Discrete Mathematical Models

Lecture 14

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Semester 2, 2024

Vectors

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$$\forall \lambda \in \mathbb{Q} \quad \lambda\mathrm{x} = \lambda(x_1, ..., x_n) = (\lambda x_1, ..., \lambda x_n).$$

Examples of vectors and vector arithmetic

- Let $\mathrm{p} = (p_1, p_2, p_3) \in \mathbb{Q}^3$ represent the state of an ecosystem with p_1, p_2, p_3 being the sizes of the populations of three different species.

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$$3\mathrm{a} = 3(a_1, ..., a_n),$$

represents to the same sound, but three times stronger.

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For matrices $\mathrm{A} = (a_{i,j})$ and $\mathrm{B} = (b_{i,j})$ in $M_n(\mathbb{Q})$, and $\lambda \in \mathbb{Q}$, we define the **sum** $\mathrm{A} + \mathrm{B}$ and **scalar product** $\lambda \mathrm{A}$ by

$$\boxed{\begin{gathered}\mathrm{A} + \mathrm{B} = (a_{i,j}) + (b_{i,j}) = (a_{i,j} + b_{i,j}). \\ \lambda \mathrm{A} = \lambda (a_{i,j}) = (\lambda a_{i,j}).\end{gathered}}$$

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Linear Functions

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Then for $m \in \mathbb{N}$ with $m \leq n$ the function F specified by

$$\begin{array}{rcl} F : \mathbb{Q}^n & \to & \mathbb{Q}^n \\ (a_1, ..., a_n) & \mapsto & (a_1, ..., a_m, 0, 0, ...0) \end{array}.$$

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Filters are linear functions. (Check!)

Linear functions: another example

Let $(p_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}^2$ represent the state of an ecosystem with two species at time n; say $p_n = (x_n, y_n)$, where x_n is the size of the population of species 1, and y_n the size of the population of species 2.

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Assume that the ecosystem evolves as follows, due to a predator-prey relationship between the two species:

$$\forall n \in \mathbb{N} \quad \begin{cases} x_{n+1} = 4x_n - y_n, \\ y_{n+1} = y_n + 2x_n. \end{cases}$$

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We will return to this example.

Multiplying a vector by a matrix: motivation

We now explore the possibility of expressing the function

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By convention, in this context, we normally write the vectors as columns. Thus

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n \end{bmatrix}$$

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Example: $$\begin{pmatrix} 2 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}.$$

Linear functions expressed using matrices

Example: $$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x - y \\ 2x + y \end{pmatrix} = F(x, y)$$

where, as we have seen, the function $F : \mathbb{Q} \to \mathbb{Q}$ so defined is linear.

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Theorem (proof omitted): To each linear function $F : \mathbb{Q}^n \to \mathbb{Q}^n$ there is a matrix $\mathrm{M} \in M_n(\mathbb{Q})$ such that

$$F(\mathrm{x}) = \mathrm{Mx} \quad \forall \mathrm{x} \in \mathbb{Q}^n.$$

Conversely, every function $F : \mathbb{Q}^n \to \mathbb{Q}^n$ defined using a matrix in this way is linear.

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$$\begin{aligned} \mathrm{M}(\mathrm{M}x) &= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \left[\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right] \\ &= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4x - y \\ 2x + y \end{pmatrix} \end{aligned}$$

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Example: For $\mathrm{M} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$:

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\mathrm{M(Mx)} &= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \left[\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right] \\
&= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4x - y \\ 2x + y \end{pmatrix} \\
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So we want $\mathrm{M}^2 = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 14 & -5 \\ 10 & -1 \end{pmatrix}$.

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For matrices $\mathsf{A} = (a_{i,j})$ and $\mathsf{B} = (b_{i,j})$ in $M_n(\mathbb{Q})$ the **product** $\mathsf{AB} = \mathsf{C} = (c_{i,j}) \in M_n(\mathbb{Q})$ is defined by

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Two Examples:

(a) First, let's check that this formula produces what we were looking for with M^2 on the previous slide:

$$M^2 = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

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(b) This example demonstrates the product formula more clearly:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{pmatrix}.$$

Identity matrices

Observe that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acts as an 'identity' in the sense that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

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So $I_1 = \begin{bmatrix} 1 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc.

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By applying the matrix product formula we can immediately establish that, for any $n \in \mathbb{N}$, the identity matrix I_n does indeed have the identity property:

$$\forall n \in \mathbb{N},\ \forall \mathrm{M} \in M_n(\mathbb{Q}) \quad \mathrm{I}_n\mathrm{M} = \mathrm{M} = \mathrm{M}\mathrm{I}_n.$$

Remark: When the value of n is clear from the context, we abbreviate I_n to just I.

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Let x be the quantity of R1, y quantity of R2.

$$\begin{cases} \text{from (a): } x = 2y \\ \text{from (b): } \frac{x}{2} + \frac{y}{3} = 5 \end{cases} \iff \begin{cases} x - 2y = 0 \\ 3x + 2y = 30 \end{cases}$$

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We can solve these equations by elimination, but consider the equivalent matrix equation

$$\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 30 \end{pmatrix}.$$

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Note the if an A^{-1} exists and $\mathsf{Ax} = \mathsf{b}$ then

$$\mathsf{x} = \mathsf{Ix} = (\mathsf{A}^{-1}\mathsf{A})\mathsf{x} = \mathsf{A}^{-1}(\mathsf{Ax}) = \mathsf{A}^{-1}\mathsf{b}.$$

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Lemma: For any $\mathrm{A}, \mathrm{B} \in M_2(\mathbb{Q})$, $\det(\mathrm{AB}) = \det(\mathrm{A}) \det(\mathrm{B})$.
Proof: Multiply out both sides.

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$$\boxed{A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}} \quad \textit{e.g.} \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}^{-1} = \frac{1}{8} \begin{pmatrix} 2 & 2 \\ -3 & 1 \end{pmatrix}$$

Proof: If A has an inverse then

$$1 = \det(I_2) = \det(AA^{-1}) = \det(A)\det(A^{-1}),$$

Calculating Inverses

A matrix A can have at most one inverse, because if B and C are both inverses then $\mathrm{BA} = \mathrm{I}$ and $\mathrm{AC} = \mathrm{I}$ and so

$$\mathrm{B} = \mathrm{BI} = \mathrm{B(AC)} = \mathrm{(BA)C} = \mathrm{IC} = \mathrm{C}.$$

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What about $n > 2$?

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What about $n > 2$? See Math1013 or Math1115.

Interesting Examples

Back to population dynamics

As a final example involving matrix multiplication and matrix inverses, we return to the simple ecosystem model

$$\forall n \in \mathbb{N} \quad \begin{cases} x_{n+1} = 4x_n - y_n, \\ y_{n+1} = y_n + 2x_n, \end{cases}$$

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We can rewrite this in the form

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This is an implicit definition of a sequence of vectors. We will use mathematical induction to establish an explicit formula, stated and proved on the next slide.

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R1: $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ $\qquad \begin{bmatrix} \text{prove by multiplying} \\ \text{out the RHS} \end{bmatrix}$

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Claim: $\forall n \in \mathbb{N}^\star \quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

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Fibonacci Sequence

Recall the Fibonacci sequence $(F_n)_{n\geq 0} = (1, 1, 2, 3, 5, 8, 13, 21, ...)$

It has implicit formula, $F_0 = 1, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$.

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For all $n \geq 1$ we have:

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\end{aligned}$$

Fibonacci Sequence

Recall the Fibonacci sequence $(F_n)_{n\geq 0} = (1, 1, 2, 3, 5, 8, 13, 21, ...)$

It has implicit formula, $F_0 = 1, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$.

For all $n \geq 1$ we have:

$$\begin{aligned}
\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} \\
&= \tfrac{1}{\sqrt{5}} \begin{pmatrix} \varphi & -\varphi^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^{-1} \end{pmatrix} \begin{pmatrix} 1 & \varphi^{-1} \\ -1 & \varphi \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} \\
&= \tfrac{1}{\sqrt{5}} \begin{pmatrix} \varphi & -\varphi^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi^n & 0 \\ 0 & -\varphi^{-n} \end{pmatrix} \begin{pmatrix} 1 & \varphi^{-1} \\ -1 & \varphi \end{pmatrix} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}
\end{aligned}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$. We can extract an explicit formula for the Fibonacci sequence!