# Discrete Mathematical Models

**Lecture 25**

Kane Townsend

Semester 2, 2024

# D2: Weighted Graphs

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|---|---|
| Text Reference (Epp) | 3ed: Chapter 11 |
| | 4ed: Chapter 10 |
| | 5ed: Chapter 10 |

Some of the work is not covered in Epp.
but is based on some examples from:
Kolman, Busby & Ross *Discrete Mathematical Structures*
Johnsonbaugh *Discrete Mathematics*

## Weighted graphs

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  - **Fibre telecommunications:** Vertices are primary routing stations; there are edges between every pair of vertices (a complete graph); weights are the costs of laying fibre between stations.
  - **The internet:** Vertices are internet nodes; edges are all direct connections between nodes; weights are times (in milliseconds) for a packet to travel across a connection.

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We will also look at a different kind of problem on a weighted **directed** graph: Maximal Flow. Details later.

# Minimal Spanning Tree

# Kruskal's algorithm for minimal spanning tree

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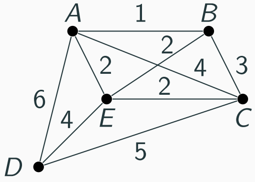
3. If adding $e$ to $T$ does not create a circuit in $T$, add $e$ to $T$ and add weight($e$) to $W$.
4. Repeat steps 2 and 3 until $T$ has $n - 1$ edges.

# Example: Applying Kruskal's algorithm

Find a minimal spanning tree for this weighted graph:



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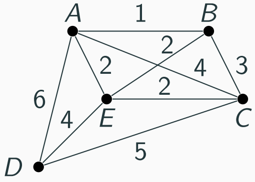


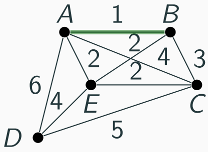


$W = 1$

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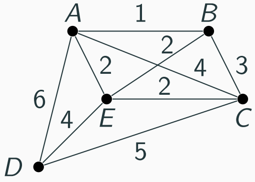


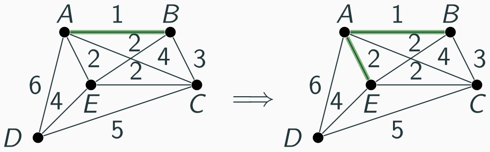


$W = 1$ $W = 3$

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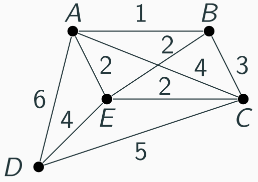
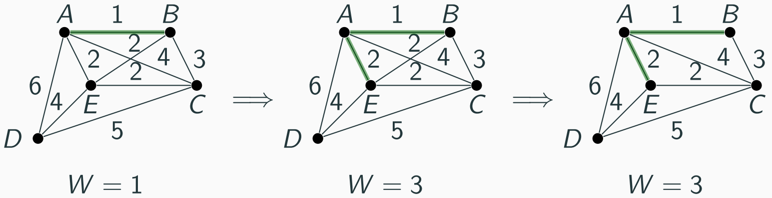
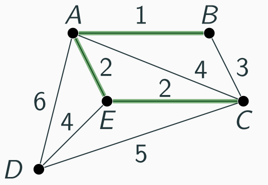


# Example: Applying Kruskal's algorithm (cont.)

$\Longrightarrow$ 


$W = 5$

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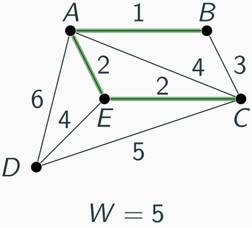
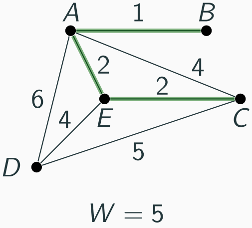
$\Longrightarrow$ 


$\Longrightarrow$ 

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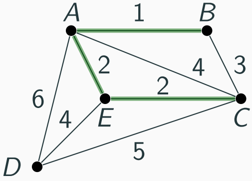


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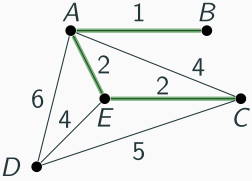


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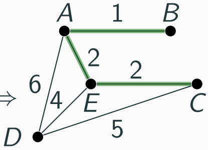


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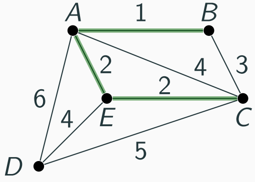


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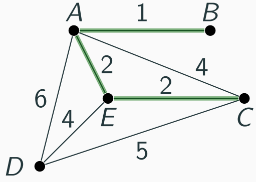


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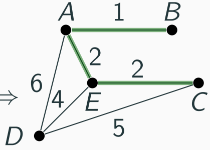


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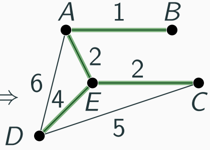


$W = 9$

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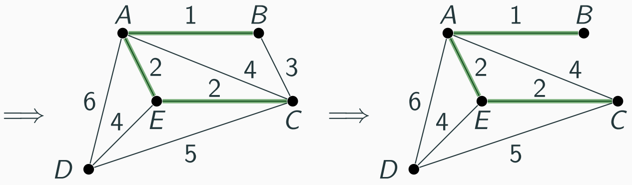
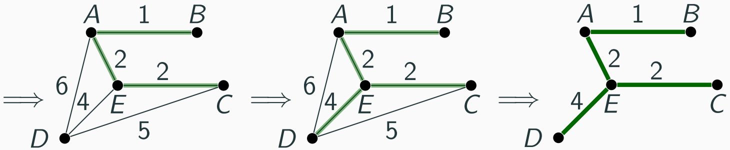



$W = 5$ $\qquad$ $W = 5$

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(The generated spanning tree)

# 'Greedy' algorithms

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- Kruskal's algorithm always succeeds! (Non-obvious theorem omitted)

  That is, it always finds a minimal spanning tree, given any weighted connected (finite) graph.

# ‘Nearest neighbour’ algorithm

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- The salesman needs to visit $n$ towns on a shortest possible circuit.
- Given: a table of distances between every pair of towns.
- **Model:** Graph $K_n$ with towns as vertices and edges weighted by the the inter-town distances.

  Find a Hamilton circuit of minimum possible total weight.

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6. Add weight$(L(n), L(1))$ to $W$. Append $L(1)$ to $L$ as $L(n+1)$.

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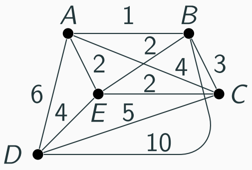
- But the Nearest Neighbour algorithm often fails to find the shortest Hamilton circuit.
- Greed doesn't always pay !!
- In fact, no efficient successful algorithm for the travelling salesman problem is known at this time. Finding one, or proving that none exists, is a major outstanding problem in mathematics.

# Example: Applying the Nearest Neighbour algorithm

Find a minimal Hamilton circuit for this weighted graph:

**Note:** This graph is as for the minimal spanning tree example but with the addition of an edge $BD$ to make it complete.



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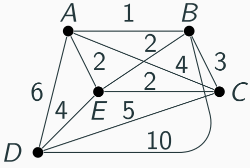


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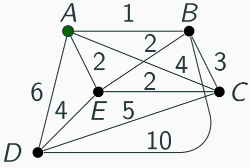


$L(1) = A, \ W = 0$

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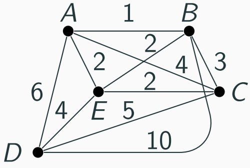


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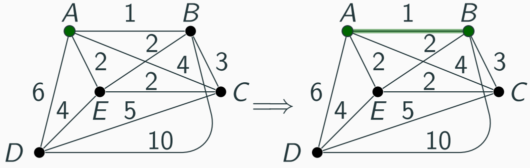


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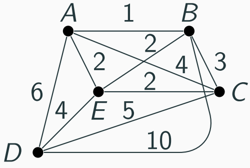


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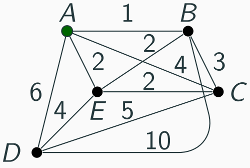


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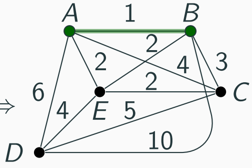
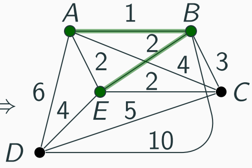
$\Longrightarrow$




$L(2) = B, \ W = 1$

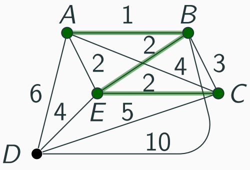
$\Longrightarrow$




$L(3) = E, \ W = 3$

# Example: Applying the Nearest Neighbour algorithm (cont.)

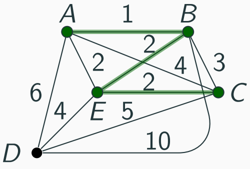
$\Longrightarrow$




$L(4) = C, \; W = 5$

## Example: Applying the Nearest Neighbour algorithm (cont.)

$\Longrightarrow$




$L(4) = C, \ W = 5$

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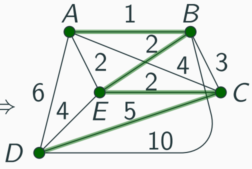


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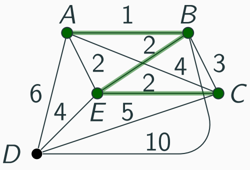


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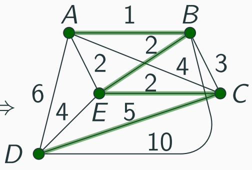


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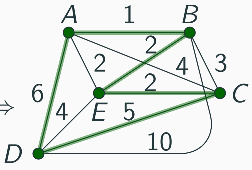


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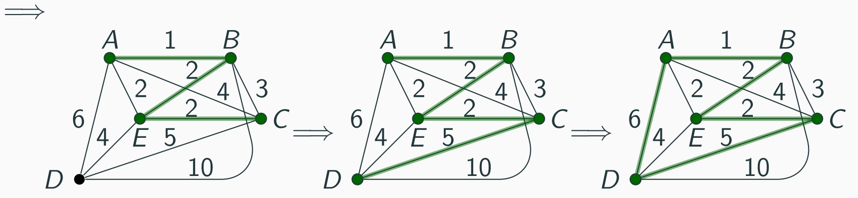


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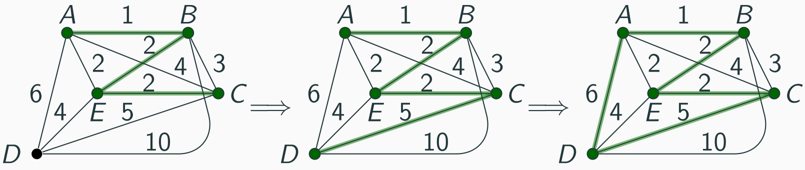


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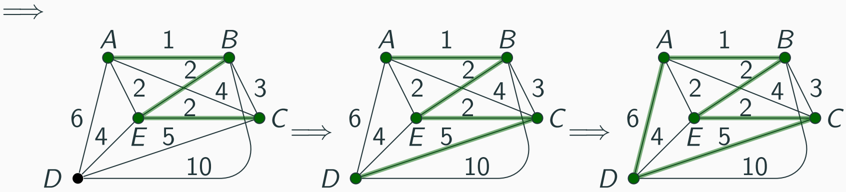


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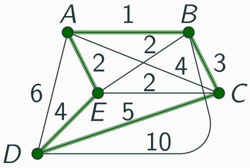


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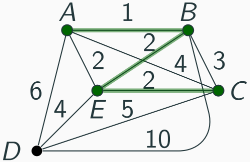


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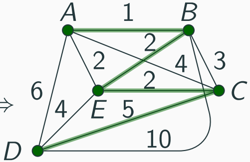


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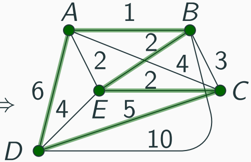


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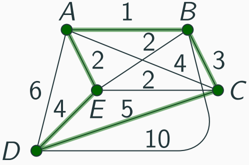


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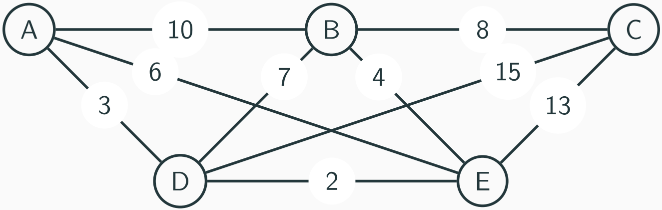
Note that Nearest Neighbour may generate this circuit if we start at $D$ instead of $A$. Then $L(2) = E$ and it just depends on the choice for $L(3)$.

# Shortest Path

# Shortest Path — Introduction

Consider this weighted graph:



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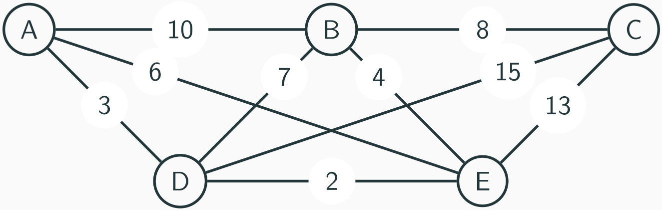


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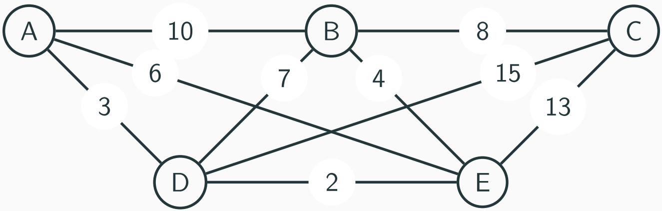


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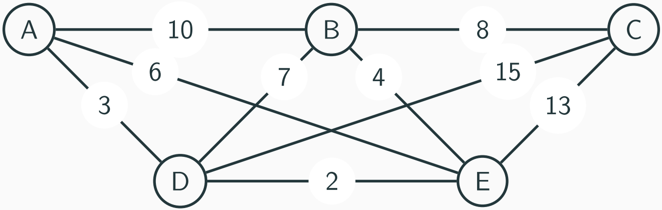


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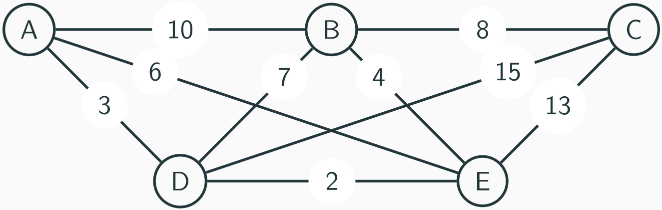


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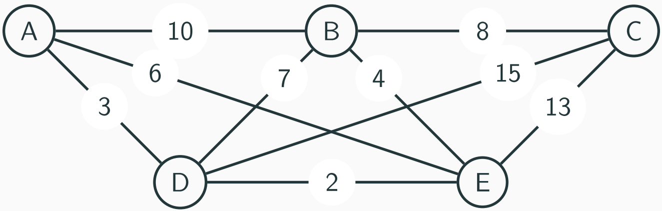


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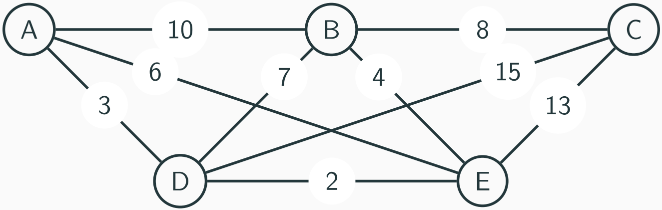


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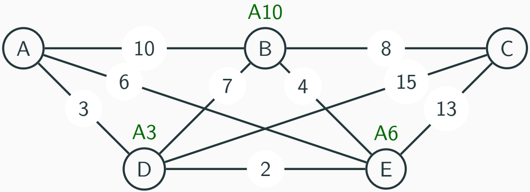
# Dijkstra's Algorithm




**Edsger Dijkstra 1930 - 2002**

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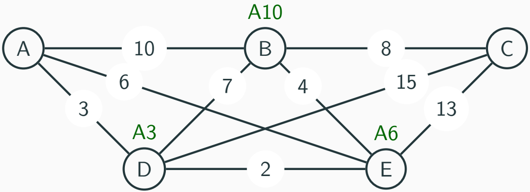


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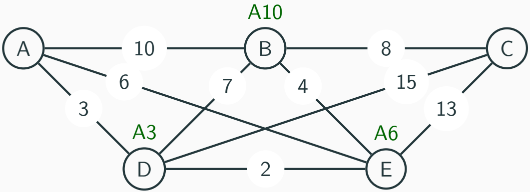


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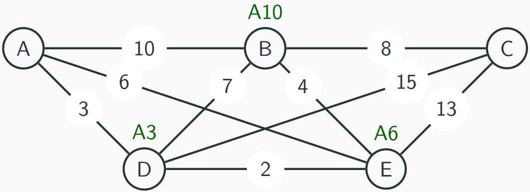


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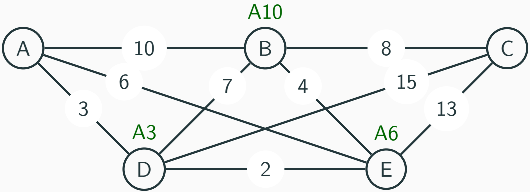


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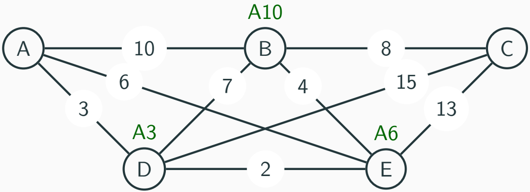


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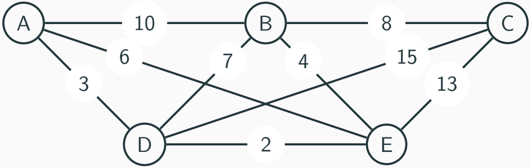
We launch straight in to demonstrating the algorithm on the above example, then we describe the algorithm.

# Example 1 — Slide 1

We seek a minimal weight path from A to C in the graph below.



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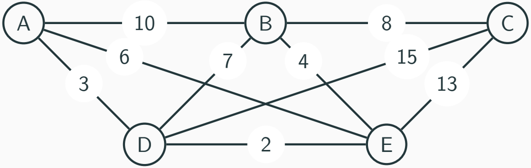


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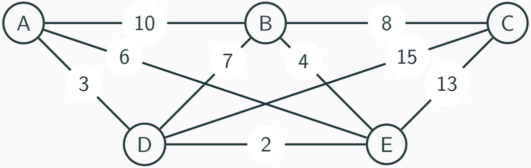


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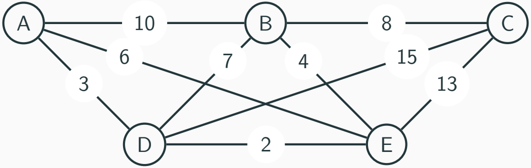


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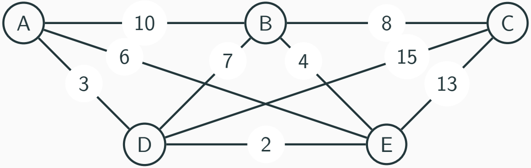


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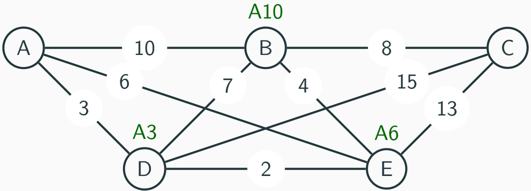

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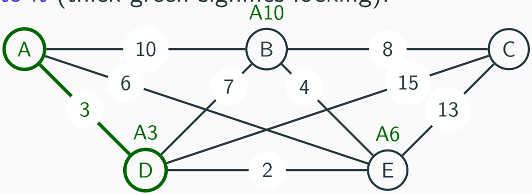

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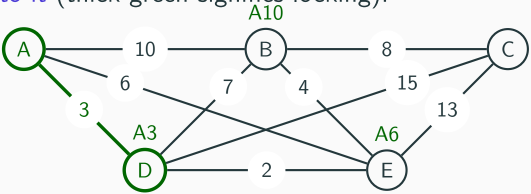


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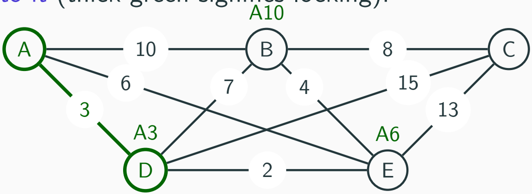


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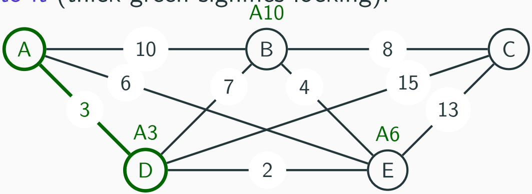


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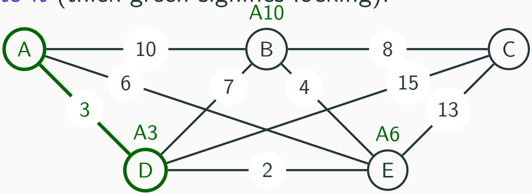


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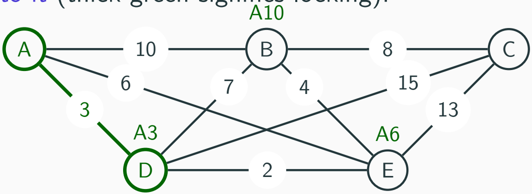


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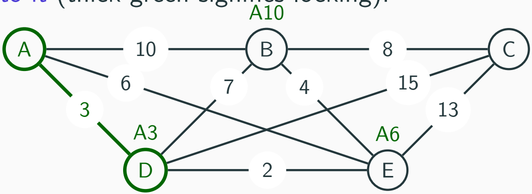


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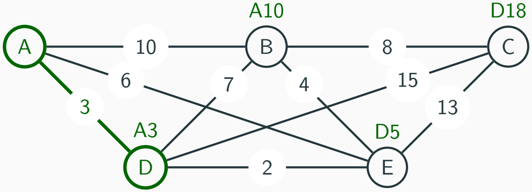
So we leave the A10 above B as it is.

The annotated graph now looks like this:



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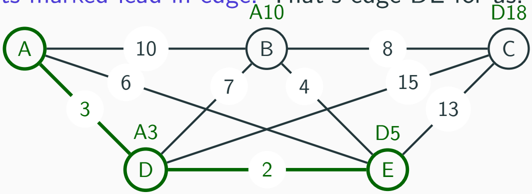

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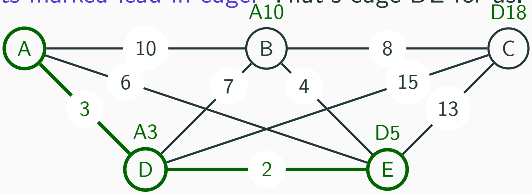


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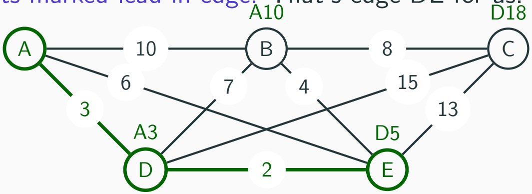


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Vertex E is adjacent to two un-locked vertices, B and C:

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Vertex E is adjacent to two un-locked vertices, B and C:

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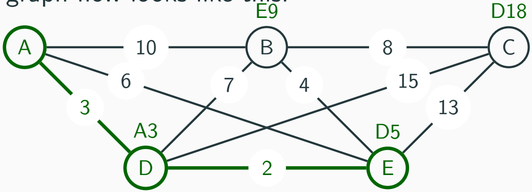
The updated graph now looks like this:



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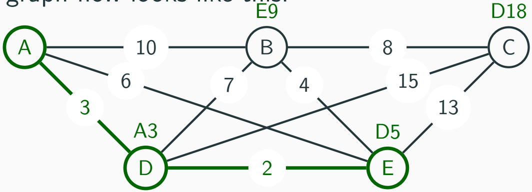
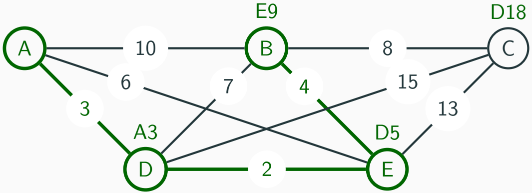
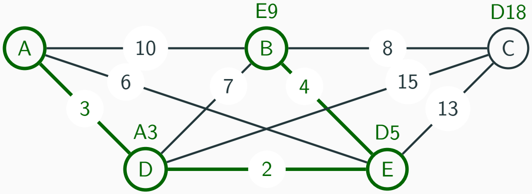



The lowest fringe value is now 9 on B, so we lock in B and its lead-in edge EB (next slide).

# Example 1 — Slide 6



# Example 1 — Slide 6




The new current vertex is the just locked-in B.

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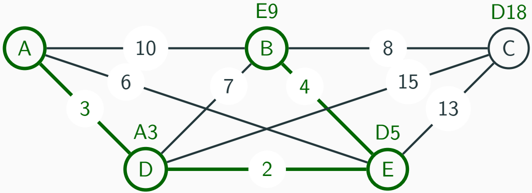


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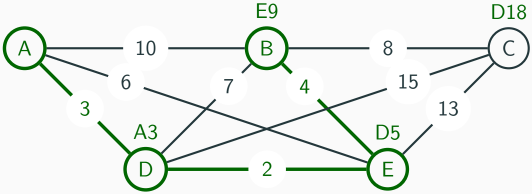


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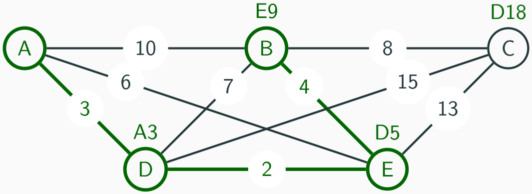


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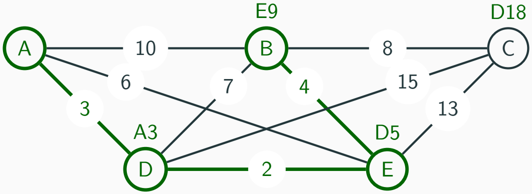


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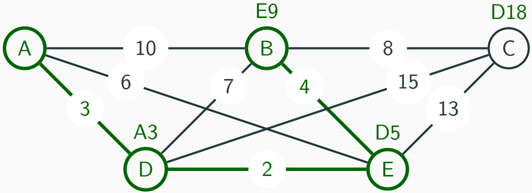


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We have now locked in the minimal distance 17 into our 'target' vertex C, so we can stop.

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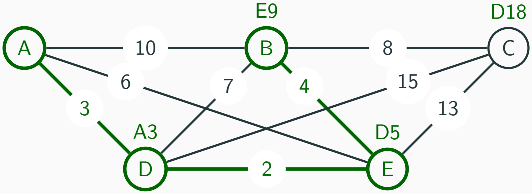


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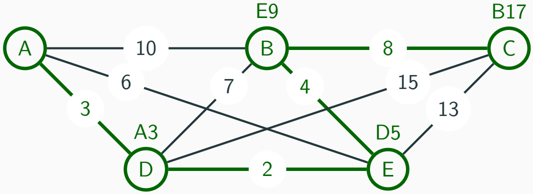
# Example 1 — Slide 7; Results and Comments



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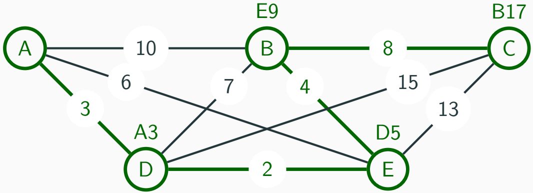


Some Observations:

- Besides the shortest path from A to C, the solution provides the shortest path to all the vertices along that path. For this example that happens to be the entire vertex set.

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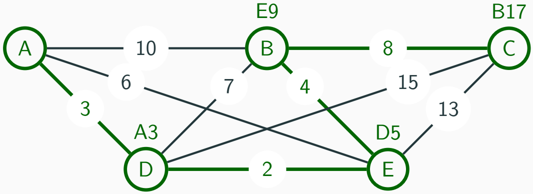


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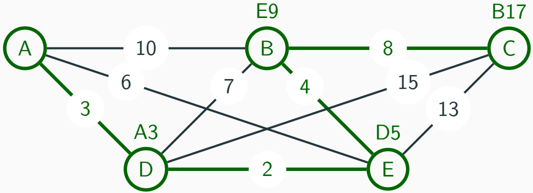


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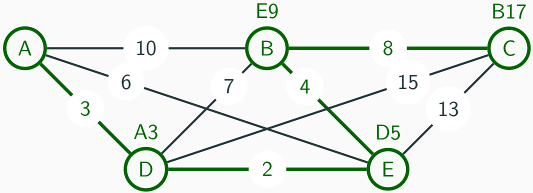


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- With all vertices locked, the solution provides a spanning tree for the graph.

# Dijkstra's Algorithm — A Formal Description

**Input:** (a) Connected simple graph $G$. Vertices A, Z from $G$.

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While $c \neq \mathsf{Z}$:

4. For each vertex $v$ adjacent to $c$ but not in $T$:
   If $v$ is unmarked (*i.e.* $M(v) = \mathsf{blank}$)
   or if $L(v) > L(c) + \mathsf{dist}(\{c, v\})$
   set $M(v) = c$, $L(v) = L(c) + \mathsf{dist}(\{c, v\})$.

5. From all marked $v \in G \setminus T$ (*i.e.* $M(v) \neq \textsf{blank}$ and $v \notin T$)
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This completes the formal description of Dijkstra's shortest path algorithm. Make some example weighted graphs and apply the algorithm for practice!