Discrete Mathematical Models

Lecture 23

Kane Townsend

Semester 2, 2024

Special simple graphs

Cycle Graphs

A **cycle graph** on $n \geq 3$ vertices is denoted by C_n.

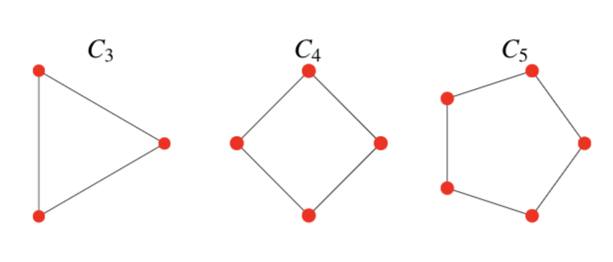

Figure 1: Wolfram Mathworld: Cycle Graph

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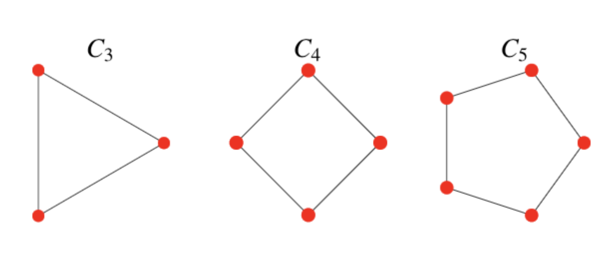

Figure 1: Wolfram Mathworld: Cycle Graph

Note that $|V(C_n)| = n = |E(C_n)|$ and that every vertex is adjacent to two other vertices.

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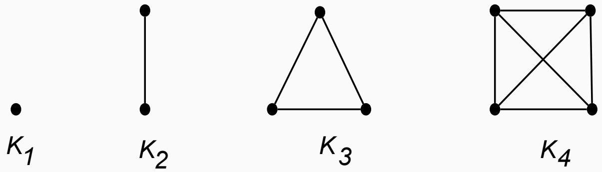
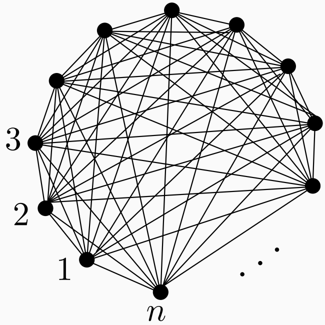


How many edges in K_n?

How many edges are there in K_n, the complete graph of order n?



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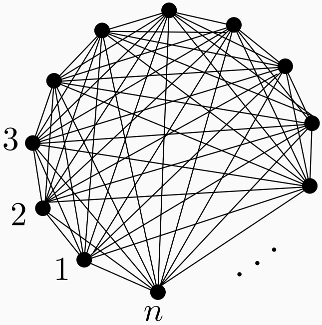

The answer is $\binom{n}{2} = \frac{n(n-1)}{2}$. *Why?*

Bipartite Graphs

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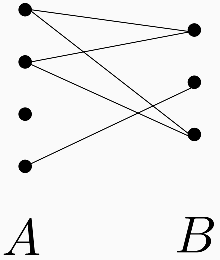

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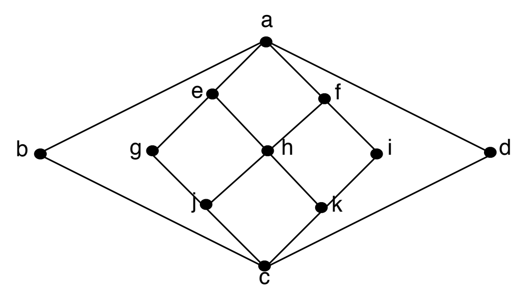

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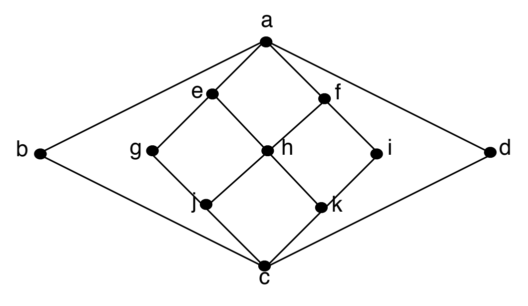
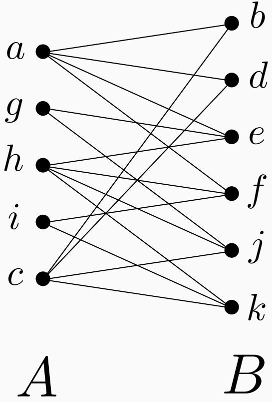


The vertex partitioning is:

$$A = \{a, g, h, i, c\} \quad B = \{b, d, e, f, j, k\}$$

Larger example continued

This is the same graph, redrawn.



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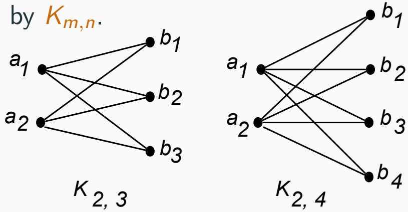

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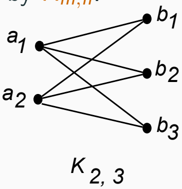
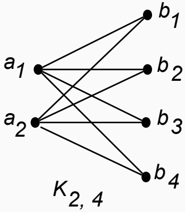
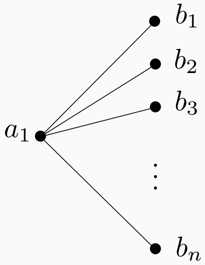




Caution: The name "Complete Bipartite Graph" is misleading. Except for $K_{1,1}$, such graphs are **not complete graphs**.
The adjective 'complete' is with respect to 'bipartite', not 'graph'.

How many edges in $K_{1,n}$?

- *How many edges in the complete bipartite graph $K_{1,n}$?*



How many edges in $K_{m,n}$?

- *How many edges in* $K_{1,n}$*?*
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$\vdots$

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Subgraphs

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- A **subgraph**, S, of a graph G, is a graph whose vertices are a subset of $V(G)$ and whose edges are a subset of $E(G)$, i.e.

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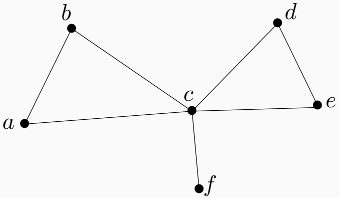
- Since S is a graph, if the edge

$$\{a, b\} \in E(S),$$

we require its endpoints to be in $V(S)$, i.e. $a, b \in V(S)$.

A subgraph example:

Let G be the graph: 

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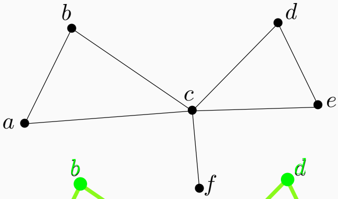
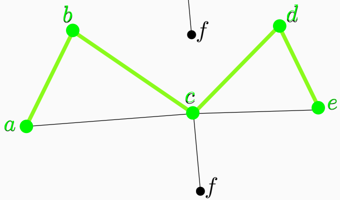


Select some edges and vertices:



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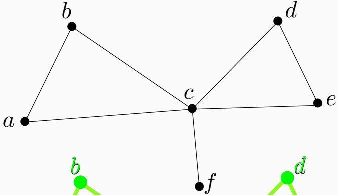

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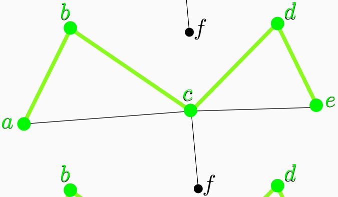
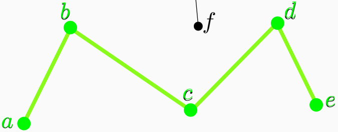


Now S is a subgraph of G:



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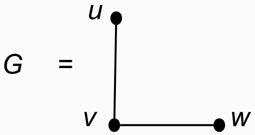

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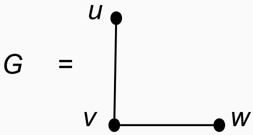

The following five graphs (numbered 1-5) are each subgraphs of G:

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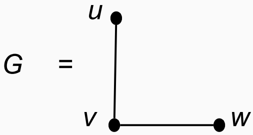

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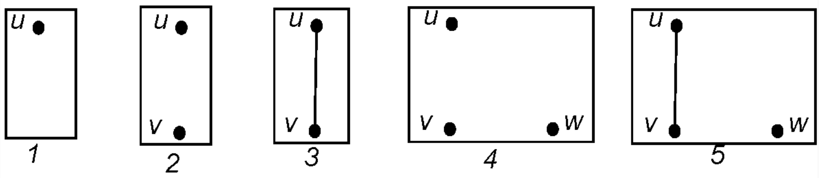

Degree

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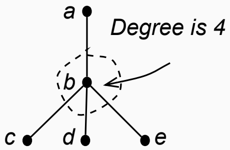

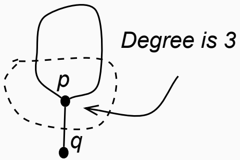

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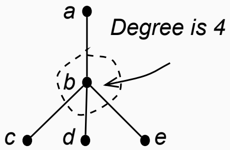

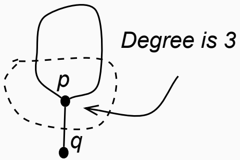

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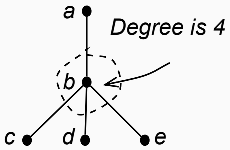

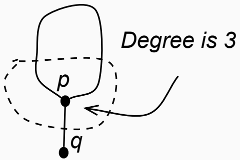

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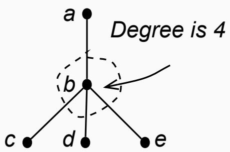

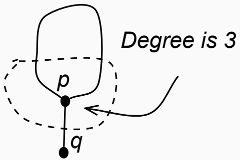

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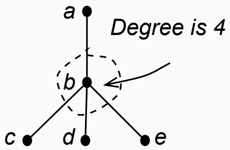

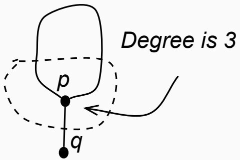

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We shall always use the 'adding two' version.

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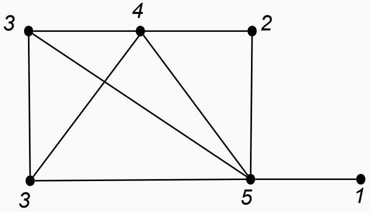

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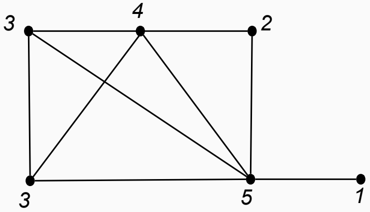

The total degree of the graph is $3 + 4 + 2 + 3 + 5 + 1 = 18$.

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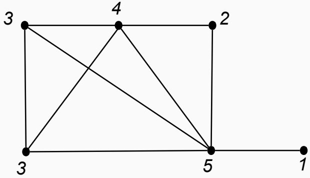

Total degree = 18
Number of edges = 9.

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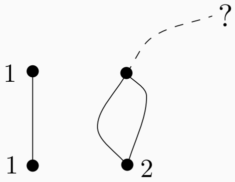
How does this corollary follow from the handshake theorem?

Example:

- The set $\{1, 1, 2, 3\}$ cannot possibly be the set of degrees of the vertices of some graph.
- However we try, we always end up with an edge that doesn't have a vertex to connect to:



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Note that

- for graphs, ab means *the same* as ba, since $\{a, b\} = \{b, a\}$; but
- for digraphs, ab is *different* from ba since $(a, b) \neq (b, a)$.

Walks, Paths and Circuits

The bridges of Königsberg

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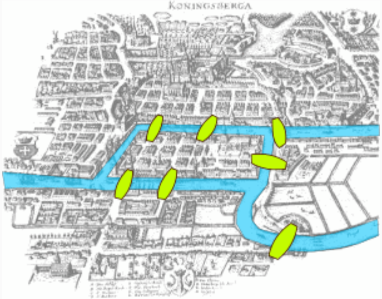

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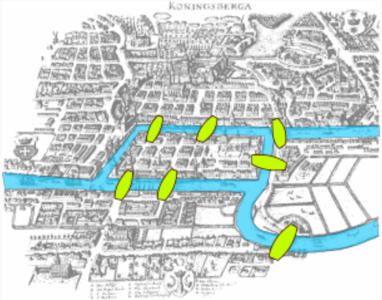
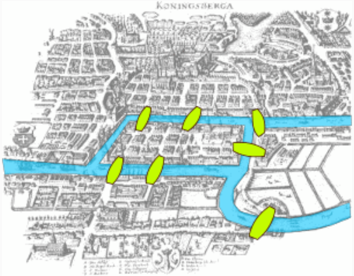


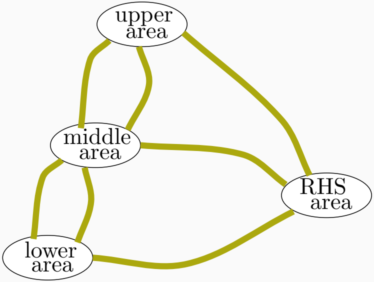
Adapted from:

http://en.wikipedia.org/wiki/Bridges_of_Konigsberg

The bridges of Königsberg

Leonard Euler realized that the task can be modeled as a problem in graph theory, which he invented for the purpose.





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Walks

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- A walk starting at v_0 and ending at v_n is said to **connect** v_0 to v_n.

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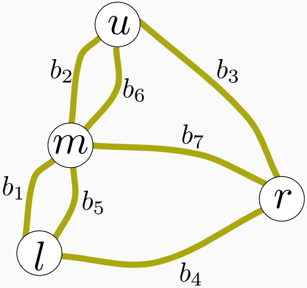
- A **trivial walk**, say v_0, contains no edges; hence has length 0.

Walks on the Königsberg Graph

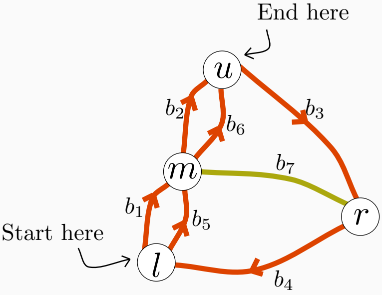
The question becomes, 'Is there a walk on the Königsberg Graph which traverses each edge exactly once?'



Walks on the Königsberg Graph

Try starting in the lower part of town, going via bridge b_1 to the middle island, then via bridge b_2 to the upper part, then via bridge b_3 to the right-most part; continuing as in the listed walk:

$$l b_1 m b_2 u b_3 r b_4 l b_5 m b_6 u$$



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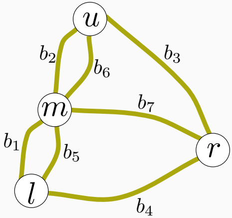

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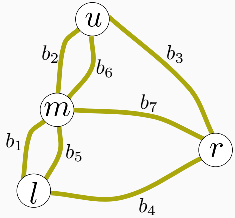

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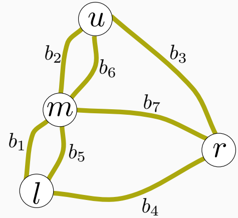

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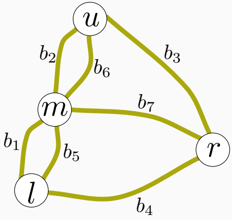

Impossibility of Königsberg Bridge Walk

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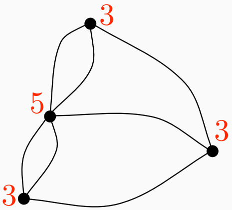
- *How? Think about the degrees of the vertices:*



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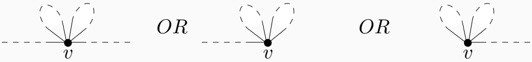


Even degree vertex in a walk

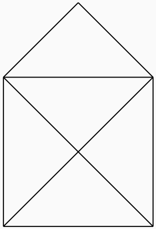
Odd degree vertex in a walk

Odd degree vertex in a walk

A related puzzle

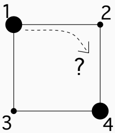
You may have come across the following puzzle:

Can you draw the following 'house' diagram without taking your pen off the paper or overwriting edges?



Can you?

Counting Walks with Adjacency Matrices

The graph 

has adjacency matrix $M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}.$

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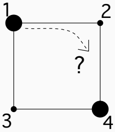
The graph 

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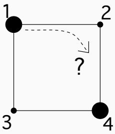
The graph  has adjacency matrix

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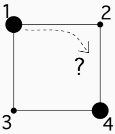

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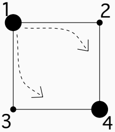

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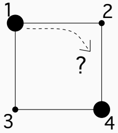
The graph 

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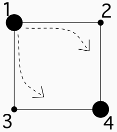

$$M^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

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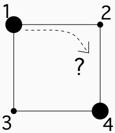
The graph 

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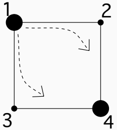

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For any graph, the number of ways to walk from vertex i to vertex j in n steps is given in terms of its adjacency matrix M by the $(i, j)^{\text{th}}$ entry of M^n.

Some special kinds of walks

Some properties that a walk on a graph may, or may not, possess are:

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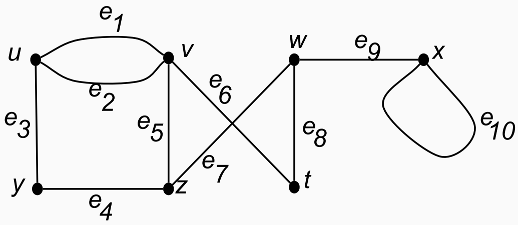
We will now look at each of these potential properties in turn, with examples using the graph G below.



Closed walks

A walk $v_0, e_1, v_1, e_2, \ldots, e_n, v_n$ is called a **closed** when $v_0 = v_n$.

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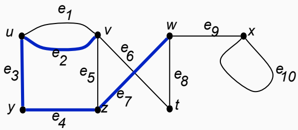

The walk

$$v \; e_2 \; u \; e_3 \; y \; e_4 \; z \; e_7 \; w$$

has length 4 and is **not** closed because $v = v_0 \neq v_n = v_4 = w$.

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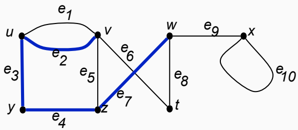

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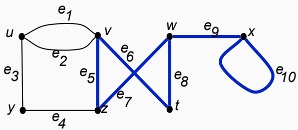

The walk

$$v\ e_6\ t\ e_8\ w\ e_9\ x\ e_{10}\ x\ e_9\ w\ e_7\ z\ e_5\ v$$

has length 7 and **is** closed because $v = v_0 = v_n = v_7 = v$.

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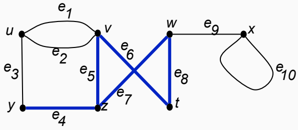

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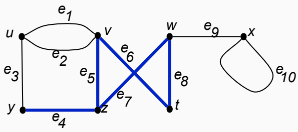

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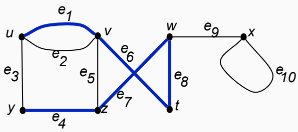

The walk

$$y\ e_4\ z\ e_7\ w\ e_8\ t\ e_6\ v\ e_1\ u$$

has length 5 and is a **simple path** because the six vertices are all different.

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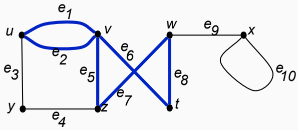

The walk

$$z\,e_7\,w\,e_8\,t\,e_6\,v\,e_1\,u\,e_2\,v\,e_5\,z$$

has length 6 and is a **circuit** as it is closed with all different edges, but is **not simple** because $v = v_3 = v_5$.

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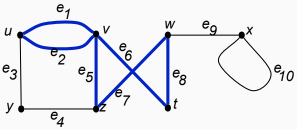

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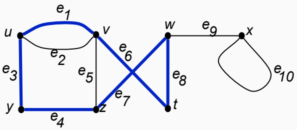

The walk

$$z\, e_7\, w\, e_8\, t\, e_6\, v\, e_1\, u\, e_3\, y\, e_4\, z$$

has length 6 and is a **simple circuit** as it is closed without repeated vertices except the first and last $z = v_0 = v_6$.

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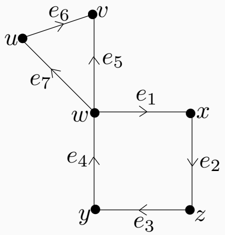

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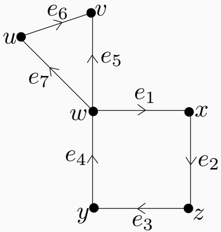

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- Note that for a simple graph or digraph (with no loops and no parallel edges), **any** walk is uniquely determined by its sequence of vertices.

Connectivity

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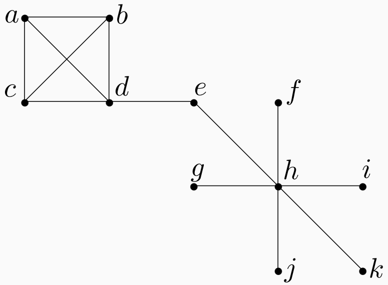
Here is an example of a connected graph:



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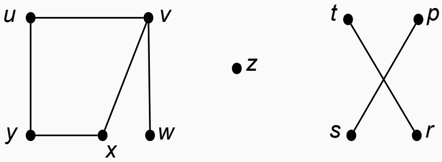

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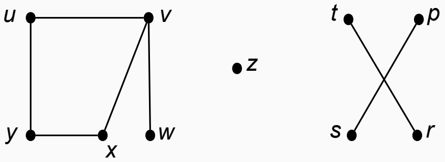

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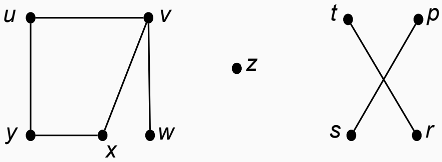
1. the subgraph with vertex set $\{u, v, w, x, y\}$ and edge set $\{\{u, v\}, \{u, y\}, \{v, w\}, \{v, x\}, \{x, y\}\}$,

Example



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1. the subgraph with vertex set $\{u, v, w, x, y\}$ and edge set $\{\{u, v\}, \{u, y\}, \{v, w\}, \{v, x\}, \{x, y\}\}$,
2. the subgraph with vertex z and no edges,

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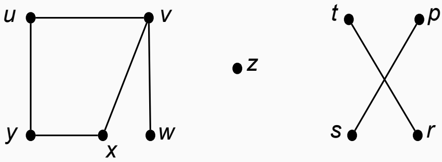

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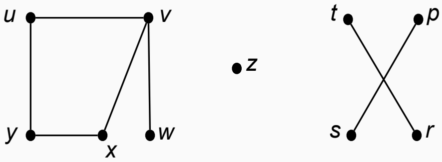

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Bridges and Cut Vertices

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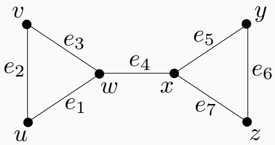
$G:$ 

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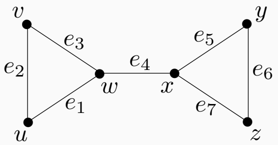

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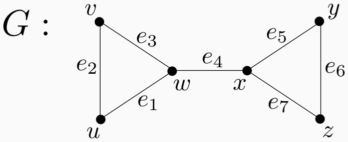

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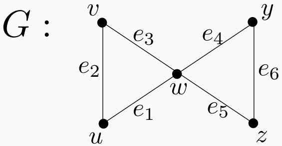

  - w is a cut vertex in G
  - v is NOT a cut vertex in G