
CS 559: Neural Networks

# Homework 1

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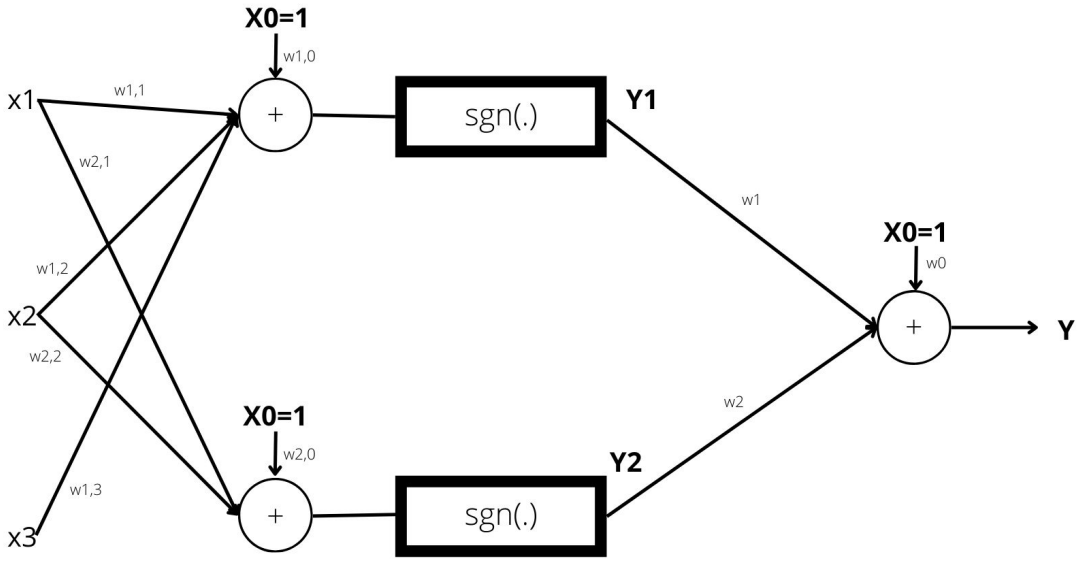
# 1 Exercise 1

In this exercise is used the sign function as activation function, sgn(x) = 1 if x > 0, sgn(x)=-1 if x < 0, and sgn(0) = 0. I will refer to this function using the simbol $\phi$.
I want to implement the logic gate expressed by the following function:

$$f(x_1, x_2, x_3) = \bar{x_1}x_2x_3 + x_1\bar{x_2}$$

The following neural network resolves the problem, now I need to find the weights.




In order to do that I study Y1, Y2 and then Y.

## 1.1 Y1

$Y_1 = \phi(x_0w_{1,0} + x_1w_{1,1} + x_2w_{1,2} + x_3w_{1,3})$

The table of the outputs is illustrated below:

| x1 | x2 | x3 | y1 |
|---|---|---|---|
| -1 | -1 | -1 | -1 |
| -1 | -1 | 1 | -1 |
| -1 | 1 | -1 | -1 |
| -1 | 1 | 1 | 1 |
| 1 | -1 | -1 | -1 |
| 1 | -1 | 1 | -1 |
| 1 | 1 | -1 | -1 |
| 1 | 1 | 1 | -1 |

From this table we can extract 8 inequalities:
1)$w_{1,0} - w_{1,1} - w_{1,2} - w_{1,3} < 0$
2)$w_{1,0} - w_{1,1} - w_{1,2} + w_{1,3} < 0$
3)$w_{1,0} - w_{1,1} + w_{1,2} - w_{1,3} < 0$
4)$w_{1,0} - w_{1,1} + w_{1,2} + w_{1,3} > 0$
5)$w_{1,0} + w_{1,1} - w_{1,2} - w_{1,3} < 0$
6)$w_{1,0} + w_{1,1} - w_{1,2} + w_{1,3} < 0$
7)$w_{1,0} + w_{1,1} + w_{1,2} - w_{1,3} < 0$
8)$w_{1,0} + w_{1,1} + w_{1,2} + w_{1,3} < 0$

To reduce the space of the possible solutions I can sum the inequalities in order to find the sign of each term.

$$\begin{gathered} 4) - w_{1,0} + w_{1,1} - w_{1,2} - w_{1,3} < 0 + \\ 8) w_{1,0} + w_{1,1} + w_{1,2} + w_{1,3} < 0 = \\ \hline 2w_{1,1} < 0 \rightarrow w_{1,1} < 0 \end{gathered}$$

$$\begin{gathered} 1) w_{1,0} - w_{1,1} - w_{1,2} - w_{1,3} < 0 + \\ 8) w_{1,0} + w_{1,1} + w_{1,2} + w_{1,3} < 0 = \\ \hline 2w_{1,0} < 0 \rightarrow w_{1,0} < 0 \end{gathered}$$

$$\begin{gathered} 2) w_{1,0} - w_{1,1} - w_{1,2} + w_{1,3} < 0 + \\ 4) - w_{1,0} + w_{1,1} - w_{1,2} - w_{1,3} < 0 = \\ \hline -2w_{1,2} < 0 \rightarrow w_{1,2} > 0 \end{gathered}$$

$$3) w_{1,0} - w_{1,1} + w_{1,2} - w_{1,3} < 0 +$$
$$4) - w_{1,0} + w_{1,1} - w_{1,2} - w_{1,3} < 0 =$$
$$-2w_{1,3} < 0 \rightarrow w_{1,3} > 0$$

Those values satisfies all the constraints:
$w_{1,0} = -5$
$w_{1,1} = -3$
$w_{1,2} = 2$
$w_{1,3} = 1$

## 1.2 Y2

$Y_2 = \phi(x_0 w_{2,0} + x_1 w_{2,1} + x_2 w_{2,2}$

The table of the outputs is illustrated below:

| x1 | x2 | y2 |
|---|---|---|
| -1 | -1 | -1 |
| -1 | 1 | -1 |
| 1 | -1 | 1 |
| 1 | 1 | -1 |

From this table we can extract 4 inequalities:
$1) w_{2,0} - w_{2,1} - w_{2,2} < 0$
$2) w_{2,0} - w_{2,1} + w_{2,2} < 0$
$3) w_{2,0} + w_{2,1} - w_{2,2} > 0$
$4) w_{2,0} + w_{2,1} + w_{2,2} < 0$

To reduce the space of the possible solutions we can sum the inequalities in order to find the sign of each term.

$$1) w_{2,0} - w_{2,1} - w_{2,2} < 0 +$$
$$4) w_{2,0} + w_{2,1} + w_{2,2} < 0 =$$
$$2w_{2,0} < 0 \rightarrow w_{2,0} < 0$$

$$\begin{array}{c} 1) w_{2,0} - w_{2,1} - w_{2,2} < 0 + \\ 3) - w_{2,0} - w_{2,1} + w_{2,2} < 0 = \\ \hline -2w_{2,1} < 0 \rightarrow w_{2,1} > 0 \end{array}$$

$$\begin{array}{c} 3) - w_{2,0} - w_{2,1} + w_{2,2} < 0 + \\ 4) w_{2,0} + w_{2,1} + w_{2,2} < 0 = \\ \hline 2w_{2,2} < 0 \rightarrow w_{2,2} < 0 \end{array}$$

Those values satisfies all the constraints:
$w_{2,0} = -3$
$w_{2,1} = 2$
$w_{2,2} = -2$

### 1.3 Y

$Y = \phi(x_0 w_0 + y_1 w_1 + y_2 w_2)$

The table of the outputs is illustrated below:

| y1 | y2 | Y |
|---|---|---|
| -1 | -1 | -1 |
| -1 | 1 | 1 |
| 1 | -1 | 1 |
| 1 | 1 | 1 |

From this table we can extract 4 inequalities:
$1) w_0 - w_1 - w_2 < 0$
$2) w_0 - w_1 + w_2 > 0$
$3) w_0 + w_1 - w_2 > 0$
$4) w_0 + w_1 + w_2 > 0$

To reduce the space of the possible solutions we can sum the inequalities in order to find the sign of each term.

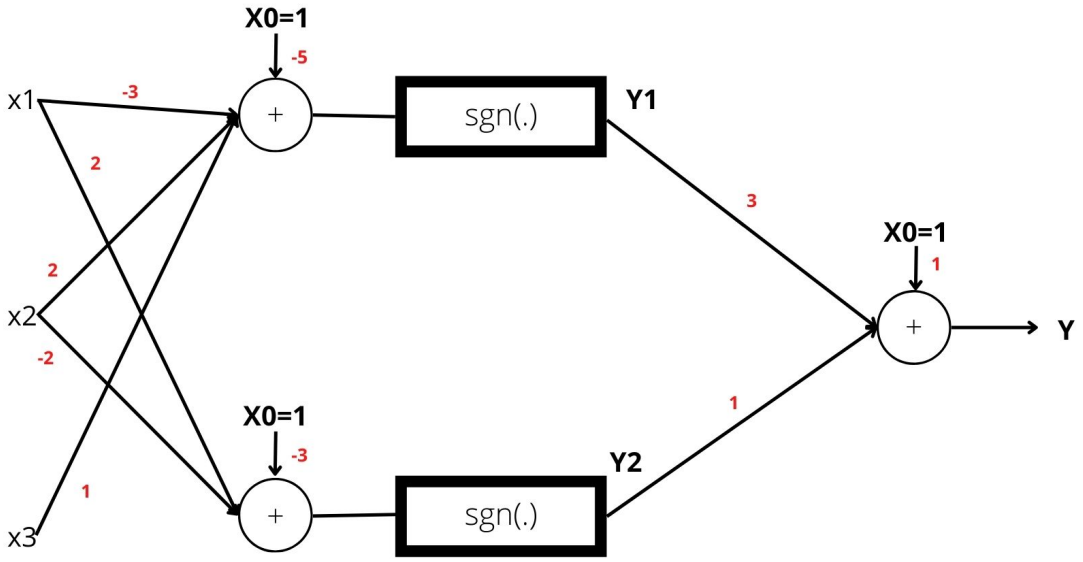
$$
\begin{array}{c}
1) - w_0 + w_1 + w_2 > 0 + \\
2) w_0 - w_1 + w_2 > 0 = \\
\hline
2w_2 > 0 \rightarrow w_2 > 0
\end{array}
$$

$$
\begin{array}{c}
2) w_0 - w_1 + w_2 > 0 + \\
3) w_0 + w_1 - w_2 > 0 = \\
\hline
2w_0 > 0 \rightarrow w_0 > 0
\end{array}
$$

$$
\begin{array}{c}
1) - w_0 + w_1 + w_2 > 0 + \\
3) w_0 + w_1 - w_2 > 0 = \\
\hline
2w_1 > 0 \rightarrow w_1 > 0
\end{array}
$$

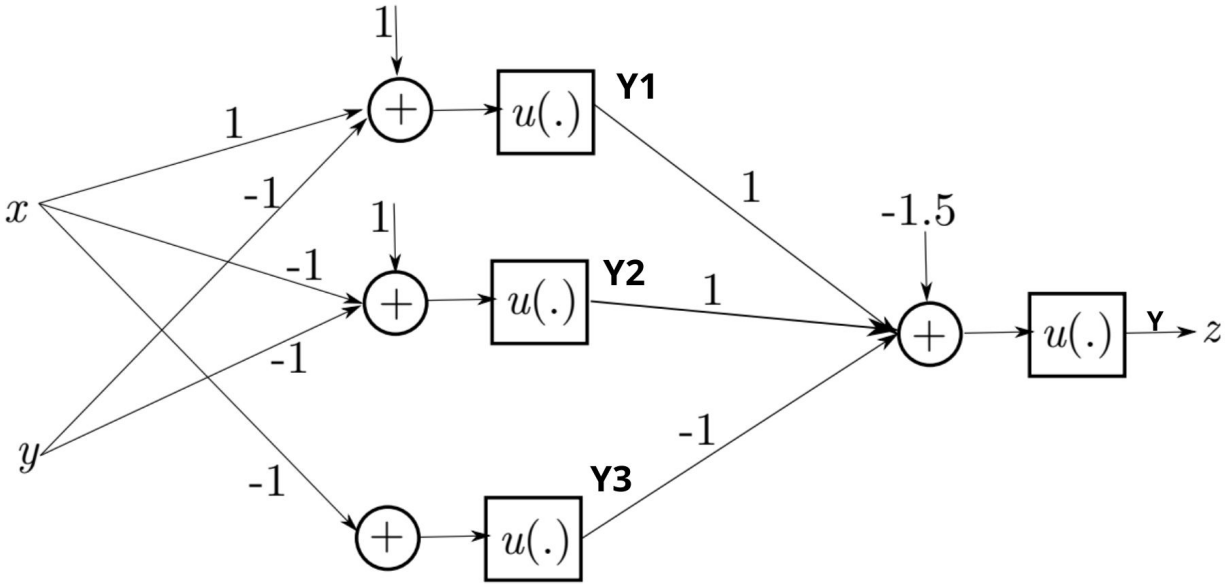
Those values satisfies all the constraints:
$w_0 = 3$
$w_1 = 3$
$w_2 = 1$

At the end we obtain the neural network showed below.




## 2 Exercise 2

In the second exercise we want to sketch the region where z=1 in the x-y plane. We consider the network showed below.




The activation function is u(x) = 1 if x >= 0 and u(x) = 0 if x < 0
$Y_1 = u(1 + x - y)$
$Y_2 = u(1 - x - y)$
$Y_3 = u(-x)$
$Z = u(-1.5 + Y_1 + Y_2 - Y3)$
We want Z=1 so:
$u(-1.5 + Y_1 + Y_2 - Y3) = 1$
$if\ -1.5 + Y_1 + Y_2 - Y3 >= 0$

Now I analize the single neuron in order to find the region where output is 0 or 1

Table 1: Neuron 1
1+x-y >= 0
y<=x+1

Table 2: Neuron 2
1-x-y >=0
y<=1-x

Table 3: Neuron 3
-x <0
x>0

| Y1 | Y2 | $Y_3$ | −1.5 | Z |
|---|---|---|---|---|
| 0 | 0 | 0 | −1.5 | 0 |
| 0 | 0 | 1 | −1.5 | 0 |
| 0 | 1 | 0 | −1.5 | 0 |
| 0 | 1 | 1 | −1.5 | 0 |
| 1 | 0 | 0 | −1.5 | 0 |
| 1 | 0 | 1 | −1.5 | 0 |
| 1 | 1 | 0 | −1.5 | 1 |
| 1 | 1 | 1 | −1.5 | 0 |

Table 4: Neuron 4

I put everything in the x-y plan and I find the part where z=1