Discrete Mathematical Models

Lecture 15

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Semester 2, 2024

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Theorem: A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q})$ has an inverse if and only if $\det(A) \neq 0$ and in this case

$$\boxed{A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}} \quad \textit{e.g.} \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}^{-1} = \frac{1}{8} \begin{pmatrix} 2 & 2 \\ -3 & 1 \end{pmatrix}$$

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What about $n > 2$?

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What about $n > 2$? See Math1013 or Math1115.

Interesting Examples

Back to population dynamics

As a final example involving matrix multiplication and matrix inverses, we return to the simple ecosystem model

$$\forall n \in \mathbb{N} \quad \begin{cases} x_{n+1} = 4x_n - y_n, \\ y_{n+1} = y_n + 2x_n, \end{cases}$$

where x_n, y_n are the populations of two species after n time steps.

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This is an implicit definition of a sequence of vectors. We will use mathematical induction to establish an explicit formula, stated and proved on the next slide.

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R1: $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ $\qquad \begin{bmatrix} \text{prove by multiplying} \\ \text{out the RHS} \end{bmatrix}$

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R2: $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \dfrac{1}{2-1} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ $\begin{bmatrix}\text{formula for} \\ \text{inverse of} \\ 2\times 2 \text{ matrix}\end{bmatrix}$

Claim: $\forall n \in \mathbb{N}^\star$ $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

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Proof:

Basis step: When $n = 0$ the RHS becomes (using R2)

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and hence the formula also holds for $n+1$.

Claim: $\forall n \in \mathbb{N}^\star$ $\quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Proof:

Basis step: When $n = 0$ the RHS becomes (using R2)

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Inductive step: Assume the explicit formula holds up to and including some particular n, and consider the case $n+1$. Then, using the implicit definition, preliminary results R1 and R2, and the inductive assumption,

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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where $\varphi = \frac{1+\sqrt{5}}{2}$. We can extract an explicit formula for the Fibonacci sequence!

C1: Counting

Cardinality

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Intervals

Example: numbers in an interval

What is the cardinality of the set of natural numbers in an interval?

Example: $S = \{150, 151, 152, ..., 160\}$

Subtract 149 from each:
$$\begin{array}{ccccc} 150 & 151 & 152 & ... & 160 \\ \downarrow & \downarrow & \downarrow & ... & \downarrow \\ 1 & 2 & 3 & ... & 11 \end{array}$$

We have made a bijection to the set $\{1, 2, 3, ..., 11\}$, so $|S| = 11$.

Example: numbers in an interval, generalized

Let $S = \{a, a+1, a+2, ..., b\} \subseteq \mathbb{N}$

A nice bijection subtracts '$a - 1$' from each element of S. We have

$$\begin{array}{ccccc} a & a+1 & a+2 & \cdots & b \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ 1 & 2 & 3 & \cdots & b-a+1 \end{array}$$

Therefore $|S| = b - a + 1$.

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Since the composition of bijections is a bijection, the answer is 26.

Countable sets

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Examples:

Finite Sets	Infinite Sets
$\{1, 2, 3\}$	$\mathbb{N}$ ... natural numbers
{red, orange, yellow, green, blue, purple}	$\mathbb{Z}$ ... integers
$\{b$: b is a book in the Hancock library$\}$	$\mathbb{Q}$ ... rational numbers
$\{s$: s is a star in the Milky Way Galaxy$\}$	$\mathbb{R}$ ... real numbers
$\{\}$	$\mathcal{P}(\mathbb{R})$ ... power set of $\mathbb{R}$

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The sets $\mathbb{N}$ and $\mathbb{P}$ are each both countable and infinite.
Such sets are called **countably infinite**.

Comparing cardinalities

Generalising from the case of finite sets, we say that two sets A and B have **the same cardinality**, written $|A| = |B|$, provided that there exists a bijection (one-to-one correspondence) from A to B.

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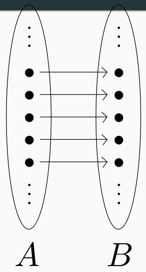

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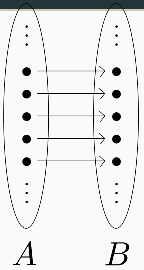

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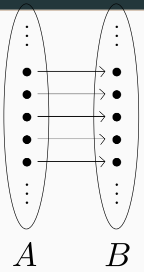

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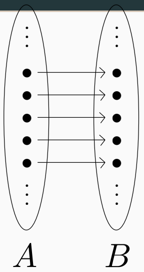

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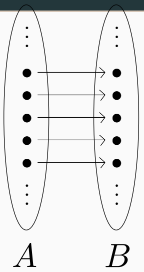

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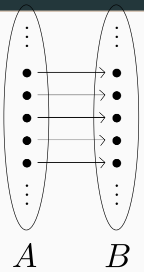

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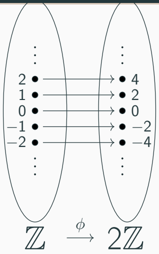

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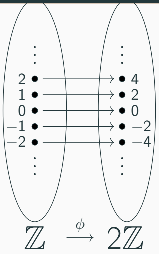

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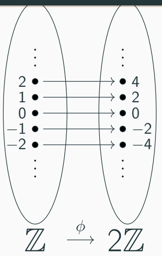

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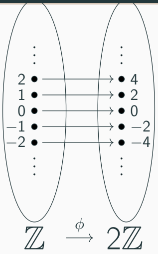

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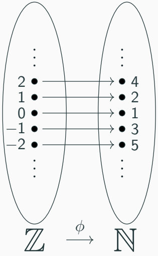
It not difficult to see that this is a bijection.



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Proof: Since every bijection has an inverse, it is enough to establish a bijection $\phi : \mathbb{N} \to S$.

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Proof: Since every bijection has an inverse, it is enough to establish a bijection $\phi : \mathbb{N} \to S$. This can be done recursively as follows:

$$\begin{aligned} \phi(1) &= \text{least member of } S \\ \phi(2) &= \text{least member of } S \setminus \{\phi(1)\} \quad (2^{\text{nd}} \text{ least member}) \\ \forall n \in \mathbb{N} \ \ \phi(n+1) &= \text{least member of } S \setminus \{\phi(1), \ldots, \phi(n)\}. \end{aligned}$$

All countably infinite sets have the same cardinality

Recall that we call a set S countably infinite if (and only if) it is infinite and there is a bijection from S to a subset of $\mathbb{N}$.
In other words, S has the same cardinality as some infinite subset of $\mathbb{N}$.
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Note: It follows from the result above (and its proof) that proving that an arbitrary infinite set S (not necessarily a subset of $\mathbb{N}$) is countably infinite amounts to showing that it can be '**well-ordered**'. This means that it is possible to order the elements of S in some (perhaps ingenious) way so that S and every subset of S has a 'least' member.

Not all infinite sets have the same cardinality

The most surprising fact we will see is that $\mathbb{Q}$ is countable. This is shown by describing a bijection from $\mathbb{N}$ to $\mathbb{Q}$. Since every bijection has an inverse that is a bijection, this suffices to show that there exists a bijection from $\mathbb{Q}$ to $\mathbb{N}$.

However, $\mathbb{R}$, the set of real numbers, is uncountable. This is proven by showing there is no surjection (and hence no bijection) from $\mathbb{N}$ to $\mathbb{R}$. It follows that there is no bijection from $\mathbb{R}$ to $\mathbb{N}$.

So we have

$$|\mathbb{N}| = |\mathbb{N}^\star| = |\mathbb{Z}| = |\mathbb{Q}|$$

but

$$|\mathbb{N}| \neq |\mathbb{R}|.$$

These observations, first made by Georg Cantor (1845-1918), were a breakthrough in mathematical thinking about infinite sets. We will have a look at the proofs next week!

Principles of counting

How do we count?

How do we count? We have a collection of counting principles. When we need to count some objects, we analyse those objects until we carefully match the situation to one of the situations in which a counting principle applies.

What makes counting hard? Matching your scenario to one of the scenarios described in a counting principle takes care, as the scenarios described in the counting principles sound similar unless you read carefully.

What is your best strategy? Understand the scenarios described in the counting principles. In particular, carefully notice how the scenarios are different. Then, when you have to count something and you think that a counting principle applies, **state which counting principle you are using and explain carefully (write it out) how you know that the situation you have is like the scenario described in the counting principle**.

The structures we count

We have already discussed sets, n-tuples, lists and sequences. In a set, order is unimportant and for elements in a set there is no notion of "how many times" the element is in the set. Lists, n-tuples and sequences are different ways to talks about the same thing—order matters and elements can appear more than once unless explicitly excluded by the language. We need one more structure, one that recognises multiplicty (how many times an element appears) but not order (there is no first element, second element etc)....

Multisets

A multiset is a ‘set’ with multiple copies of elements allowed and acknowledged.

Multisets

A multiset is a 'set' with multiple copies of elements allowed and acknowledged. An example is $\{c, b, a, c, a\}$, which has 2 a's, 1 b and 2 c's.

As for ordinary sets, order is irrelevant: $\{c, b, a, c, a\} = \{a, a, b, c, c\}$.

But the multiplicities **do** matter.

Formally, a **size-r multiset** is a set S together with a 'multiplicity function' $m : S \to \mathbb{N}$, where,

$$\forall s \in S \quad m(s) = \text{number of copies of } s \quad \text{and} \qquad r = \textstyle\sum_{s \in S} m(s).$$

So, for example, $\{c, b, a, c, a\}$ has size $r = 2 + 1 + 2 = 5$.

Principles of counting

Bijections preserve cardinality If A and B are finite sets and there exists a bijection $f : A \to B$, then $|A| = |B|$.
TO USE THIS PRINCIPLE: Count something easier, and exhibit a bijection between the set you which to count and the set you have counted.

The Pigeonhole Principle If $k + 1$ or more pigeons occupy k pigeonholes, then at least one pigeonhole must contain two or more pigeons.

The Generalised Pigeonhole Principle If N objects are classified in k disjoint categories, then at least one category must contain $\lceil \frac{N}{k} \rceil$ objects. ($\lceil \frac{N}{k} \rceil$ means the least integer that is greater than or equal to $\frac{N}{k}$)

Permutations There are $n!$ ways to arrange n distinct objects in a list.

r-**Permutations** There are

$$P(n, r) = \frac{n!}{(n-r)!}$$

ways to select and order r out of n distinct objects.

Combinations There are

$$C(n, r) = \binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

ways to choose a set of r objects from a set of n objects (that is, to select r out of n distinct objects when the order in which objects are selected in not important). The notation $\binom{n}{r}$ is read "n choose r."

Principles of counting

Multisets (Stars and Bars) There are $\binom{r+n-1}{r}$ size-r multisets with members from a set of size n. That is, there are $\binom{r+n-1}{r}$ ways to arrange a list of r stars and $n-1$ bars.

Inclusion-Exclusion If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

The Sum Rule If A is a finite set and $\{A_1, A_2, \ldots, A_m\}$ is a partition of A, then $|A| = |A_1| + |A_2| + \cdots + |A_m|$.

The Product Rule If $A_1, A_2, \ldots, A_m$ are finite sets, then $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|$.

Example 1

Prove the following: If A is a non-empty finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

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IDEA: A subset of A corresponds to making one of two choices for each element of A: include it in the subset or not. These choices could be recorded as a bit-string. So elements of $\mathcal{P}(A)$ can be counted by counting bit-strings...

Proof: Let A be a non-empty finite set. Let $n = |A|$. Let B be the set of n-bit strings. When representing integers, the smallest element of B represents 0 and the largest represents $2^n - 1$; it follows that $|B| = 2^n$. Since **bijections preserve cardinality**, to prove the result, it is enough to exhibit a bijection from B to $\mathcal{P}(A)$. Let $f : A \to \{1, 2, \ldots, n\}$ be a bijection. Let $g : B \to \mathcal{P}(A)$ be the function defined by the rule $b_1 b_2 \ldots b_n \mapsto \{a \in A \mid b_{f(a)} = 1\}$. We leave the reader to verify that g is a bijection. $\square$

An example illustrating the functions in Example 1

Let $A = \{\text{cat}, \text{dog}, \text{chicken}\}$. Then $n = 3$ and

$$B = \{000, 001, 010, 011, 100, 101, 110, 111\}.$$

Let $f : A \to \{1, 2, 3\}$ be the function defined by

$$f(\text{cat}) = 1, f(\text{dog}) = 2, f(\text{chicken}) = 3.$$

Then

$$g(011) = \{\text{dog}, \text{chicken}\}$$
$$g(101) = \{\text{cat}, \text{chicken}\}$$
$$g(000) = \emptyset$$
$$g(100) = \{\text{cat}\}$$