Discrete Mathematical Models

Lecture 24

Kane Townsend

Semester 2, 2024

Euler paths and circuits

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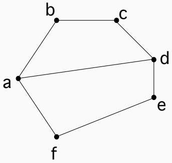

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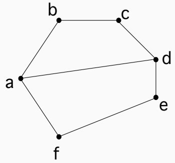

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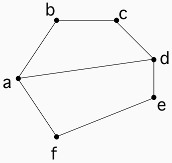

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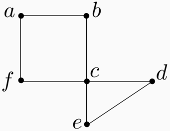

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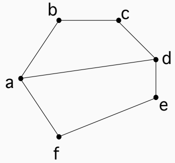

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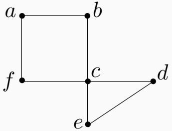

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  The algorithm easily adapts to this case.

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- Always leave an edge to return to the start vertex as the last step.

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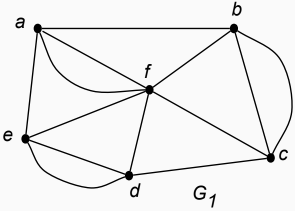
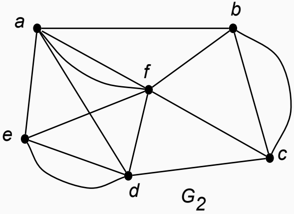
4. Travel that edge, coming to the next vertex.
5. Repeat steps 2 - 4 until all edges have been traversed, and you are back to the starting vertex.

Candidate Graphs for Fleury's Algorithm

The graph G_1 below satisfies the criterion that all vertices have even degree, so it contains an Euler circuit and Fleury's algorithm can be used to find that circuit.





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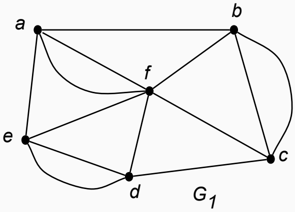
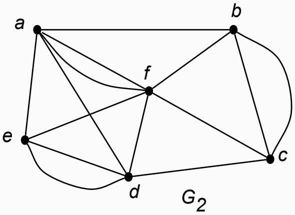
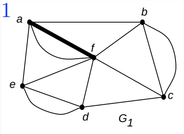


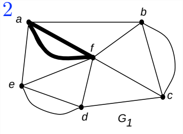


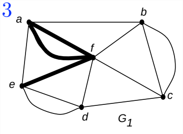
The graph G_2 has two vertices of odd degree. Fleury's algorithm can be modified to find an Euler path in this graph. The only modification needed is that the first vertex must be one of the vertices of odd degree.

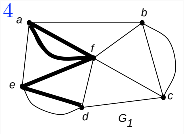
Fleury's Algorithm example in pictures

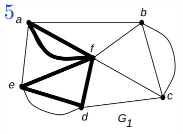
Start at f (just because we feel like it!)

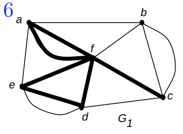


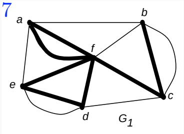








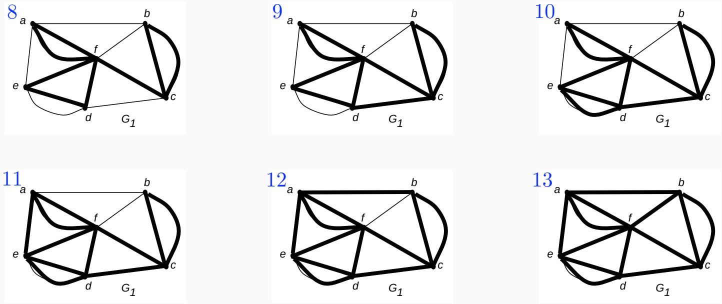




Notice that from here
we MAY NOT step to f
but either a or c is allowed

Fleury's Algorithm example in pictures

At step 9 we're forced to go to *d*; and then all steps are forced.



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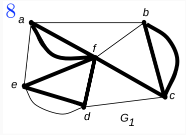

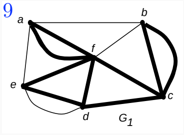

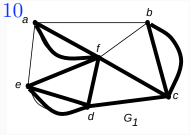

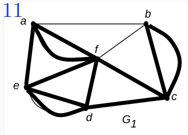

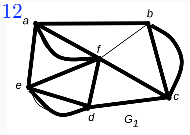

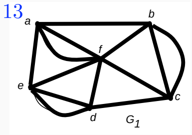

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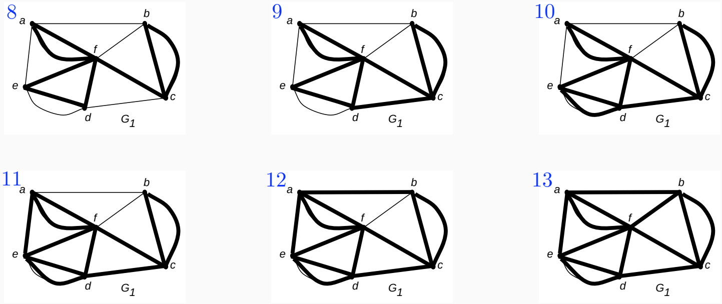

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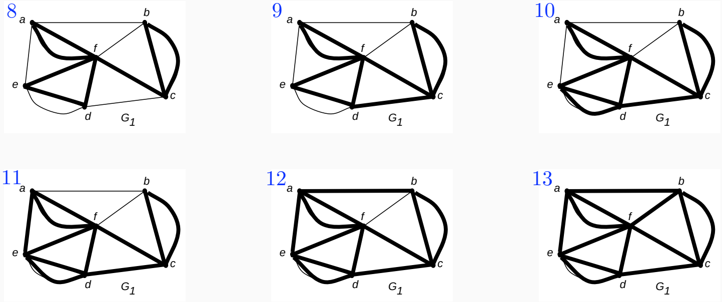

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(As mentioned earlier, we do not call this path an Euler *path*, because, by convention, Euler paths are not closed.)

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- In any implementation, we never have to back-track, so the algorithm is quite fast; as are some other algorithms to solve this problem.
- By contrast, the following problem – that of finding a *Hamilton* path or circuit – has no known 'fast' algorithm.

Applications of Euler Paths and Circuits: Optimal logic gate ordering, reconstructing DNA sequencing, optimising delivery services.

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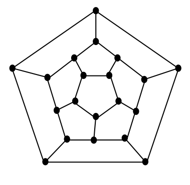
Note: By convention, a Hamilton path must be open, *i.e* not a circuit.

- This graph has a Hamilton Circuit. *Can you find one?*



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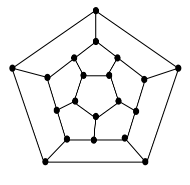
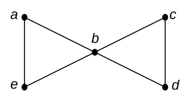


- This graph has no Hamilton circuit. *Why not?*



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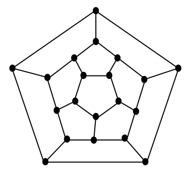

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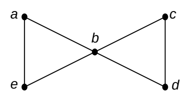

Applications of Hamilton Circuit and Paths: Updating servers and you only want one server down at a time, scheduling problems, route optimisation.

Types of Walks on Graphs – Summary

For a walk $v_0,\ e_1,\ v_1,\ e_2,\ \ldots,\ e_n,\ v_n$ on a graph G:

	Properties				
Name:	**closed**	—	**Euler**	**simple**	**Hamilton**
Description:	—	no repeated edges	uses all edges	no repeated vertices	uses all vertices
Requirement:	$v_0 = v_n$	$i \neq j \implies e_i \neq e_j$	$\forall e \in E(G)\ \exists i\ e_i = e$	$i \neq j \implies v_i \neq v_j$ ($v_0 = v_n$ OK)	$\forall v \in V(G)\ \exists i\ v_i = v$
path		✓			
simple path		✓		✓	
Euler path		✓	✓		
Hamilton path		✓		✓	✓
closed walk	✓				
circuit	✓	✓			
simple circuit	✓	✓		✓	
Euler circuit	✓	✓	✓		
Hamilton circuit	✓	✓		✓	✓

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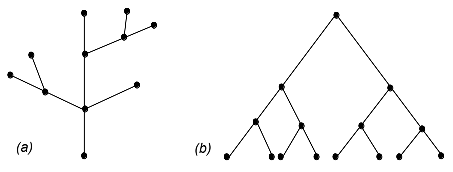

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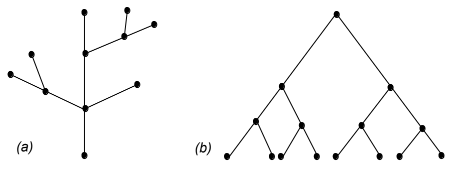
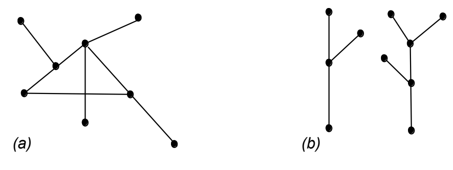


- Examples of **non-trees**



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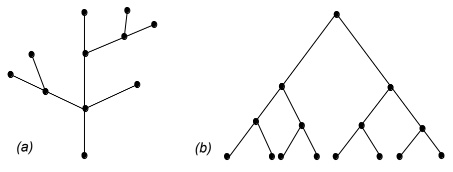

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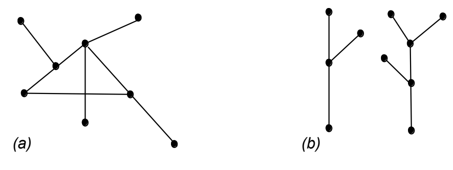

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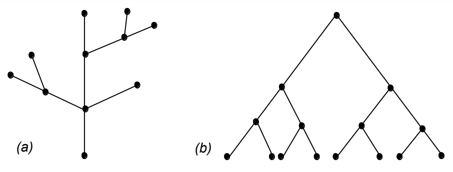

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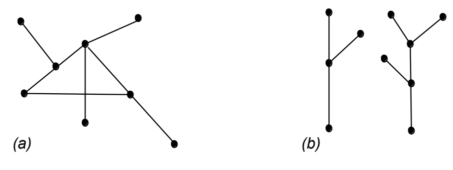

  - Example (a) contains a circuit, so is not a tree.
  - Example (b) is not a tree. *Why not?*

Forests and leaves

- A **forest** is a disjoint collection of trees.

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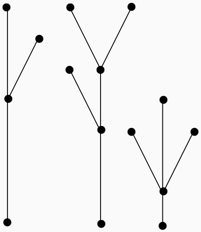

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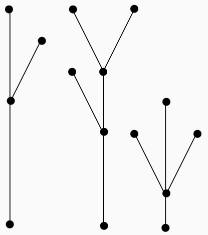

A **leaf** of a tree or forest is a vertex v with $\deg(v) = 1$.

Trees as Models

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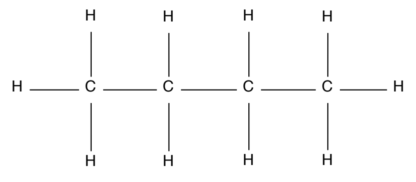
- In the representation of such molecules we use C to represent a carbon atom and H to represent a hydrogen atom. These will be used instead of dots for the vertices.

Examples of Hydrocarbon Graphs

- Butane:



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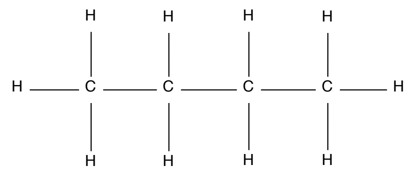
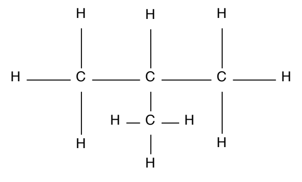


- Isobutane:



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- **Saturated hydrocarbon** molecules contain the maximum number of hydrogen atoms for a given number of carbon atoms.

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- You will explore a proof of this formula in a Workshop in the coming weeks.

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(vi) T contains no non-trivial circuits, but the addition of any new edge (connecting an existing pair of vertices) creates a simple circuit.

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- To prove the Theorem, we should show that any statement in the list is derivable from any other statement. One way to do this is to show the chain of implications:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i).$$

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  Lemma A: Any tree that has more than one vertex has at least one vertex of degree 1.

  Lemma B: If G is any connected graph, C is a non-trivial circuit in G, and one of the edges of C is removed, then the subgraph that remains is connected.

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- Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks, and so on, can be solved using models that involve graphs.
- A subgraph of a connected graph that provides a unique path between any two vertices is a *spanning tree*. These trees have applications in many fields, including engineering.

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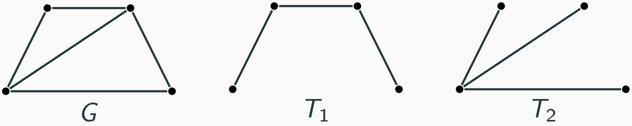

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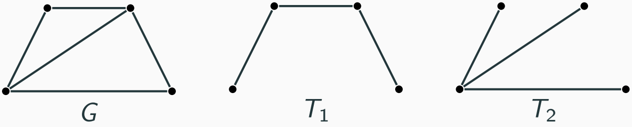

How many more spanning trees can you find for G?

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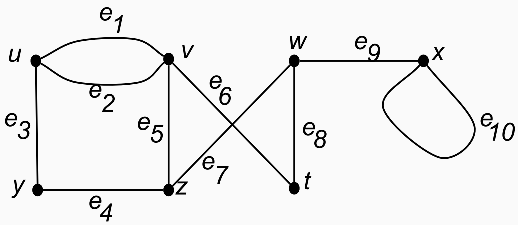
For computer implementation it is necessary to **also**:
— at step 1 initialize a 'pool of potential edges' P to $E(G)$,
— at step 2 ensure the picked edge e comes from P and
— after step 3 remove e from P (whether it contributes to T or not).

Building a spanning tree: example

To demonstrate the spanning tree algorithm we will use a graph we have seen before:



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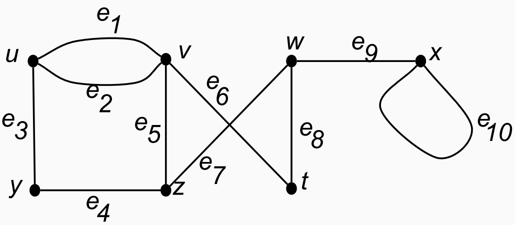

Initialize T to

- vertex set $V(T) = \{u, v, w, x, y, z, t\}$,
- edge set $E(T) = \{\}$:

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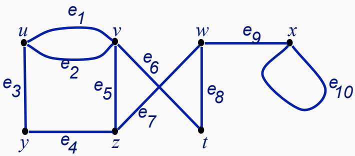

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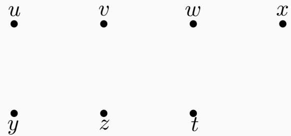

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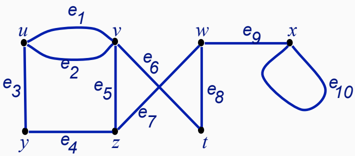

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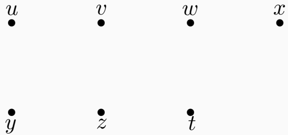

$E(T) = \{e_1\}$

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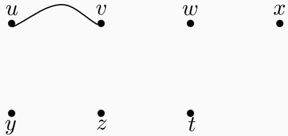

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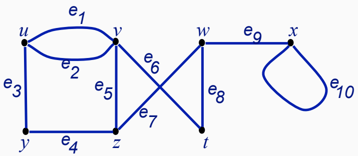

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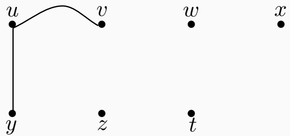

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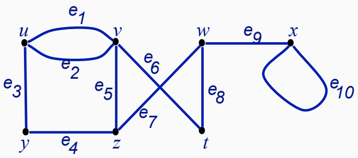

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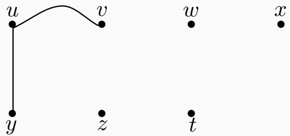

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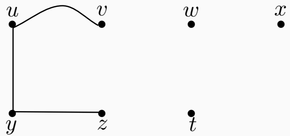

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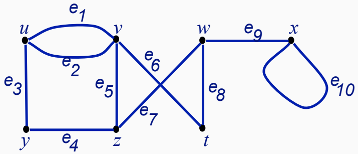

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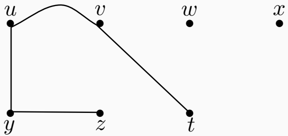

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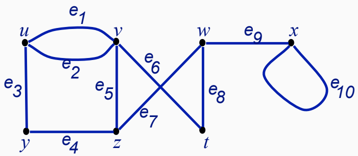

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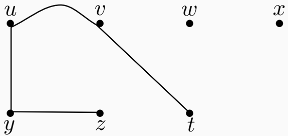

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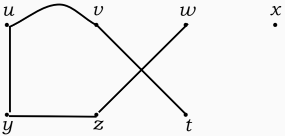

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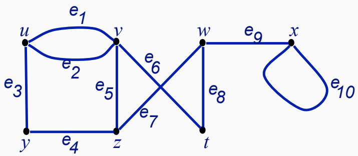

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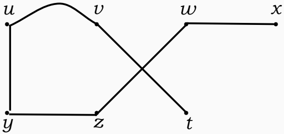

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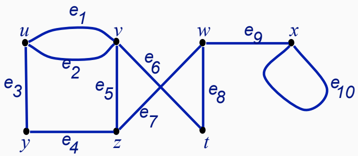

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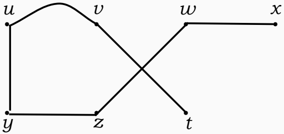

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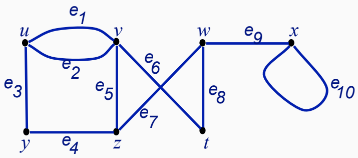

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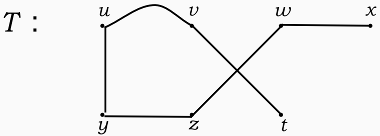

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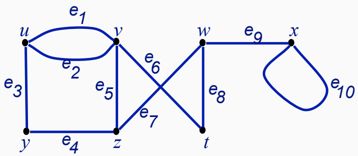

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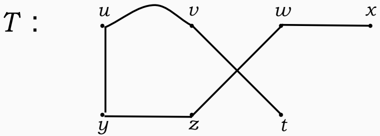

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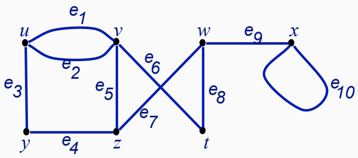
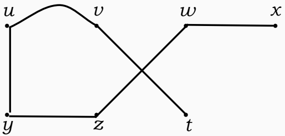
Can you make a spanning tree in which the longest path has length 4?

Completed spanning tree for



$E(T) = \{e_1, e_3, e_4, e_6, e_7, e_9\}$:

T :

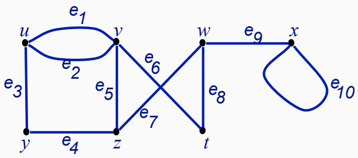


0ur tree now has $7 - 1 = 6$ edges, so we are done (as you can see).

As it happens, this tree has no branching and so contains a path of length 6. That just results from our order of choosing edges.

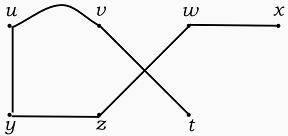
Can you make a spanning tree in which the longest path has length 4? Length 3?

Completed spanning tree for



$E(T) = \{e_1, e_3, e_4, e_6, e_7, e_9\}$:

T :



0ur tree now has $7 - 1 = 6$ edges, so we are done (as you can see).

As it happens, this tree has no branching and so contains a path of length 6. That just results from our order of choosing edges.

Can you make a spanning tree in which the longest path has length 4? Length 3?

END OF SECTION D1