Discrete Mathematical Models

Lecture 12

Kane Townsend

Semester 2, 2024

Section B: Digital Information (cont.)

Section B2: Sequences, Induction, Sorting (cont.)

From implicit to explicit definitions; Example 4

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$r = 1.03$, $d = -10$ and $c = 2 \times 10^4$.

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So the \$10 annual fee over 10 years costs the investment \$114.64.

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It takes very little extra analysis to generalise the previous example:

Mixed Geometric-Arithmetic Sequence	
Implicit Definition	Explicit Definition
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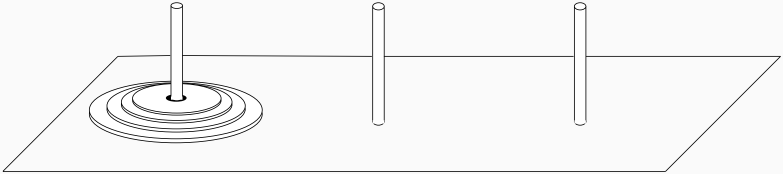
Counting moves: Towers of Hanoi

"The Towers of Hanoi" is a puzzle with 3 pegs and a number of punctured discs of decreasing sizes initially on the leftmost peg.



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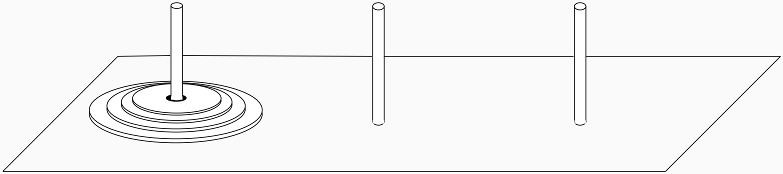

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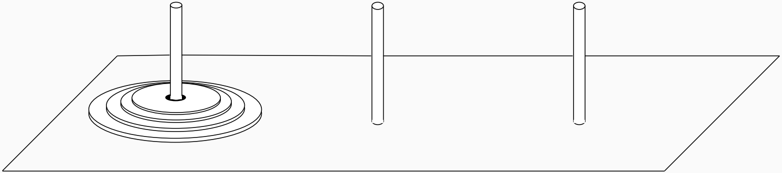

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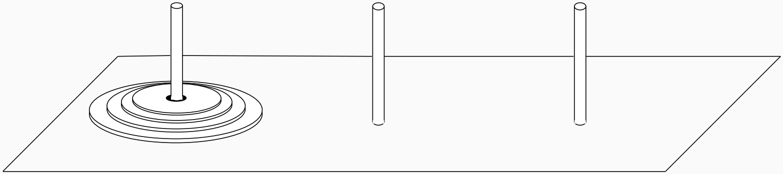

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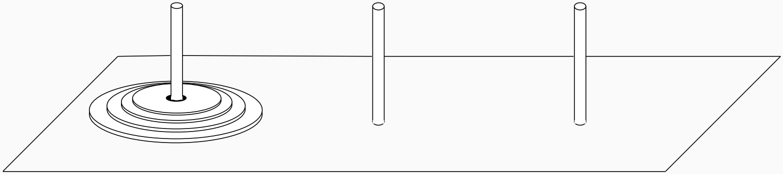

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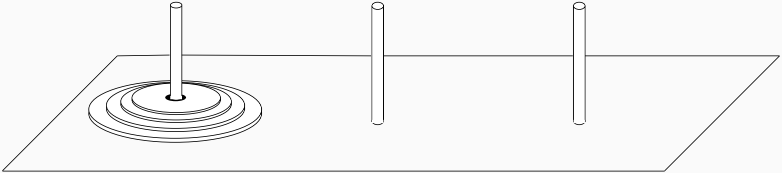

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At one move per second, how fast can you solve a puzzle with 64 discs?

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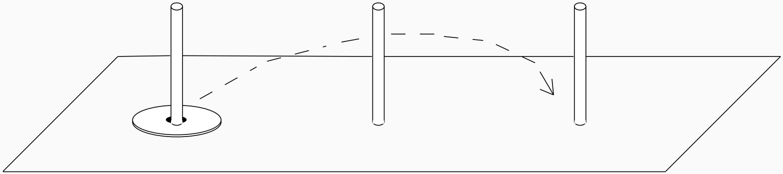

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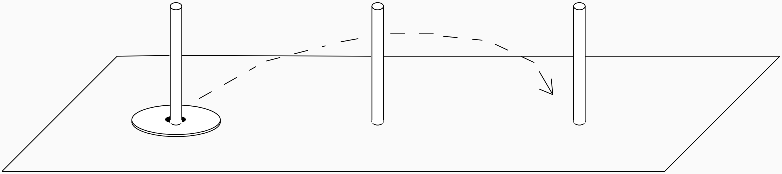

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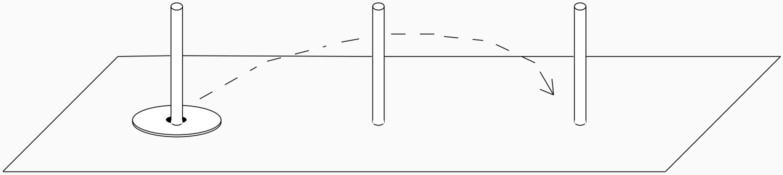

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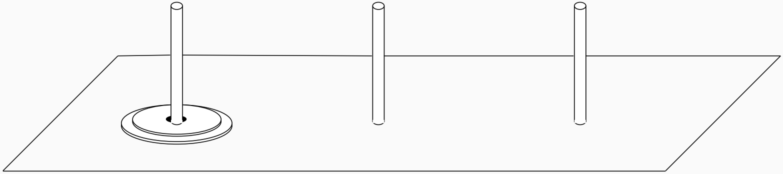

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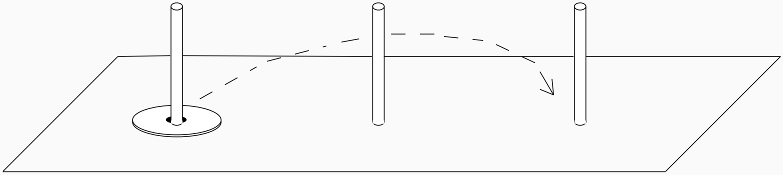

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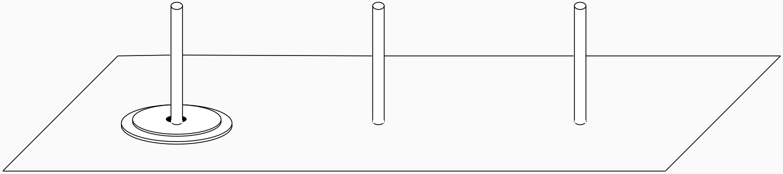

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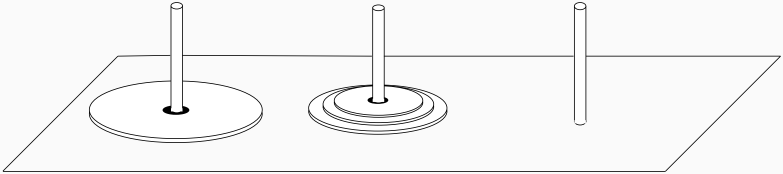

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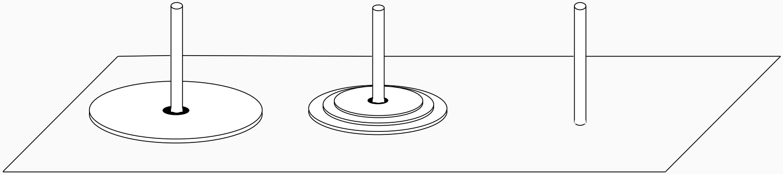

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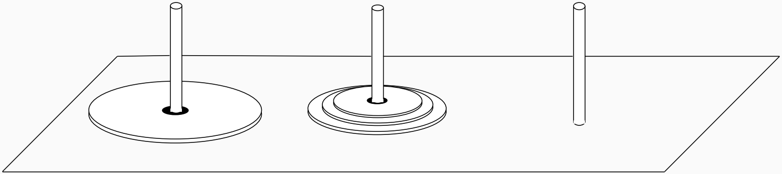

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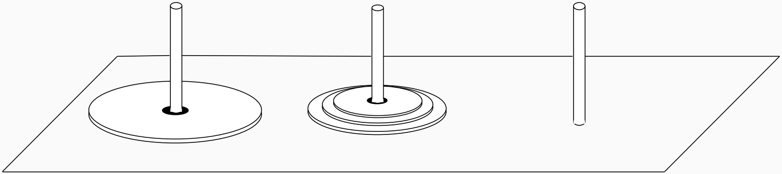

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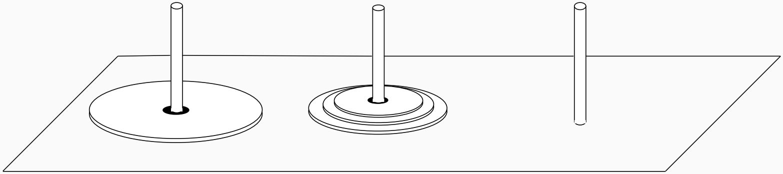

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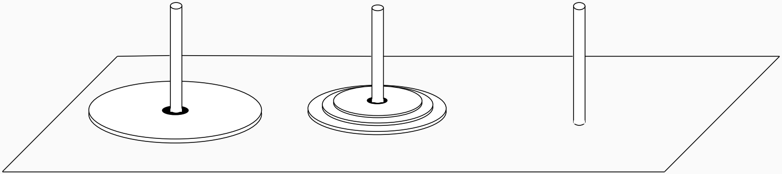

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In particular $x_{64} = (2^{64} - 1)$ seconds $\sim 5.8 \times 10^{11}$ years.

Sorting

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Example:

$(x_n)_{n\in\{1,\ldots,5\}} =$ Jane, Fred, Jo, Jane, Ann
$(y_n)_{n\in\{1,\ldots,5\}} =$ Ann, Fred, Jane, Jane, Jo (in alphabetical order)

Sorting preliminaries

An **index set** I is a set of the form

$$I = \{i \in \mathbb{N}^\star \,:\, s \leq i \leq f\} = \{s, \ldots, f\}$$

where $s, f \in \mathbb{N}^\star,\ s \leq f$, are the **start index** and the **finish index**.

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$$\pi = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 5 \end{pmatrix} \text{ means } \begin{array}{l} I = \{3, 4, 5, 6\} \\ \pi(3){=}6, \ \pi(4){=}4, \ \pi(5){=}3, \ \pi(6){=}5 \end{array}.$$

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Example: (names example recast using an index permutation)

If $(x_n)_{n\in\{1,\ldots,5\}}$ = Jane, Fred, Jo, Jane, Ann

then $(x_{\pi(n)})_{n\in\{1,\ldots,5\}}$ = Ann, Fred, Jane, Jane, Jo

where $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix}$ sorts the sequence into alphabetical order.

Least element algorithm

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Example: $(x_i)_{1..5} = (J, O, E)$ with index set $\{1, 2, 3\}$. Our index function begins as $\pi(i) = i$. We want to know how to rearrange (J, O, E) so that the smallest letter with respect to alphabetical order can be first.

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We begin with our marker $m = 1$ and index $i = 1 + 1 = 2$, and we compare $x_1 = J$ and $x_{1+1} = x_2 = O$. Since J is before O in the alphabet we keep $m = 1$ as our marker (least in sequence tested so far). We increase our index by 1, $i = 2 + 1 = 3$.

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We now put our marker in the first place by modifying π to $\pi(1) = 3$, $\pi(2) = 2$ and $\pi(3) = 1$.

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Example: $(s = 1,\ f = 6)$

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before	$\pi(i)$	1	2	3	4	5	6
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i	2	3	4	5	6	7
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	i	1	2	3	4	5	6
after	$\pi(i)$	**5**	2	3	4	**1**	6
after	$x_{\pi(i)}$	**B**	D	C	E	**F**	C

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We complete stop the algorithm since we reached the end of our indexing.

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Loop: If $s = n$ stop.

Run least element algorithm on $(x_{\pi(i)})_{s..n}$

$s \leftarrow s + 1$

Repeat loop

Selection sort algorithm

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Output: Modification to π, so that $(x_{\pi(i)})_{1..n}$ is in increasing order $x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}$.
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Example:

	i:	1	2	3	4	5	6
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	$x_{\pi(i)}$	B	**C**	**D**	E	F	C
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	$x_{\pi(i)}$	B	C	**C**	E	F	**D**
$s = 4$							
After 4th iteration	$\pi(i)$	5	3	6	**2**	1	**4**
	$x_{\pi(i)}$	B	C	C	**D**	F	**E**

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	$x_{\pi(i)}$	B	**C**	**D**	E	F	C
$s = 3$							
After 3rd iteration	$\pi(i)$	5	3	**6**	4	1	**2**
	$x_{\pi(i)}$	B	C	**C**	E	F	**D**
$s = 4$							
After 4th iteration	$\pi(i)$	5	3	6	**2**	1	**4**
	$x_{\pi(i)}$	B	C	C	**D**	F	**E**
$s = 5$							
After final iteration	$\pi(i)$	5	3	6	2	**4**	**1**
	$x_{\pi(i)}$	B	C	C	D	**E**	**F**

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So:	1st	iteration uses	$n-1$	comparisons
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That is: $\quad \forall \boldsymbol{n} \in \quad \boldsymbol{T_n} = \dfrac{\boldsymbol{n(n-1)}}{2}.$

Other sorting algorithms

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As with Selection Sort, Merge Sort makes use of a sub-algorithm, which we treat first.