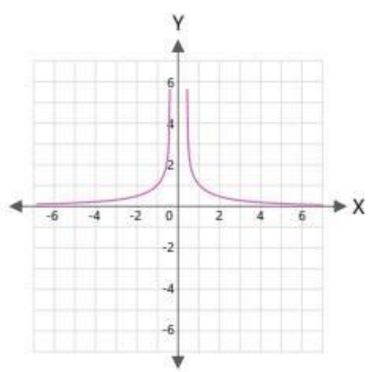
  $= -\frac{1}{x^2}$

- Generally, if $f(x) = x^n$,

  then $\frac{df(x)}{dx} = \frac{d}{dx}x^n = nx^{n-1}$



$f(x) = \frac{1}{x}$



$f'^{(x)} = -\frac{1}{x^2}$

MULTIPLICATION BY A CONSTANT

- If $f(x) = 4x^2$,

  then $\frac{df(x)}{dx} = \frac{d}{dx}(4x^2)$

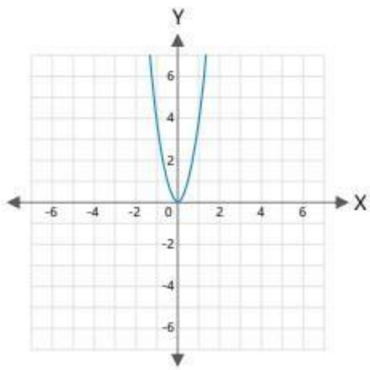
  $= 4\frac{d}{dx}(x^2)$

  $= 4(2x)$

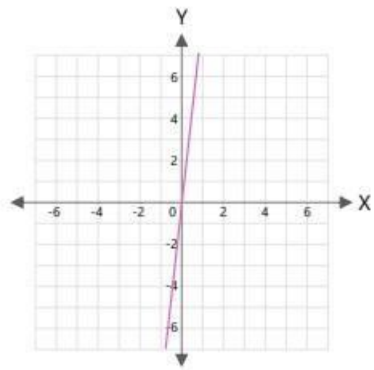
  $= 8x$

- Generally, for a constant c,

  $\frac{d(cf(x))}{dx} = c\frac{d(f(x))}{dx}$



$f(x) = 4x^2$



$f'(x) = 8x$

SUM RULE

- If $f(x) = 2x + x^2$,

  then $\frac{df(x)}{dx} = \frac{d}{dx}(2x + x^2)$

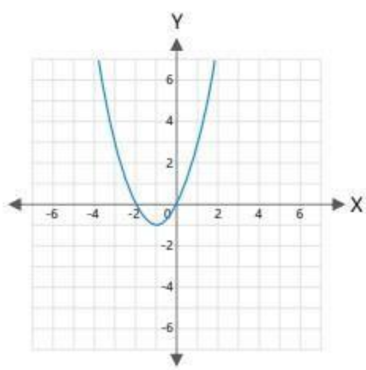
  $= \frac{d}{dx}(2x) + \frac{d}{dx}(x^2)$

  $= 2\frac{d}{dx}(x) + \frac{d}{dx}(x^2)$

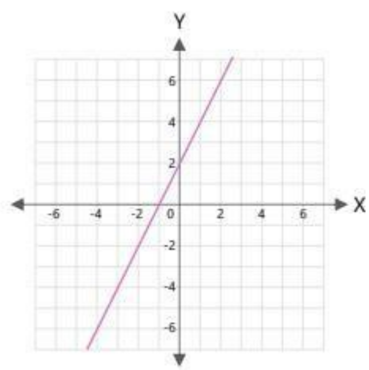
  $= 2 + 2x$

- Generally, if $f(x) = a(x) + b(x)$,

  then $\frac{df(x)}{dx} = \frac{da(x)}{dx} + \frac{db(x)}{dx}$



$f(x) = 2x + x^2$



$f'(x) = 2 + 2x$

PRODUCT RULE

- If $f(x) = x(x^2+1)$,

  then $\frac{df(x)}{dx} = (x^2+1)\frac{dx}{dx} + x\frac{d(x^2+1)}{dx}$

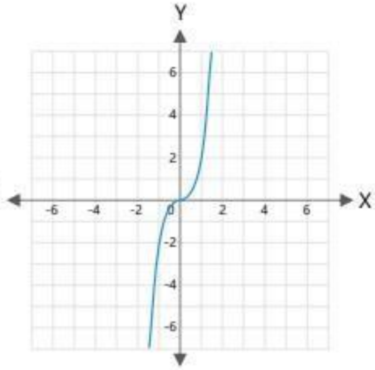
  $= (x^2+1)1 + x\left(\frac{dx^2}{dx} + \frac{d(1)}{dx}\right)$

  $= (x^2+1) + x(2x)$

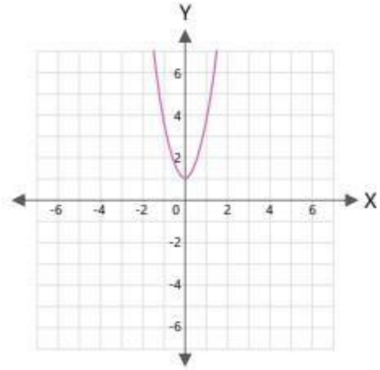
  $= 3x^2 + 1$

- Generally, if $f(x) = a(x)b(x)$,

  then $\frac{df(x)}{dx} = \frac{da(x)}{dx}b(x) + a(x)\frac{db(x)}{dx}$



$f(x) = x(x^2+1)$



$f'(x) = 3x^2 + 1$

Chain Rule

- The chain rule helps find the derivatives of composite functions.
- $h(x) = f(g(x)) = \sin(x^2+1)$
- $f(x) = \sin(x) \Rightarrow \frac{df(x)}{dx} = \frac{d(\sin(x))}{dx} = \cos(x)$
- $g(x) = x^2 + 1 \Rightarrow \frac{df(x)}{dx} = \frac{d(x^2+1)}{dx} = 2x$
- $\frac{dh(x)}{dx} = \frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} . \frac{dg(x)}{dx} = \frac{d\sin(x^2+1)}{dg(x^2+1)} . \frac{d(x^2+1)}{dx} = \cos(x^2+1) . 2x =$

  $2x.\cos(x^2+1)$

- Chain rule is represented in many ways as follows:
  - $\frac{dh(x)}{dx} = \frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}$
  - $\frac{dh}{dx} = \frac{df(g)}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$
  - $h'(x) = f\big(g(x)\big)' = f'\big(g(x)\big).\, g'(x)$
- We can have multiple nested functions:
  - $\frac{dh(x)}{dx} = \frac{da(b(c(x)))}{dx} = \frac{da(b(c(x)))}{db(c(x))} \cdot \frac{db(c(x))}{dc(x))} \cdot \frac{dc(x)}{dx}$
  - $\frac{dh}{dx} = \frac{da}{db} \cdot \frac{db}{dc} \cdot \frac{dc}{dx}$

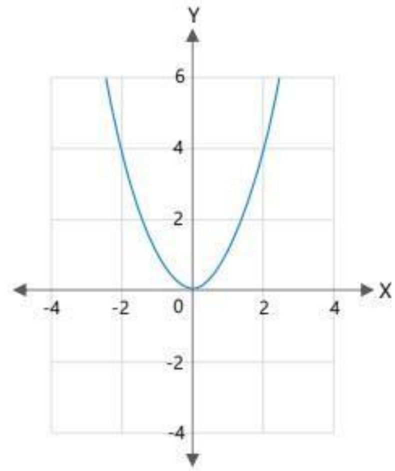
Extrema

Absolute Extrema

- Suppose the profit earned by a company is a function of the selling price of a product. To find the selling price at which the company earns maximum profit, we need to find the maximum value attained by the function. We are often interested in finding the largest and smallest values attained by a function.

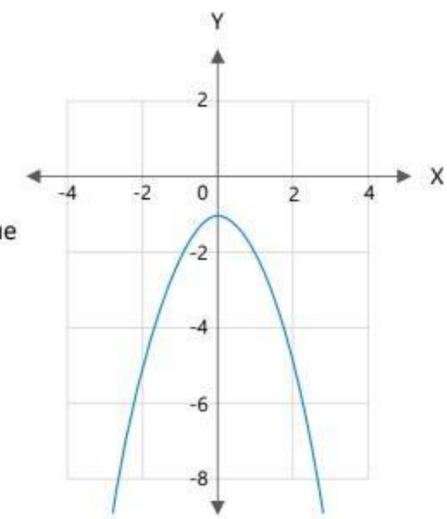
ABSOLUTE EXTREMA

- Consider the function: $f(x) = x^2$
  - It does not have a largest value but has the smallest value at $x = 0$
  - We get the absolute minimum of $f(x) = 0$ when $x = 0$



$f(x) = x^2$

- Consider the function: $f(x) = -x^2 - 1$
  - It does not have the smallest value but has the largest value at $x = 0$
  - We get the absolute maximum of $f(x) = -1$ when $x = 0$

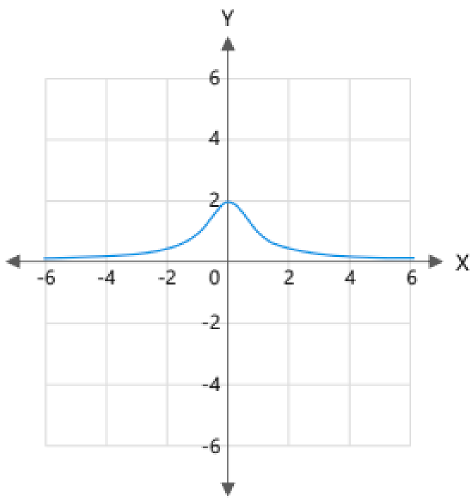


$f(x) = -x^2 - 1$

It is not necessary for every function to have an absolute maximum or absolute minima.

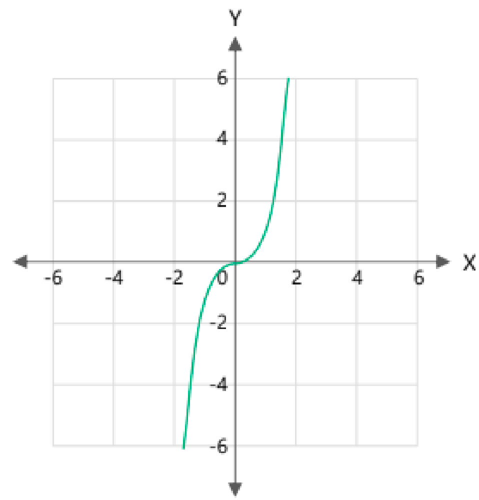
Consider these examples, where we do not have absolute extrema.

$$f(x) = \frac{1}{x^2 + 0.5}$$



- Absolute maximum of $f(x) = 2$ at $x = 0$
- No absolute minimum

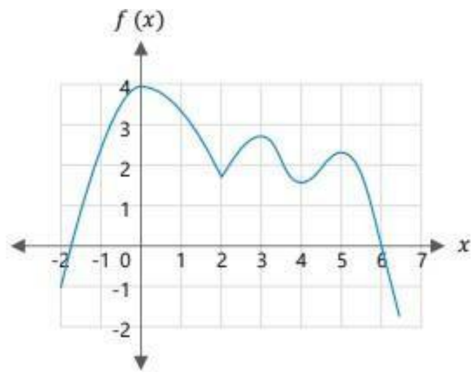
$$f(x) = x^3$$



- No absolute maximum
- No absolute minimum

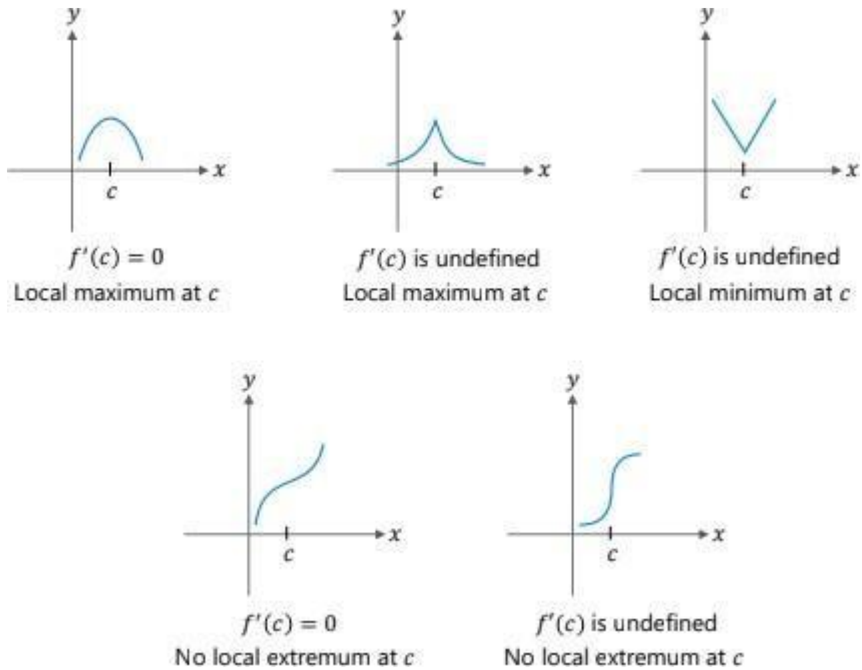
Local Extrema

- $f(x)$ does not attain absolute maxima at $x = 3$ and $x = 5$, but the value attained at $x = 3$ and $x = 5$ is the highest around the points close to $x = 3$ and $x = 5$. We refer to these points as local maxima
- Also, though we do not have an absolute minimum, we have local minima at $x = 2$ and $x = 4$. Points $x = 2$ and $x = 4$ are the lowest in a small interval around them



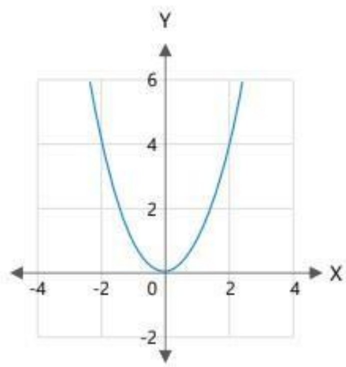
Critical Points

- To be a local extremum, the derivative of the function at those points can be 0 or undefined. These points are the critical points and are used to find the maximum and minimum values of a function. If $f'(c) = 0$ or $f'(c)$ is undefined, then c is a critical point.

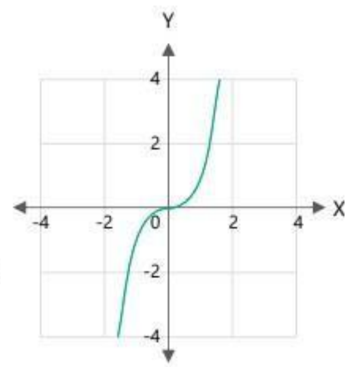


It is not necessary for every critical point to be a point of a local minimum or a local maximum. Some critical points are also saddle points.

- $f(x) = x^2$
  - $f'(x) = 2x$
  - At $x = 0, f'(x) = 0$
  - At $x = 0$, the function attains a minima
- $f(x) = x^3$
  - $f'(x) = 3x^2$
  - At $x = 0, f'(x) = 0$
  - At $x = 0$, the function attains neither a maxima nor a minima



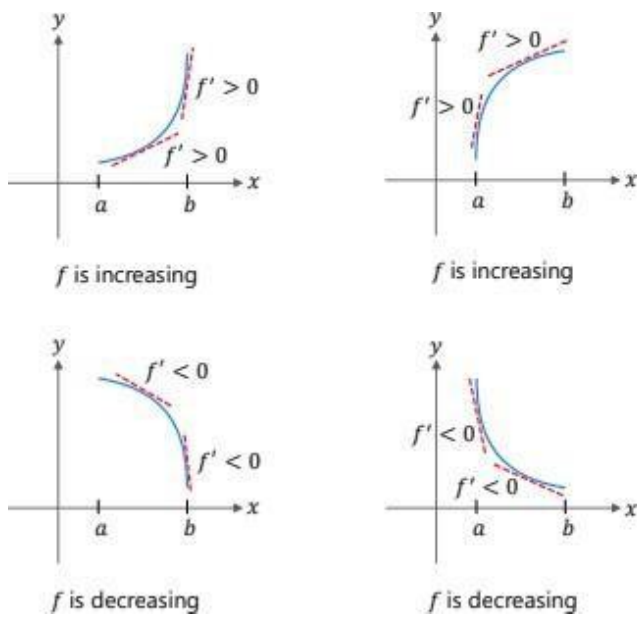
$f(x) = x^2$



$f(x) = x^3$

Increasing and Decreasing Functions

- Moving along the *x*-axis, if the slope of the function between *a* and *b* is greater than 0, then that function is said to be increasing. The sign of f' is positive.
- Moving along the *x*-axis, if the slope of the function is less than 0, then that function is said to be decreasing. The sign of f' is negative.

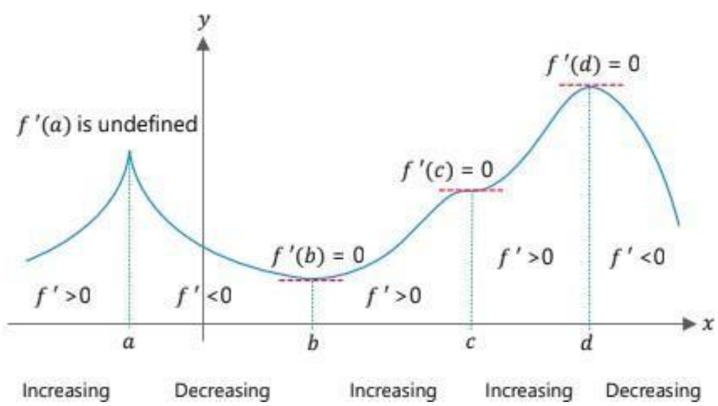


The First Derivative Test

- If a continuous function f has a local extremum, then it must occur at a critical point c.
- A function has a local extremum at the critical point c if and only if the derivative, f', switches sign as x increases through c.
- To test whether a function has a local extremum at a critical point c, we must determine the sign of $f'(x)$ to the left and right of c.
- This result is known as the first derivative test.

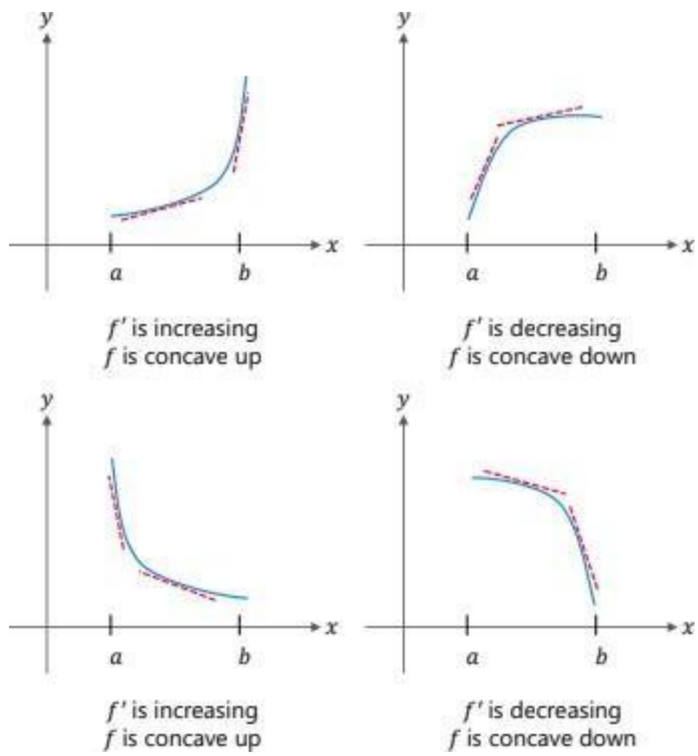
Consider this example to understand the test better.

- Function f has four critical points: a, b, c and d. Function f has local maxima at a and d, and a local minimum at b. Function f does not have a local extremum at c. The sign of f' changes at all local extrema



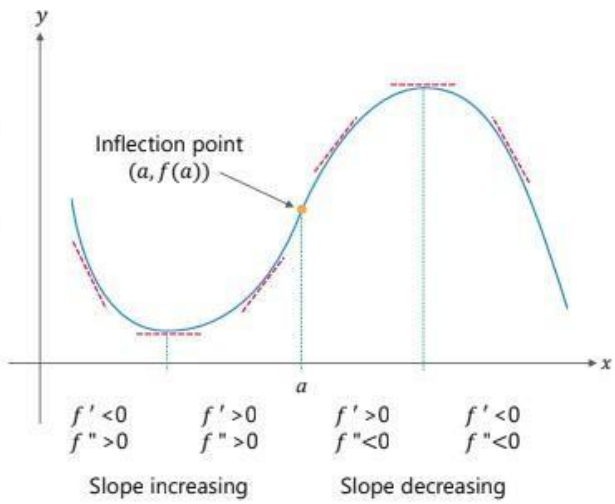
Concavity

- The notion of whether a function curves upwards or downwards is called concavity.



Inflection Points

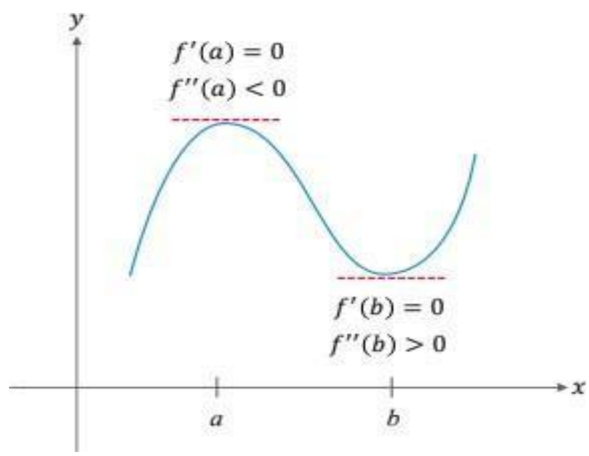
- The point at which the concavity changes is the inflection point
- a is an inflection point, because at that point the graph changes from concave up to a concave down – this happens when the second derivative changes signs
- Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$
- Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞)
- The point $(a, f(a))$ is an inflection point of f



- The chain rule helps find the derivatives of composite functions.
- $h(x) = f(g(x)) = \sin(x^2 + 1)$
- $f(x) = \sin(x) \Rightarrow \frac{df(x)}{dx} = \frac{d(\sin(x))}{dx} = \cos(x)$
- $g(x) = x^2 + 1 \Rightarrow \frac{df(x)}{dx} = \frac{d(x^2+1)}{dx} = 2x$
- $\frac{dh(x)}{dx} = \frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} . \frac{dg(x)}{dx} = \frac{d\sin(x^2+1)}{dg(x^2+1)} . \frac{d(x^2+1)}{dx} = \cos(x^2 + 1) . 2x =$ $2x. \cos(x^2 + 1)$

Second Derivative Test

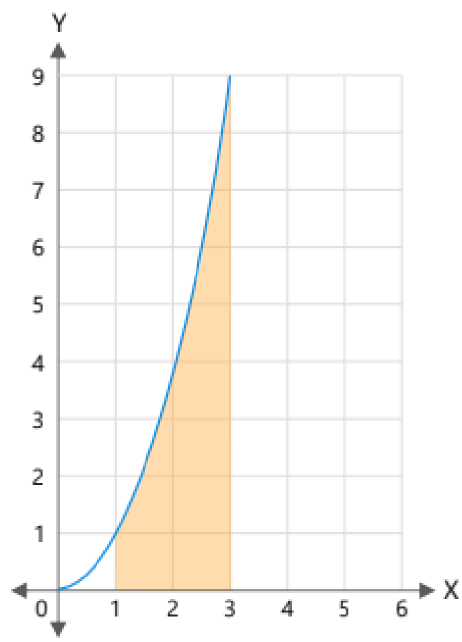
- The second derivative test can be used to determine whether a function has a local extremum at a critical point.
- If the second derivative is negative, the slope goes from increasing to decreasing we get a local maximum.
- If the second derivative is positive, the slope goes from decreasing to increasing we get a local minimum.



Integration

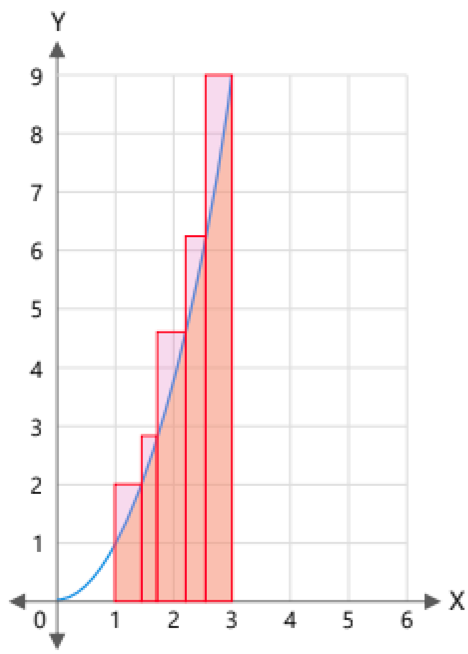
Introduction to Integration

- A farmer is interested in selling his farm to a buyer. His farm is the region bounded by $y = x^2$, $x = 1$, $x = 3$ and $y = 0$ in the graph below (each square in the graph represents 1 acre of land). The price of the farmland is fixed at ₹53 lakh per acre. To calculate the price of the farm, both the parties need to agree upon its exact area. They acknowledge the fact that they only know how to calculate the area of rectangles. So, how should they calculate the actual area of the farm?



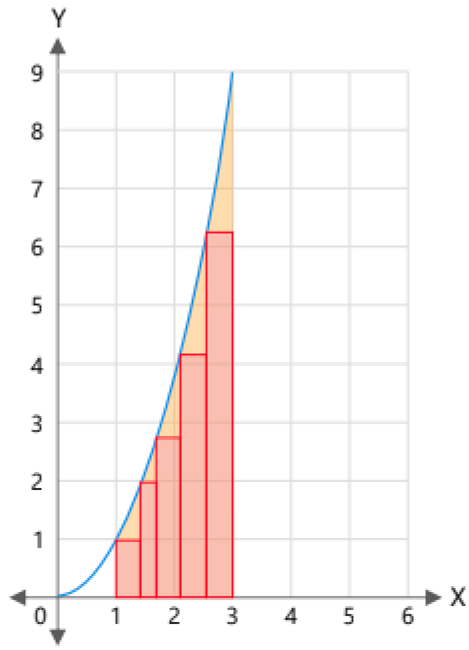
Area of farmland

- The farmer wants to increase his profits; so, he shows the buyer the calculation of the farm’s area using rectangles. Notice how the rectangles overestimate the actual area of the farm.



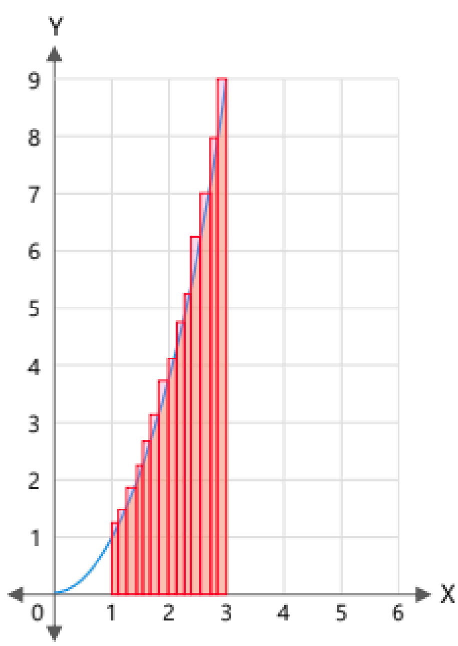
Farmer’s calculation

- The buyer wants to decrease his losses; so, he shows the farmer the calculation of the farm’s area using rectangles. Notice how the rectangles underestimate the actual area of the farm.

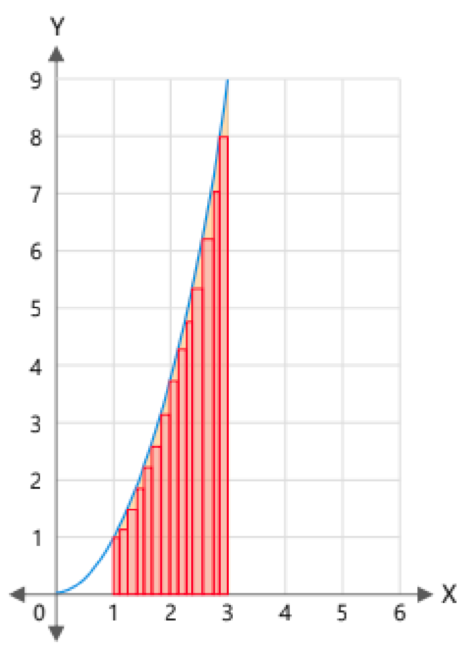


Buyer’s calculation

- They both agree that the error due to the overestimation or underestimation will reduce if they use a greater number of rectangles.



Farmer’s calculation



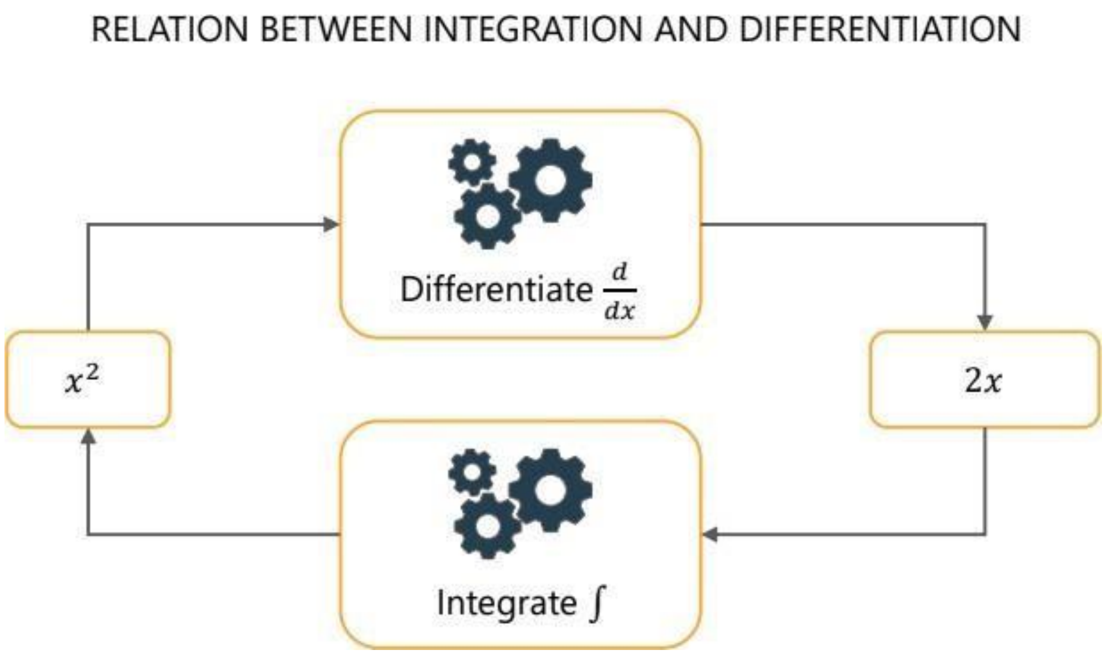
Buyer’s calculation

Taking an infinite number of infinitesimally small rectangles allows us to get the exact area under a curve. Nevertheless, adding the area of infinitely many rectangles is not an efficient method to calculate the total area.

Here is how we obtain the area between a curve defined by a function $f(x)$ and the ***x***-axis on a closed interval of $[a, b]$:

- First, we approximate the area under the curve using known shapes, such as rectangles, for which we know how to compute the area.
- By using smaller rectangles, we get closer and closer approximations to the area.
- Taking an infinite number of infinitesimally small rectangles allows us to obtain the exact area under the curve.
- This infinite sum is denoted with $\int_a^b f(x)\, dx$, which is also called a definite integral.

Introduction to Antiderivatives



- Antiderivatives are the opposite of derivatives.
- An antiderivative is a function, which reverses what the derivative does.
- Antiderivatives are important for solving problems where we know a derivative of a function but not the function itself.
- All continuous functions have antiderivatives.

Indefinite Integrals

- Consider $F(x) = x^2$; then $F'(x) = f(x) = 2x$. Therefore, x^2 is an antiderivative of $2x$.
- The derivative of any constant C is 0; so, $x^2 + C$ is also an antiderivative of $2x$, and we write $\int 2x\, dx = x^2 + C$.
- If $\int f(x)\, dx = F(x) + C$, then $\int f(x) dx$ is called the **indefinite integral** or the **antiderivative** of f.

Definite Integrals

- An indefinite integral is a family of functions

$$\int 2x\, dx = x^2 + C$$

- Definite integrals have start and end values

$$\int_a^b f(x)\, dx$$

- A definite integral is a number

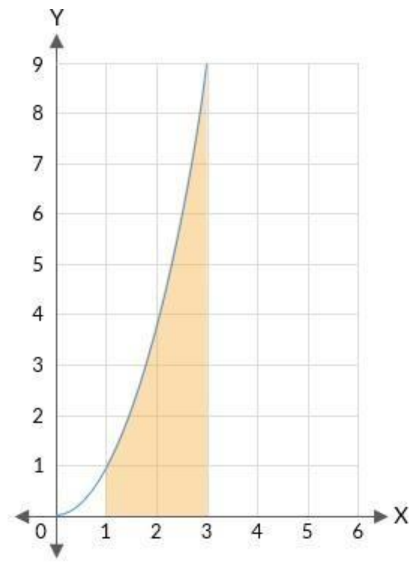
$$\int_1^3 2x\, dx = x^2 + C$$

At $a = 1$: $\int 2x\, dx = 1^2 + C$
At $b = 3$: $\int 2x\, dx = 3^2 + C$
Subtracting the two values : $\int_1^3 2x\, dx = (9 + C) - (1 + C) = 8 \Rightarrow \int_1^3 2x\, dx = 8$

- If a function $f(x)$ is continuous on the interval $[a, b]$, then the function f is integrable on $[a, b]$

Using the explanation above, we can solve the farmer–buyer problem.

- Area = $\int_1^3 x^2\,dx = \frac{x^3}{3} + C$
  - At a = 1: $\int x^2\,dx = \frac{1^3}{3} + C = \frac{1}{3} + C$
  - At b = 3: $\int x^2\,dx = \frac{3^3}{3} + C = 9 + C$
  - Subtracting the two values:
  - $\int_1^3 x^2\,dx = (9 + \mathrm{C}) - \left(\frac{1}{3} + \mathrm{C}\right) = \frac{26}{3} \Rightarrow \int_1^3 x^2\,dx = 8\frac{2}{3}$
  - Area ≈ 8.66... acres
  - Price of the farm ≈ 8.66 × 53 ≈ ₹ 4.59 cr



Area of farmland

To calculate $\int_1^2 e^x dx$, we will follow these steps:

At a = 1: $\int e^x dx = e^1 + C$

At b = 2: $\int e^x dx = \; e^3 + C$

Subtracting the two values: $\int_1^2 e^x dx = (\, e^3 + C) - (e^1 + C) = e^3 - e^1$