Discrete Mathematical Models

Lecture 17

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Semester 2, 2024

Unordered Selection (cont.)

Stars and bars

[**Multisets (Stars and Bars)**] There are $\binom{r+n-1}{r}$ size-r multisets with members from a set of size n. That is, there are $\binom{r+n-1}{r}$ ways to arrange a list of r stars and $n-1$ bars.

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If the stars and bars were all distinct than there would be $(r+n-1)!$ ways to permute them. However, we are overcounting, since there are r indistinguishable stars and (n-1) indistinguishible bars the total number of arrangements is

$$\frac{(r+n-1)!}{r! \times (n-1)!}$$

This is exactly $\binom{r+n-1}{r}$.

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For example for $n = 3$ there are $\binom{7}{5} = \binom{7}{2} = 21$ such 5-tuples:

33333 33332 33331 33322 33321 33311 33222 33221 33211 33111
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So the number of integers is $\displaystyle\binom{14}{4}-5=\frac{14\times13\times12\times11}{4\times3\times2\times1}-5=996.$

Selecting summary

Select r from n	With repetition	Without repetition
Ordered	n^r	$\frac{n!}{(n-r)!}$
Unordered	$\binom{r+n-1}{r}$	$\binom{n}{r}$

Warning: Questions are often harder and more subtle than just applying these rules directly.

Question: What are the set and function interpretations of each?

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- There is one way to choose the first bit (1)
- There are two ways to choose the second bit (0 or 1)
- There are two ways to choose the third bit (0 or 1)

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- There are two ways to choose the eighth bit (0 or 1)

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Product Rule: Task 1 can be done in $1 \times 2^7 = 128$ ways.

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Task 2: Construct a string of length 8 that ends with ‘00’.

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- There are two ways to choose the second bit (0 or 1)

  ⋮

- There are two ways to choose the sixth bit (0 or 1)
- There is one way to choose the seventh bit (0)
- There is one way to choose the eighth bit (0)

Bit-String Example of Inclusion-Exclusion

Task 2: Construct a string of length 8 that ends with '00'.

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- There are two ways to choose the sixth bit (0 or 1)
- There is one way to choose the seventh bit (0)
- There is one way to choose the eighth bit (0)

Product Rule: Task 2 can be done in $2^6 \times 1^2 = 64$ ways.

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Is the answer $128+64 = 196$?

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Task 3: Construct a string of length 8 that both starts with '1' and ends with '00'.

- There is one way to choose the first bit (1)
- There are two ways to choose each of the the second to sixth bit (0 or 1)
- There are two ways to choose the sixth bit (0 or 1)
- There is one way to choose the seventh bit (0)
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Product Rule: Task 3 can be done in $1 \times 2^5 \times 1^2 = 32$ ways.

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Finally, the number of ways to construct a bit string of length 8 that starts with ‘1’ or ends with ‘00’ is equal to:

No. ways to do Task 1 + No. ways to do Task 2 − No. ways to do Task 3

That is:

$$128 + 64 - 32 = 160$$

Note: An alternative, and quite different, way to solve this problem is to use **complementary counting**;

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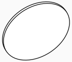
Can you see how to get the $1 \times 2^5 \times 3$?

C2: Probability

Equal likelihood models

A thought experiment:



- Toss a coin.
- What are the possible outcomes?
- 'Heads' or 'Tails'
- What is the probability of 'Heads'?
- We say it is

$$\mathbb{P}(\text{Heads}) = \frac{1}{2}. \qquad \text{Why?}$$

Methods of assigning probabilities

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Method 1: Use relative frequencies (empirical experiment)

Method 2: Use a model (combination of prior knowledge, guessing, deduction)

- Eg. assume equally likely outcomes

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- There is much more to be said on ‘relative frequencies’, but for this course we will focus on making ‘models’.

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  for

  Heads or Tails.

A model for coin tossing: equal likelihood

The two possibilities are just as likely as each other.

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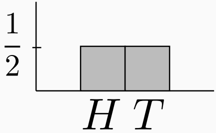
We can represent this situation graphically as



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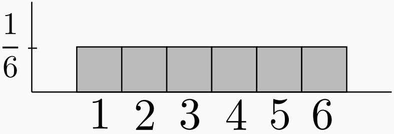
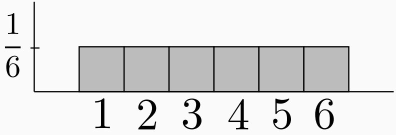
- Sample space: $\{1, 2, 3, 4, 5, 6\}$

An equal likelihood model for die-tossing



What's an event?

An equal likelihood model for die-tossing



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Eg. an event is

$$\{3, 6\}$$

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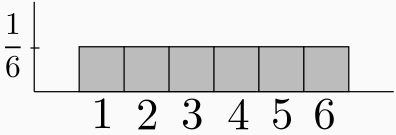

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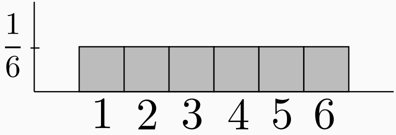

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$$\boxed{\mathbb{P}(\{3,6\}) = \frac{|\{3,6\}|}{|\{1,2,3,4,5,6\}|} = \frac{2}{6} = \frac{1}{3}}$$

Equal Likelihood for finite Sample Spaces

Generalising from the previous example we have:

- Let S be a finite sample space in which all outcomes are equally likely.
- Let E be an event in S.
- Then the probability of the event E is

$$\mathbb{P}(E) = \frac{|E|}{|S|}$$

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where $|E|$ is the number of outcomes in E, and
$|S|$ is the number of outcomes in S.

Properties of Probability

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$$\mathbb{P}(\text{not } E) = 1 - \mathbb{P}(E).$$

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- The sum of the probabilities of all outcomes in the sample space is 1.
- '$\mathbb{P}(E) = 1$' implies E is certain to occur.*
- '$\mathbb{P}(E) = 0$' implies E is impossible.*

*For infinite sets, this isn't necessarily true. 'Measure theory' explains why.

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$$\mathbb{P}(\{1\}) + \ldots + \mathbb{P}(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

Probability Rules

The Sum Rule

Sum Rule: If events $E_1, ..., E_n$ are mutually disjoint, i.e. $E_i \cap E_j = \emptyset$ for all $i \neq j$, then

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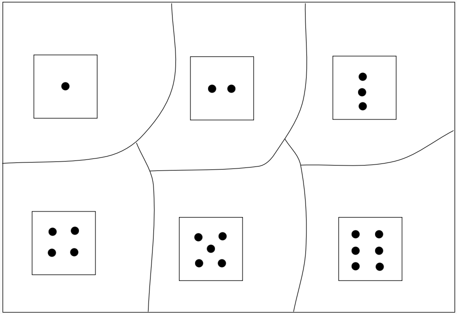
Disjoint events exclude one another in the sense that they cannot happen at the same time.

Sum Rule for probability: another die-tossing example

What is the probability that the outcome from a single toss of a die is an odd number?

The six possible outcomes are all disjoint (cannot occur simultaneously).



Thus the sum rule applies.

- We assign equal probabilities to each of these disjoint events (Why?)
- Six possible outcomes in total $\rightarrow$ each has probability $\frac{1}{6}$ of occurring.
- The probability that the die lands with an odd number up is

$$
\begin{aligned}
& \Pr(⚀) + \Pr(⚂) + \Pr(⚄) \\
= \quad & \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\
= \quad & \frac{1}{2}
\end{aligned}
$$

by the sum rule.

Sum Rule for probability. Example: Non-zero numbers

Let $R_n = \{-n, .., -2, -1, 0, 1, 2, ..., n\}$.

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$$\begin{aligned} &\mathbb{P}(\text{the number is negative}) + \mathbb{P}(\text{the number is positive}) \\ =&\frac{n}{2n+1} + \frac{n}{2n+1} = \frac{2n}{2n+1}. \end{aligned}$$

The Product Rule

- Product Rule: If events $E_1, ..., E_n$ are 'independent' of each other; then the probability of composite event 'E_1 and E_2 and ... and E_n' is

$$\mathbb{P}(E_1 \times E_2 \times ... \times E_n) = \mathbb{P}(E_1) \times \mathbb{P}(E_2) \times ...\mathbb{P}(E_n).$$

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- To see what we mean by 'independent', consider a procedure that can be broken down into successive tasks, each of which could be done in a number of ways. If the choice of the way to do any one task had no influence on the choice of ways to do any other of the tasks, then the tasks would be independent.

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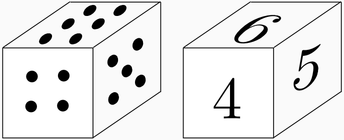
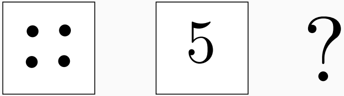
- A formal definition of independence will be given later.

Product Rule probability example: Tossing two dice



- What is the probability that the outcome from tossing a pair of dice is '4' for the first die and '5' for the second die i.e.



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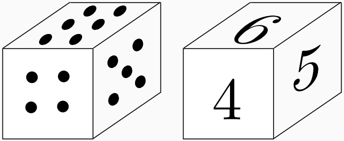

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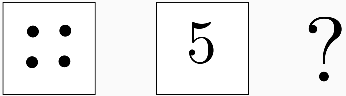

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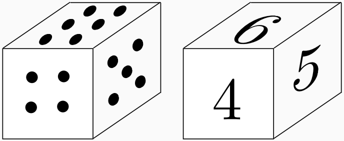

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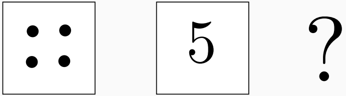

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- Hence the product rule applies.

$$
\begin{aligned}
&\Pr\left(\boxed{\text{⚃}}\ \boxed{5}\right)\\
=\ &\Pr\left(\boxed{\text{⚃}}\right) \times \Pr\left(\boxed{5}\right)\\
=\ &\frac{1}{6} \times \frac{1}{6}\\
=\ &\frac{1}{36}
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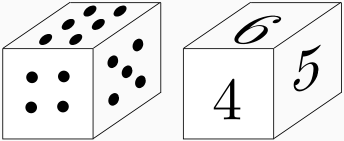

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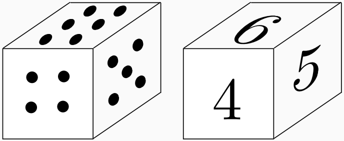

- To obtain an odd total, either
    - the first die must give odd and the second die even; or
    - the first die must give even and the second die odd.
- These two possibilities are **disjoint**, so the sum rule applies:

  $\mathbb{P}(\text{odd total}) = \mathbb{P}(\text{1st odd, 2nd even}) + \mathbb{P}(\text{1st even, 2nd odd})$

- But now consider $\mathbb{P}(\text{1st odd, 2nd even})$. The events

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$$\mathbb{P}(\text{1st odd, 2nd even}) = \mathbb{P}(\text{1st odd}) \times \mathbb{P}(\text{2nd even})$$
$$= \frac{3}{6} \times \frac{3}{6} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

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- Putting it all together,

$$\mathbb{P}(\text{odd total}) = \mathbb{P}(\text{1st odd, 2nd even}) + \mathbb{P}(\text{1st even, 2nd odd})$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$