# Discrete Mathematical Models

**Lecture 18**

Kane Townsend

Semester 2, 2024

# The Product Rule (with some clarity)

Suppose you run $n$ experiments with sample spaces $S_1, S_2, ..., S_n$. For each $1 \leq i \leq n$ let $E_i \subseteq S_i$ be an event. If the events $E_1, ..., E_n$ are 'independent' of each other; then the probability of composite event '$E_1$ and $E_2$ and ... and $E_n$' is

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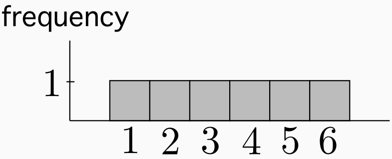
# Density Functions

# Frequency Histograms

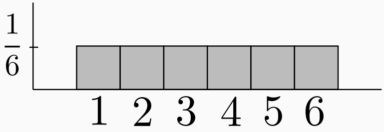
- One way to visualize all possible outcomes of an experiment together is to draw a frequency histogram.
- We have already seen some simple examples, like tossing a die with equally likely possible outcomes: $1, 2, 3, 4, 5, 6$:



# Probability Density Functions

- The Probability Density Function (or just Density) is obtained from a Frequency Histogram by normalizing. We divide the vertical axis by the total number of outcomes.

- Continuing the die-tossing example, we have




*What is the area under the curve?* *Why?*

# Cumulative Probability Distribution Functions

- The Cumulative Probability Distribution Function (or Distribution) is obtained from the Density Function by graphing cumulative totals.

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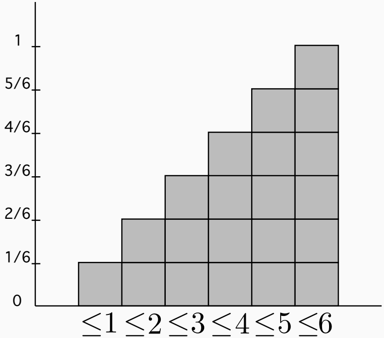
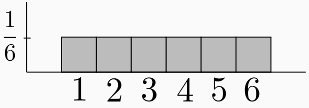



- We will only use of cumulative distributions when looking up probability values in tables or online.

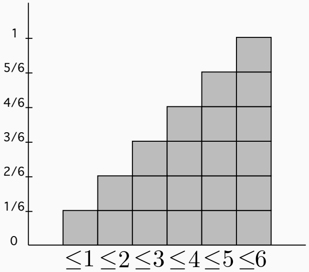
# Uniform Distribution

- When every event has the same probability the resulting densities and distributions are called 'uniform'. Examples:

- Uniform density:




- Uniform distribution:



# Tossing two coins:

- Some more interesting densities and distributions are obtained by considering events which combine several outcomes.

- For example, tossing two coins. A neat way to list all possible outcomes is to expand

$$\begin{aligned} & (T+H)(T+H) \\ = \quad & TT + TH + HT + HH \end{aligned}$$

- *What is the sample space?*

$$\{TT, TH, HT, HH\}$$

# Tossing two coins:

- Now consider events:
  - $E_0$: 'No heads'
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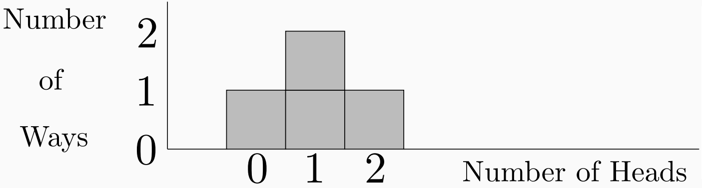
$$E_0 = \{TT\}, \qquad E_1 = \{TH, HT\}, \qquad E_2 = \{HH\}$$

**Frequency Histogram:**



# Two fair coins: Density and Distribution Functions

- Assuming a fair coin (equally likely outcomes), divide out by the size of the sample space to get density function.

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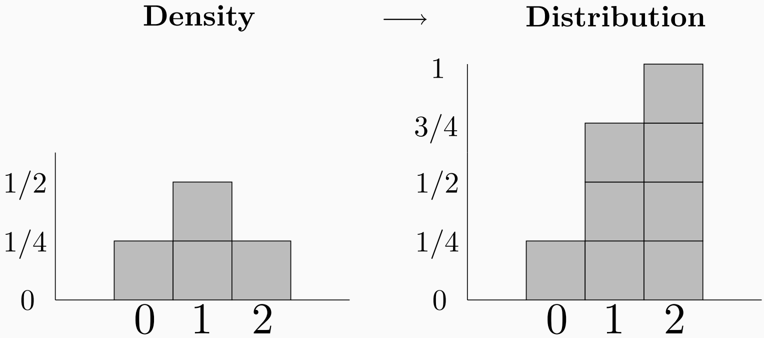

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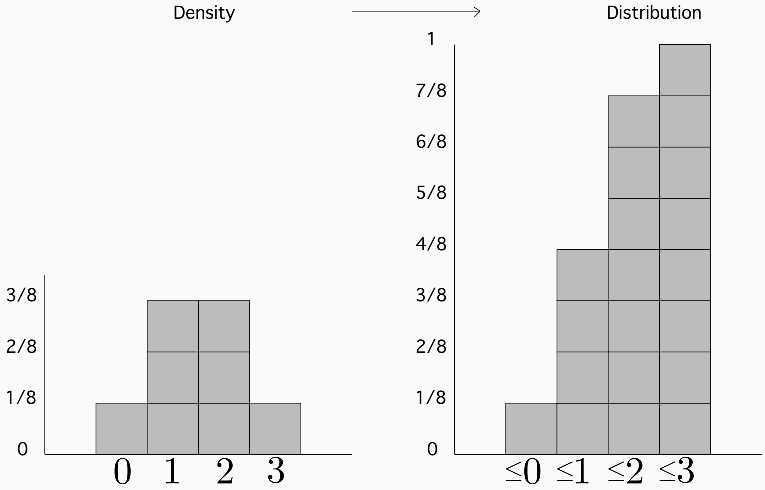
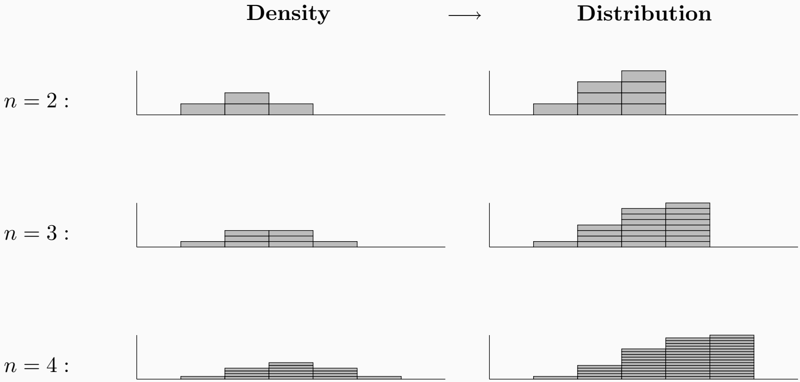
A: $\mathbb{P}(E_0) = \frac{1}{8}$; $\mathbb{P}(E_1) = \frac{3}{8}$; $\mathbb{P}(E_2) = \frac{3}{8}$; $\mathbb{P}(E_3) = \frac{1}{8}$.

# Three Fair Coins: Density and Distribution Functions



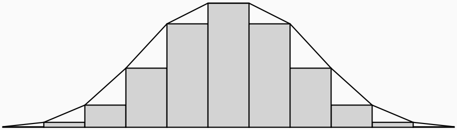
# Binomial Probability Distributions

The family of functions that come from coin-tossing are all examples of binomial densities/distributions:



# Bell-like curves for large $n$

As $n$ gets larger and larger these **binomial probability density functions** get closer and closer to the famous Bell Curve:



which is the so-called **'Normal' Probability Density Function**. This convergence to the normal probability function is a result of the **Central Limit Theorem**.

# Binomial Theorem

# Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & + & 2 & & 1 & & \\
 & 1 & & 3 & + & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

## Pascal's Triangle

- Frequencies in Coin-Tossing are numbers in Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

- Each row is generated by expanding a binomial, eg:

$$(y+x)^4 = y^4 + 4y^3x + 6y^2x^2 + 4yx^3 + x^4.$$

# Pascal's Triangle

- We've seen these numbers before in 'combinations': $\binom{n}{k}$:

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

# The Binomial Theorem

- The Binomial Theorem states that

$$(y+x)^n = \binom{n}{0} y^n x^0 + \binom{n}{1} y^{n-1} x^1 + \cdots + \binom{n}{n} y^0 x^n$$

and gives the rows of Pascal's Triangle in its coefficients.

# Idea of Proof of Binomial Theorem:

$$(y+x)(y+x)(y+x)$$

$$= yyy + yyx + yxy + yxx + xyy + xyx + xxy + xxx$$

$$= \underbrace{yyy}_{\binom{3}{0}\ x\text{'s}} + \underbrace{yyx + yxy + xyy}_{\binom{3}{1}\ x\text{'s}} + \underbrace{yxx + xyx + xxy}_{\binom{3}{2}\ x\text{'s}} + \underbrace{xxx}_{\binom{3}{3}\ x\text{'s}}$$

$$\square\square\square \quad \square\square\boxtimes \quad \square\boxtimes\square \quad \boxtimes\square\square \quad \square\boxtimes\boxtimes \quad \boxtimes\square\boxtimes \quad \boxtimes\boxtimes\square \quad \boxtimes\boxtimes\boxtimes$$

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  What is the total size of the sample space?

- I.e. what is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{n}?$$

- The binomial theorem gives a neat way to find the sum.
- Set $x = y = 1$ in $(x + y)^n$. Then

$$\binom{n}{0}1^n 1^0 + \binom{n}{1}1^{n-1}1^1 + \cdots + \binom{n}{n}1^0 1^n = (1+1)^n$$

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- So, for example, the probability of obtaining exactly three heads from six tosses of a fair coin is

$$\frac{\binom{6}{3}}{2^6} = \frac{\frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}}{64} = \frac{20}{64} = 516.$$

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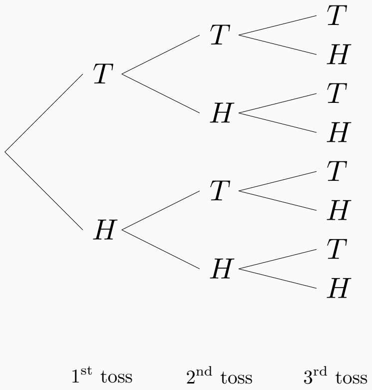
- Examples will be found in worksheet questions.

# Binomial Distribution

# A tree representation of Coin-tossing

- Another way to list all the outcomes of an event is to draw a Tree Diagram of the Possibilities

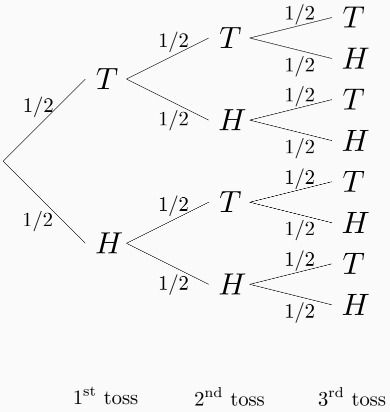



Outcome:

$TTT$
$TTH$
$THT$
$THH$
$HTT$
$HTH$
$HHT$
$HHH$

# Three tosses of a fair coin

- This allows us to deal with fair coins, as before:




Outcome:

$$\Pr(TTT) = 1/8$$
$$\Pr(TTH) = 1/8$$
$$\Pr(THT) = 1/8$$
$$\Pr(THH) = 1/8$$
$$\Pr(HTT) = 1/8$$
$$\Pr(HTH) = 1/8$$
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# Three tosses of a fair coin

Collecting possibilities from the tree and using the sum rule gives

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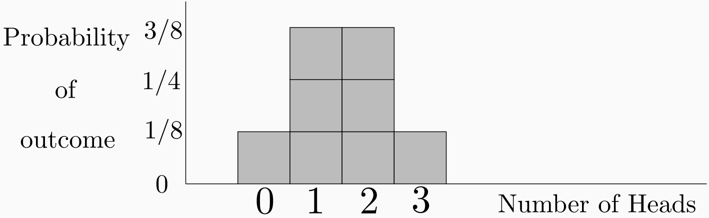

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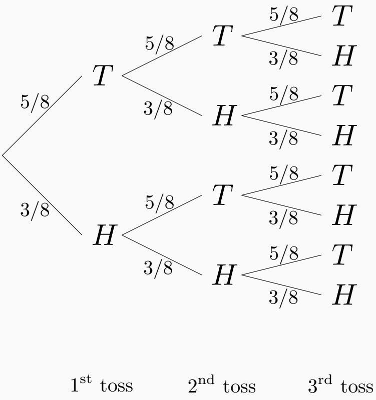


Outcome:

$\Pr(TTT) = 125/512$

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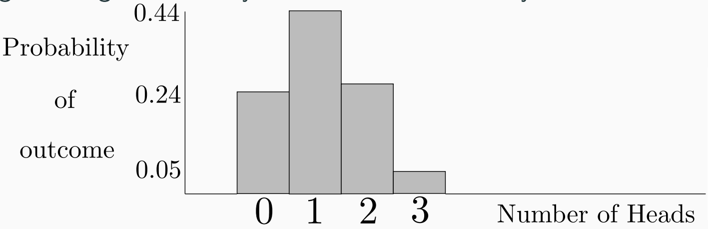

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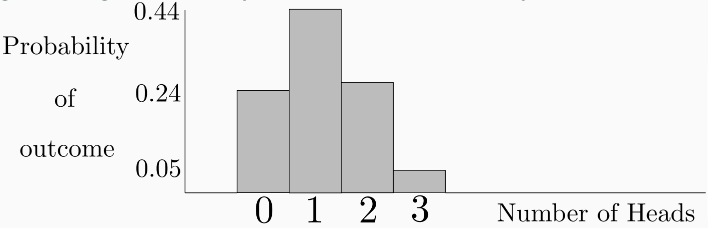


The general binomial density function for $n$ trials (*e.g.* tosses) with probability $p$ of a success (*e.g.* head) on each trial is given by

$$\boxed{\mathbb{P}(k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}.}$$

# “Fun” Problems

# Another general probability example

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The associated event probabilities are

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$$\mathbb{P}(\{M, A\}) = \tfrac{5}{10}, \quad \mathbb{P}(\{A, C\}) = \tfrac{7}{10}, \quad \mathbb{P}(\{M, C\}) = \tfrac{8}{10}, \quad \mathbb{P}(S) = 1\ .$$

# The Monty Hall problem

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The contestant chooses one door.

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Thus

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c) \sim 0.97.$$

There is a 97% chance that two people will have the same birthday.