Discrete Mathematical Models

Lecture 5

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Section A2: Sets (continued)

Making new sets from old

Suppose that A and B are subsets of a universe U.

The **union** of A and B, denoted $A \cup B$, is the set

$$\{x \in U \mid (x \in A) \vee (x \in B)\}.$$

The **intersection** of A and B, denoted $A \cap B$, is the set

$$\{x \in U \mid (x \in A) \wedge (x \in B)\}.$$

The **difference** of B minus A, or B **without** A, denoted $B - A$ or $B \setminus A$, is the set

$$\{x \in U \mid (x \in B) \wedge (x \notin A)\}.$$

Making new sets from old

Suppose that A and B are subsets of a universe U.

The **complement** of A (in U), denoted A^c, is the set

$$\{x \in U \mid x \notin A\}$$

The complement of A cannot be understood unless the universe of discourse has been communicated.

The **symmetric difference** of A and B, denoted $A \triangle B$, is the set

$$\{x \in U \mid (x \in A) \oplus (x \in B)\}.$$

Some examples

Suppose that the universe of discourse is the set $\mathbb{Z}$ and let

$$\begin{aligned} &O \text{ be the set of odd integers} \\ &E \text{ be the set of even integers} \\ &P \text{ be the set of primes} \\ &C \text{ be the set of composite numbers.} \end{aligned}$$

A **composite number** is a positive integer that can be formed by multiplying two smaller positive integers.

Find simple expressions for: $O \cup E$, $O \cap E$, $E \cap P$, $O \cap P$, $P \cup C$, O^c, P^c, $E \triangle P$, $(O \triangle P) \cap \mathbb{Z}^+$

Some examples

$$
\begin{aligned}
O \cup E &= \\
O \cap E &= \\
E \cap P &= \\
O \cap P &= \\
P \cup C &= \\
O^c &= \\
P^c &= \\
E \triangle P &= \\
(O \triangle P) \cap \mathbb{Z}^+ &=
\end{aligned}
$$

Some examples

$$
\begin{aligned}
O \cup E &= \mathbb{Z} \\
O \cap E &= \emptyset \\
E \cap P &= \{2\} \\
O \cap P &= P \setminus \{2\} \\
P \cup C &= \{2, 3, 4, 5, \dots, \} \\
&= \{x \in \mathbb{Z} \mid x \geq 2\} \\
O^c &= E \\
P^c &= \{\dots, -3, -2, -1, 0, 1\} \cup C \\
&= \{z \in \mathbb{Z} \mid z \leq 1\} \cup C \\
E \triangle P &= (E \cup P) \setminus \{2\} \\
(O \triangle P) \cap \mathbb{Z}^+ &= (O \cap C) \cup \{1, 2\}.
\end{aligned}
$$

Using logic to prove set identities

Since the set operations $\cup$, $\cap$, $\setminus$, $\subseteq$, ^{c} and $\triangle$ are defined using logical connectives, logical equivalences can be used to proved set theoretic identities (an identity is a relationship that holds not matter which substitutions are made for the variables).

An example

Let A, B and C be subsets of a universe U. Then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof.

$$\begin{aligned}
& x \in A \cap (B \cup C) & \\
\Leftrightarrow & (x \in A) \wedge (x \in B \cup C) & \text{Defn of } \cap \\
\Leftrightarrow & (x \in A) \wedge (x \in B \vee x \in C) & \text{Defn of } \cup \\
\Leftrightarrow & ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C)) & \text{Distr.} \\
\Leftrightarrow & (x \in A \cap B) \vee (x \in A \cap C) & \text{Defn of } \cap \\
\Leftrightarrow & x \in (A \cap B) \cup (A \cap C) & \text{Defn of } \cup
\end{aligned}$$

$\square$

Another way to construct a new set from an old set

For any set A, the power set of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

For example, if $A = \{1, 2, 3\}$, then

$$\mathcal{P}(A) \qquad = \qquad \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Q: If A has n elements, how many elements does $\mathcal{P}(A)$ have?

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Q: If A has n elements, how many elements does $\mathcal{P}(A)$ have?

A: $\mathcal{P}(A)$ has 2^n elements... for reasons we will explain when we discuss counting techniques later in the course.

Cartesian products: Another way to make new sets from old

Order and multiplicity

In sets, there is no sense of the order in which elements appear and there is no idea of how many times an elements appears.

However, in many situations the order in which data appears is important, and the same data sometimes appears multiple times.

We now look at a construction that allows us to represent order and multiplicity.

Ordered n-tuples (see p.11 of our optional text).

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple** $(x_1, x_2, \ldots, x_n)$ consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair**, and an ordered 3-tuple is called an **ordered triple**.

Two ordered n-tuples are **equal** when their elements match up exactly in order. Symbolically:

$$
\begin{aligned}
&(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \\
\Leftrightarrow &(x_1 = y_1) \wedge (x_2 = y_2) \wedge \cdots \wedge (x_n = y_n).
\end{aligned}
$$

An ordered m-tuple and an ordered n-tuple cannot be equal if $m \neq n$.

Examples

$(a, b, c) \neq (b, c, a)$ because their first elements differ.

$(a, a, b, c) \neq (a, b, c)$ because one is an ordered 4-tuple and the other is an ordered triple.

The elements in ordered n-tuples do not need to be of the same type. For example, $(\text{cat}, \text{car}, 1, \$)$ is an ordered 4-tuple.

We are, however, usually interested in sets of ordered n-tuples where all of the elements in, say, the i-th position are of the "same type"...

Cartesian product

Given (not necessarily distinct sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$, $\ldots$, $a_n \in A_n$.

$$\{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

This last expression may be read aloud as:

Cartesian product

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$$\{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

This last expression may be read aloud as: "the set of all ordered n-tuples with elements a_1, a_2, through, a_n such that a_1 comes from A_1, a_2 comes from A_2, through a_n comes from A_n."

A word about the notation just used

The expression

$$\{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

does not appear to conform to the rules of set-builder notation we laid out in the last lecture because

- the domain part introduces variables but does not specify a domain for each;
- the "predicate" does not appear to be a single predicate.

We can fix the second concern easily by making a rule that in a predicate, each comma is read and understood as "and". It is usually better to use $\wedge$.

What I have written is an entirely standard way to describe a Cartesian product, even though it seems like a poor use of set-builder notation.

Examples

Let

$$\begin{aligned} A &= \{\text{cat}, \text{dog}, \text{chicken}\} \\ B &= \{\text{yes}, \text{no}\} \\ C &= \{100, 300\} \end{aligned}$$

Then

$$\begin{aligned} A \times B = \{&(\text{cat}, \text{yes}), (\text{cat}, \text{no}), (\text{dog}, \text{yes}), (\text{dog}, \text{no}), \\ &(\text{chicken}, \text{yes}), (\text{chicken}, \text{no})\}. \end{aligned}$$

and

$$C \times C = \{(100, 100), (100, 300), (300, 100), (300, 300)\}.$$

Strings and languages

Let A be a set. A **string** (or **word**) of length n over (the **alphabet**) A is an ordered n-tuple, written without parentheses or commas, in which every element is taken from A. The **null string** (or **empty word**) over A is the "string" with no characters. The null string over A is denoted λ and said to have length 0. We write A^* for the set of all strings over the alphabet A. Any subset of A^* is called a **language**.

Example

Let $A = \{0, 1\}$. Then a string over A is called a bit string.

The set $\mathcal{L} = \{w \in A^* \mid w \text{ has length } 8\}$ is the language of all bit strings of length 8 (that is, the language of all bytes).

Partitions: A structure for recognising that a classification works well

Motivation

A common task in any discipline (science, mathematics, philosophy, humanities, ...) is that of classifying things of a certain type into various sub-types. Thanks to our development of set theoretic tools, we have a way to formalise what it means for such a classification scheme to work really well.

Q: What properties do you think an excellent classification scheme will have?

Example

Which, if any, of the following classification schemes works well?

- We classify each integer as positive, negative or 0.
- We classify each song on the charts as pop, rock or hip-hop.
- We classify each student enrolled in this course as a mathematician or a computer scientist or a physicist or an other.

Disjoint sets

Sets A, B are called **disjoint** when $A \cap B = \emptyset$.

Given a set of sets $\mathcal{S}$, the sets in $\mathcal{S}$ are said to be **pairwise disjoint** when

$$\forall A, B \in \mathcal{S} \;\; A \neq B \Rightarrow A \cap B = \emptyset.$$

An example

Let P be the set of prime numbers, C the set of composite numbers, and E be the set of even integers.

Example: Let $\mathcal{A} = \{\{1\}, P, C\}$. Since

$$\{1\} \cap P = \emptyset, \{1\} \cap C = \emptyset \text{ and } P \cap C = \emptyset,$$

the sets in $\mathcal{A}$ are pairwise disjoint.

Example: Let $\mathcal{B} = \{\{1\}, P, E \cap \mathbb{N}\}$. Since $(E \cap \mathbb{N}) \cap P = \{2\}$, the sets in $\mathcal{B}$ are not pairwise disjoint.

Partitions

Let S be a set and $\mathcal{A} \subseteq \mathcal{P}(S)$ (so $\mathcal{A}$ is a set, the elements of which are subsets of S). We say that $\mathcal{A}$ is a **partition** of S when each of the following statements is true:

1. $\emptyset \notin \mathcal{A}$
2. every element of S is an element of some set in $\mathcal{A}$ (that is, $\forall s \in S \; \exists A \in \mathcal{A} \; s \in A$)
3. the sets in $\mathcal{A}$ are pairwise disjoint.

Q: Do you agree or disagree that the three properties listed in the definition of a partition are a reasonable interpretation of what it means for a classification scheme (that classifies the elements of S) to be 'excellent.'

Examples

- $\mathcal{A} = \{\{1\}, P, C\}$ is a partition of $\mathbb{N}$
- $\mathcal{B} = \{\{1\}, P, E \cap \mathbb{N}\}$ is not a partition of $\mathbb{N}$ because the sets in $\mathcal{B}$ are not pairwise disjoint.
- $\mathcal{A} = \{\{1\}, P, C\}$ is not a partition of $\mathbb{Z}_{\geq 0}$ because $0 \in \mathbb{Z}_{\geq 0}$ but 0 is not in any set in $\mathcal{A}$.
- Let P, C, E, O be as above. Then $\{P \cap C, P \cap E, P \cap O\}$ is not a partition of P, because $P \cap C = \emptyset$.
- Let P, C, E, O be as above. Then $\{\{1\}, P \cap E, P \cap O, C\}$ is a partition of $\mathbb{N}$.