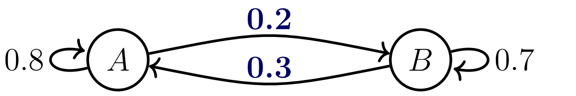
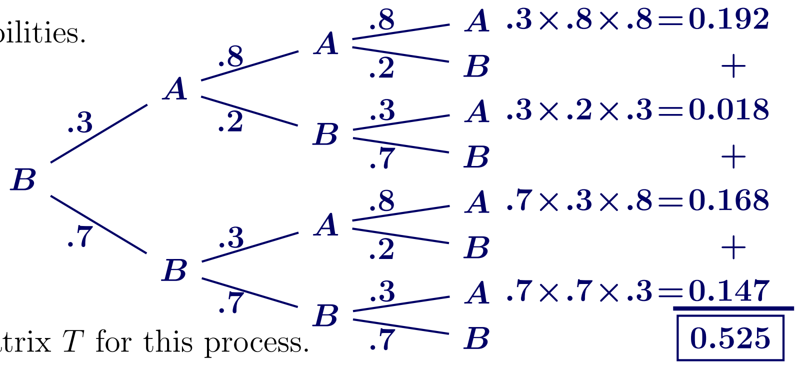
1. A Markov process has two states A and B with transition graph at right.



(a) Write in the missing probabilities.

(b) Suppose the system is initially in state B.
Use a tree diagram to find the probability that the system will be in state A after three steps.



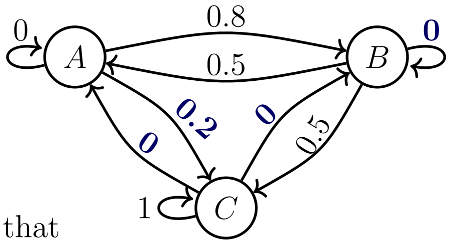
(c) Write out the transition matrix T for this process.

$$\boldsymbol{T} = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

(d) Use T to recalculate your answer to (b).

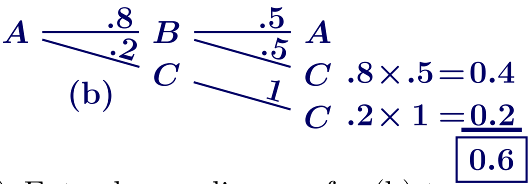
$\boldsymbol{TTT}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boldsymbol{TT}\begin{bmatrix} .3 \\ .7 \end{bmatrix} = \boldsymbol{T}\begin{bmatrix} .45 \\ .55 \end{bmatrix} = \begin{bmatrix} .525 \\ .475 \end{bmatrix}$, **so probability system in state $\boldsymbol{A}$ is 0.525.**

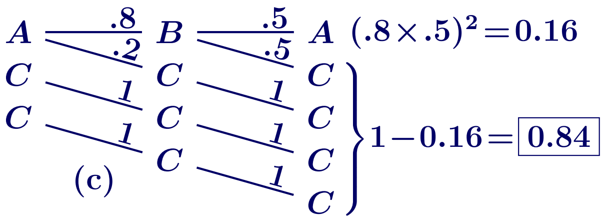
2. A Markov process has three states A, B and C with transition graph at right.



(a) Write in all the missing probabilities.

(b) Suppose the system is initially in state A.
Use a tree diagram to find the probability that the system will be in state C after two steps.
To simplify your diagram, leave out branches that have zero probability.





(c) Extend your diagram for (b) to cover four steps. What is the probability that the system will be in state C after four steps?

(d) Find the probability that the system will be in state C after ten steps starting from A. Do not use a diagram. Generalise from (c) and use complementary probability.

$$\boldsymbol{1 - (.8\times.5)^5 = 1 - 0.01024 = 0.98976.}$$

(e) As for (d), but starting from B.

$$\boldsymbol{1 - (.5\times.8)^5 = 1 - 0.01024 = 0.98976.}$$

(f) Guess the long-term probability that the system will be in state C, no matter what state the system starts in.

$$\boldsymbol{1 - (0.4)^n \to 1 - 0 = \boxed{1} \text{ as } n \to \infty.}$$

(g) Write out the transition matrix T for this process.

$$\boldsymbol{T} = \begin{bmatrix} 0 & .5 & 0 \\ .8 & 0 & 0 \\ .2 & .5 & 1 \end{bmatrix}$$

(h) Calculate T^2 and $T^4 = (T^2)^2$ and use them to confirm your answers to (b) and (c).

$$\boldsymbol{T^2} = \begin{bmatrix} 0 & .5 & 0 \\ .8 & 0 & 0 \\ .2 & .5 & 1 \end{bmatrix}\begin{bmatrix} 0 & .5 & 0 \\ .8 & 0 & 0 \\ .2 & .5 & 1 \end{bmatrix} = \begin{bmatrix} .4 & 0 & 0 \\ 0 & .4 & 0 \\ \boxed{.6} & .6 & 1 \end{bmatrix}$$

$$\boldsymbol{T^4} = \begin{bmatrix} .4 & 0 & 0 \\ 0 & .4 & 0 \\ .6 & .6 & 1 \end{bmatrix}\begin{bmatrix} .4 & 0 & 0 \\ 0 & .4 & 0 \\ \boxed{.6} & .6 & 1 \end{bmatrix} = \begin{bmatrix} .16 & 0 & 0 \\ 0 & .16 & 0 \\ \boxed{.84} & .84 & 1 \end{bmatrix}$$

(i) Convert your answer to (f) to a steady state vector S and confirm that answer by verifying that $TS = S$. $\boldsymbol{S} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 & .5 & 0 \\ .8 & 0 & 0 \\ .2 & .5 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

3. Ari is an innovative mathematics teacher. Once a week he sets up his classroom as four activity 'stations' labelled 1, 2, 3 and 4. Students spend 15 minutes at each station. In order to mix up the students, at change-over time Ari randomly divides the groups at stations 1 - 3 into two subgroups, as equally-sized as possible, and randomly sends one subgroup to the next station ($i \to i+1$) and the other subgroup to the station beyond ($i \to i+2$ except $3 \to 1$). Owing to the nature of the activity at station 2, Ari needs to limit the numbers at that station, so he starts with a smaller group there and at changeover time all students at station 4 move only to station 1.

(a) Compile a transition T matrix representing this (Markov) process. The states are the stations and entry t_{ij} of T specifies, for a student at station i, the probability that, at change-over, the student will move to station j.

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{.5} & \mathbf{1} \\ \mathbf{.5} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{.5} & \mathbf{.5} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{.5} & \mathbf{.5} & \mathbf{0} \end{bmatrix}$$

(b) Ari starts with six students at each of stations 1, 3 and 4, and five at station 2. The class lasts an hour. How many students will there be at each station when the class ends? [There are several possible answers here, since odd-sized groups cannot be equally subdivided. Flip a coin to decide to which station each larger subgroup goes.]

$$\boldsymbol{T}\begin{bmatrix}6\\5\\6\\6\end{bmatrix} = \begin{bmatrix} 0 & 0 & .5 & 1 \\ .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & .5 & .5 & 0 \end{bmatrix}\begin{bmatrix}6\\5\\6\\6\end{bmatrix} = \begin{bmatrix}9\\3\\5.5\\5.5\end{bmatrix} \to \begin{bmatrix}9\\3\\6\\5\end{bmatrix}^{\star} \qquad \boldsymbol{T}\begin{bmatrix}9\\3\\6\\5\end{bmatrix} = \begin{bmatrix} 0 & 0 & .5 & 1 \\ .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & .5 & .5 & 0 \end{bmatrix}\begin{bmatrix}9\\3\\6\\5\end{bmatrix} = \begin{bmatrix}8\\4.5\\6\\4.5\end{bmatrix} \to \begin{bmatrix}8\\4\\6\\5\end{bmatrix}^{\star}$$

$\star$: Total number of students stays at 23.

(c) Verify that the steady state is eight students at station 1, four at station 2, six at station 3 and five at station 4.

$$\boldsymbol{T}\begin{bmatrix}8\\4\\6\\5\end{bmatrix} = \begin{bmatrix} 0 & 0 & .5 & 1 \\ .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & .5 & .5 & 0 \end{bmatrix}\begin{bmatrix}8\\4\\6\\5\end{bmatrix} = \begin{bmatrix}8\\4\\6\\5\end{bmatrix}$$

4. A certain Markov Process has transition matrix T at right. Using a computer some powers of T were calculated and are shown below to three decimal places.

$$T = \begin{bmatrix} .6 & .3 & .2 & .1 \\ .2 & .3 & 0 & .2 \\ 0 & .2 & .2 & .1 \\ .2 & .2 & .6 & .6 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} .44 & .33 & .22 & .20 \\ .22 & .19 & .16 & .20 \\ .06 & .12 & .10 & .12 \\ .28 & .36 & .52 & .48 \end{bmatrix} \quad T^4 = \begin{bmatrix} .335 & .306 & .276 & .276 \\ .204 & .200 & .199 & .197 \\ .092 & .099 & .105 & .106 \\ .368 & .396 & .421 & .421 \end{bmatrix} \quad T^8 = \begin{bmatrix} .302 & .300 & .299 & .299 \\ .200 & .200 & .200 & .200 \\ .100 & .100 & .100 & .100 \\ .398 & .400 & .401 & .401 \end{bmatrix}.$$

Use the powers of T to guess a steady state vector for the process, and then prove your guess is correct. **Columns of $\boldsymbol{T^8}$ are all approximately [.3 .2 .1 .4].**

$$\textbf{So guess } \boldsymbol{S} = \begin{bmatrix}.3\\.2\\.1\\.4\end{bmatrix}. \qquad \textbf{Then } \boldsymbol{TS} = \begin{bmatrix} .6 & .3 & .2 & .1 \\ .2 & .3 & 0 & .2 \\ 0 & .2 & .2 & .1 \\ .2 & .2 & .6 & .6 \end{bmatrix}\begin{bmatrix}.3\\.2\\.1\\.4\end{bmatrix} = \begin{bmatrix}.3\\.2\\.1\\.4\end{bmatrix} = \boldsymbol{S}.$$

5. A certain Markov Process has transition matrix T at right. Use a matrix calculation tool such as `https://matrixcalc.org/en/` to calculate T^{16} to three decimal places.

$$T = \begin{bmatrix} .4 & .4 & 0 & .7 \\ .1 & .1 & .3 & .2 \\ 0 & .2 & .4 & .1 \\ .5 & .3 & .3 & 0 \end{bmatrix}$$

Use T^{16} to guess a steady state vector for the process, and then prove your guess is correct. (If your matrix tool doesn't have a powering function but does have a multiplication function you could first calculate $T \times T = T^2$ then $T^2 \times T^2 = T^4$ and so on. Depending on the tool, with lots of cut-and-paste the only matrix you may need to enter is T.)

$$\boldsymbol{T}^{16} \approx \begin{bmatrix} .450 & .450 & .450 & .450 \\ .150 & .150 & .150 & .150 \\ .100 & .100 & .100 & .100 \\ .300 & .300 & .300 & .300 \end{bmatrix}; \ \boldsymbol{S} = \begin{bmatrix}.45\\.15\\.1\\.3\end{bmatrix}. \ \boldsymbol{TS} = \begin{bmatrix} .4 & .4 & 0 & .7 \\ .1 & .1 & .3 & .2 \\ 0 & .2 & .4 & .1 \\ .5 & .3 & .3 & 0 \end{bmatrix}\begin{bmatrix}.45\\.15\\.1\\.3\end{bmatrix} = \begin{bmatrix}.45\\.15\\.1\\.3\end{bmatrix} = \boldsymbol{S}.$$

6. Calculate the steady state vector S for the Markov process of Question 1. Do this by hand, using the matrix inverse method with short cut to solve $TS = S$.

$$\boldsymbol{T-I}=\begin{bmatrix}-.2 & .3\\ .2 & -.3\end{bmatrix}.\ \textbf{Solve}\ \begin{bmatrix}-.2 & .3\\ 1 & 1\end{bmatrix}\boldsymbol{S}=\begin{bmatrix}0\\1\end{bmatrix}.$$

$$\boldsymbol{S}=\begin{bmatrix}-.2 & .3\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{-.2-.3}\begin{bmatrix}1 & -.3\\ -1 & -.2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}.6\\.4\end{bmatrix}.$$

7. Recalculate the steady state vector for the Markov process of Question 5 by using the 'Gauss-Jordan Elimination' function in the 'Matrix Reshish'[1] computer application `https://matrix.reshish.com`. Specify your input to, and output from, Reshish.

As for Q6, solve $(\boldsymbol{T-I})\boldsymbol{S}=0$ with last row replaced by all 1's.

The input augmented matrix : $\begin{bmatrix}-0.6 & 0.4 & 0 & 0.7 & 0\\ 0.1 & -0.9 & 0.3 & 0.2 & 0\\ 0 & 0.2 & -0.6 & 0.1 & 0\\ 1 & 1 & 1 & 1 & 1\end{bmatrix}$, **Output:** $\begin{matrix}x_1=0.45\\ x_2=0.15\\ x_3=0.1\\ x_4=0.3\end{matrix}$, **so** $\boldsymbol{S}=\begin{bmatrix}0.45\\0.15\\0.1\\0.3\end{bmatrix}$.

8. Let T be an $n\times n$ stochastic matrix (rows are probability vectors) and $\mathbf{v}$ a column probability n-vector. Prove that $T\mathbf{v}$ is always also a probability vector. Try this first for $n = 2$ and then for $n = 3$. Do it for general n if your algebra is up to it.

For any n the entries of T and v are non-negative, so same is true for Tv.

For $n=2$ let $\boldsymbol{T}=\begin{bmatrix}p_1 & q_1\\ p_2 & q_2\end{bmatrix}$, $\mathrm{v}=\begin{bmatrix}p_0\\ q_0\end{bmatrix}$. **Then** $\boldsymbol{T}\mathrm{v}=\begin{bmatrix}p_1p_0+p_2q_0\\ q_1p_0+q_2q_0\end{bmatrix}$ **and entries sum to**
$(p_1p_0+p_2q_0)+(q_1p_0+q_2q_0)=(p_1+q_1)p_0+(p_2+q_2)q_0=p_0+q_0=1.$

For $n=3$, $\boldsymbol{T}=\begin{bmatrix}p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\\ p_3 & q_3 & r_3\end{bmatrix}$, $\mathrm{v}=\begin{bmatrix}p_0\\ q_0\\ r_0\end{bmatrix}$, $\boldsymbol{T}\mathrm{v}=\begin{bmatrix}p_1p_0+p_2q_0+p_3r_0\\ q_1p_0+q_2q_0+q_3r_0\\ r_1p_0+r_2q_0+r_3r_0\end{bmatrix}$ **and entries sum to**

$$(p_1p_0+p_2q_0+p_3r_0)+(q_1p_0+q_2q_0+q_3r_0)+(r_1p_0+r_2q_0+r_3r_0)$$
$$=(p_1+q_1+r_1)p_0+(p_2+q_2+r_2)q_0+(p_3+q_3+r_3)r_0=p_0+q_0+r_0=1.$$

9. *Hardy-Weinburg Equilibrium*: Consider a gene that has two forms, or *alleles*, A and a. Each individual has two of these genes and so has *genotype* AA, Aa or aa.
Assume that an individual's genotype consists of a random selection of one each of its parents' alleles. So, for example, the offspring of parents who are both Aa has a 50% chance of also being Aa and a 25% chance each of being AA and aa.
Assume further that mating partners are chosen at random.
Let π_{AA}, π_{Aa} and π_{aa} be the proportions of each genotype in a breeding colony.

(a) Explain why $p = \pi_{AA}+\pi_{Aa}/2$ and $q = \pi_{aa}+\pi_{Aa}/2$ are the probabilities that a random allele chosen from a random individual is A or a respectively.

$$\begin{aligned}\mathbb{P}(\text{allele is } A) &= \mathbb{P}(\text{genotype is } AA)\mathbb{P}(A \text{ chosen from } AA)\\ &\quad+\mathbb{P}(\text{genotype is } Aa)\mathbb{P}(A \text{ chosen from } Aa)\\ &\quad+\mathbb{P}(\text{genotype is } aa)\mathbb{P}(A \text{ chosen from } aa)\\ &= \pi_{AA}\times 1+\pi_{Aa}\times {}^1\!/_2+\pi_{aa}\times 0 = \pi_{AA}+\pi_{Aa}/2\end{aligned}$$

Calculation of $\mathbb{P}$(allele is a) is similar.

(b) Explain why the parent-to-offspring transition matrix is given by $T = \begin{array}{c|ccc} & AA & Aa & aa\\ \hline AA & p & p/2 & 0\\ Aa & q & 1/2 & p\\ aa & 0 & q/2 & q\end{array}$.

Here are explanations for two representative sample entries:

AA parent gives A with prob. 1, mate gives A with prob. p $\therefore t_{11}=1p=p$.
Aa parent gives A with prob. $1/2$, mate gives a with prob. q and
Aa parent gives a with prob. $1/2$, mate gives A with prob. p $\therefore t_{22}=q/2+p/2=1/2$.

(c) Show that the steady state vector is $S=\begin{bmatrix}p^2\\ 2pq\\ q^2\end{bmatrix}$.

$$\boldsymbol{TS}=\begin{bmatrix}p^3+p^2q\\ p^2q+pq+pq^2\\ pq^2+q^3\end{bmatrix}=\begin{bmatrix}p^2(p+q)\\ pq(p+1+q)\\ q^2(p+q)\end{bmatrix}=\boldsymbol{S}\ \textbf{as}\ p+q=1.$$

(d) Show that S is always achieved in just one transition step.

[1] When entering decimal values, Reshish requires a digit before the decimal point. *e.g.* enter '0.4', not '.4'.

$$\boldsymbol{T}\begin{bmatrix}\pi_{AA}\\ \pi_{Aa}\\ \pi_{aa}\end{bmatrix}=\begin{bmatrix}p & \frac{p}{2} & 0\\ q & \frac{p}{2}+\frac{q}{2} & p\\ 0 & \frac{q}{2} & q\end{bmatrix}\begin{bmatrix}\pi_{AA}\\ \pi_{Aa}\\ \pi_{aa}\end{bmatrix}=\begin{bmatrix}p(\pi_{AA}+\frac{1}{2}\pi_{Aa})\\ q(\pi_{AA}+\frac{1}{2}\pi_{Aa})+p(\pi_{aa}+\frac{1}{2}\pi_{Aa})\\ q(\pi_{aa}+\frac{1}{2}\pi_{Aa})\end{bmatrix}=\begin{bmatrix}pp\\ qp+pq\\ qq\end{bmatrix}=\boldsymbol{S}.$$