Discrete Mathematical Models

Lecture 19

Kane Townsend

Semester 2, 2024

Next Weeks Quiz Topics

Your Week 8 quiz will be on:

- Matrices
- Counting

Binomial Distribution (Revision)

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a. What is the probability of rolling a 5 exactly three times.
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a. What is the probability of rolling a 5 exactly three times.
b. What is the probability of rolling a 3 or 5 at most one time.

Solutions:

a. $\binom{8}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^5$
b. $\binom{8}{0}\left(\frac{2}{6}\right)^0\left(\frac{4}{6}\right)^8 + \binom{8}{1}\left(\frac{2}{6}\right)^1\left(\frac{4}{6}\right)^7$

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Example:

$S = \{\text{H,T}\}^3$ = set of outcomes of tossing three coins.

$X((a, b, c))$ = number of H's amongst a, b, c.

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Relative to a probability density function $\mathbb{P} : S \to \mathbb{Q}_+$ the **expected value** $\mathbb{E}(X)$ of a random variable X is defined by

$$\mathbb{E}(X) = \sum_{s \in S} \mathbb{P}(s) X(s) = \sum_{a \in \mathrm{Range}(X)} \mathbb{P}(\{X = a\}) a$$

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Example(cont.):

$$\mathbb{E}(X) = (\tfrac{1}{8})0 + (\tfrac{3}{8})1 + (\tfrac{3}{8})2 + (\tfrac{1}{8})3 = \tfrac{12}{8} = 1.5.$$

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Thus the expected value of X is just the average number of heads obtained when three coins are tossed.

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$X : S \to \mathbb{Q}$ defined by $X(j) = \begin{cases} \text{10-2=8} & \text{if } j = 6, \\ -2 & \text{otherwise.} \end{cases}$

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$$\mathbb{E}(X) = \sum_{j=1}^{6} \frac{1}{6} \times X(j) = 5\left(\frac{1}{6} \times -2\right) + \left(\frac{1}{6} \times 8\right) = \frac{-2}{6} = -\frac{1}{3}.$$

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On average you expect to lose \$1/3 per game. In other words if you play this game 30 times, you should expect to lose $30(\frac{1}{3}) = 10$ dollars.

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- $G = \{\text{HT,TH,HH}\}$ (at least one Head), $\mathbb{P}(G) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$,
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  G, K are **not** independent (again as we would expect) since
  $$\mathbb{P}(G \cap K) = \mathbb{P}(\{\mathrm{HT,TH}\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq \frac{9}{16} = \frac{3}{4} \times \frac{3}{4} = \mathbb{P}(G) \times \mathbb{P}(K).$$

Independent random variables

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Then for *any* $a, b \in \{0, 1\}$:

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and hence the events $\{X = a\}, \{Y = b\}$ are independent because

$$\mathbb{P}(\{X = a\} \cap \{Y = b\}) = \tfrac{1}{4} = \tfrac{1}{2} \times \tfrac{1}{2} = \mathbb{P}(\{X = a\}) \times \mathbb{P}(\{Y = b\}).$$

Thus, by the above definition, X, Y are independent.

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Table 1: Definition of X and Y

s	1	2	3	4	5	6
$s \bmod 2 = X(s)$	1	0	1	0	1	0
$s \bmod 3 = Y(s)$	1	2	0	1	2	0

Table 2: Probabilities

a	0	1	2
$\mathbb{P}(\{X = a\})$	$\frac{1}{2}$	$\frac{1}{2}$	0
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The columns in Table 1 are all different and cover all possible combinations of values of X, Y. This ensures that each pair of values $(X,Y) = (a,b)$ relates to a unique s, and hence has probability $\mathbb{P}(s)$ $(= \frac{1}{6})$.

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When $(a, b) = (0, 0)$, we have $s = 6$. When $(a, b) = (0, 1)$, we have $s = 4$.
When $(a, b) = (0, 2)$, we have $s = 2$. When $(a, b) = (1, 0)$, we have $s = 3$.
When $(a, b) = (1, 1)$, we have $s = 1$. When $(a, b) = (1, 2)$, we have $s = 5$.

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When $(a, b) = (1, 1)$, we have $s = 1$. When $(a, b) = (1, 2)$, we have $s = 5$.
Using Table 2 it now follows that, for any $a \in \{0, 1\}$ $b \in \{0, 1, 2\}$ the events $\{X = a\}, \{Y = b\}$ are independent because

$$\mathbb{P}(\{X = a\} \cap \{Y = b\}) = \frac{1}{6} = \frac{1}{2} \times \frac{1}{3} = \mathbb{P}(\{X = a\}) \times \mathbb{P}(\{Y = b\}).$$

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Toss a regular fair die. $S = \{1, \ldots, 6\}$, $\mathbb{P}(i) = \frac{1}{6}$, $i = 1, \ldots, 6$.

Let $Y, Z : S \to \mathbb{Q}$ be random variables as follows:

Table 1: Definition of Y and Z

s	1	2	3	4	5	6
$s \bmod 3 = Y(s)$	1	2	0	1	2	0
$s \bmod 4 = Z(s)$	1	2	3	0	1	2

Table 2: Probabilities

a	0	1	2	3
$\mathbb{P}(\{Y = a\})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
$\mathbb{P}(\{Z = a\})$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

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Notice that, in the Table 1, many potential columns are not present.

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Notice that, in the Table 1, many potential columns are not present. For example there is no value of s for which $Y(s)=0$ and $Z(s)=0$. Using this, and also Table 2, we now have

$$\mathbb{P}(\{Y=0\} \cap \{Z=0\}) = 0 \neq \frac{1}{3} \times \frac{1}{6} = \mathbb{P}(\{Y = 0\}) \times \mathbb{P}(\{Z = 0\}),$$

and so the events $\{Y = 0\}$, $\{Z = 0\}$ are not independent.

It follows that the random variables Y, Z are not independent.

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and so the events $\{Y = 0\}$, $\{Z = 0\}$ are not independent.

It follows that the random variables Y, Z are not independent.

Challenge: Are the random variables X, Z independent?

Conditional Probability and Bayes' Theorem

Conditional Probability and Bayes' Theorem

Reference: §9.9 of our optional text

Conditional Probability

(Theme: Use all of the information you have.)

Definition
Consider a probability experiment with sample space S. If $A, B \subseteq S$ and $\mathbb{P}(A) \neq 0$, then the **conditional probability of** B **given** A, denoted $\mathbb{P}(B|A)$, is

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

An example

P: I toss two fair coins but only I can see the outcome. You ask “Did they both come up tails?" I say “No."

What is the probability that both coins came up heads?

A: The probability experiment is to toss two fair coins.

An outcome will be recorded as a two-letter string using only H's and T's, with the first letter recording the result of tossing the first coin and the second letter the result of tossing the second coin. For example, the outcome HT records that the first coin come up 'heads' and the second coin came up ‘tails’.

The sample space is the set $S = \{HH, HT, TH, TT\}$

Example (cont.)

Since the coins are 'fair', the outcomes are equally likely. We then have that $\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{|E|}{4}$ for each event $E \subseteq S$.

Let A be the event that we did not have both coins coming up tails; that is, $A = \{HH, HT, TH\}$. Let B be the event that both coins came up heads; that is, $B = \{HH\}$. We compute

$$
\begin{aligned}
&\text{The probability that both coins came up heads given that they} \\
&\text{did not both come up tails} \\
=&\mathbb{P}(B|A) \qquad \text{(translating into notation)} \\
=&\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \qquad \text{(defn of conditional prob.)} \\
=&\frac{\mathbb{P}(\{HH\})}{\mathbb{P}(\{HH, HT, TH\})} \\
=&\frac{1/4}{3/4} = \frac{1}{3}. \qquad \square
\end{aligned}
$$

An example

P: A pair of fair 6-sided dice, one red and one blue, are rolled. What is the probability that the sum of the numbers showing face up is 8, given that both of the numbers are even?

Proof: The probability experiment is to roll a pair of fair 6-sided dice, one red and one blue.

An outcome will be recorded as an element of $\{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$, with the first element recording the result of the red die and the second digit the result of rolling the blue die. For example, the outcome $(2, 4)$ records that we rolled a 2 on the red die and a 4 on the blue die.

The sample space is the set $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$. By the product rule for counting, $|S| = 6^2 = 36$.

example (cont.)

Since the dice are 'fair', the outcomes are equally likely. We then have that $\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{|E|}{36}$ for each event $E \subseteq S$.

Let B be the event that the sum of the numbers showing face up is 8; that is, $B = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$. Let A be the event that both of the numbers rolled are even; that is,

$$A = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}.$$

We compute:

$$
\begin{aligned}
&\text{The probability that the sum of the numbers showing face up is 8} \\
&\text{given that both numbers are even} \\
=&\mathbb{P}(B|A) \qquad \text{(translating into notation)} \\
=&\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \qquad \text{(defn of conditional prob.)} \\
=&\frac{\mathbb{P}(\{(2,6),(4,4),(6,2)\})}{\mathbb{P}(\{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\})} \\
=&\frac{3/36}{9/36} \\
&= \frac{1}{3}. \qquad \square
\end{aligned}
$$

A lemma

Lemma

For any probability experiment with sample space S, and for any events $A, B \subseteq S$, if $\mathbb{P}(A) \neq 0$ then

$$\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A).$$

Proof.
Consider a probability experiment with sample space S. Let $A, B \subseteq S$. Suppose that $\mathbb{P}(A) \neq 0$. Since $\mathbb{P}(A) \neq 0$, the conditional probability $\mathbb{P}(B|A)$ is defined. The definition gives

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

Multiplying both sides by $\mathbb{P}(A)$ gives $\mathbb{P}(B|A)\mathbb{P}(A) = \mathbb{P}(A \cap B)$. $\square$

Theorem (Bayes' Theorem)

For any probability experiment with sample space S, for any $n \in \mathbb{N}$, for any partition $\{B_1, B_2, \ldots, B_n\}$ of S and for any event $A \subseteq S$, if $\mathbb{P}(A) \neq 0$ and for all $i \in \{1, 2, \ldots, n\}$ we have $\mathbb{P}(B_i) \neq 0$, then for all $k \in \{1, 2, \ldots, n\}$ we have

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)}$$

Proof

Consider a probability experiment with sample space S. Let $n \in \mathbb{N}$, let $\{B_1, B_2, \ldots, B_n\}$ be a partition of S and let $A \subseteq S$. Suppose that $\mathbb{P}(A) \neq 0$ and for all $i \in \{1, 2, \ldots, n\}$ we have $\mathbb{P}(B_i) \neq 0$. Let $k \in \{1, 2, \ldots, n\}$. Now

$$\begin{aligned}
\mathbb{P}(B_k|A) &= \frac{\mathbb{P}(B_k \cap A)}{\mathbb{P}(A)} && \text{(By defn of } \mathbb{P}(B_k|A)\text{)} \\
&= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A)} && \text{(Lemma noting } \mathbb{P}(B_k) \neq 0\text{)} \\
&= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A \cap S)} && \text{(Because } A \cap S = A\text{)} \\
&= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A \cap (B_1 \cup B_2 \cup \cdots \cup B_n))}
\end{aligned}$$

(because $\{B_1, ..., B_n\}$ is a partition of S)

Proof (cont.)

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}((A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n))} \quad (\cap \text{ distributes over } \cup)$$

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A \cap B_1) + \mathbb{P}(A \cap B_1) + \cdots + \mathbb{P}(A \cap B_n)} \quad \text{(Applying the sum rule}$$

which is OK because $B_1, \ldots, B_n$ are mutually disjoint)

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) \cdots + \mathbb{P}(A|B_n)\mathbb{P}(B_n)}$$

(Applying the lemma n times, which is OK because $\mathbb{P}(B_i) \neq 0$

for $i \in \{1, 2, \ldots, n\}$)

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)} \quad \text{(Using } \Sigma \text{ notation)} \qquad \square$$

Applications of Bayes' Theorem

- Solving Monty Hall problem
- Drug testing
- Disease testing
- Defective item rates

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- Solving Monty Hall problem
- Drug testing
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A **false positive** means that a patient gets a positive test of having the disease when they do not have the disease.

A **false negative** means that a patient gets a negative test of having the disease when they do have the disease.

Example 9.9.3 from Epp.

Consider a medical test that screens for a disease found in 5 people in 1,000. Suppose that the false positive rate is 3% and the false negative rate is 1%. Then 99% of the time a person who has the condition tests positive for it, and 97% of the time a person who does not have the condition tests negative for it.

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a. What is the probability that a randomly chosen person who tests positive for the disease actually has the disease?
b. What is the probability that a randomly chosen person who tests negative for the disease does not in fact have the disease?

Solution: Consider a random person from those screened. Let A be the event they test positive, B_1 the event they have the disease, B_2 the event they do not have the disease. Then:

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a. What is the probability that a randomly chosen person who tests positive for the disease actually has the disease?
b. What is the probability that a randomly chosen person who tests negative for the disease does not in fact have the disease?

Solution: Consider a random person from those screened. Let A be the event they test positive, B_1 the event they have the disease, B_2 the event they do not have the disease. Then:

$$\mathbb{P}(A|B_1) = 0.99, \quad \mathbb{P}(A^c|B_1) = 0.01, \quad \mathbb{P}(A^c|B_2) = 0.97, \quad \mathbb{P}(A|B_2) = 0.03.$$

Also because 5 people in 1,000 have the disease,

$$\mathbb{P}(B_1) = 0.005 \text{ and } \mathbb{P}(B_2) = 0.995.$$

Example 9.9.3 from Epp. (Cont.)

A person tests positive, B_1 person has disease, B_2 the event they do not have the disease. Then: $\mathbb{P}(A^c|B_1) = 0.99, \mathbb{P}(A|B_1) = 0.01, \mathbb{P}(A^c|B_2) = 0.97, \mathbb{P}(A|B_2) = 0.99, \mathbb{P}(B_1) = 0.005$ and $\mathbb{P}(B_2) = 0.995$.

a. By Bayes' Theorem

$$\begin{aligned}\mathbb{P}(B_1|A) &= \frac{\mathbb{P}(A|B_1)\mathbb{P}(B_1)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2)} \\ &= \frac{(0.99)(0.005)}{(0.99)(0.005) + (0.03)(0.995)} \\ &\approx 0.1422 \approx 14.2\%.\end{aligned}$$

Example 9.9.3 from Epp. (Cont.)

A person tests positive, B_1 person has disease, B_2 the event they do not have the disease. Then: $\mathbb{P}(A^c|B_1) = 0.99, \mathbb{P}(A|B_1) = 0.01, \mathbb{P}(A^c|B_2) = 0.97, \mathbb{P}(A|B_2) = 0.99, \mathbb{P}(B_1) = 0.005$ and $\mathbb{P}(B_2) = 0.995$.

a. By Bayes' Theorem

$$\begin{aligned}\mathbb{P}(B_1|A) &= \frac{\mathbb{P}(A|B_1)\mathbb{P}(B_1)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2)}\\ &= \frac{(0.99)(0.005)}{(0.99)(0.005) + (0.03)(0.995)}\\ &\approx 0.1422 \approx 14.2\%.\end{aligned}$$

b. By Bayes' Theorem

$$\begin{aligned}\mathbb{P}(B_2|A^c) &= \frac{\mathbb{P}(A^c|B_2)\mathbb{P}(B_2)}{\mathbb{P}(A^c|B_1)\mathbb{P}(B_1) + \mathbb{P}(A^c|B_2)\mathbb{P}(B_2)}\\ &= \frac{(0.97)(0.995)}{(0.01)(0.005) + (0.97)(0.995)}\\ &\approx 0.999948 \approx 99.995\%.\end{aligned}$$