Discrete Mathematical Models

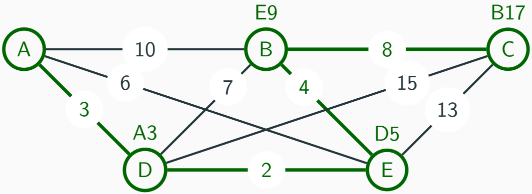
Lecture 26

Kane Townsend

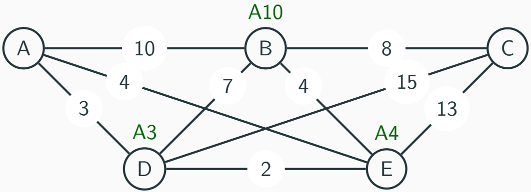
Semester 2, 2024

Dijkstra's Algorithm again

Last lecture we found a minimal path from A to C in the following weighted graph:



We now consider the same problem but with an adjustment to the graph:



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A **weighted digraph** is a digraph G together with a **weight function** $\text{weight} : E(G) \to \mathbb{Q}_+$.

Transport Networks

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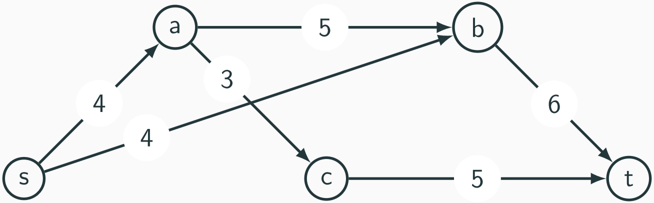

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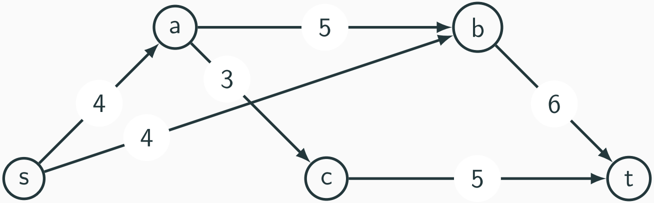

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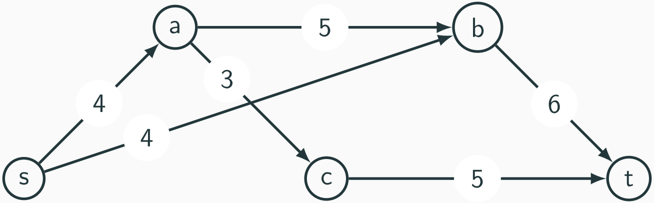

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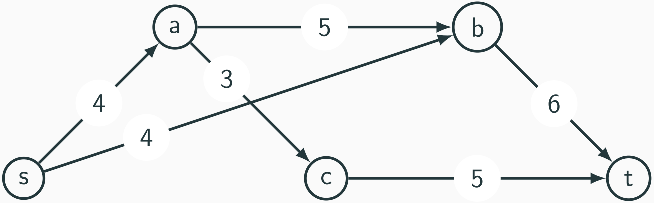

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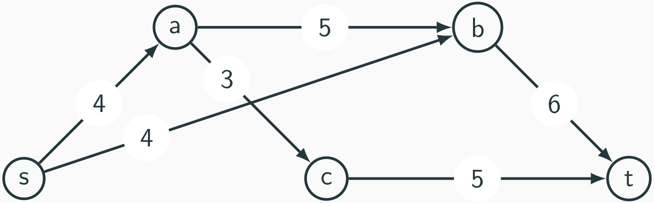

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- Every edge lies on some simple (directed) path from s to t.

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(2) In each edge, flow direction = edge direction.

(3) Total flow into a node equals total flow out, except for nodes s, t. [$\forall v \in V(D) \backslash \{s, t\} \;\; \sum_{e \in v_{\mathbf{in}}} F(e) = \sum_{e \in v_{\mathbf{out}}} F(e)$, where $v_{\text{in}}, v_{\text{out}}$ are the sets of edges coming **in** to, and **out** of, v, respectively.]

Finding a maximum flow

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At stage i, flow F_i is constructed as $F_i = F_{i-1} + f_i$, where the incremental flow f_i is based on a constant $k_i \in \mathbb{Q}^+$ and a simple path p_i from s to t:

$$f_i(e) = \begin{cases} k_i & \text{for every edge } e \text{ on the path } p_i \\ 0 & \text{for evey other edge } e. \end{cases}$$

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To depict a flow I will follow the capacity value $C(e)$ on each (directed) edge e with the flow value $F(e)$ for that edge. For example

represents a flow of 3 in the edge from x to y, with spare capacity 2.

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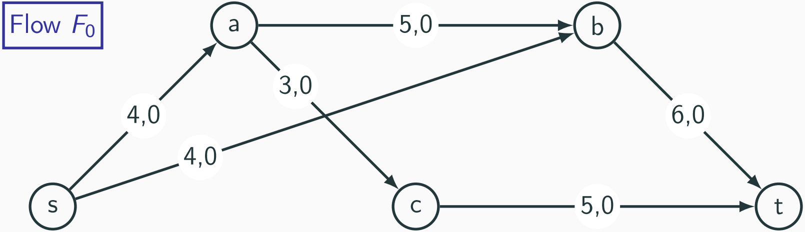
Vertex labelling algorithm, Example 1: Stage 1: F_0 to F_1



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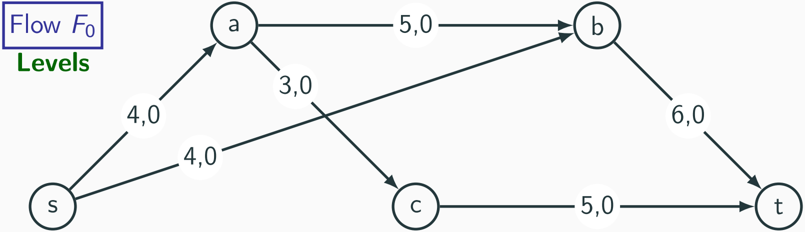

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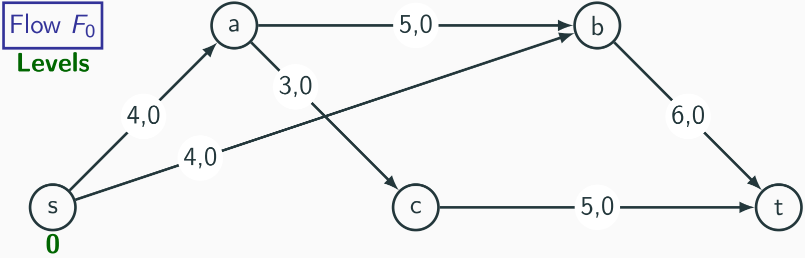

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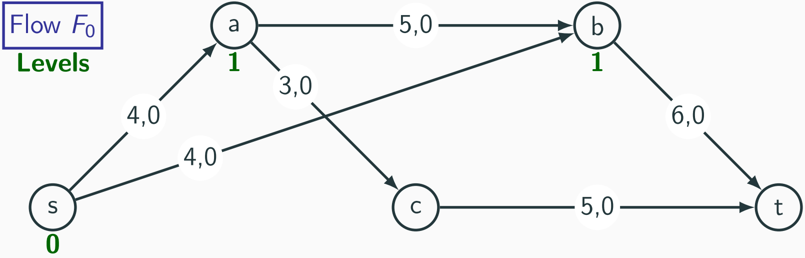

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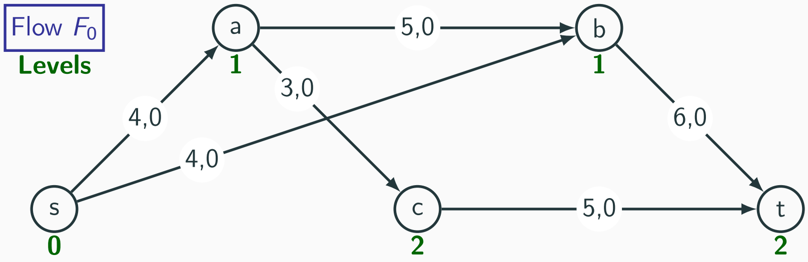

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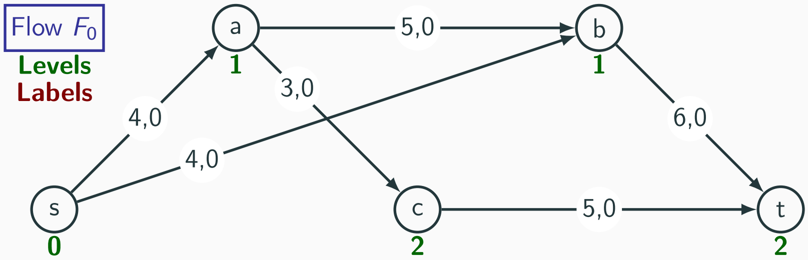

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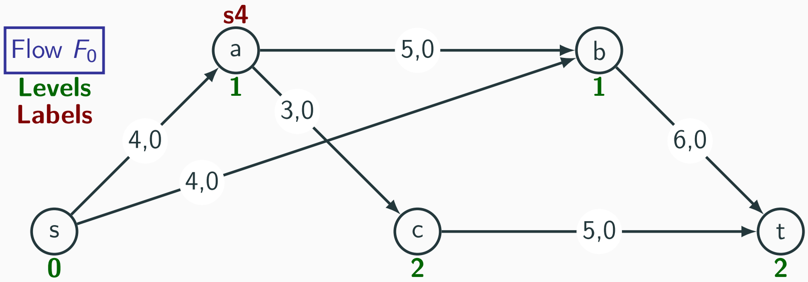

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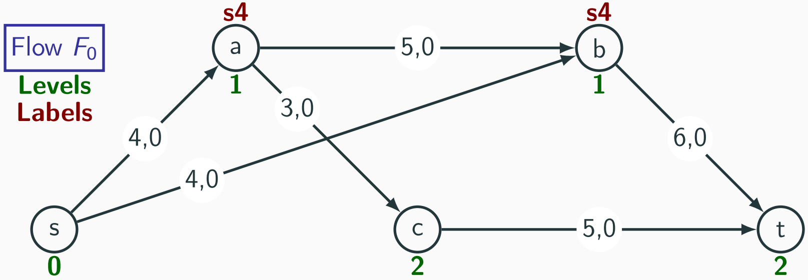

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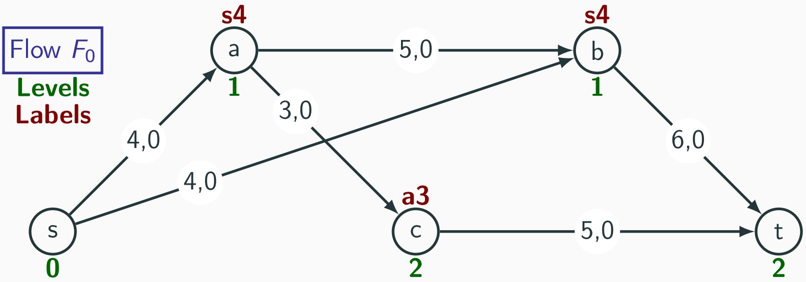

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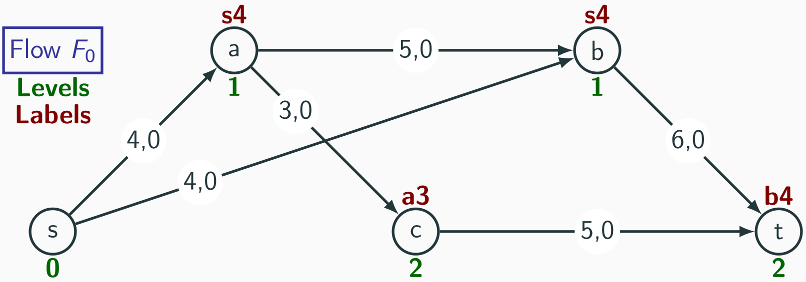

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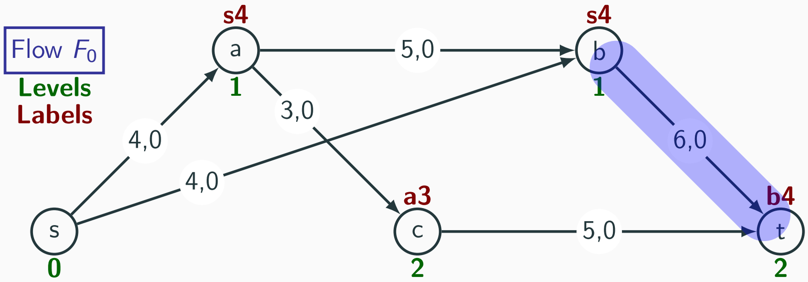

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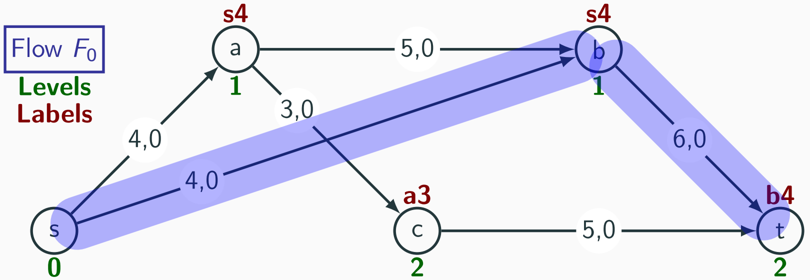

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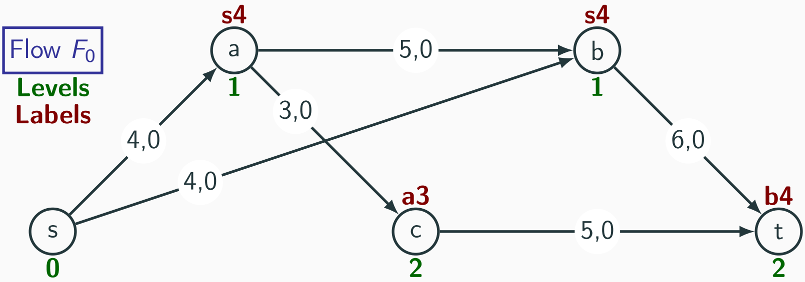

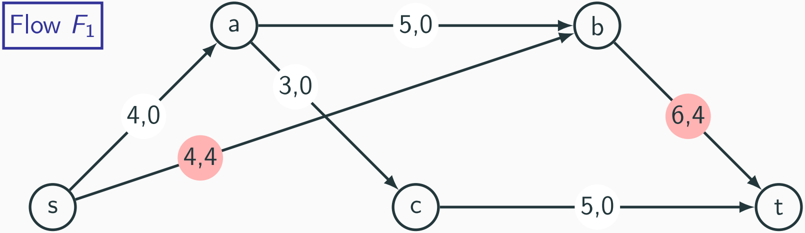

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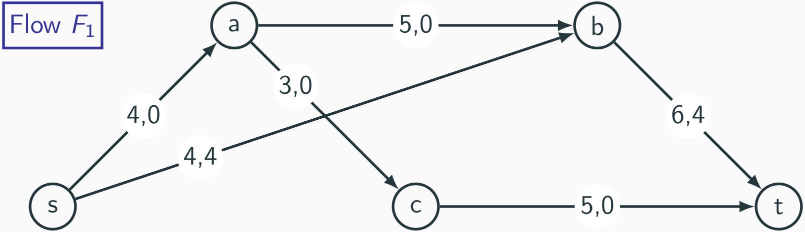

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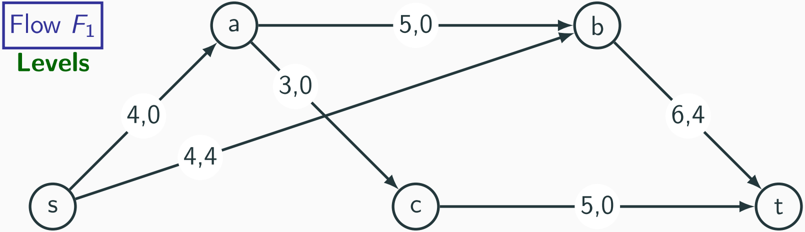

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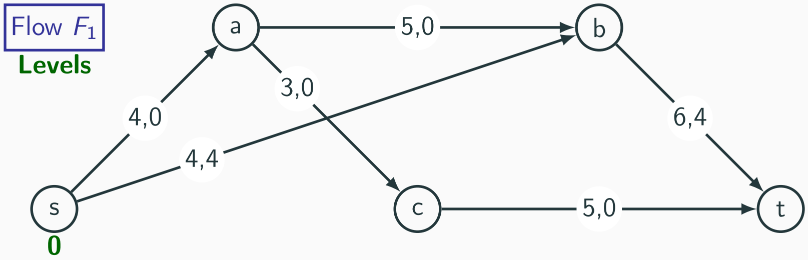

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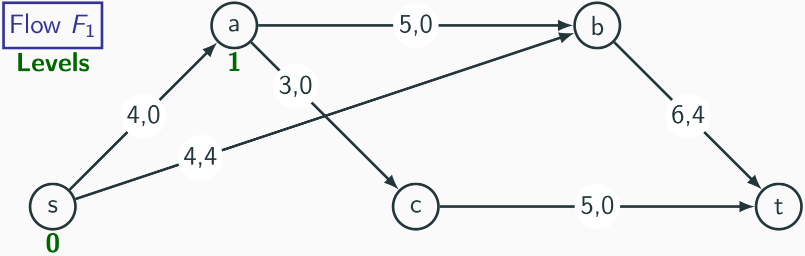

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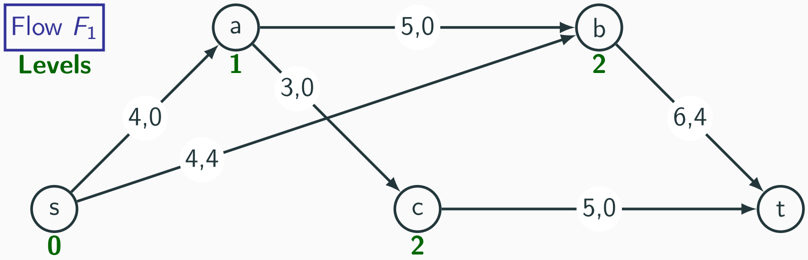

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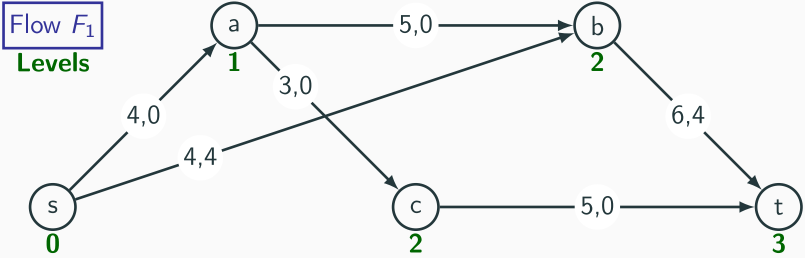

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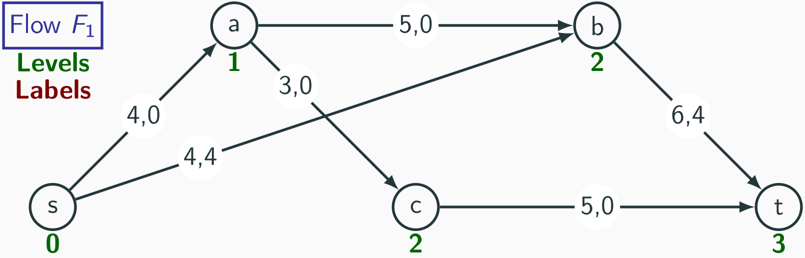

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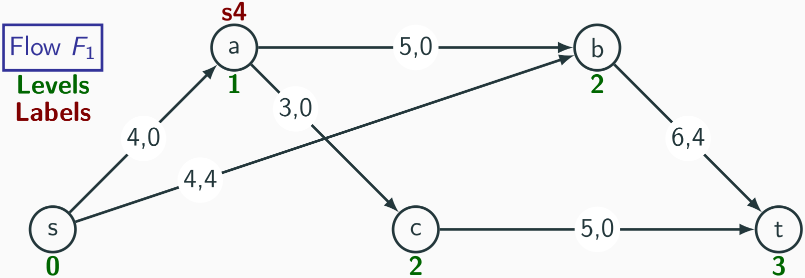

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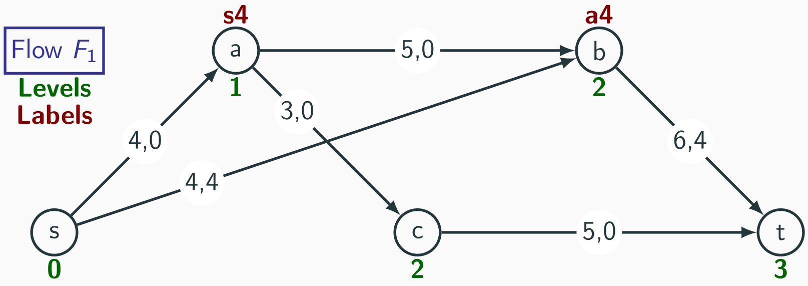

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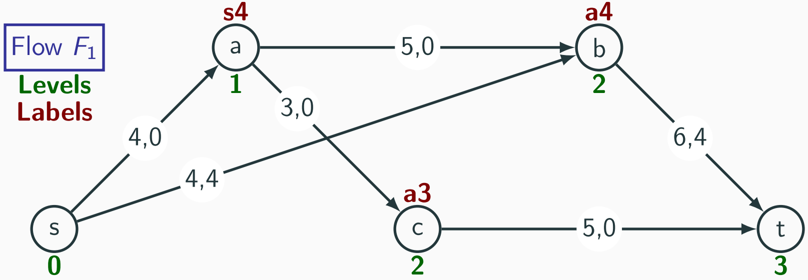

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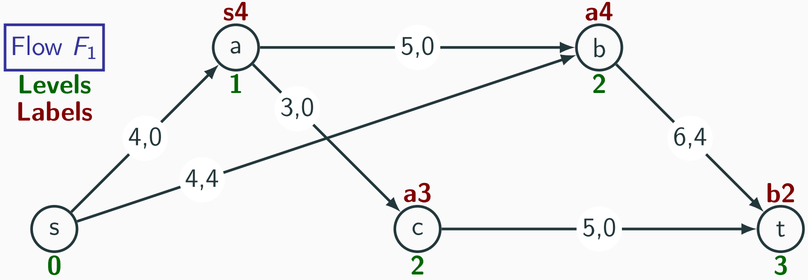

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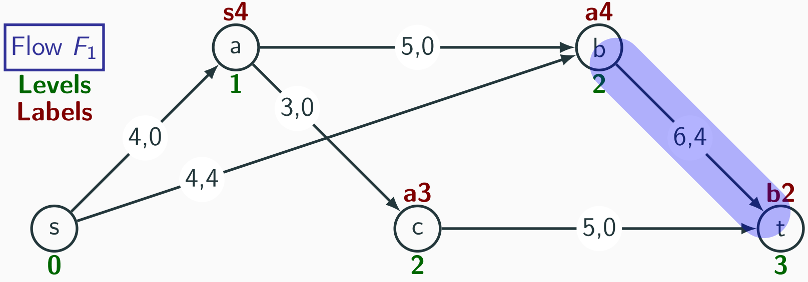

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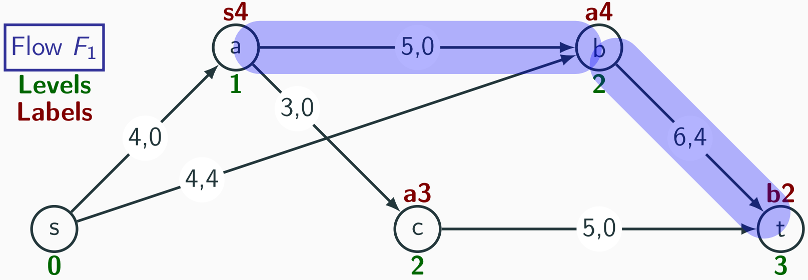

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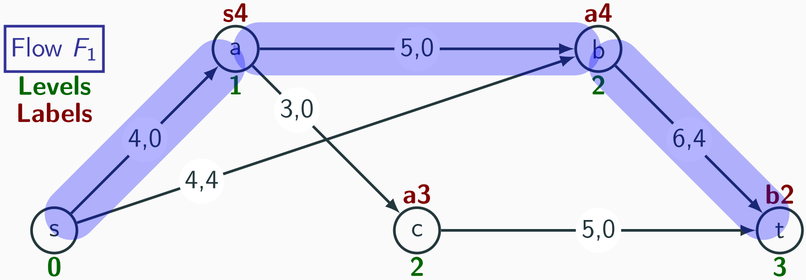

The incremental flow f_2

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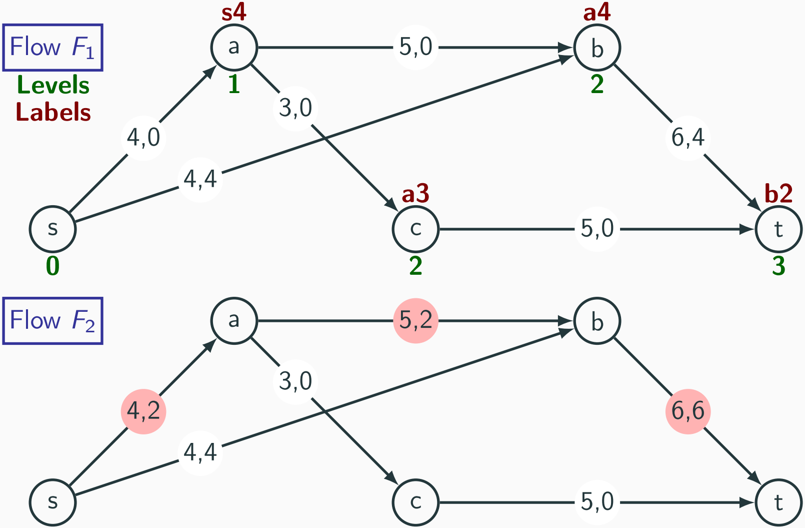
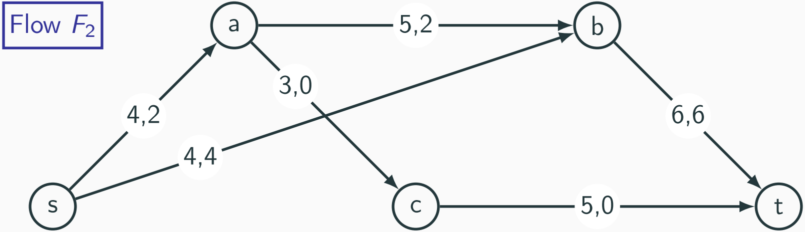


Vertex labelling algorithm, Example 1: Stage 3: F_2 to F_3



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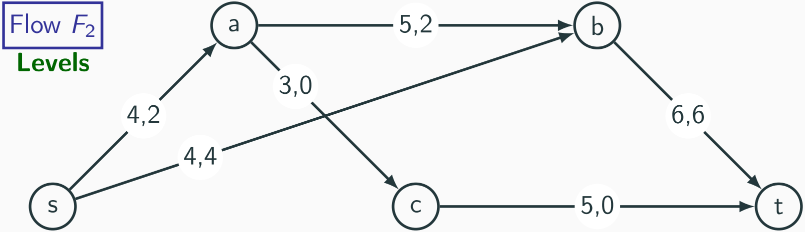

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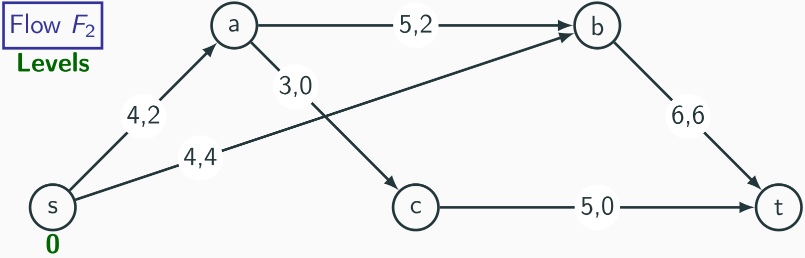

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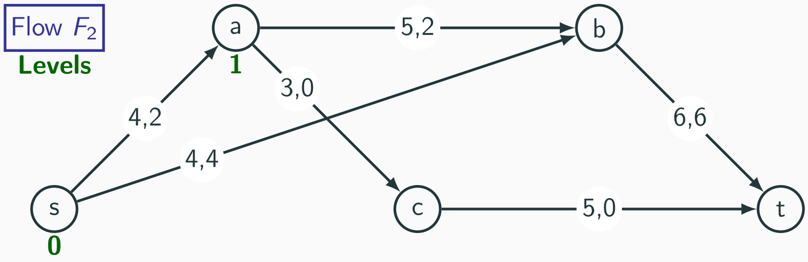

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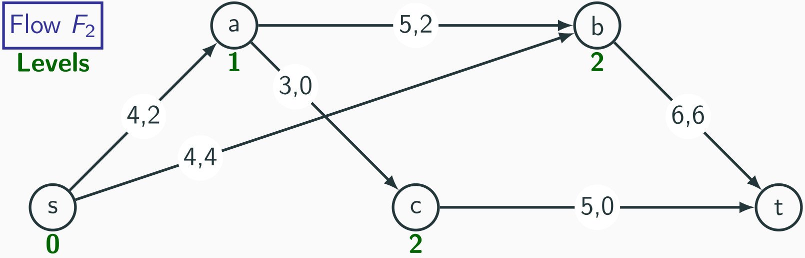

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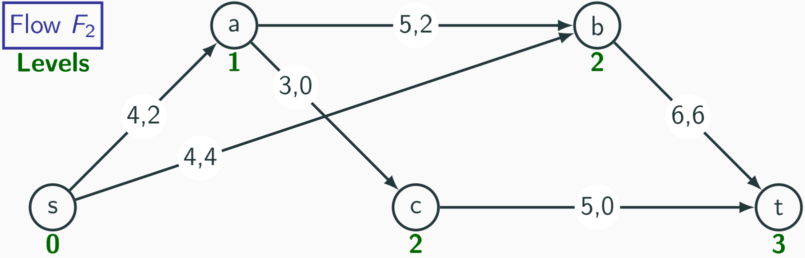

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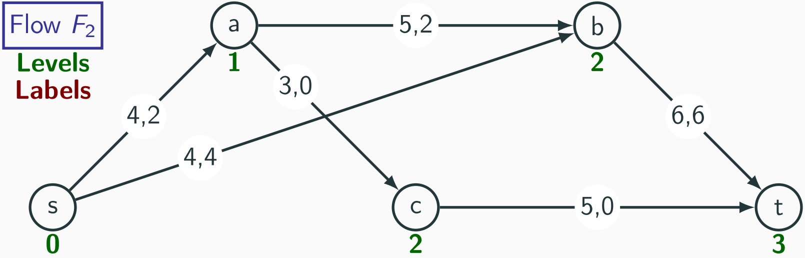

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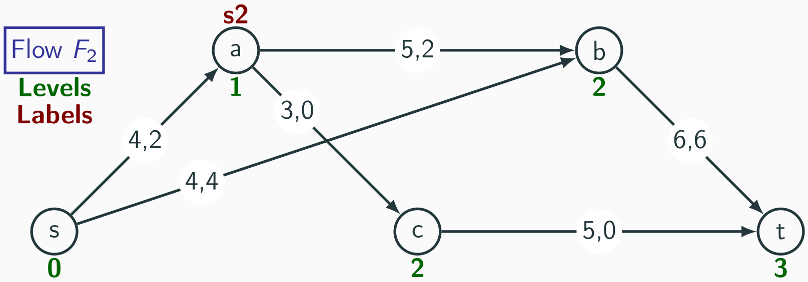

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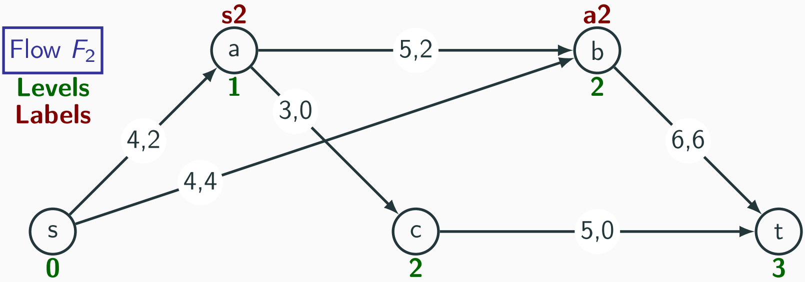

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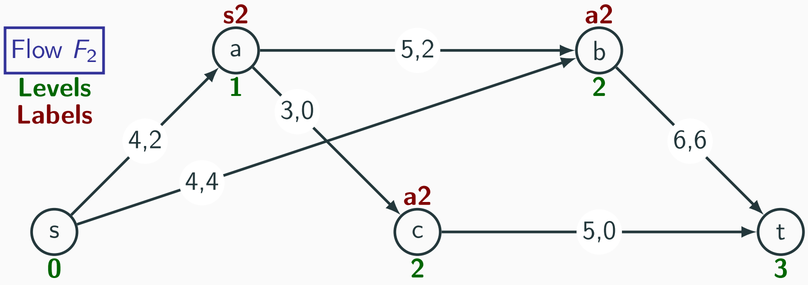

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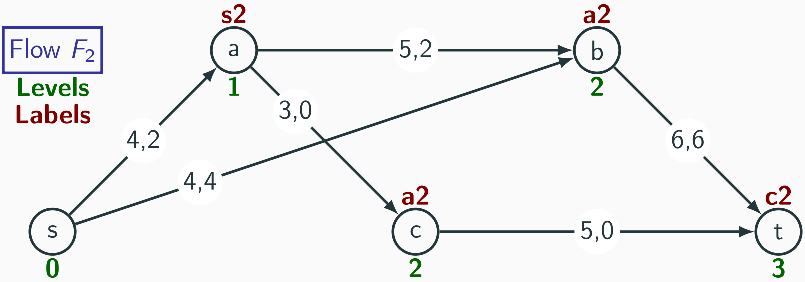

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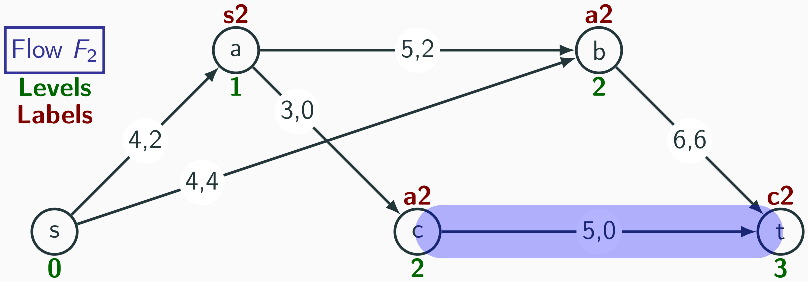

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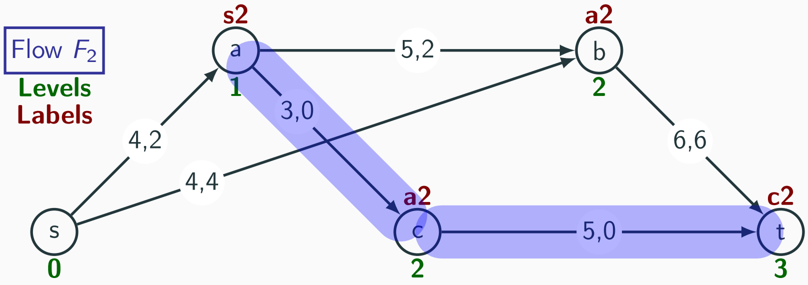

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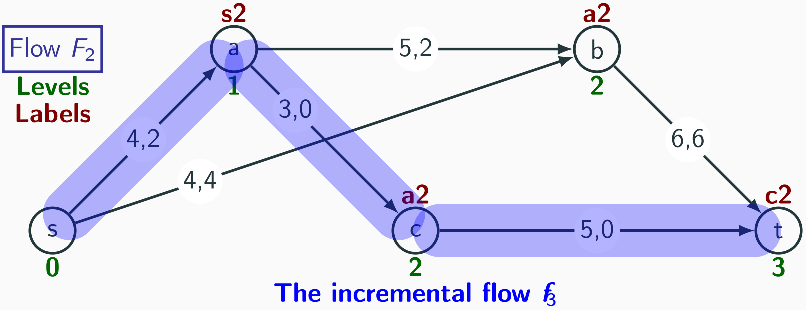

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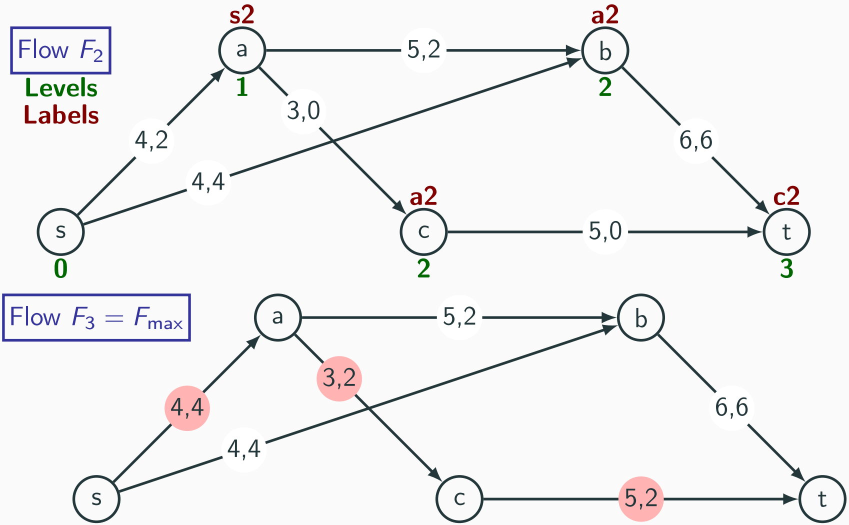

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3. Next slide....

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5. Let p_i be the path $\mathrm{u}_0\mathrm{u}_1 \ldots \mathrm{u}_n$ where $\mathrm{u}_n = \mathrm{t}$ and for $0 < j \leq n$ u_j has label $\mathrm{u}_{j-1}k_j$.
   Define f_i to be the incremental flow on p_i with flow value k_n.

End of Method