Discrete Mathematical Models

Lecture 28

Kane Townsend

Semester 2, 2024

Cut capacities and flow values (PROPER)

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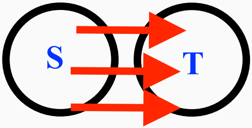

We will prove that cut capacity is at least the flow value, max flow value eq min cut capacity and the complete labelling algorithm finds max flow while giving us a minimal cut.

Cut capacity is at least the flow value

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Let F be a flow and $K = E(D) \cap (S \times T)$ be a cut in D. The cut capacity of C is at least the flow value of F. That is, $C(K) \geq v(F)$.

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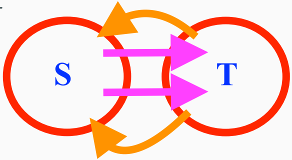
Proof.

$$v(F) = \sum_{i \in V(D)} F(s,i) = \sum_{j \in S} \sum_{i \in T} F(j,i) - \sum_{j \in S} \sum_{i \in T} F(i,j)$$
$$\leq \sum_{j \in S} \sum_{i \in T} F(j,i) \leq \sum_{j \in S} \sum_{i \in T} C(j,i) = C(K).$$



□

Explanations:

i. By definition of flow value.
ii. By conservation of flow and $S \cup T = V(D)$.
iii. By removing a negative term.
iv. Since $F(j,i) \leq C(j,i)$ for all $(j,i) \in E(D)$.
v. Definition of capacity of K.

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Let F be a flow and $K = V(D) \cap (S \times T)$ a cut of D. If equality holds in the previous theorem, then the flow is maximal and the cut is minimal. This occurs if and only if

- $F(i,j) = C(i,j)$ *for all* $i \in S, j \in T$ *and*
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Carefully looking at the proof of the previous theorem we notice that equality holds precisely when

$$\sum_{j \in S} \sum_{i \in T} F(i,j) = 0 \quad \text{and} \quad \sum_{j \in S} \sum_{i \in T} F(j,i) = \sum_{j \in S} \sum_{i \in T} C(j,i)$$

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Finding a minimal cut

Theorem (Complete algorithm works and finds minimal cut)
The Complete Algorithm produces a maximal flow. Furthermore, if we take the cut $K = E(D) \cap (S \times T)$, where S is the labelled vertices at the termination of the Complete Algorithm and T is the unlabelled vertices at the termination of the Complete Algorithm, then K is minimal.

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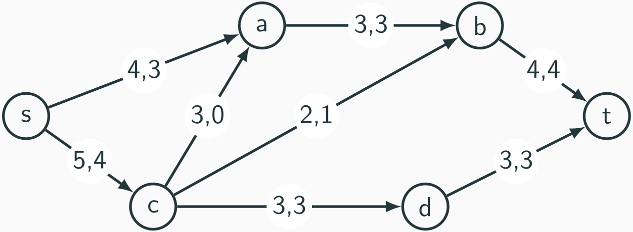
Go check all the previous examples that you can find the minimal cut by running the algorithm and applying this theorem!

Example 2: Finding min cut via Complete Labelling Algorithm

Flow F_3



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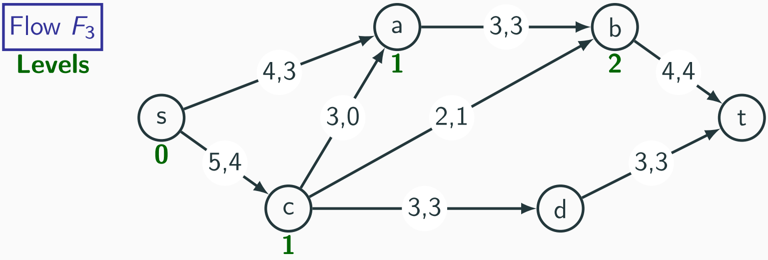

No level can be assigned to d or t !

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Matching

A matching problem

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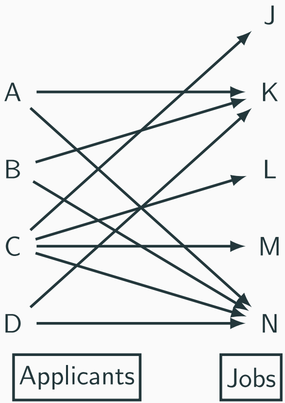

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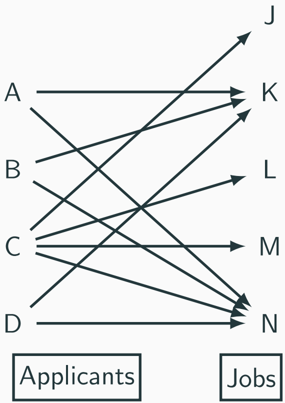

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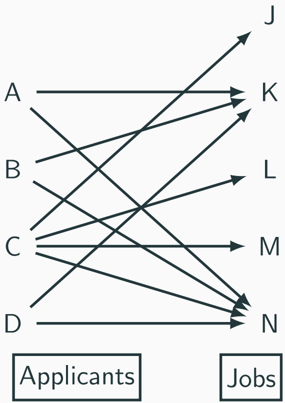

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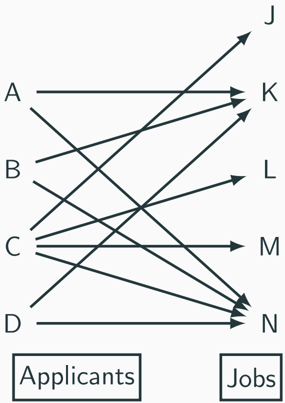

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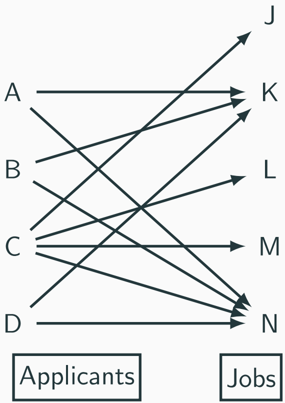

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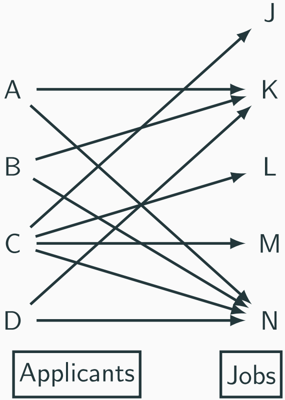

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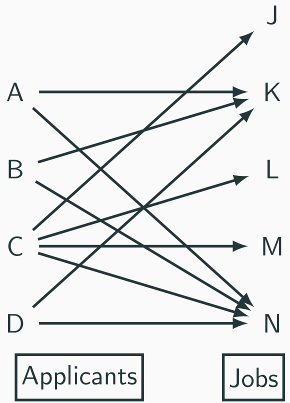

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More generally, given a relation $R \subseteq S \times T$ a **matching problem** seeks a **maximal matching function** (or just a 'matching') $m \subseteq R$.
This is a **injective** (one-to-one) function $f : S' \to T$ with domain $S' \subseteq S$ as large as possible subject to m being an injective subset of R.

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A solution to the max flow problem provides the matching:

$$\boxed{m = \{(x, y) \in S \times T \ :\ F_{\max}((x, y)) = 1\}.}$$

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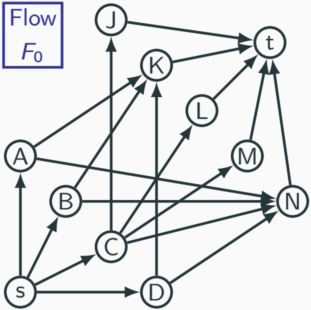

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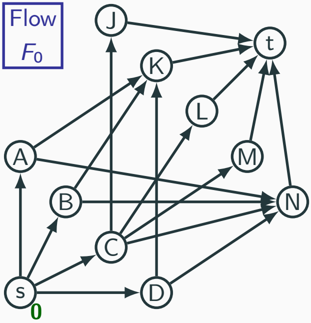

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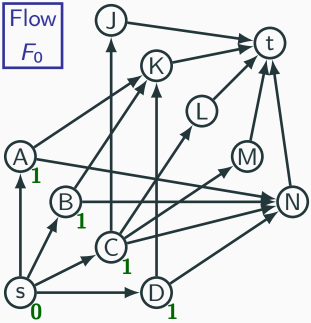

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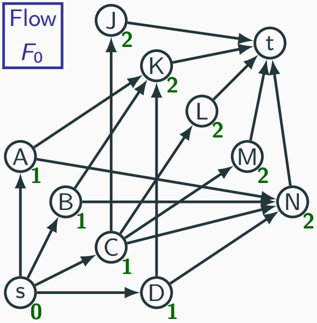

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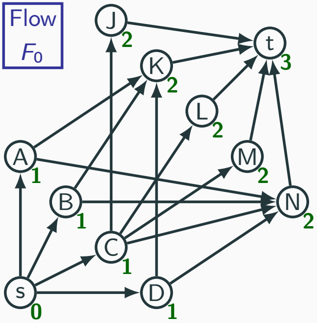

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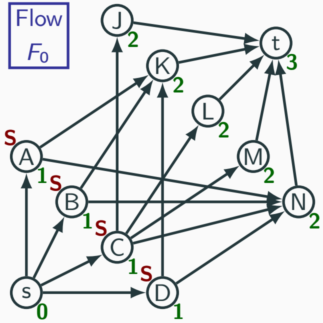

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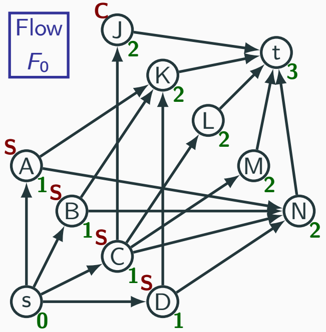

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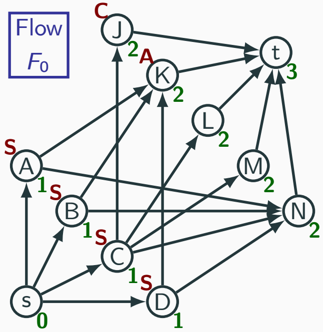

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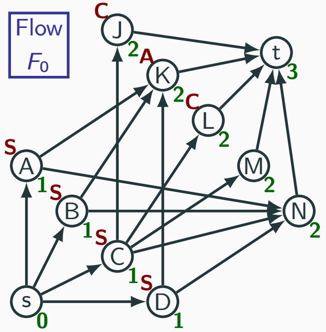

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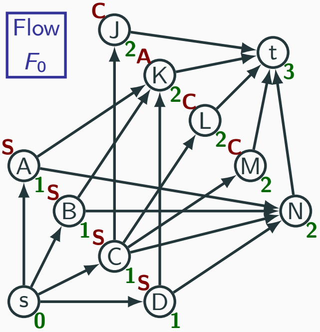

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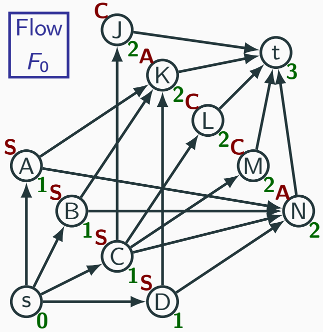

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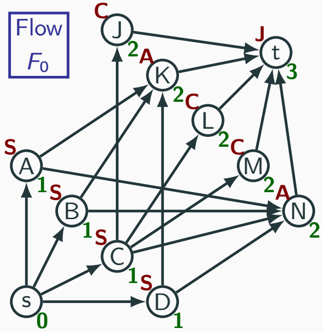
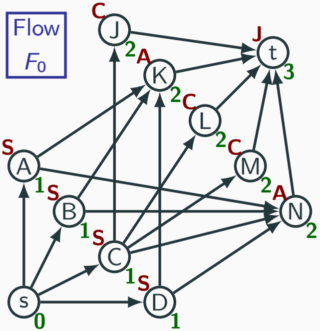


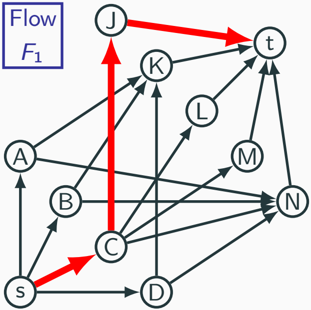
Here is how vertex labelling would be used on the Johnsonbaugh problem. This is just for demonstration, since, as we've seen, it's easily solved by eye.

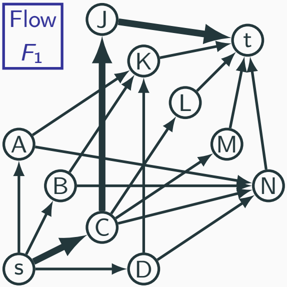
With all edge capacities 1, edge flows are either 0 or 1.

- Edge capacities not marked.
- (Non-zero) Edge flows marked by edge thickening.
- Potential flow values not indicated on labels.



⇒




Flow
F_1
J
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L
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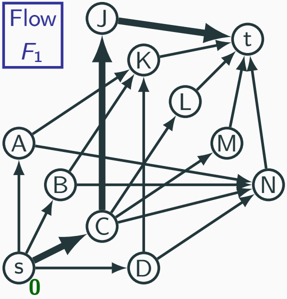
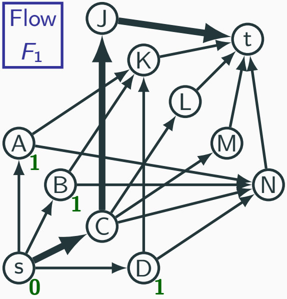
M
B
N
C
s
D

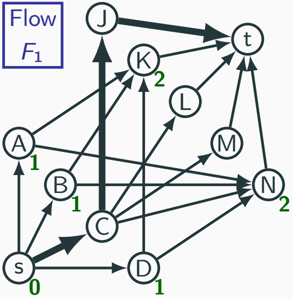

Flow
F_1
J
t
K
L
A
M
B
N
C
s
D
0

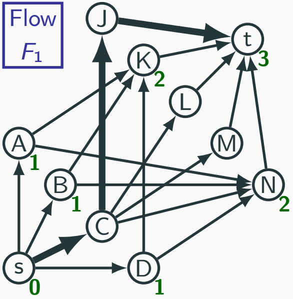
Flow
F_1



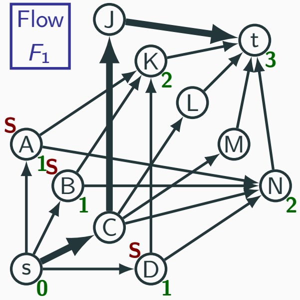
Flow

F_1





Flow
F_1
J
t
3
K
2
L
A
1
M
B
1
N
2
C
s
0
D
1


Flow
F_1
J
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s
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S
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1
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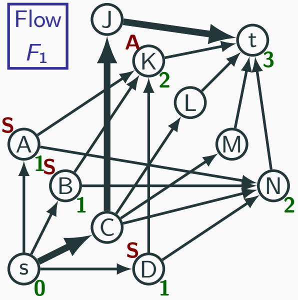
1
2
2
3


Flow
F_1
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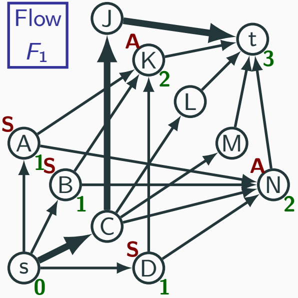

Flow
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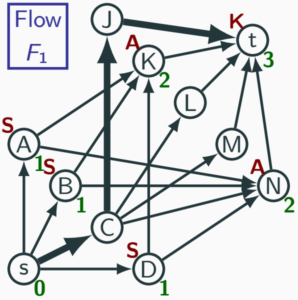
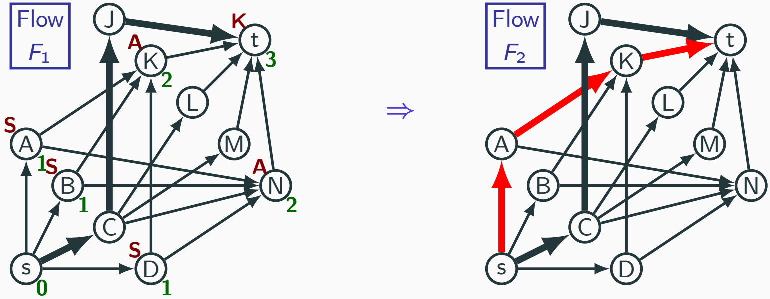
Flow F_1





Flow
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C
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⇒
Flow
F_2
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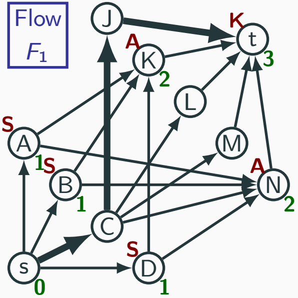
C
s
D


Flow
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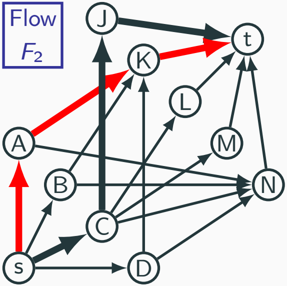
⇒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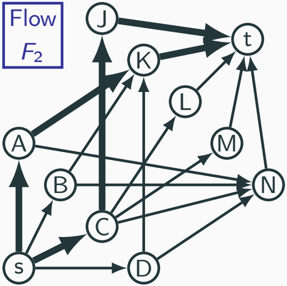
Flow
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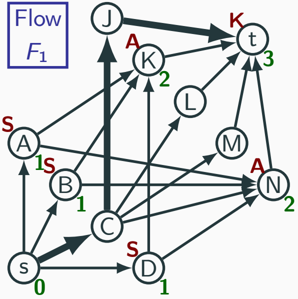
Flow F_1
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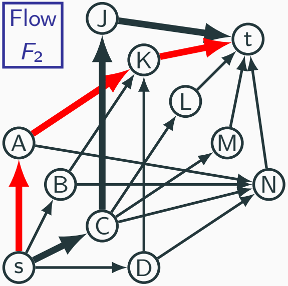
⇒


Flow F_2
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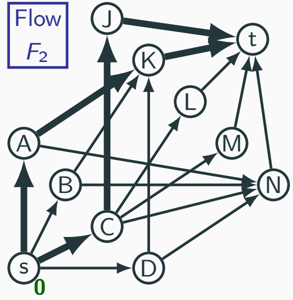
Flow F_2
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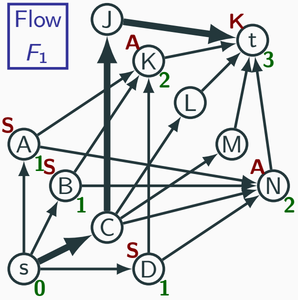
Flow
F_1
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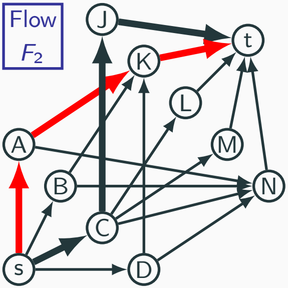
1

⇒


Flow
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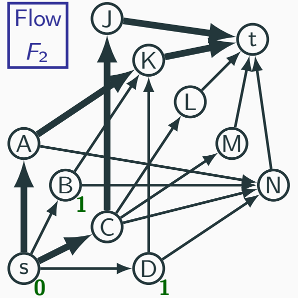
Flow
F_2
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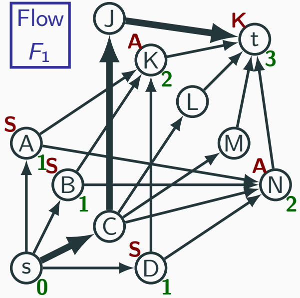
1


Flow F_1
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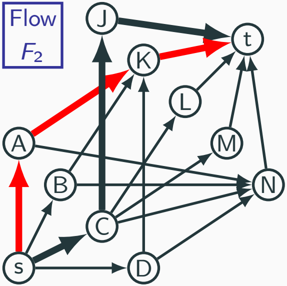
0
1

⇒


Flow F_2
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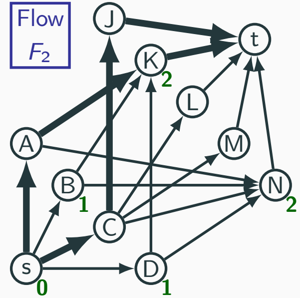
Flow F_2
J
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C
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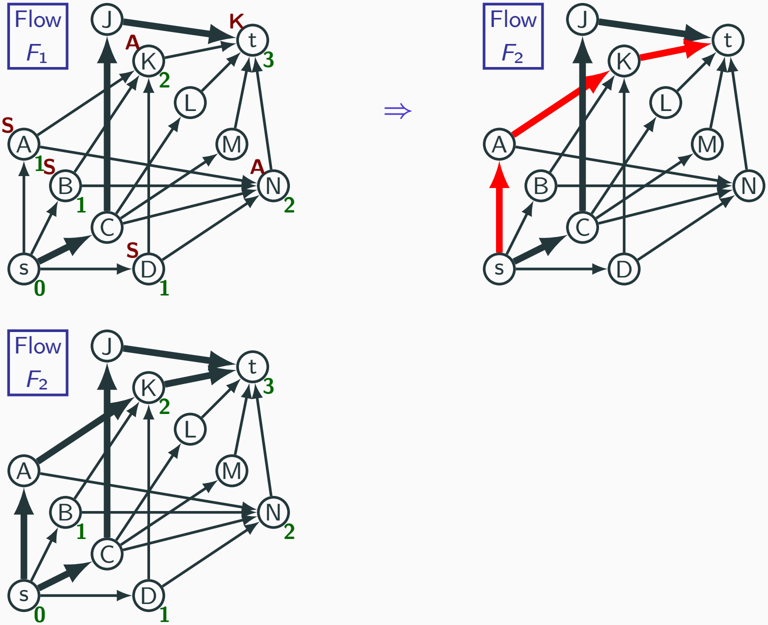
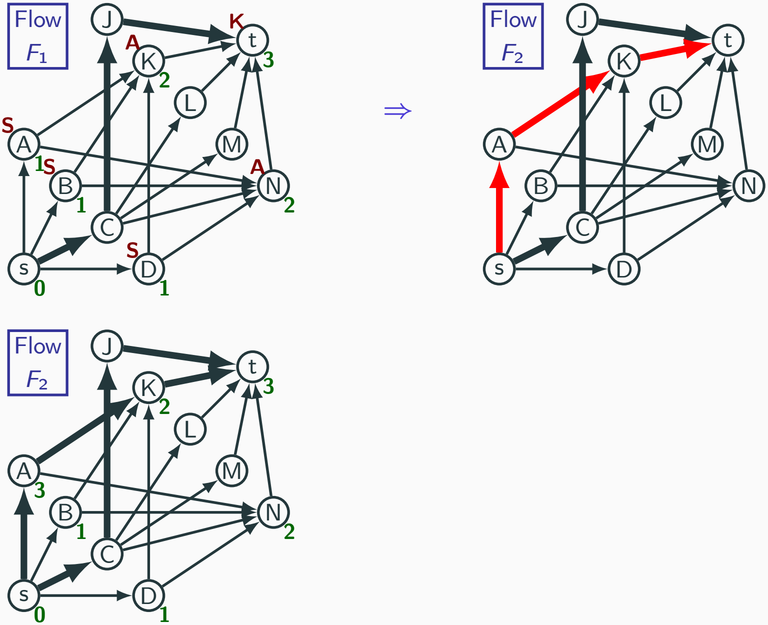
1



Flow F_1
J t K A L M S N S B C D s 0 1 1 1 2 2 3
⇒
Flow F_2
J t K L A M B N C s D
Flow F_2
J t K L A M B N C s D 0 1 1 2 2 3


Flow F_1
⇒
Flow F_2
Flow F_2
s
A
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C
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J
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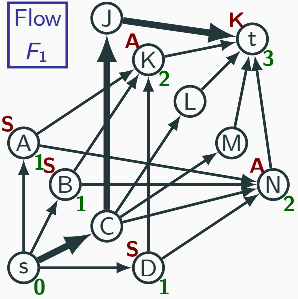
N
t


Flow
F_1
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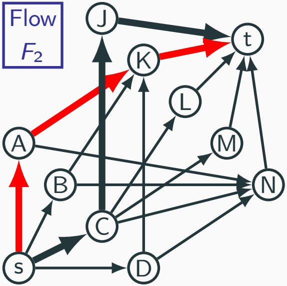
⇒


Flow
F_2
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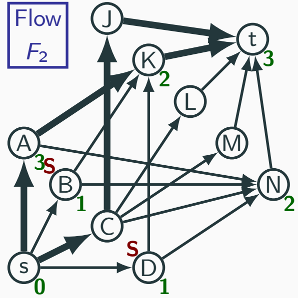
Flow
F_2
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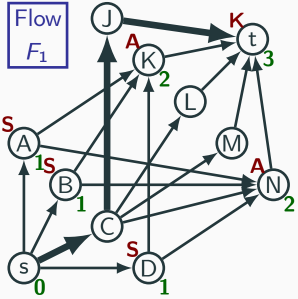
Flow
F_1
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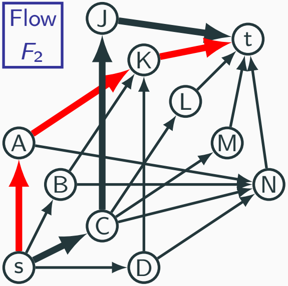
C
S
s
D
0
1

$\Rightarrow$


Flow
F_2
J
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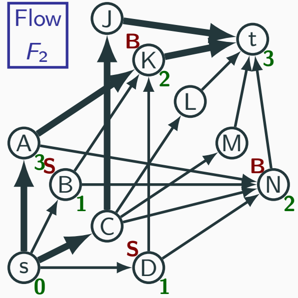
Flow
F_2
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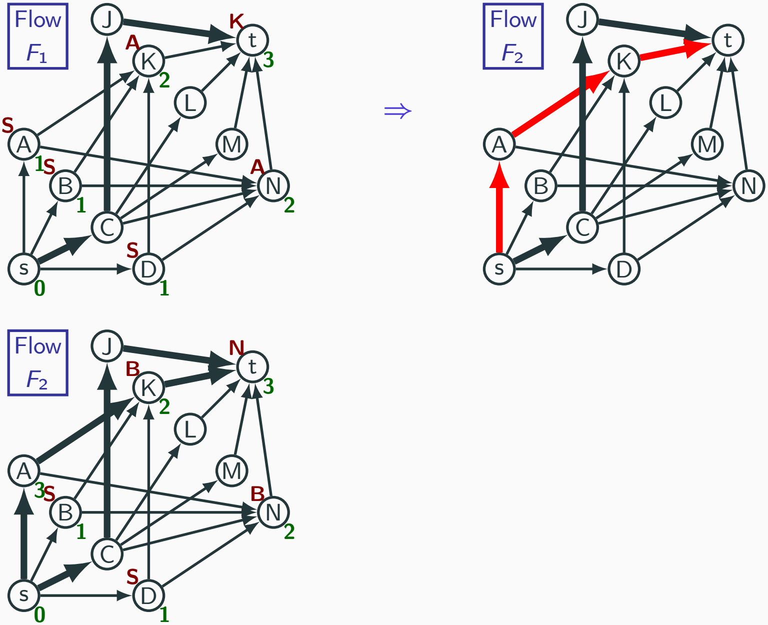
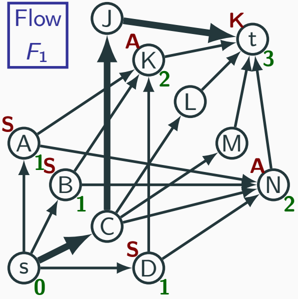
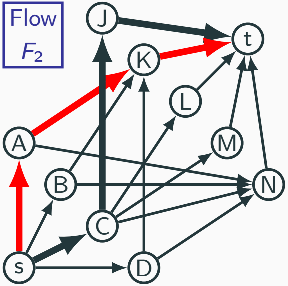
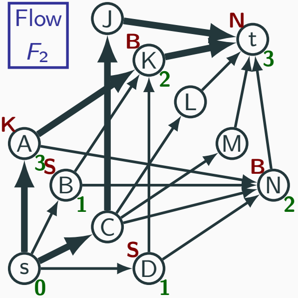
1

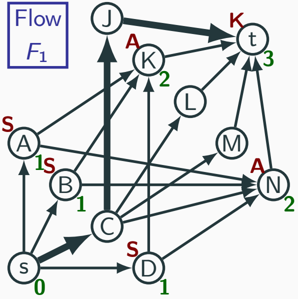

Flow F_1
⇒
Flow F_2
Flow F_2


Flow
F_1
J t K L M A B C D N s
K A S S A S
3 2 1 1 2 0 1

⇒


Flow
F_2
J t K L M A B C D N s


Flow
F_2
J t K L M A B C D N s
N B K S B S
3 2 3 1 2 0 1


Flow
F_1
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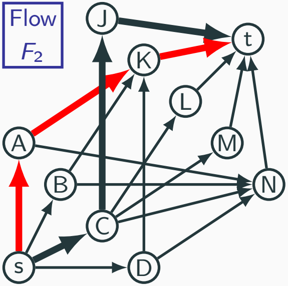
⇒


Flow
F_2
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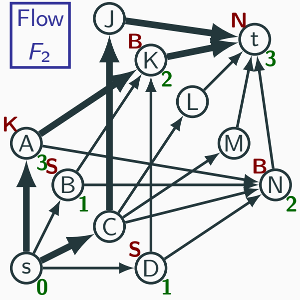
Flow
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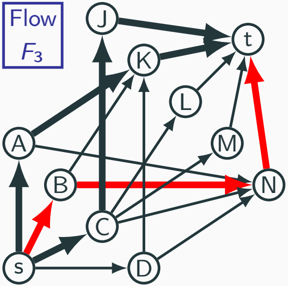
⇒


Flow
F_3
J
K
t
A
L
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s
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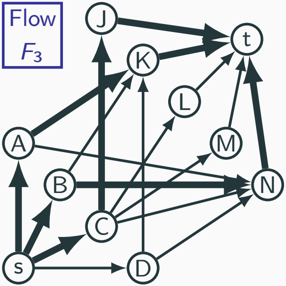
Flow
F_3
J
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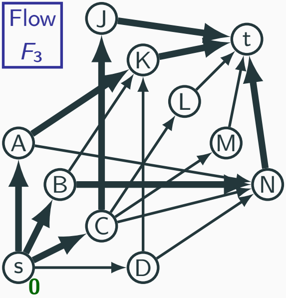
D


Flow
F_3
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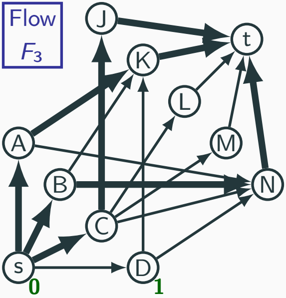
Flow
F_3
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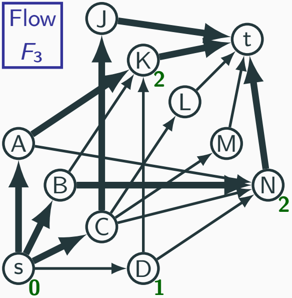

Flow
F_3
J
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2
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C
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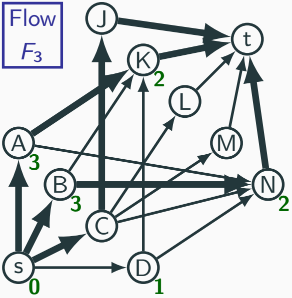
Flow
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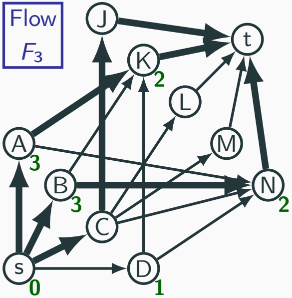
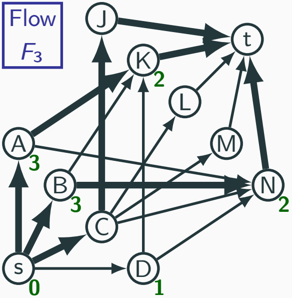


$\Rightarrow$

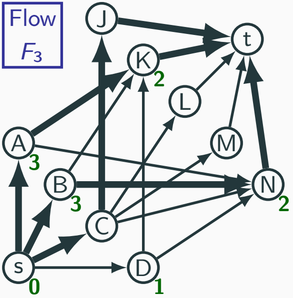
Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.



$\Rightarrow$

Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.

The matchings are indicated by the edges between $\{A, B, C, D\}$ and $\{J, K, L, M, N\}$ that have flow 1 (the thick edges).



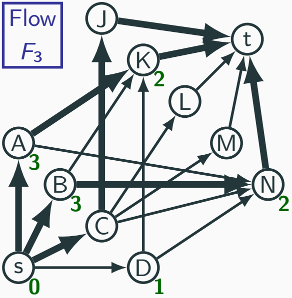
$\Rightarrow$

Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.

The matchings are indicated by the edges between $\{A, B, C, D\}$ and $\{J, K, L, M, N\}$ that have flow 1 (the thick edges).

So the matching is

$$m = \{(A, K), (B, N), (C, J)\}.$$



$\Rightarrow$

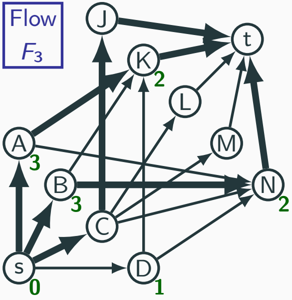
Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.

The matchings are indicated by the edges between $\{A, B, C, D\}$ and $\{J, K, L, M, N\}$ that have flow 1 (the thick edges).

So the matching is

$m = \{(A, K), (B, N), (C, J)\}$.

As remarked earlier, the above solution is very easy to get without the use of the vertex labelling algorithm. We have just matched the first member of T with the first 'available' member of S, then the next member of T with the next 'available' member of S and so on.



$\Rightarrow$

Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.

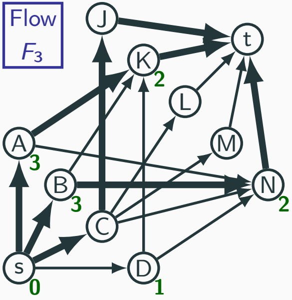
The matchings are indicated by the edges between $\{A, B, C, D\}$ and $\{J, K, L, M, N\}$ that have flow 1 (the thick edges).

So the matching is

$m = \{(A, K), (B, N), (C, J)\}$.

As remarked earlier, the above solution is very easy to get without the use of the vertex labelling algorithm. We have just matched the first member of T with the first 'available' member of S, then the next member of T with the next 'available' member of S and so on.

The benefit of using the algorithm lies in its ability to 'undo' initial pairings and replace them by new ones when this becomes necessary to enable later pairings.



$\Rightarrow$

Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.

The matchings are indicated by the edges between $\{A, B, C, D\}$ and $\{J, K, L, M, N\}$ that have flow 1 (the thick edges).

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The final example shows how it does this in the simplest possible case.

Vertex labelling for matching; Example 2

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for the relation $R = \{(a, p), (a, q), (b, p)\}$.

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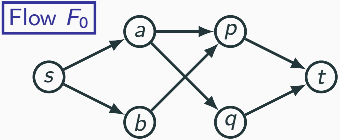

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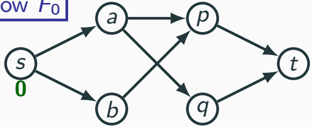

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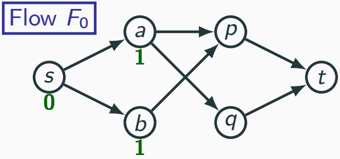

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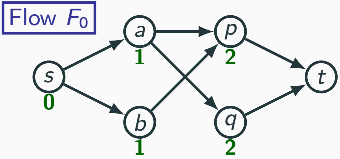

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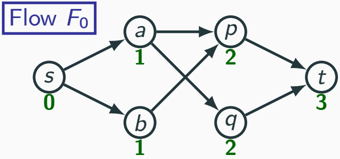

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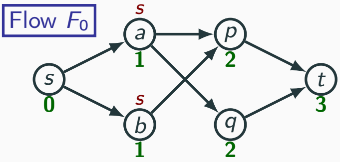

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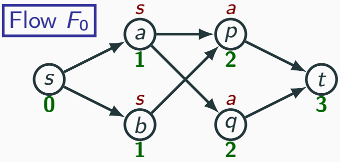

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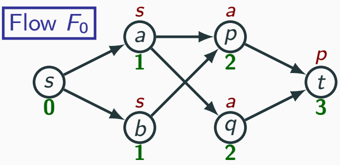

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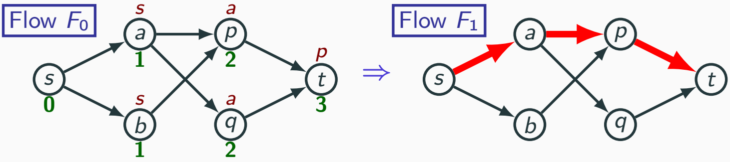

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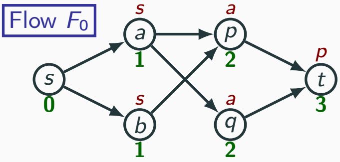

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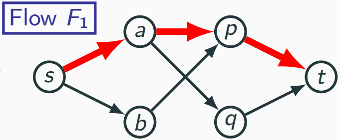

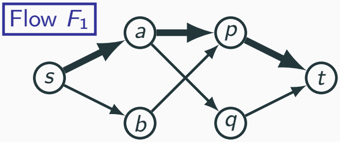

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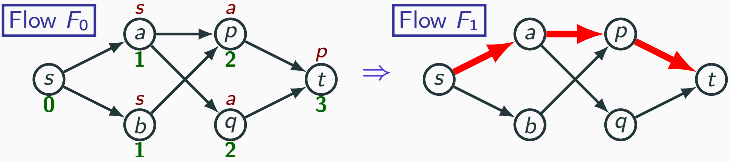

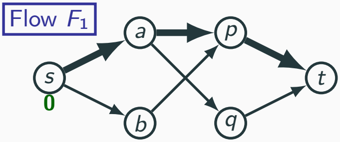

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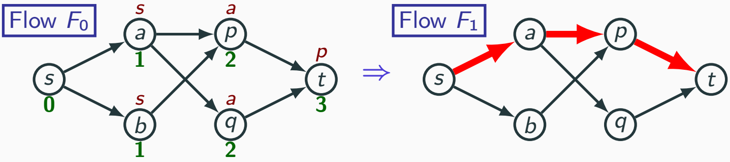

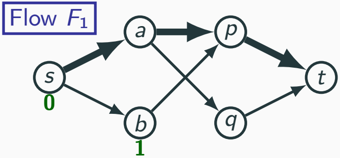

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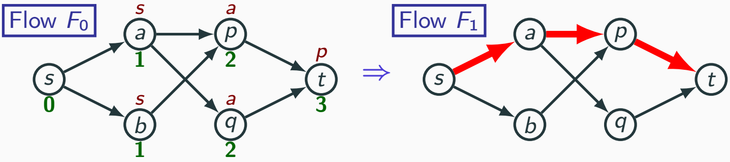

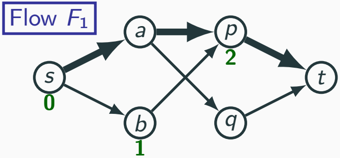

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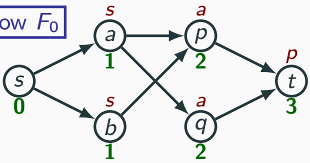

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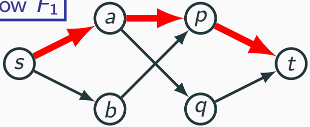

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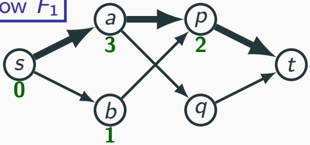

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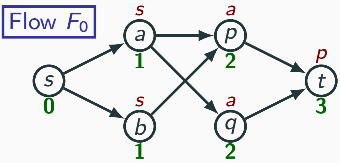

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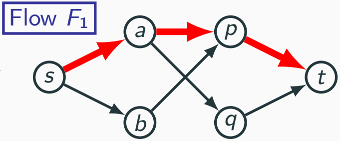

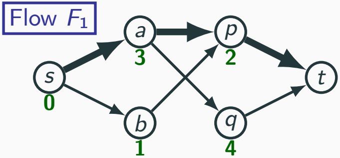

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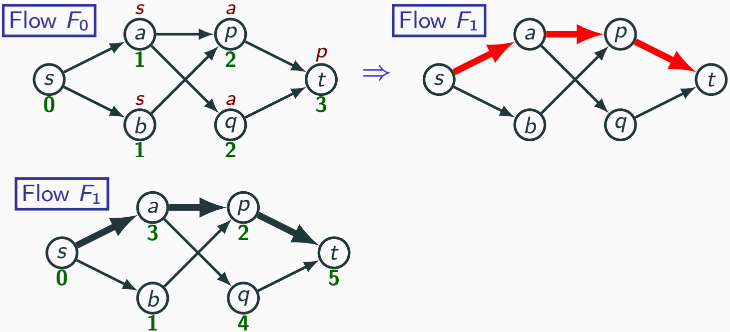

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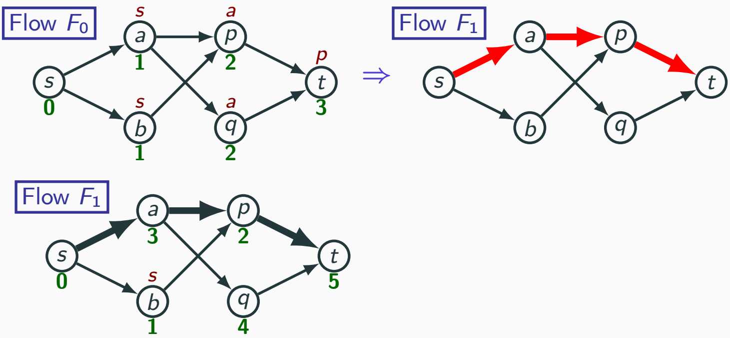

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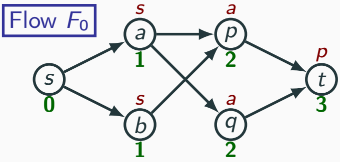

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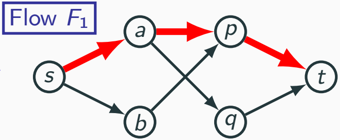

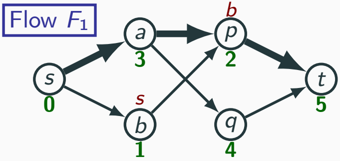

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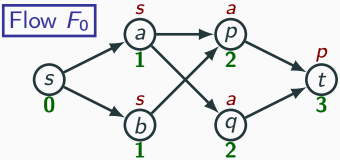

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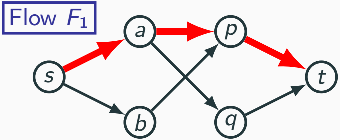

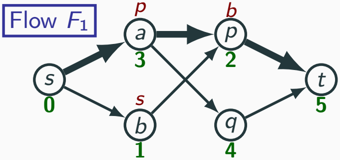

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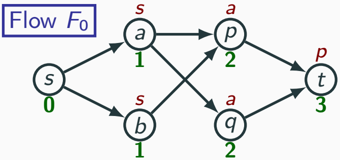

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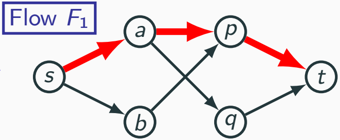

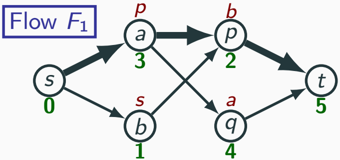

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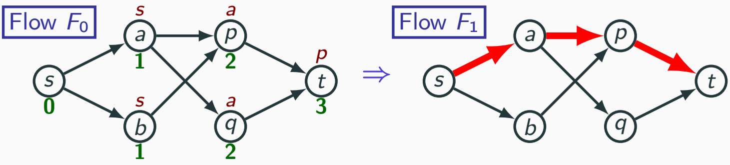

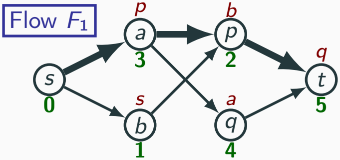

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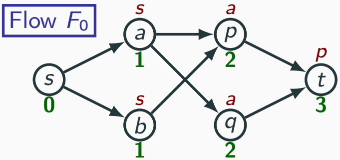

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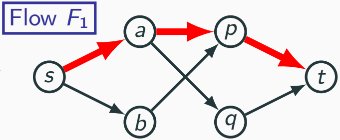

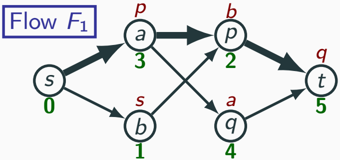

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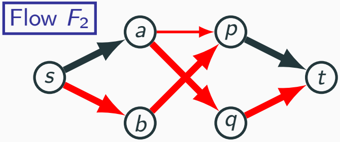

End of Weighted directed graphs

END OF SECTION D2

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Next week the Quiz 9 (Week 11) will be on 'D1: Introduction to Graph Theory' (Lectures 22,23,24).