Discrete Mathematical Models

Lecture 13

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Collaboration Statement for Assignment 1

I have posted an example collaboration statement on the Wattle page. For example:

The assignment I have submitted has been produced according to the rules on Page 1 of the MATH?005 Assignment 1. The solutions produced have been written/typed on my own. I have not used AI to prepare my solutions. I have provided references to the sources I have used where appropriate.

Student ID:

Sorting (Cont.)

Merge Algorithm

The **merge algorithm** takes two sequences in S, already ordered according to some $\leq$ and outputs an ordered sequence that merges the two input sequences.

Example: Merge $(x_i)_{1..2} = (1, 4)$ and $(y_j)_{1..3} = (2, 3, 5)$.

Compare $x_1 = 1$ and $y_1 = 2$. Since $1 < 2$ we assign $z_1 = x_1 = 1$.

Compare $x_2 = 4$ and $y_1 = 2$. Since $2 < 4$, we assign $z_2 = y_1 = 2$.

Compare $x_2 = 4$ and $y_2 = 3$. Since $3 < 4$, we assign $z_3 = y_2 = 3$.

Compare $x_2 = 4$ and $y_3 = 5$. Since $4 < 5$, we assign $z_4 = x_2 = 4$.

There is no x_3 so we assign $z_5 = y_3 = 5$.

We have reached the end of both $(x_i)_{1..2}$ and $(y_j)_{1..3}$. So we end the merging and we have $(1, 2, 3, 4, 5)$ as our merged list.

Merge algorithm

Input: Two lists (sequences) $(a_i)_{i \in \{1,\dots,n\}} \subseteq S$ and $(b_j)_{j \in \{1,\dots,p\}} \subseteq S$ pre-sorted according to an ordering rule "$\leq$" on S.

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Output: In-order list (sorted sequence) $(z_k)_{k \in \{1,\dots,n+p\}}$ that merges the two input lists.

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Method:

$i, j, k \leftarrow 1$. [Initialize the indices for the a-, b- and z- lists]

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$i, j, k \leftarrow 1$. [Initialize the indices for the a-, b- and z- lists]

Loop: If $k = n+p+1$ stop. [there will be $n+p$ items in the z-list]

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If $i = n+1$ then [$z_k \leftarrow b_j$, $j \leftarrow j+1$] [a-list empty; take from b-list]

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Else if $a_i < b_j$ then [$z_k \leftarrow a_i$, $i \leftarrow i+1$] [a-list item less; take it]

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Else if $a_i < b_j$ then [$z_k \leftarrow a_i$, $i \leftarrow i+1$] [a-list item less; take it]

Else [$z_k \leftarrow b_j$, $j \leftarrow j+1$] [else take item from b-list]

$k \leftarrow k+1$ [prepare to add next item to z-list]

Repeat loop.

Merge algorithm: example of execution

Example: Merge $(a_i)_{1..3} = (1,3,7)$ and $(b_j)_{1..5} = (2,3,6,8,9)$.

After iteration	i	j	k	a_i	b_j	$(z_1,\ldots,z_{k-1})$
0	1	1	1	$(1,3,7)$	$(2,3,6,8,9)$	$()$

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0	1	1	1	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$()$
1	2	1	2	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	(1)

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Example: Merge $(a_i)_{1..3} = (1, 3, 7)$ and $(b_j)_{1..5} = (2, 3, 6, 8, 9)$.

After iteration	i	j	k	a_i	b_j	$(z_1, \dots, z_{k-1})$
0	1	1	1	$(\color{red}{1}, 3, 7)$	$(\color{red}{2}, 3, 6, 8, 9)$	$()$
1	2	1	2	$(1, \color{red}{3}, 7)$	$(\color{red}{2}, 3, 6, 8, 9)$	(1)
2	2	2	3	$(1, \color{red}{3}, 7)$	$(2, \color{red}{3}, 6, 8, 9)$	$(1, 2)$

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1	2	1	2	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	(1)
2	2	2	3	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2)$
3	2	3	4	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3)$

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2	2	2	3	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2)$
3	2	3	4	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3)$
4	3	3	5	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3)$

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3	2	3	4	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3)$
4	3	3	5	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3)$
5	3	4	6	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3, 6)$

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After iteration	i	j	k	a_i	b_j	$(z_1, \ldots, z_{k-1})$
0	1	1	1	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$()$
1	2	1	2	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	(1)
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4	3	3	5	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3)$
5	3	4	6	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3, 6)$
6	4	4	7	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3, 6, 7)$

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4	3	3	5	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3)$
5	3	4	6	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3, 6)$
6	4	4	7	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3, 6, 7)$
7	4	5	8	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3, 6, 7, 8)$

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4	3	3	5	$(1, 3, \color{red}{7})$	$(2, 3, \color{red}{6}, 8, 9)$	$(1, 2, 3, 3)$
5	3	4	6	$(1, 3, \color{red}{7})$	$(2, 3, 6, \color{red}{8}, 9)$	$(1, 2, 3, 3, 6)$
6	4	4	7	$(1, 3, 7)$	$(2, 3, 6, \color{red}{8}, 9)$	$(1, 2, 3, 3, 6, 7)$
7	4	5	8	$(1, 3, 7)$	$(2, 3, 6, 8, \color{red}{9})$	$(1, 2, 3, 3, 6, 7, 8)$
8	4	6	9	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3, 6, 7, 8, 9)$

Merge Sort Algorithm

Consider the sequence $(4, 2, 3, 1)$.

Sort this sequence into increasing order.

Apply merge algorithm for $\{(4), (2)\}$ and $\{(3), (1)\}$.

This gives $(2, 4)$ and $(1, 3)$.

Now apply the merge algorithm on $\{(2, 4), (1, 3)\}$.

This gives $(1, 2, 3, 4)$.

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(For lists whose length is not a power of 2, see workshop question.)

Output: In-order list (sorted sequence) $(z_n)_{n \in \{1,\ldots,2^r\}}$ that is a rearrangement of the input list.

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Method: There are r steps.

If $r < 3$ adjust the description below accordingly.

- Step 1: Apply the Merge algorithm 2^{r-1} times with inputs $\{(x_1),(x_2)\}$, $\{(x_3),(x_4)\}$,...,$\{(x_{2^r-1}),(x_{2^r})\}$.
  This gives 2^{r-1} in-order lists of length 2.

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- Step 2: Apply the Merge algorithm 2^{r-2} times with pairs of these lists as input.
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  This gives 2^{r-2} in-order lists of length $2 \times 2 = 2^2$.
- Steps 3 to r: Continue in this vein until you have just one $(= 2^{r-r})$ in-order list with 2^r elements.

Merge sort: example

Example: merge sort $(1, 2, 6, 1, 7, 9, 4, 5)$.

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Merging $\{(7), (9)\}$ gives $(7, 9)$.
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Merging $\{(4), (5)\}$ gives $(4, 5)$.

Step 2:
Merging $\{(1, 2), (1, 6)\}$ gives $(1, 1, 2, 6)$.
Merging $\{(7, 9), (4, 5)\}$ gives $(4, 5, 7, 9)$.

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Step 2:
Merging $\{(1, 2), (1, 6)\}$ gives $(1, 1, 2, 6)$.
Merging $\{(7, 9), (4, 5)\}$ gives $(4, 5, 7, 9)$.

Step 3:
Merging $\{(1, 1, 2, 6), (4, 5, 7, 9)\}$ gives $(1, 1, 2, 4, 5, 6, 7, 9)$.

Merge sort: counting comparisons

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$(1)\quad(2)\quad(6)\quad(1)\quad(7)\quad(9)\quad(4)\quad(5)$	
$(1,2)\quad(1,6)\quad(7,9)\quad(4,5)$	$1{+}1{+}1{+}1{=}4$ comps
$(1,1,2,6)\quad(4,5,7,9)$	$3{+}2{=}5$ comps
$(1,1,2,4,5,6,7,9)$	6 comps
TOTAL:	15 comparisons

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(1) (2) (6) (1) (7) (9) (4) (5)	
$(1, 2)$ $(1, 6)$ $(7, 9)$ $(4, 5)$	$1{+}1{+}1{+}1{=}4$ comps
$(1, 1, 2, 6)$ $(4, 5, 7, 9)$	$3{+}2{=}5$ comps
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Note for example that when merging $(7, 9)$ and $(4, 5)$ only 2 comparisons are used:

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(1)	(2)	(6)	(1)	(7)	(9)	(4)	(5)	
$(1, 2)$		$(1, 6)$		$(7, 9)$		$(4, 5)$		1+1+1+1=4 comps
$(1, 1, 2, 6)$				$(4, 5, 7, 9)$				3+2=5 comps
$(1, 1, 2, 4, 5, 6, 7, 9)$								6 comps
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7 and 4 are compared; 4 is transferred

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$(1, 2)\quad(1, 6)\quad(7, 9)\quad(4, 5)$	$1+1+1+1=4$ comps
$(1, 1, 2, 6)\quad(4, 5, 7, 9)$	$3+2=5$ comps
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$$
\begin{array}{ll}
(1) \quad (2) \quad (6) \quad (1) \quad (7) \quad (9) \quad (4) \quad (5) & \\
(1,2) \qquad (1,6) \qquad (7,9) \qquad (4,5) & 1{+}1{+}1{+}1{=}4 \text{ comps} \\
(1,1,2,6) \qquad\qquad (4,5,7,9) & 3{+}2{=}5 \text{ comps} \\
(1,1,2,4,5,6,7,9) & 6 \text{ comps} \\
\text{TOTAL:} & 15 \text{ comparisons}
\end{array}
$$

Note for example that when merging $(7, 9)$ and $(4, 5)$ only 2 comparisons are used:

7 and 4 are compared; 4 is transferred

7 and 5 are compared; 5 is transferred

7 and 9 are transferred without comparison (other list exhausted.)

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However an upper bound is given by the number of *transfers*.

Let T_r denote the number of transfers required to sort a sequence of length 2^r using Merge sort.

Now $2^{r-1} + 2^{r-1} = 2^r$ transfers are required for the Merge algorithm to merge two sequences of length 2^{r-1},

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So $T_1 = 2, \quad T_2 = 2^2 + 4 = 8, \quad T_3 = 2^3 + 2\times 8 = 3\times 2^3,$

$T_4 = 2^4 + 2 \times (3\times 2^3) = 4\times 2^4, \quad \ldots$

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Claim: $\forall \boldsymbol{r} \in \mathbb{N} \quad \boldsymbol{T_r = r2^r}$. Verify by induction!
(The sequence $(T_r)_{r\in\mathbb{N}}$ is neither geometric, arithmetic nor mixed.)

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For short lists on high speed computers, slower speed may not matter.

B3: Matrices

What is a matrix (plural: matrices) ?

Definition: Let S be a set, and $m, n \in \mathbb{N}$.

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$$\mathrm{A} = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} \pi/2 \\ -\pi/2 \end{bmatrix} \qquad \mathrm{C} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

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Indexing

A generic member of $M_{m \times n}(S)$ is written

$$\mathsf{A} = (a_{i,j}) = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix}$$

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- A matrix $(a_{i,j}) \in M_n(\mathbb{Q})$ can define a weighted relation. Let us consider 4 companies, called 1,2,3,4, and let $a_{i,j}$ be the money ($) received by i from j in a year.

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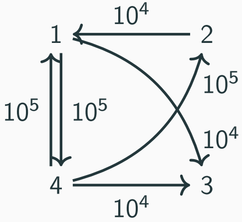

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A vector $\mathrm{x} \in \mathbb{Q}^n$ can be viewed as

an element of $M_{1\times n}(\mathbb{Q})$; x is then called a **row vector**

or as an element of $M_{n\times 1}(\mathbb{Q})$; x is then called a **column vector**.

The **sum** of two vectors, $\mathrm{x} + \mathrm{y}$, is defined element-wise:

$$\mathrm{x} + \mathrm{y} = (x_1, ..., x_n) + (y_1, ..., y_n) = (x_1 + y_1, ..., x_n + y_n).$$

(When viewed as row or column vectors, x and y must be the same shape.)

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$$\forall\lambda \in \mathbb{Q} \quad \lambda\mathrm{x} = \lambda(x_1, ..., x_n) = (\lambda x_1, ..., \lambda x_n).$$

Examples of vectors and vector arithmetic

- Let $\mathrm{p} = (p_1, p_2, p_3) \in \mathbb{Q}^3$ represent the state of an ecosystem with p_1, p_2, p_3 being the sizes of the populations of three different species.

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$$3\mathrm{a} = 3(a_1, ..., a_n),$$

represents to the same sound, but three times stronger.

Next Lecture

We will learn more about matrix arithmetic and see what type of processes can be modelled using matrices!