Discrete Mathematical Models

Lecture 16

Kane Townsend

Semester 2, 2024

C1: Counting (cont.)

Multisets (correction)

A multiset is a 'set' with multiple copies of elements allowed and acknowledged (the multiplicities **do** matter).

Formally, a **size-r multiset** is a set S together with a 'multiplicity function' $m : S \to \mathbb{N}^*$, where,

$$\forall s \in S \quad m(s) = \text{number of copies of } s \quad \textit{and} \quad r = \sum_{s \in S} m(s).$$

Multisets (correction)

A multiset is a 'set' with multiple copies of elements allowed and acknowledged (the multiplicities **do** matter).

Formally, a **size-r multiset** is a set S together with a 'multiplicity function' $m : S \to \mathbb{N}^*$, where,

$$\forall s \in S \quad m(s) = \text{number of copies of } s \quad \textit{and} \quad r = \sum_{s \in S} m(s).$$

For example, let $S = \{a, b, c, d\}$ and $m : S \to \mathbb{N}^*$, where $m(a) = 2, m(b) = 1, m(c) = 2, m(d) = 0$. It is common to write such a size-5 multiset with notation $\{c, b, a, c, a\} = \{a, a, b, c, c\}$.

Rationals are countable

0					
	$\pm\frac{1}{1}$	$\pm\frac{1}{2}$	$\pm\frac{1}{3}$	$\pm\frac{1}{4}$	$\dots$
	$\pm\frac{2}{1}$	$\pm\frac{2}{2}$	$\pm\frac{2}{3}$	$\pm\frac{2}{4}$	$\dots$
	$\pm\frac{3}{1}$	$\pm\frac{3}{2}$	$\pm\frac{3}{3}$	$\pm\frac{3}{4}$	$\dots$
	$\pm\frac{4}{1}$	$\pm\frac{4}{2}$	$\pm\frac{4}{3}$	$\pm\frac{4}{4}$	$\dots$
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Rationals are countable

0					
	$\pm\frac{1}{1}$	$\pm\frac{1}{2}$	$\pm\frac{1}{3}$	$\pm\frac{1}{4}$	$\dots$
	$\pm\frac{2}{1}$	$\pm\frac{2}{2}$	$\pm\frac{2}{3}$	$\pm\frac{2}{4}$	$\dots$
	$\pm\frac{3}{1}$	$\pm\frac{3}{2}$	$\pm\frac{3}{3}$	$\pm\frac{3}{4}$	$\dots$
	$\pm\frac{4}{1}$	$\pm\frac{4}{2}$	$\pm\frac{4}{3}$	$\pm\frac{4}{4}$	$\dots$
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Notes: Table contains all rationals (some are repeated) and there are finitely many rationals on any diagonal (down to up and left to right).

Rationals are countable

0					
	$\pm\frac{1}{1}$	$\pm\frac{1}{2}$	$\pm\frac{1}{3}$	$\pm\frac{1}{4}$	$\dots$
	$\pm\frac{2}{1}$	$\pm\frac{2}{2}$	$\pm\frac{2}{3}$	$\pm\frac{2}{4}$	$\dots$
	$\pm\frac{3}{1}$	$\pm\frac{3}{2}$	$\pm\frac{3}{3}$	$\pm\frac{3}{4}$	$\dots$
	$\pm\frac{4}{1}$	$\pm\frac{4}{2}$	$\pm\frac{4}{3}$	$\pm\frac{4}{4}$	$\dots$
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Notes: Table contains all rationals (some are repeated) and there are finitely many rationals on any diagonal (down to up and left to right). Define the bijection $f : \mathbb{N} \to \mathbb{Q}$ by:

- $f(1) = 0$
- $f(2) = \frac{1}{1}$, $f(3) = -\frac{1}{1}$
- $f(4) = \frac{2}{1}$, $f(5) = -\frac{2}{1}$, $f(6) = \frac{1}{2}$, $f(7) = -\frac{1}{2}$
- $f(8) = \frac{3}{1}$, $f(9) = -\frac{3}{1}$, $f(10) = \frac{1}{3}$, $f(11) = -\frac{1}{3}$

Reals are uncountable

We will prove that the real numbers between $(0, 1)$ are uncountable by a contradiction. Assume the real numbers between $(0, 1)$ are countable.

Reals are uncountable

We will prove that the real numbers between $(0,1)$ are uncountable by a contradiction. Assume the real numbers between $(0,1)$ are countable.

Note that we assume that the real numbers are all expressible as infinite strings in decimals starting with $0.\ldots$. We decide to never express a real number using $\overline{9}$ ($0.1\overline{0} = 0.0\overline{9}$).

Reals are uncountable

We will prove that the real numbers between $(0, 1)$ are uncountable by a contradiction. Assume the real numbers between $(0, 1)$ are countable.

Note that we assume that the real numbers are all expressible as infinite strings in decimals starting with $0.\ldots$. We decide to never express a real number using $\overline{9}$ ($0.1\overline{0} = 0.0\overline{9}$).

Since we have assumed the real numbers between $(0, 1)$ are countable we have a bijection $f : \mathbb{N} \to (0, 1)$. Write $f(1), f(2), f(3), \ldots$ over each other:

$$f(1) = 0.d_{1,1}d_{1,2}d_{1,3}d_{1,4}\ldots$$

$$f(2) = 0.d_{2,1}d_{2,2}d_{2,3}d_{2,4}\ldots$$

$$f(3) = 0.d_{3,1}d_{3,2}d_{3,3}d_{3,4}\ldots$$

$$f(4) = 0.d_{4,1}d_{4,2}d_{4,3}d_{4,4}\ldots$$

Let $x = 0.e_1e_2e_3e_4\ldots$ where each $e_i \in \{1, 2, 3, 4, 5, 6, 7, 8\} \setminus \{d_{i,i}\}$.

Clearly x is not in our list above but is a real number between $(0, 1)$.

Basic set counting rules

Basic Set Counting

1. **Inclusion-Exclusion** If A and B are finite sets, then
$$|A \cup B| = |A| + |B| - |A \cap B|.$$

2. **The Sum Rule** If A is a finite set and $\{A_1, A_2, \ldots, A_m\}$ is a partition of A, then $|A| = |A_1| + |A_2| + \cdots + |A_m|$.

The Product Rule If $A_1, A_2, \ldots, A_m$ are finite sets, then
$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|.$$

1. $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$ then $|A \cup B| = 3 + 3 - 1 = 5$.

2. A is partitioned by B, C and D where $|B| = 4, |C| = 5$ $|D| = 7$. Then $|A| = 4 + 5 + 7 = 16$.

3. Let A be the set of alphabet letters and B be the set of decimal digits. What is $|A^3 \times B^2 \times A|$? This is $26^3 \times 10^2 \times 26$. (Maybe counting how many car number plates of form Letter Letter Letter Digit Digit Letter).

Counting functions between finite sets A and B

Claim: The number of functions $f : A \to B$ is given by $|B|^{|A|}$.

Proof: For each element $a \in A$ there is $|B|$ choices of $f(a)$, so to define function f is the same as choosing $f(a)$ for each $a \in A$. In other words the set of functions $f : A \to B$ is in bijection with $B^{|A|}$.

Since **bijections preserve cardinality** and by the **product rule**, the number of functions is given by $|B|^{|A|}$.

Example 6

Prove the following: If A is a non-empty finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

Example 6

Prove the following: If A is a non-empty finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

IDEA: A subset of A corresponds to making one of two choices for each element of A: include it in the subset or not. so we claim: there is a bijection between the subsets of A and the functions $f : A \to \{0, 1\}$. If we prove this then we know the number of subsets is $2^{|A|}$ by the previous slide.

Example 6

Prove the following: If A is a non-empty finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

IDEA: A subset of A corresponds to making one of two choices for each element of A: include it in the subset or not. so we claim: there is a bijection between the subsets of A and the functions $f : A \to \{0, 1\}$. If we prove this then we know the number of subsets is $2^{|A|}$ by the previous slide.

Proof: Let B be a subset of A. We define $f_B : A \to \{0, 1\}$, where

$$f_B(a) = \begin{cases} 0 & \text{if } a \notin B, \\ 1 & \text{if } a \in B. \end{cases}$$

Each function $f : A \to \{0, 1\}$ arises in this way for a unique subset B. So there is a bijection from the subsets of A and the functions $f : A \to \{0, 1\}$.
$\square$

Pigeon holes

Pigeonhole Principle

[**The Pigeonhole Principle**] If $k+1$ or more pigeons occupy k pigeonholes, then at least one pigeonhole must contain two or more pigeons.

[**The Generalised Pigeonhole Principle**] If N objects are classified in k disjoint categories, then at least one category must contain at least $\lceil \frac{N}{k} \rceil$ objects. ($\lceil \frac{N}{k} \rceil$ means the least integer that is greater than or equal to $\frac{N}{k}$)

Example 1 (Pigeonhole Principle)

Prove the following: If there are 11 players in a soccer team that wins $12-0$, there must be at least one player in the team who scored more than once (assuming no own-goals).

Example 1 (Pigeonhole Principle)

Prove the following: If there are 11 players in a soccer team that wins $12 - 0$, there must be at least one player in the team who scored more than once (assuming no own-goals).

Proof: We shall apply the pigeonhole principle.

Each goal is assigned to a player. The goals scored are the 'pigeons' and the players are the pigeonholes. Since there are 12 goals (12 pigeons) and only 11 players (11 pigeonholes), at least one player must be assigned at least two goals (thinking about a scoresheet may also be helpful). $\square$

Example 2 (Generalised Pigeonhole principle)

Prove the following: If a molecule can exist in 2 different configurations, and you have 10^9 such molecules, at least half of them must be in the same configuration.

Example 2 (Generalised Pigeonhole principle)

Prove the following: If a molecule can exist in 2 different configurations, and you have 10^9 such molecules, at least half of them must be in the same configuration.

Proof: We shall apply the generalised pigeonhole principle. We have 10^9 objects (the molecules) to be classified into 2 disjoint categories (the configurations). By the generalised pigeonhole principle, at least one category contains $\lceil 10^9/2 \rceil$ objects (molecules). Note that $\lceil 10^9/2 \rceil \geq 10^9/2$; that is, $\lceil 10^9/2 \rceil$ is at least half of 10^9. Hence at least one configuration is taken by at least $10^9/2$ molecules. $\square$

Example 3 (Generalised Pigeonhole principle)

Show that, in any set of a thousand words, there must be at least 39 words that start with the same letter.

Example 3 (Generalised Pigeonhole principle)

Show that, in any set of a thousand words, there must be at least 39 words that start with the same letter.

Proof: We shall apply the generalised pigeonhole principle. Each word can be classified by its first letter, and such categories are disjoint. Since there are 1000 words and 26 categories, at least one category (configuration) contains at least $\lceil 1000/26 \rceil = \lceil 38.46 \rceil = 39$ words. $\square$

Example 4 (Epp(4ed) Q9.4.33) (Challenging pigeonhole principle)

Let A be a set of six distinct positive integers each of which is less than 15. Show that there must be two distinct nonempty proper subsets of A whose elements when added up give the same sum.

Example 4 (Epp(4ed) Q9.4.33) (Challenging pigeonhole principle)

Let A be a set of six distinct positive integers each of which is less than 15. Show that there must be two distinct nonempty proper subsets of A whose elements when added up give the same sum.

IDEAS: The phrase '*there must be two distinct subsets*' in the second sentence of the question suggests that the subsets should play the role of the pigeons.

Example 4 (Epp(4ed) Q9.4.33) (Challenging pigeonhole principle)

Let A be a set of six distinct positive integers each of which is less than 15. Show that there must be two distinct nonempty proper subsets of A whose elements when added up give the same sum.

IDEAS: The phrase '*there must be two distinct subsets*' in the second sentence of the question suggests that the subsets should play the role of the pigeons.

Then the phrase '*give the same sum*' suggests that the possible sums should play the role of the pigeon holes.

Example 4 (Epp(4ed) Q9.4.33) (Challenging pigeonhole principle)

Let A be a set of six distinct positive integers each of which is less than 15. Show that there must be two distinct nonempty proper subsets of A whose elements when added up give the same sum.

IDEAS: The phrase '*there must be two distinct subsets*' in the second sentence of the question suggests that the subsets should play the role of the pigeons.

Then the phrase '*give the same sum*' suggests that the possible sums should play the role of the pigeon holes. So

$$A = \{a, b, c, d, e, f \in \mathbb{N} \,:\, a < b < c < d < e < f < 15\}$$

$$\begin{aligned} \text{Pigeons:} &\quad \text{subsets of } A \\ \text{Pigeon holes:} &\quad \text{possible element sums of subsets of } A \end{aligned}$$

Example 4 (Epp(4ed) Q9.4.33) (Challenging pigeonhole principle)

Let A be a set of six distinct positive integers each of which is less than 15. Show that there must be two distinct nonempty proper subsets of A whose elements when added up give the same sum.

IDEAS: The phrase '*there must be two distinct subsets*' in the second sentence of the question suggests that the subsets should play the role of the pigeons.

Then the phrase '*give the same sum*' suggests that the possible sums should play the role of the pigeon holes. So

$$A = \{a, b, c, d, e, f \in \mathbb{N} \, : \, a < b < c < d < e < f < 15\}$$

$$\begin{array}{rl} \text{Pigeons:} & \text{subsets of } A \\ \text{Pigeon holes:} & \text{possible element sums of subsets of } A \end{array}$$

Now we have to count the pigeons and pigeon holes.

Example 5 (cont)

Proof: First we show that there are 62 distinct proper nonempty subsets of A (pigeons). Since $|A| = 6$, we have that $|\mathcal{P}(A)| = 2^6 = 64$. Since $\emptyset$ is empty, and A is not proper, there are 62 distinct proper nonempty subsets of A.

Now we show that there are only 60 different possible element sums among these nonempty proper subsets. A proper subset of A contains at most 5 elements. The elements of A are selected from $1, 2, \ldots, 14$. The most a sum of at most 5 elements from these integers can be is $10 + 11 + 12 + 13 + 14 = 60$. The least the sum can be is 1. Hence the element sum of a proper nonempty subset of A is between 1 and 60. (Thus there are 60 pigeonholes).

By the pigeonhole principle, at least one element sum is taken by two distinct nonempty proper subsets of A. $\square$

Example 5 (Another tough pigeon)

Given a set of 52 distinct integers, show that there must be two whose sum or difference is divisible by 100.

Example 5 (Another tough pigeon)

Given a set of 52 distinct integers, show that there must be two whose sum or difference is divisible by 100.

Proof: We label the 52 distinct integers $a_1, a_2, \ldots, a_{52}$. For each i such that $1 \leq i \leq 52$, let $A_i = a_i \bmod 100$. We consider two cases.

Case: There exist i, j such that $i < j$ and $A_i = A_j$.
In this case,

$$A_i = A_j \Leftrightarrow a_i \equiv a_j \pmod{100} \Leftrightarrow (a_i - a_j) \text{ is divisible by 100.}$$

In this case, there are two integers whose difference is divisible by 100 and we are done.

Example 5 (cont.)

Case: There does not exist i, j such that $i < j$ and $A_i = A_j$.
In this case, the integers $A_1, A_2, \ldots, A_{52}$ are distinct. We shall apply the pigeonhole principle, with the integers $A_1, A_2, \ldots, A_{52}$ playing the role of pigeons. We label 51 'pigeonholes' as shown, so that each is labelled by two not necessarily distinct two-digit numbers:

00	01	02	03		49	50
$\square$	$\square$	$\square$	$\square$	....	$\square$	$\square$
00	99	98	97		51	50

Example 5 (cont.)

Case: There does not exist i, j such that $i < j$ and $A_i = A_j$.
In this case, the integers $A_1, A_2, \ldots, A_{52}$ are distinct. We shall apply the pigeonhole principle, with the integers $A_1, A_2, \ldots, A_{52}$ playing the role of pigeons. We label 51 'pigeonholes' as shown, so that each is labelled by two not necessarily distinct two-digit numbers:

00	01	02	03		49	50
$\square$	$\square$	$\square$	$\square$	....	$\square$	$\square$
00	99	98	97		51	50

Observe that, for each box, the pair of integers labelling the box have a sum that is divisible by 100.

Example 5 (cont.)

Case: There does not exist i, j such that $i < j$ and $A_i = A_j$.
In this case, the integers $A_1, A_2, \ldots, A_{52}$ are distinct. We shall apply the pigeonhole principle, with the integers $A_1, A_2, \ldots, A_{52}$ playing the role of pigeons. We label 51 'pigeonholes' as shown, so that each is labelled by two not necessarily distinct two-digit numbers:

00	01	02	03		49	50
$\square$	$\square$	$\square$	$\square$	....	$\square$	$\square$
00	99	98	97		51	50

Observe that, for each box, the pair of integers labelling the box have a sum that is divisible by 100.

There are now exactly 51 'pigeonholes', and 52 numbers in the list $A_1, \ldots, A_{52}$. So at least two of the numbers in the list $A_1, \ldots, A_{52}$ label the same pigeonhole. This means that at least two of the numbers in the list $A_1, \ldots, A_{52}$ have a sum that is divisible by 100. $\square$

Ordered Selection

Permutations

[**Permutations**] There are $n!$ ways to arrange n distinct objects in a list.

[r-**Permutations**] There are

$$P(n, r) = \frac{n!}{(n-r)!}$$

ways to select and order r out of n distinct objects.

Theorem: If A and B are sets with $|A| = r$ and $|B| = n$, then the number of injective functions $f : A \to B$ is $P(n, r) = \frac{n!}{(n-r)!}$.

Example 7 (Permutations)

Andy, Beth and Cai are standing in line.
In how many different orders could this queue be arranged?

Example 7 (Permutations)

Andy, Beth and Cai are standing in line.
In how many different orders could this queue be arranged?

Proof: Arranging Andy, Beth and Cai in a line is like arranging 3 distinct objects in a list. Thus there are $3! = 3 \times 2 \times 1 = 6$ ways to do this. $\square$

Checking our answer (unnecessary, but instructive): We can represent an "outcome" by listing three letters in order: the first letter represents the first person in the line; the second letter represents the second person in the line; and the third letter represents the third person in the line. The entire set of outcomes, containing 6 elements, is represented below:

$$\{ABC, ACB, BAC, BCA, CAB, CBA\}$$

Example 8 (r-Permutations)

A pet show awards 1st, 2nd and 3rd prizes.

Example 8 (r-Permutations)

A pet show awards 1st, 2nd and 3rd prizes. There are 5 entrants:

Rachel the Rabbit	Charles the Chicken	Tilly the Terrier
Bob the Bilby	Karen the Kangaroo	

Example 8 (r-Permutations)

A pet show awards 1st, 2nd and 3rd prizes. There are 5 entrants:

Rachel the Rabbit	Charles the Chicken	Tilly the Terrier
Bob the Bilby	Karen the Kangaroo	

In how many ways can the prizes be handed out?

Example 8 (r-Permutations)

A pet show awards 1st, 2nd and 3rd prizes. There are 5 entrants:

Rachel the Rabbit	Charles the Chicken	Tilly the Terrier
Bob the Bilby	Karen the Kangaroo	

In how many ways can the prizes be handed out?

Answer: Handing out the prizes is like selecting and ordering 3 out of 5 distinct objects. Thus there are

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 5 \times 4 \times 3 = 60$$

ways to hand out the prizes. $\square$

Checking our answer to Example 8

We can represent an "outcome" by listing three letters in order: the first letter represents the pet awarded first prize; the second letter represents the pet awarded second prize; and the third letter represents the pet awarded third prize. The entire set of outcomes, containing 60 elements, is represented below:

{ BCK,	BCR,	BCT,	BKC,	BKR,	BKT,
BRC,	BRK,	BRT,	BTB,	BTC,	BTR,
CBK,	CBR,	CBT,	CKB,	CKR,	CKT,
CRB,	CRK,	CRT,	CTB,	CTK,	CTR,
KBC,	KBR,	KBT,	KCB,	KCR,	KCT,
KRB,	KRC,	KRT,	KTB,	KTC,	KTR,
RBC,	RBK,	RBT,	RCB,	RCK,	RCT,
RKB,	RKC,	RKT,	RTB,	RTC,	RTK,
TBC,	TBK,	TBR,	TCB,	TCK,	TCR,
TKB,	TKC,	TKR,	TRB,	TRC,	TRK, }

Example 9: Permutations with overcounting

How many distinguishable ways can the letters of the word

$$\text{MILLIMICRON}$$

be arranged?

Example 9: Permutations with overcounting

How many distinguishable ways can the letters of the word

$$\mathbf{MILLIMICRON}$$

be arranged?

If we were to distinguish between like letters using labels, as in

$$M_1 I_1 L_1 L_2 I_2 M_2 I_3 C\, R\, O\, N$$

Example 9: Permutations with overcounting

How many distinguishable ways can the letters of the word

$$\mathbf{MILLIMICRON}$$

be arranged?

If we were to distinguish between like letters using labels, as in

$$M_1 I_1 L_1 L_2 I_2 M_2 I_3 C\,R\,O\,N$$

there would be $\quad 11! = 39\,916\,800 \quad$ different arrangements.

Example 9: Permutations with overcounting

How many distinguishable ways can the letters of the word

$$\mathbf{MILLIMICRON}$$

be arranged?

If we were to distinguish between like letters using labels, as in

$$M_1 I_1 L_1 L_2 I_2 M_2 I_3 C\, R\, O\, N$$

there would be $11! = 39\,916\,800$ different arrangements.

Now we must compensate for the over-counting induced by this distinguishing between indistinguishable arrangements.

Example 9: Permutations with overcounting

How many distinguishable ways can the letters of the word

$$\mathbf{MILLIMICRON}$$

be arranged?

If we were to distinguish between like letters using labels, as in

$$M_1 I_1 L_1 L_2 I_2 M_2 I_3 C\,R\,O\,N$$

there would be $\quad 11! = 39\,916\,800 \quad$ different arrangements.

Now we must compensate for the over-counting induced by this distinguishing between indistinguishable arrangements.

Since MILLIMICRON has 2 M's, 3 I's and 2 L's the true answer is :

Example 9: Permutations with overcounting

How many distinguishable ways can the letters of the word

$$\textbf{MILLIMICRON}$$

be arranged?

If we were to distinguish between like letters using labels, as in

$$M_1 I_1 L_1 L_2 I_2 M_2 I_3 C\,R\,O\,N$$

there would be $\quad 11! = 39\,916\,800 \quad$ different arrangements.

Now we must compensate for the over-counting induced by this distinguishing between indistinguishable arrangements.

Since MILLIMICRON has 2 M's, 3 I's and 2 L's the true answer is :

$$\frac{11!}{2!\,3!\,2!} = 1\,663\,200.$$

Unordered Selection

Combinations

[**Combinations**] There are

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

ways to choose a set of r objects from a set of n objects (that is, to select r out of n distinct objects when the order in which objects are selected in not important). The notation $\binom{n}{r}$ is read "n choose r."

Combinations

[**Combinations**] There are

$$C(n, r) = \binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

ways to choose a set of r objects from a set of n objects (that is, to select r out of n distinct objects when the order in which objects are selected in not important). The notation $\binom{n}{r}$ is read "n choose r."

In other words $\binom{n}{r}$ counts the number of k-element subsets of an n-element set.

Example 10 (Combinations)

A different pet show does not give 1st, 2nd and 3rd prizes. Instead, three "ribbon-winning" pets are chosen to travel to the district show. There are 5 entrants:

Rachel the Rabbit	Charles the Chicken	Tilly the Terrier
Bob the Bilby	Karen the Kangaroo	

In how many ways can the ribbon-winners be chosen?

Example 10 (Combinations)

A different pet show does not give 1st, 2nd and 3rd prizes. Instead, three "ribbon-winning" pets are chosen to travel to the district show. There are 5 entrants:

Rachel the Rabbit	Charles the Chicken	Tilly the Terrier
Bob the Bilby	Karen the Kangaroo	

In how many ways can the ribbon-winners be chosen?

Answer: Selecting the ribbon-winning pets is like choosing a set of 3 objects (ribbon-winning pets) from a set of 5 objects (the pets in the show). Thus there are

$$C(5,3) = \binom{5}{3} = \frac{P(5,3)}{3!} = \frac{5!}{3!(5-3)!} = \frac{120}{6 \times 2} = 10$$

ways to choose the ribbon-winners. □

Checking our answer to Example 10

Each "outcome" may be represented by a set containing the ribbon-winning pets. The ten sets representing the ten different outcomes are listed below:

{Rachel, Charles, Tilly},	{Rachel, Charles, Bob},
{Rachel, Charles,Karen},	{Rachel, Tilly, Bob},
{Rachel, Tilly, Karen},	{Rachel, Bob, Karen},
{Charles, Tilly, Bob},	{Charles, Tilly, Karen},
{Charles, Bob, Karen},	{Tilly, Bob, Karen}