Discrete Mathematical Models

Lecture 20

Kane Townsend

Semester 2, 2024

Applications of Bayes' Theorem

Theorem (Bayes' Theorem)
For any probability experiment with sample space S, for any $n \in \mathbb{N}$, for any partition $\{B_1, B_2, \ldots, B_n\}$ of S and for any event $A \subseteq S$, if $\mathbb{P}(A) \neq 0$ and for all $i \in \{1, 2, \ldots, n\}$ we have $\mathbb{P}(B_i) \neq 0$, then for all $k \in \{1, 2, \ldots, n\}$ we have

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)}$$

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- Drug testing
- Disease testing
- Defective item rates

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A **false positive** means that a patient gets a positive test of having the disease when they do not have the disease.

A **false negative** means that a patient gets a negative test of having the disease when they do have the disease.

Example 9.9.3 from Epp.

Consider a medical test that screens for a disease found in 5 people in 1,000. Suppose that the false positive rate is 3% and the false negative rate is 1%. Then 99% of the time a person who has the condition tests positive for it, and 97% of the time a person who does not have the condition tests negative for it.

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a. What is the probability that a randomly chosen person who tests positive for the disease actually has the disease?
b. What is the probability that a randomly chosen person who tests negative for the disease does not in fact have the disease?

Solution: Consider a random person from those screened. Let A be the event they test positive, B_1 the event they have the disease, B_2 the event they do not have the disease. Then:

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Solution: Consider a random person from those screened. Let A be the event they test positive, B_1 the event they have the disease, B_2 the event they do not have the disease. Then:

$$\mathbb{P}(A|B_1) = 0.99, \quad \mathbb{P}(A^c|B_1) = 0.01, \quad \mathbb{P}(A^c|B_2) = 0.97, \quad \mathbb{P}(A|B_2) = 0.03.$$

Also because 5 people in 1,000 have the disease,

$$\mathbb{P}(B_1) = 0.005 \text{ and } \mathbb{P}(B_2) = 0.995.$$

Example 9.9.3 from Epp. (Cont.)

A person tests positive, B_1 person has disease, B_2 the event they do not have the disease. Then: $\mathbb{P}(A|B_1) = 0.99, \mathbb{P}(A^c|B_1) = 0.01, \mathbb{P}(A^c|B_2) = 0.97, \mathbb{P}(A|B_2) = 0.03, \mathbb{P}(B_1) = 0.005$ and $\mathbb{P}(B_2) = 0.995$.

a. By Bayes' Theorem

$$\begin{aligned}
\mathbb{P}(B_1|A) &= \frac{\mathbb{P}(A|B_1)\mathbb{P}(B_1)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2)} \\
&= \frac{(0.99)(0.005)}{(0.99)(0.005) + (0.03)(0.995)} \\
&\approx 0.1422 \approx 14.2\%.
\end{aligned}$$

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b. By Bayes' Theorem

$$\begin{aligned}
\mathbb{P}(B_2|A^c) &= \frac{\mathbb{P}(A^c|B_2)\mathbb{P}(B_2)}{\mathbb{P}(A^c|B_1)\mathbb{P}(B_1) + \mathbb{P}(A^c|B_2)\mathbb{P}(B_2)} \\
&= \frac{(0.97)(0.995)}{(0.01)(0.005) + (0.97)(0.995)} \\
&\approx 0.999948 \approx 99.995\%.
\end{aligned}$$

C3: Markov Processes (not covered in textbook)

Introduction

Markov processes

Markov processes are about probabilities. We consider

- the **state** of a system, amongst a finite number of possibilities
- the **probability** of moving between states in one time-step,
- and the probable state after **many** time-steps.
- We often don't make a sharp distinction between **proportions** and **probabilities** as you will see in the examples.

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  This works well for large samples but you may need to be careful with small samples.

Example 1

Introductory Example

Adapted from ‘Finite Mathematics’, Maki & Thompson:

A freelance computer network consultant, let’s call her Cathy, is employed in weekly contracts. Each week she is either:

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(a) If she's employed this week, then next week she'll be employed with probability 0.8 and unemployed with probability 0.2.

(b) If she's unemployed this week, then next week she'll be employed with probability 0.6 and unemployed with probability 0.4.

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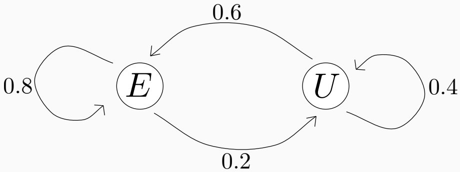
- A **Transition Diagram** encodes the transition probabilities:



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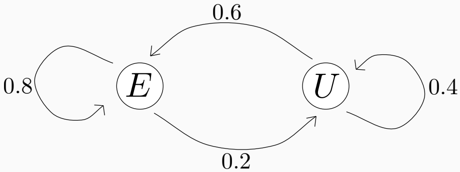
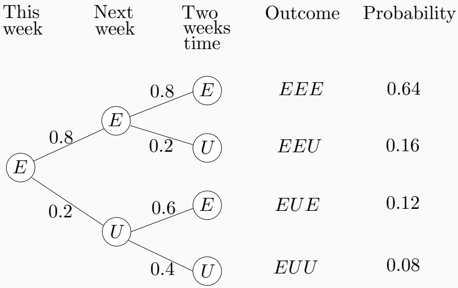


It is a property of a Markov Process that the probability of stepping from one state to another *only depends on the current state*.

Two time-steps

If Cathy is employed this week, what is the probability that she will be employed two weeks from now? We can use a tree:



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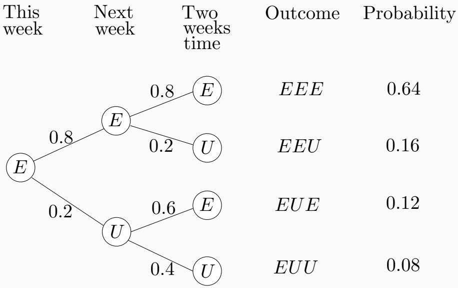

From the tree diagram, the probability that Cathy will be employed two weeks from now is

$\Pr(\text{EEE or EUE}) = \Pr(\text{EEE}) + \Pr(\text{EUE}) = 0.64 + 0.12 = 0.76.$

Two time-steps on the transition diagram

Starting employed, then employment after two weeks can be shown on the transition diagram as

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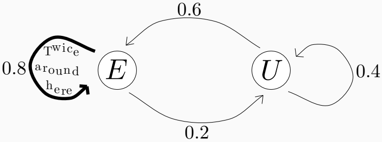

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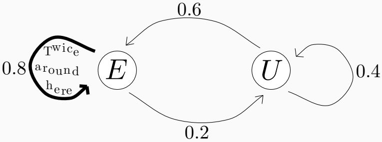

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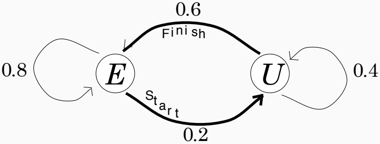
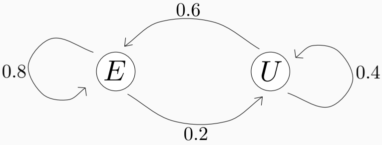


Transition Matrix

The information in Cathy's transition diagram



can be encoded in the transitition matrix

$$\begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$$

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This can be expressed as:

$$\begin{aligned} x_1 &= T\mathrm{x}_0 \\ &= \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \end{aligned}$$

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In Week 2, *i.e.* after two time-steps, Cathy's chances of work are given by the state vector x_2.

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n time-steps

$$\text{Continuing:}\quad \begin{aligned} \mathrm{x}_3 &= T\mathrm{x}_2 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.76 \\ 0.24 \end{bmatrix} = \begin{bmatrix} 0.752 \\ 0.248 \end{bmatrix} \\ \mathrm{x}_4 &= T\mathrm{x}_3 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.752 \\ 0.248 \end{bmatrix} = \begin{bmatrix} 0.7504 \\ 0.2496 \end{bmatrix} \\ \mathrm{x}_5 &= T\mathrm{x}_4 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.7504 \\ 0.2496 \end{bmatrix} = \begin{bmatrix} 0.75008 \\ 0.24992 \end{bmatrix} \\ \vdots \quad & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{aligned}$$

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$$\text{Thus:} \quad \begin{aligned} \mathrm{x}_1 &= T\mathrm{x}_0 \\ \mathrm{x}_2 &= T\mathrm{x}_1 = TT\mathrm{x}_0 = T^2\mathrm{x}_0 \\ \mathrm{x}_3 &= T\mathrm{x}_2 = TTT\mathrm{x}_0 = T^3\mathrm{x}_0 \\ \vdots \quad & \quad \vdots \qquad\qquad \vdots \\ \mathrm{x}_n &= T^n\mathrm{x}_0 \end{aligned}$$

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$$
\begin{aligned}
T^2 = TT &= \begin{bmatrix} 0.76 & 0.72 \\ 0.24 & 0.28 \end{bmatrix} \\
T^3 = TT^2 &= \begin{bmatrix} 0.752 & 0.744 \\ 0.248 & 0.256 \end{bmatrix} \\
T4 = TT^3 &= \begin{bmatrix} 0.7504 & 0.7488 \\ 0.2496 & 0.2512 \end{bmatrix} \\
T^5 = TT^4 &= \begin{bmatrix} 0.75008 & 0.74976 \\ 0.24992 & 0.25024 \end{bmatrix} \\
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$$\vdots \qquad \vdots \qquad \qquad \vdots$$

So we can guess that: $T^n \approx \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix}$ for large values of n.

The significance of T^n for large n

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Notice that the columns of this matrix are equal, and that

$$\begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}.$$

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So, irrespective of the initial state, in the long term the state vector becomes approximately $\begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$. This means

$\boxed{\text{No matter what, eventually Cathy will be employed 75\% of the time.}}$

Definitions, Terminology, Results

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Theorem (Perron-Frobenius)
Let T be a positive stochastic matrix. Then there is a unique normalised steady state vector v with respect to T. Moreover, T^n converges to $[v \; v]$ as $n \to \infty$ and so for any initial vector v_0 with entries summing to c, $v_n = T^n v_0$ converges to cv.

More definitions

A (discrete) **Markov process** is a system that has:

- a finite number k of states,
- a sequence of time steps $n \in \mathbb{N}^*$,
- probabilities of moving from a state to another state (including itself) that depends only on your current state.

Hence, probabilities of being in a particular state at time $n \geq 1$ depend on

(i) its state at the $(n-1)$-th time step, and

(ii) a fixed stochastic matrix $T \in M_k(\mathbb{Q}_+)$ called the **transition matrix** of the process.

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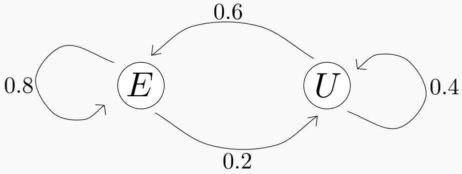
A **transition diagram** is a complete weighted directed graph with k vertices representing the states of the system and the edge from the j-th vertex to the i-th vertex labelled with the probability T_{ij}.

Cathy's Example is a Markov Process

Transition diagram:



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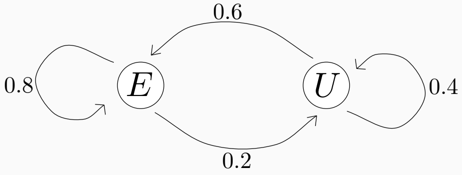

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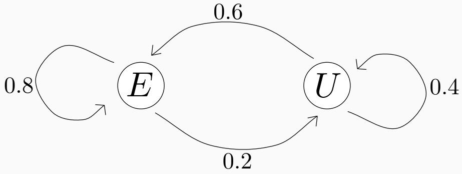

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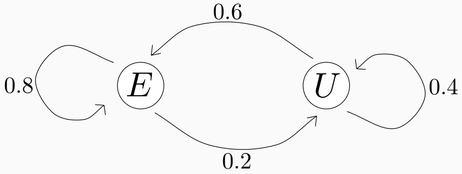

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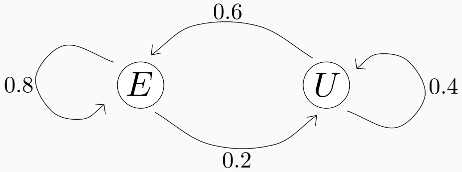

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Hence, $v_n = T^n v_0$ converges to $\begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$ as $n \to \infty$

Furthermore, the normalised steady state vector is $\begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$.

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- Because of this, Markov processes are said to “have no memory".

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- One way to find a steady state vector of a Markov process is to do as we did in the example - namely multiply together enough copies of T to see the higher powers tending to a limit.
- There are more direct methods of finding steady state vectors that use linear algebra. We will cover this in the next lecture using a larger example.