Discrete Mathematical Models

Lecture 11

Kane Townsend

Semester 2, 2024

Section B: Digital Information (cont.)

Section B2: Sequences, Induction, Sorting

Sequences

Text Reference (Epp)

3ed: Sections	4.1-4, 8.1-3 (Sequences and induction), 9.3,5 (Sorting)
4ed: Sections	5.1-4,6-8, (Sequences and induction), 11.3,5 (Sorting)
5ed: Sections	5.1-4,6-7, (Sequences and induction), 11.3,5 (Sorting)

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In practice we sometimes leave out the parentheses and speak of "the sequence a_1, a_2, a_3" or "the sequence $a_0, a_1, a_2, \ldots$."

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3. U set of users. $(u_n)_{n \in \{1,2,3,4,5\}} \subseteq U$: a list of 5 users.

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Proofs about sequences

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Continuing to argue in this manner gives $P(n)$ for all $n \in \mathbb{N}$.

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and so the formula is also correct for $n+1$.

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$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \ldots + a_k, \qquad \prod_{n=1}^{k} a_n = a_1 \times a_2 \times a_3 \times \ldots \times a_k.$$

Examples: (1) $\sum_{n=1}^{10} n = 1 + 2 + 3 + 4 + \ldots + 9 + 10 = 55.$

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(3) $\prod_{n=1}^{5} n = 1 \times 2 \times 3 \times 4 \times 5 = 5! = 120.$

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(3) $\prod\limits_{n=1}^{5} n = 1\times 2\times 3\times 4\times 5 = 5! = 120.$

(4) $\prod\limits_{n=1}^{8} n^2 = 4\times 9\times 16\times\ldots\times 64 = 1\,625\,702\,400.$

Series

Given a sequence $(a_n)_{n \in I} \subseteq \mathbb{Q}$, we define a **series** as $\sum_{n \in I} a_n$.

Given a sequence we can also consider **partial sums** of the series, which is just a series associated to some 'subsequence'.

We will now consider a few important examples of sequences and series:

- Geometric Sequence and Series
- Arithmetic Sequence and Series
- Mixed Geometric-Arithmetic Sequence

how we can calculate their terms and their associated series.

Geometric Sequences and Series

Slide 6 is a special case of a

Geometric Sequence	
Implicit Definition	Explicit Definition
$a_k = a$ (a is the **first term**) $a_{n+1} = ra_n$, $\forall n \geq k$ (r is the **common ratio**)	$\forall n \geq k$ $a_n = ar^{n-k}$

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When terms or a sequence are summed, we get a **series**. In particular

Geometric Series	
Series of N terms	**Sum of N terms**
$\sum_{n=k}^{k+(N-1)} ar^{n-k} = a + ar + \cdots + ar^{N-1}$ [Usually $k=0$ or $k=1$.]	$\begin{cases} \dfrac{a(1-r^N)}{(1-r)} & \text{if } r \neq 1 \\ Na & \text{if } r = 1 \end{cases}$

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Example: $k = 0, \ a = 6, \ r = \frac{1}{2}, \ N = 5$:

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A sequence is defined implicitly by $$\begin{cases} b_{n+1} = b_n + 5 \quad \forall n \in \mathbb{N}, \\ b_1 = 0. \end{cases}$$

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and so the formula is also correct for $n+1$.

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Example: $1+3+5+7+\cdots+99 = 50\left(\frac{1+99}{2}\right) = 2500.$

In this example $a=1, d=2, k=0, N=50$, check yourself!

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   So $N = 8$ *i.e.* eight years.

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Question: What will the capital be in 10 years?

Answer: $\forall n \in \mathbb{N}^\star \; c_n = 2 \times 10^4 \times 1.03^n$.

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Question: What will the capital be in 10 years?

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$$d_{120} = 2\times 10^4 \times 1.0025^{120} = \$26\,987.71.$$ **Slightly better!**

More applications of sequences

Better models

Compound Interest: If the bank is charging a fee of \$10 per year, the compound interest model becomes

$$\begin{cases} c_{n+1} = 1.03c_n - 10 \quad \forall n \in \mathbb{N}^\star, \\ c_0 = 2 \times 10^4. \end{cases}$$

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The sequences defined by these models are neither geometric nor arithmetic, but are a generalisation of both.
We need to start again from scratch.

From implicit to explicit definitions; Example 3

We seek an explicit formula for the population given by the implicit formula at right, where d and p are shorthand for 10^3 and 5×10^6.

$$\begin{cases} p_{n+1} = 2p_n + d \ \forall n \in \mathbb{N}^\star, \\ p_0 = p. \end{cases}$$

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and so the formula is also correct for $n+1$.