**1.*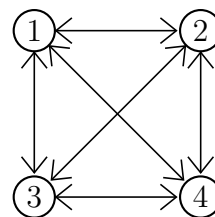* Jeremy is taking random walks on the digraph shown at right. At any vertex Jeremy has three choices as to where to go next:

- go to the vertex diagonally across the square.
- go to the adjacent vertex around the square in either a clockwise or an anticlockwise direction;

He takes the first option with probability one half but his probability for the other options is equal to one tenth of the destination vertex number. For example, if he is at vertex 1, his probabilities of next visiting vertices 2, 3 and 4 are 0.2, 0.3 and 0.5 respectively.

(a) List all the walks of length 2 from vertex 1 to vertex 2 and the probabilities associated with each of them. Hence find the probability that a walk of length 2 that starts at vertex 1 finishes at vertex 2.

Solution:

$1 - 3 - 2 : p = 0.3 \times 0.5 = 0.15$

$1 - 4 - 2 : p = 0.5 \times 0.2 = 0.10$ Probability $= \cdot 15 + \cdot 10 = \boxed{\cdot 25}$

(b) List all the walks of length 3 from vertex 1 to vertex 2 (there are a lot!) and the probabilities associated with each of them. Hence find the probability that a walk of length 3 that starts at vertex 1 finishes at vertex 2.

Solution: $1 - 2 - 1 - 2 : p = \cdot 2 \times \cdot 1 \times \cdot 2 = \cdot 004$
$1 - 2 - 3 - 2 : p = \cdot 2 \times \cdot 5 \times \cdot 5 = \cdot 050$
$1 - 2 - 4 - 2 : p = \cdot 2 \times \cdot 4 \times \cdot 2 = \cdot 016$

$1 - 3 - 1 - 2 : p = \cdot 3 \times \cdot 1 \times \cdot 2 = \cdot 006$
$1 - 3 - 4 - 2 : p = \cdot 3 \times \cdot 4 \times \cdot 2 = \cdot 024$

$1 - 4 - 1 - 2 : p = \cdot 5 \times \cdot 5 \times \cdot 2 = \cdot 050$
$1 - 4 - 3 - 2 : p = \cdot 5 \times \cdot 3 \times \cdot 5 = \cdot 075$

Probability $= \cdot 004 + \cdots + \cdot 075 = \boxed{\cdot 225}$

(c) Compile Jeremy's transition matrix T. Check that it is stochastic.

Solution: $T = \begin{bmatrix} 0 & \cdot 1 & \cdot 1 & \cdot 5 \\ \cdot 2 & 0 & \cdot 5 & \cdot 2 \\ \cdot 3 & \cdot 5 & 0 & \cdot 3 \\ \cdot 5 & \cdot 4 & \cdot 4 & 0 \end{bmatrix}$.   All entries are non-negative.
Each column sums to 1.
So T is stochastic.

(d) Calculate T^2 (by hand or computer) and use it to check your answer to (a).

Solution: $T^2 = \begin{bmatrix} \cdot 30 & \cdot 25 & \cdot 25 & \cdot 05 \\ \mathbf{\cdot 25} & \cdot 35 & \cdot 10 & \cdot 25 \\ \cdot 25 & \cdot 15 & \cdot 40 & \cdot 25 \\ \cdot 20 & \cdot 25 & \cdot 25 & \cdot 45 \end{bmatrix}$.

(e) Calculate T^3 (by hand or computer) and use it to check your answer to (b). If the answers don't agree, you probably missed some of the walks.

Solution: $T^3 = \begin{bmatrix} \cdot 150 & \cdot 175 & \cdot 175 & \cdot 275 \\ \mathbf{\cdot 225} & \cdot 175 & \cdot 300 & \cdot 225 \\ \cdot 275 & \cdot 325 & \cdot 200 & \cdot 275 \\ \cdot 350 & \cdot 325 & \cdot 325 & \cdot 225 \end{bmatrix}$.

(f) Use T to verify that, if Jeremy keeps walking, then in the long run his visits to any vertex v will constitute a fraction $(v + 5)/30$ of his visits to all vertices. (Thus, for example, 20% of his visits will be to vertex 1 and 30% to vertex 4.)

Solution: The claim is that the steady state vector is $S = \begin{bmatrix} 6/30 \\ 7/30 \\ 8/30 \\ 9/30 \end{bmatrix}$.

We need to check that $TS = S$:

$$TS = \begin{bmatrix} 0 & \cdot1 & \cdot1 & \cdot5 \\ \cdot2 & 0 & \cdot5 & \cdot2 \\ \cdot3 & \cdot5 & 0 & \cdot3 \\ \cdot5 & \cdot4 & \cdot4 & 0 \end{bmatrix} \frac{1}{30}\begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 0 & \cdot1 & \cdot1 & \cdot5 \\ \cdot2 & 0 & \cdot5 & \cdot2 \\ \cdot3 & \cdot5 & 0 & \cdot3 \\ \cdot5 & \cdot4 & \cdot4 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} = S.$$

2. Anton the ant is wandering around the structure shown at right, made from twelve length 1 struts joined at six vertices 1,...,6, making a regular octahedron with the struts as edges. He walks at a steady pace along the edges and at each vertex makes a random (equal-probability) choice as to which edge to walk next (this includes the possibility of retracing the previous edge). (See questions for diagram!).

(a) Anton starts at vertex 1. Explain why, after a walk of length two, he is twice as likely to be at vertex 6 than at vertex 5.

Solution:

Since four edges meet at each vertex, the probability of Anton taking any one of them is 1/4. Thus all walks of length 2 starting at vertex 1 have probability $(1/4)^2 = 1/16$. But there are four walks of length 2 from vertex 1 to vertex 6 but only two from vertex 1 to vertex 5.

(b) Again starting from vertex 1, what is the probability that after a walk of length 3 (not necessarily using distinct edges), Anton is back at vertex 1?

Solution:

Walks of length 3 from vertex 1 back to vertex 1 are
1231, 1251, 1321, 1341, 1431, 1451, 1541, 1521.

So the probability is $8(1/4)^3 = \mathbf{1/8}$.

(c) Write out Anton's transition matrix T. Write it in the form $T = fM$ where f is a fraction and M is a symmetric matrix of zeros and ones.

Solution:

$$T = \begin{bmatrix} 0 & 1/4 & 1/4 & 14 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

(d) Making good use of the simple nature of M to save work, calculate T^2, writing it in similar simple form. Use T^2 to check your answer to (a).

Solution:

$$T^2 = \tfrac{1}{16}\begin{bmatrix} 4 & 2 & 2 & 2 & 2 & 4 \\ 2 & 4 & 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 & 4 & 2 \\ 4 & 2 & 2 & 2 & 2 & 4 \end{bmatrix} = \tfrac{1}{8}\begin{bmatrix} 2 & 1 & 1 & 1 & \mathbf{1} & \mathbf{2} \\ 1 & 2 & 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

(e) Calculate $T^4 = (T^2)^2$. Use a computer if you like, but it is not really necessary.

Solution:

$$T^4 = (T^2)^2 = \tfrac{1}{64}\begin{bmatrix} 12 & 10 & 10 & 10 & 10 & 12 \\ 10 & 12 & 10 & 12 & 10 & 10 \\ 10 & 10 & 12 & 10 & 12 & 10 \\ 10 & 12 & 10 & 12 & 10 & 10 \\ 10 & 10 & 12 & 10 & 12 & 10 \\ 12 & 10 & 10 & 10 & 10 & 12 \end{bmatrix} = \tfrac{1}{32}\begin{bmatrix} 6 & 5 & 5 & 5 & 5 & 6 \\ 5 & 6 & 5 & 6 & 5 & 5 \\ 5 & 5 & 6 & 5 & 6 & 5 \\ 5 & 6 & 5 & 6 & 5 & 5 \\ 5 & 5 & 6 & 5 & 6 & 5 \\ 6 & 5 & 5 & 5 & 5 & 6 \end{bmatrix}.$$

(f) Suppose Anton has been wandering for a long time. By comparing T, T^2 and T^4, and by considering the symmetry inherent in the situation, estimate the proportion of Anton's vertex visits that have been to vertex 1.

Solution:

The entries in the matrix are getting closer to one another, suggesting that higher powers of T will have entries essentially all equal. Since T must remain stochastic, row sums must be 1, so it appears that:

$$T^n \longrightarrow \tfrac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \text{ as } n \longrightarrow \infty, \quad \text{implying that } S = \tfrac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

This looks plausible in view of the complete symmetry of the graph.

It suggests that $1/6$ of all Anton's visits will be to vertex 1.

(g) Support your answer to (f) by demonstrating a steady state vector for T.

Solution:

$$TS = \tfrac{1}{4}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tfrac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \tfrac{1}{24}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \tfrac{1}{24}\begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} = \tfrac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = S.$$

3. A tiny web graph has four pages A, B, C, D. Every page has a link to every other page except for pages C and D, neither of which has link to the other.

(a) Draw the web graph.

Solution:
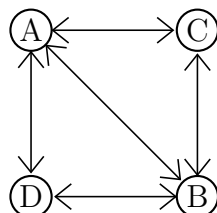
(b) Write out the basic transition matrix, with probabilities assigned according to the Google PageRank algorithm.

Solution:

$$T = \begin{bmatrix} 0 & 1/3 & 1/2 & 1/2 \\ 1/3 & 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 \end{bmatrix}.$$

(c) Calculate the page ranks of each page.
Hint: By symmetry, pages A and B will have the same rank, as will pages C and D. So there will only be two unknowns.

Solution:

Suppose A, B have page rank p, C, D page rank q. Then $S = \begin{bmatrix} p \\ p \\ q \\ q \end{bmatrix}$.

We need to solve $TS = S$.
We use $(T - I)S = 0$ with the last row replaced by 1's:

$$\begin{bmatrix} -1 & 1/3 & 12 & 1/2 \\ 1/3 & -1 & 1/2 & 1/2 \\ 1/3 & 1/3 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p \\ p \\ q \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \implies \begin{matrix} -2/3p+q=0 \\ -2/3p+q=0 \\ 2/3p-q=0 \\ 2p+2q=1 \end{matrix} \implies \begin{matrix} q=2/3p \\ p=1/2-q \\ =1/2-2/3p \\ \therefore 5/3p=1/2 \text{ so } p=\cdot3, \ q=\cdot2. \end{matrix}$$

or by solving with a computer. Thus A, B have page rank $\cdot3$ and C, D have page $\cdot2$.

4. Repeat Question 3 with the links from A and B to C removed.
It will no longer be true that C and D have the same rank, so there will now be three equations in three unknowns. However one equation gives the value of one of the unknowns immediately, leading to a 2×2 system.

Solution:
Suppose A, B have page rank p and C and D have rank q and r respectively.
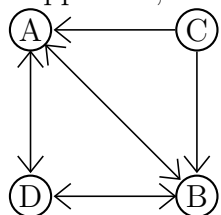
$$T = \begin{bmatrix} 0 & 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} \implies \begin{bmatrix} -1 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 & 1/2 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p \\ p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \implies$$

$$\begin{matrix} -p/2+q/2+r/2=0 \\ -p/2+q2+r/2=0 \\ -q=0 \\ 2p+q+r=1 \end{matrix} \implies \begin{matrix} q=0 \\ \therefore p=r \\ \therefore 3p=1 \\ \therefore p=1/3 = r. \end{matrix}$$

Thus C has rank 0 and A, B, D each have rank $1/3$. Looking at the diagram, this is what you should expect. There are no links into C so it has rank 0 and can be ignored. The remaining three pages all have links to each other, so none is more important than the others and hence they have equal rank.

5. Repeat Question 3 but this time with the links **to** A and B **from** C removed.
Again C and D will have different ranks, leading to a 3×3 system, but one equation involves only two unknowns, allowing one unknown to be eliminated from the other two equations, thus generating a 2×2 system.

Solution:
There are no links from C so we introduce links from C to all other pages (constructing the webgraph G^+). Suppose A, B have page rank p and C and D have rank q and r respectively.
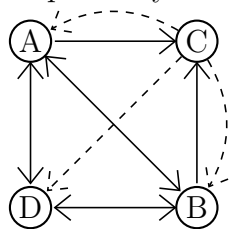
$$T = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/2 \\ 1/3 & 0 & 1/3 & 1/2 \\ 1/3 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \implies \begin{bmatrix} -1 & 1/3 & 1/3 & 1/2 \\ 1/3 & -1 & 1/3 & 1/2 \\ 1/3 & 1/3 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p \\ p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \implies$$

$$\begin{array}{ll} -2p/3 + q/3 + r/2 = 0 & q = 2p/3 \\ -2p/3 + q/3 + r/2 = 0 & \therefore r = 8p/9 \\ 2p/3 - q = 0 \implies & \therefore 2p + 2p/3 + 8p/9 = 1 \\ 2p + q + r = 1 & \therefore p = 9/32, \ q = 3/16, \ r = 1/4. \end{array}$$

Thus A, B each have rank $9/32$, C has rank $3/16$ and D has rank $1/4$.

6. Repeat Question 3 with 90% damping. This means solving the equation

$$M\!PR = PR$$

with $\alpha = {}^1\!/_{10}$. Symmetry still applies, so PR still involves only two unknowns, so this 4×4 system boils down to only 2×2, and hence can be solved without the need for Gaussian elimination. However the fractions get a little nastier so you may prefer to use a computer. You should find that the page rank fractions have denominator 64 (when expressed in lowest terms).

Solution:
$M = ((\alpha/n)U + (1 - \alpha)T)$. So

$$M = \frac{1}{40}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \frac{9}{10}\begin{bmatrix} 0 & 1/3 & 1/2 & 1/2 \\ 1/3 & 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/40 & 13/40 & 19/40 & 19/40 \\ 13/40 & 1/40 & 19/40 & 19/40 \\ 13/40 & 13/40 & 1/40 & 1/40 \\ 13/40 & 13/40 & 1/40 & 1/40 \end{bmatrix}.$$

Then by using the 'shortcut method' and a matrix calculator we find:

$$PR = \begin{bmatrix} 19/64 \\ 19/64 \\ 13/64 \\ 13/64 \end{bmatrix}$$

A, B have rank $19/64$ and C, D have rank $13/64$.

7. Repeat Question 4 with 80% damping. Considerations similar to those for the previous question regarding solution method will apply. You should find D has rank $43/140$.

Solution:
$M = ((\alpha/n)U + (1-\alpha)T)$. So

$$M = \frac{1}{20}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \frac{4}{5}\begin{bmatrix} 0 & 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/20 & 9/20 & 9/20 & 9/20 \\ 9/20 & 1/20 & 9/20 & 9/20 \\ 1/20 & 1/20 & 1/20 & 1/20 \\ 9/20 & 9/20 & 1/20 & 1/20 \end{bmatrix}.$$

Then by using the 'shortcut method' and a matrix calculator we find:
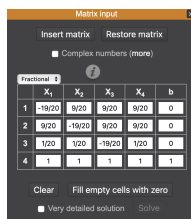
$$PR = \begin{bmatrix} 19/28 \\ 9/28 \\ 1/20 \\ 43/140 \end{bmatrix}$$

8. Again repeat Question 4 with 80% damping, but this time use two steps of the iterative method

$$P_0 = (1/n)\mathbf{1}; \quad P_k = \alpha P_0 + (1-\alpha)TP_{k-1}, \ k \geq 1.$$

That is, calculate P_0, then P_1 and finally P_2, and use P_2 as an approximation for PR. Compare the result to your answer to question 7.

Solution:

$$P_0 = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cdot25 \\ \cdot25 \\ \cdot25 \\ \cdot25 \end{bmatrix}.$$

$$P_1 = \frac{2}{10}\begin{bmatrix} \cdot25 \\ \cdot25 \\ \cdot25 \\ \cdot25 \end{bmatrix} + \frac{8}{10}\begin{bmatrix} 0 & 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cdot25 \\ \cdot25 \\ \cdot25 \\ \cdot25 \end{bmatrix} = \begin{bmatrix} \cdot05 \\ \cdot05 \\ \cdot05 \\ \cdot05 \end{bmatrix} + \cdot8\begin{bmatrix} 3/8 \\ 3/8 \\ 0 \\ 2/8 \end{bmatrix} = \begin{bmatrix} \cdot35 \\ \cdot35 \\ \cdot05 \\ \cdot25 \end{bmatrix}.$$

$$P_2 = \begin{bmatrix} \cdot05 \\ \cdot05 \\ \cdot05 \\ \cdot05 \end{bmatrix} + \cdot8\begin{bmatrix} 0 & 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cdot35 \\ \cdot35 \\ \cdot05 \\ \cdot25 \end{bmatrix} = \begin{bmatrix} \cdot05 \\ \cdot05 \\ \cdot05 \\ \cdot05 \end{bmatrix} + \cdot8\begin{bmatrix} \cdot325 \\ \cdot325 \\ 0 \\ \cdot35 \end{bmatrix} = \begin{bmatrix} \cdot31 \\ \cdot31 \\ \cdot05 \\ \cdot33 \end{bmatrix}.$$

We see that P_2 is already within one or two percentage points of PR.

9. The adjacency matrix A for a twelve page web graph is given at right.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Write out the corresponding transition matrix T.

(b) Using a computer, find the page ranks of each page using the Google PageRank algorithm without damping.

(c) Repeat part (b) but using 60% damping.

Solution:

(a) $T = \begin{bmatrix} 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & 0 \end{bmatrix}'$ (note the transpose symbol)

(b) We need to solve the equation $TS = S$, where S is the PageRank (steady state) vector. We use the 'shortcut method' to find S.

Page:	1	2	3	4	5	6	7	8	9	10	11	12
Rank:	2.6%	3.2%	2.6%	3.2%	5.9%	8.0%	4.9%	11.1%	13.3%	4.2%	12.4%	28.5%

to 1dp.

(c) We need to solve the equation $MPR = PR$, where PR is the PageRank (steady state) vector. We use the 'shortcut method' to find PR. Note that

$$M = (0.4/12)U + 0.6T = (1/30)U + (3/5)T$$

	Matrix input												
	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	b
1	-29/30	1/30	1/30	1/30	1/30	1/30	1/30	1/30	1/30	1/30	1/30	29/330	0
2	11/60	-29/30	1/30	1/30	1/30	1/30	1/30	1/30	1/30	1/30	1/30	29/330	0
3	1/30	1/30	-29/30	1/30	1/30	1/30	1/30	1/30	1/30	1/30	1/30	29/330	0
4	11/60	1/30	1/30	-29/30	1/30	1/30	1/30	1/30	1/30	1/30	1/30	29/330	0
5	11/60	1/3	1/30	7/30	-29/30	1/30	1/30	1/30	1/30	1/30	1/30	29/330	0
6	1/30	1/3	2/3	1/30	23/150	-29/30	1/30	1/30	1/30	1/30	1/30	29/330	0
7	1/30	1/30	1/30	7/30	23/150	1/30	-29/30	1/30	1/30	1/30	1/30	29/330	0
8	11/60	1/30	1/30	7/30	23/150	1/3	7/30	-29/30	1/30	1/30	1/30	29/330	0
9	1/30	1/30	1/30	1/30	23/150	1/3	1/30	1/3	-29/30	1/30	1/30	29/330	0
10	1/30	1/30	1/30	1/30	1/30	1/30	7/30	1/30	1/30	-29/30	1/30	29/330	0
11	1/30	1/30	1/30	1/30	1/30	1/30	1/30	1/3	1/30	2/3	-29/30	29/330	0
12	1	1	1	1	1	1	1	1	1	1	1	1	1

Insert matrix   Restore matrix
☐ Complex numbers (more)

Fractional

Clear   Fill empty cells with zero   ☐ Very detailed solution   Solve

Solution set:

x_1 = 725000/16647309
x_2 = 833750/16647309
x_3 = 725000/16647309
x_4 = 833750/16647309
x_5 = 416875/5549103
x_6 = 4753100/49941927
x_7 = 347275/5549103
x_8 = 1834250/16647309
x_9 = 973137920000000/8523422208000001
x_{10} = 933365/16647309
x_{11} = 11198437/99883854
x_{12} = 18709867/99883854

Page:	1	2	3	4	5	6	7	8	9	10	11	12
Rank:	4.4%	5.0%	4.4%	5.0%	7.5%	9.5%	6.3%	11.0%	11.4%	5.6%	11.2%	18.7%

to 1dp.