Throughout this exam we write $\mathbb{N}$ for the set $\{1, 2, 3, \dots\}$ and we write $\mathbb{Z}_{\geq 0}$ for the set $\{0, 1, 2, 3, \dots\}$.

Problem 1 (10 marks)

(a) For each of the following sentences, circle one of the words STATEMENT/PREDICATE/NEITHER to indicate the nature of the sentence.

(i) $\forall t \in \mathbb{Z} \;\; t \geq t^2$ STATEMENT /PREDICATE /NEITHER

(ii) $\exists n \in \mathbb{N} \;\; x^2 - \frac{x}{2} = n$ STATEMENT /PREDICATE /NEITHER

(iii) Google's PageRank algorithm is a clever example of discrete mathematical modelling. STATEMENT /PREDICATE /NEITHER

(iv) For any simple undirected graph G, the total degree of G equals twice the number of edges in G. STATEMENT /PREDICATE /NEITHER

(b) Consider the following statement:

> "Today is not Tuesday nand today is not Tuesday, nand, I am enjoying my exam nand I am enjoying my exam."

Write a statement that is logically equivalent to the statement under consideration, but which only uses the logical structure of an implication. Your statement should be written in English and you should use a truth table to justify that your statement is logically equivalent to the statement under consideration.

(c) Consider the predicate

$$P(f): \quad \forall a_1 \in \mathbb{N} \ \ \forall a_2 \in \mathbb{N} \ \ \forall b \in \mathbb{N} \ \ (((a_1, b) \in f \land (a_2, b) \in f) \Rightarrow (a_1 = a_2)),$$

defined over the domain of functions from $\mathbb{N}$ to $\mathbb{N}$.

(i) Without using the symbol $\neg$, write down a predicate $Q(f)$, defined over the domain of functions from $\mathbb{N}$ to $\mathbb{N}$, such that $Q(f) \equiv \neg P(f)$

(ii) Write down an example of a function $g : \mathbb{N} \to \mathbb{N}$ for which $P(g)$ is true. Justify your answer.

(iii) Write down an example of a function $h : \mathbb{N} \to \mathbb{N}$ for which $P(h)$ is false. Justify your answer.

(d) (i) In no more than two sentences, explain what it means to say that a compound statement c is in **disjunctive normal form**.

(ii) In no more than three sentences, describe an application of the following fact to circuit design:

> Every statement that is not a contradiction is logically equivalent to a statement that is in disjunctive normal form.

Problem 2 (10 marks)

(a) For each $i \in \mathbb{N}$, we write $M_i = \{\ldots, -3i, -2i, -i, 0, i, 2i, 3i, \ldots\}$.

(i) Use set-builder notation to describe M_5.

(ii) Either prove or disprove the following statement:

$$\forall n \in \mathbb{N} \ \ (11^n - 6 \in M_5).$$

(b) Let E denote the set of even integers.

(i) Let A and B be sets. Write down what it means to say that A and B have the same cardinality.

(ii) Write down the rule for a bijection $f : \mathbb{N} \to E$, or explain how you know that no such bijection exists.

(iii) What fact about E does your answer to (ii) demonstrate?

(c) Either use the logical equivalences below and the definitions of set operations to prove, or provide a counterexample to disprove, the following statement:

For any universal set U, and any sets $A, B, C, \in \mathcal{P}(U)$, $A \cup (B \setminus C) = B \cup (A \setminus C)$.

Given any statement variables p, q, and r, a tautology t and a contradiction c, the following hold.

1. Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:	$p \wedge t \equiv p$	$p \vee c \equiv p$
5. Negation laws:	$p \vee \neg p \equiv t$	$p \wedge \neg p \equiv c$
6. Double negative law:	$\neg(\neg p) \equiv p$	
7. Idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. Universal bound laws:	$p \vee t \equiv t$	$p \wedge c \equiv c$
9. De Morgan's laws:	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
10. Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. Negations of t and c:	$\neg t \equiv c$	$\neg c \equiv t$

(d) Throughout this part of the problem we consider the following statement:

For any $x \in \mathbb{Z}$, if $x \not\equiv 2 \pmod 3$ or $x \not\equiv 2 \pmod 5$, then $x \not\equiv 2 \pmod{75}$.

(i) Write down the logical structure of a proof of the statement under consideration that proceeds directly.

(ii) Write down the logical structure of a proof of the statement under consideration that proceeds via the contrapositive.

(iii) Prove or disprove the statement under consideration.

Problem 3 (10 marks)

(a) A web-banking password is always 8 characters long and it always comprises two upper-case letters from the standard English alphabet, two digits, and four lower-case letters from the standard English alphabet. How many different passwords can be created that follow these rules?
Give your answer in the form of an expression that uses only arithmetic operations (possibly including factorial and exponentiation) and numbers.

(b) Consider a probability experiment in which we roll a six-sided die. The outcome of the experiment is the number facing up on the die after it is rolled. Let $S = \{1, 2, 3, 4, 5, 6\}$ and let $X, Y : S \to \{0, 1\}$ be random variables such that for all $s \in S$ we have

$$X(s) = s \bmod 2, \text{ and } Y(s) = \begin{cases} 0 \text{ if } s \leq 3, \\ 1 \text{ otherwise.} \end{cases}$$

Determine whether or not X, Y are independent. Justify your answer.

(c) A 4-tuple $(x_1, x_2, x_3, x_4) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is **valid** if $x_1 + x_2 + x_3 + x_4 = 50$. If a valid 4-tuple is chosen at random, what is the probability that it will contain only the numbers 5, 10, 15 and 20 in some order?
Give your answer in the form of an expression that uses only arithmetic operations (possibly including factorial and exponentiation) and numbers.

In an excellent response, each of the following will be clearly identified: the experiment, the sample space, how probabilities will be computed, and any events of interest.

Problem 4 (10 marks)

(a) Your friend tries to define the term "graph" and writes the following:

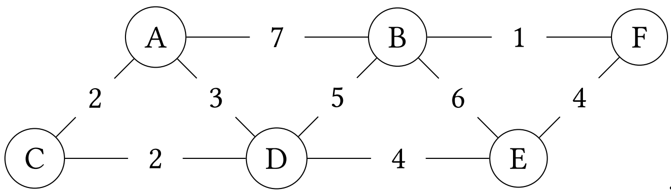
> "A graph G is an ordered pair $G = (V(G), E(G))$ comprising: a set of vertices $V(G)$; and a set of edges $E(G)$, with each edge being a set of 2 vertices."

Looking at our lecture notes, you see that we defined the term "graph" as follows:

> "A graph G is an ordered pair $G = (V(G), E(G))$ comprising: a set of vertices $V(G)$; and a multiset of edges $E(G)$, with each edge being a size-2 multiset of vertices."

In no more than three sentences and using appropriate examples, identify the effective difference(s) between the two definitions.

(b) Let G be the weighted graph



.

Draw a minimal spanning tree for G in the space below.

(c) Let

$$E = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid K_{m,\,n} \text{ has an Euler circuit and } m + n \geq 3\}$$
$$H = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid K_{m,\,n} \text{ has a Hamilton circuit and } m + n \geq 3\}.$$

(i) Describe H in set-roster notation. Prove that your description gives H.

(ii) Use set-roster notation to describe $E \cap H$. Justify your answer.

(d) In this problem we consider the vertex-labelling algorithm described in the course (pseudocode for the algorithm is given on page 21).

(i) In no more than three sentences, describe a physical system that may be reasonably modelled by a transport network. In your description, clearly state what it is about the physical system that the vertices and directed edges and capacities represent.

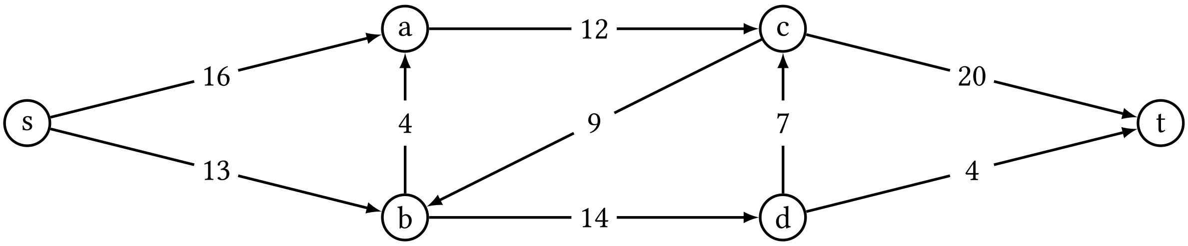
(ii) Is the vertex-labelling algorithm an example of a greedy algorithm? Justify your answer.

(iii) The following is a quote from our lecture slides:

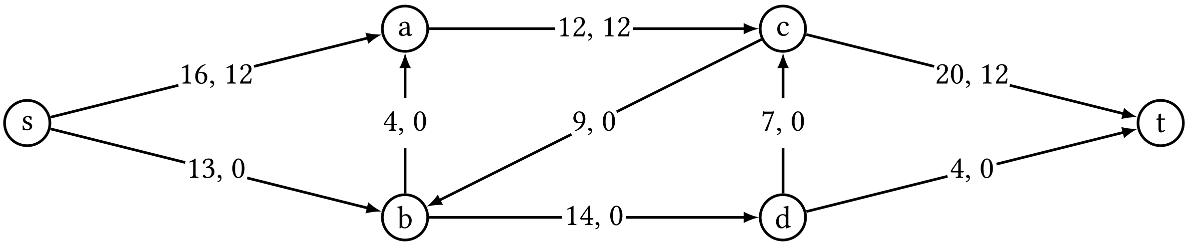
> "Given a relation $R \subseteq S \times T$, a **matching problem** seeks an injective function $f : S' \to T$ with domain $S' \subseteq S$ as large as possible subject to m being an injective subset of R."

In no more than three sentences explain how a matching problem for $R \subseteq S \times T$ may be converted to the maximum flow problem for a transport network (so that the vertex-labelling algorithm may be used to solve the matching problem).

(iv) Indicate a minimum cut on the following transport network.



(v) Use the vertex-labelling algorithm described in the course to find a maximum flow function for the transport network in part (iv). The first incremental flow f_1 is shown in the first row of the table at the bottom of the page, and the cumulative flow F_1 is shown in the graph below. Write down the subsequent incremental flows in the table (use only as many rows as you need).



incremental flow label	path of incremental flow	volume of incremental flow
f_1	$s\ a\ c\ t$	12

Problem 5 (10 marks)

(a) We consider population movement within a geographic region comprising a city, its suburbs, and the hinterland. We aim to understand the relative populations of the city, the suburbs, and the hinterland as they will change over time. Suppose that the annual migration between the three parts of the geographic region obeys the following. Each year: 4% of the city residents move to the suburbs and 2% move to the hinterland; 7% of the suburban residents move to the city and 2% move to the hinterland; 4% of hinterland residents move to the city and 6% move to the suburbs; all other residents stay in the region they were in.

(i) Draw a transition diagram for a Markov process model of the population movement.

(ii) Write down a transition matrix T for your model.

(iii) List all of the properties a 3×1 vector $\mathbf{v}$ must satisfy in order to be a steady state of your model?

(b) In this problem we consider applying Google's PageRank algorithm to a small webgraph W. The adjacency matrix for W is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

(i) We note that A is a 7×7 matrix. What does this tell us about the web represented by W?

(ii) We note that the entry in row 2 column 4 of the matrix A is 1. What does this tell us about the web represented by W?

(iii) Write down the basic transition matrix T when Google's PageRank algorithm is applied to W.

(iv) Write down an expression for the modified transition matrix M when Google's PageRank algorithm is applied to W. Use a damping factor of 85% (that is, $\alpha = 0.15$).

(v) In no more than three sentences, explain how to use the matrix M to rank the pages in our small web in order of "awesomeness" (or importance).

Vertex-labelling algorithm for finding a maximum flow function for a transport network

Input: Transport network D with capacity function C.
Output: A maximum flow function $F_{\max}$ for the network.
Method: Initialise F to the zero flow F_0. Initialize i to 1.

For $i = 1, 2, \ldots$ carry out stage i below to attempt to build an incremental flow f_i.

If stage i succeeds, define $F_i = F_{i-1} + f_i$ and proceed to stage $i+1$.

If stage i fails, define $F_{\max} = F_{i-1}$ and stop.

Stage i:

(a) If $i > 1$, mark up the amended edge flows for F_{i-1}.

(b) Mark up the levels for F_{i-1}, as explained below.

(c) If t is assigned a level, stage i will succeed, so continue.

If not, then stage i fails, so return above to define $F_{\max}$ and terminate.

(d) Mark up labels for F_{i-1} as follows until t is labelled:

  (i) Label each level 1 vertex v with sk_{v}, where $k_{\mathrm{v}} = S((s,v))$. (see below for definition of S)

  (ii) If t has level 2 or more now work through the level 2 vertices in alphabetical order, labelling each vertex v with uk_u, where

  - u is the alphabetically earliest level 1 vertex with $(u,v) \in E(D)$ and $S((u,v)) > 0$,
  - k_{u} is the minimum of $S((u,v))$ and the value part of u's label.

  (iii) If t has level 3 or more now work through the level 3 vertices in a similar manner and so on.

(e) Let p_i be the path $\mathrm{u}_0\mathrm{u}_1 \ldots \mathrm{u}_n$ where $\mathrm{u}_n = \mathrm{t}$ and for $0 < j \leq n$ u_j has label $\mathrm{u}_{j-1}k_j$.

Define f_i to be the incremental flow on p_i with flow value k_n.

End of Method
Levels and labels: At each stage of the vertex labelling algorithm levels and labels are associated afresh with the vertices of the network.

The **level** of a vertex is determined iteratively as follows:

- The source vertex s always has level 0.
- If $e = (s, x)$ and $S(e) > 0$ then x has level 1.
- If x has level n and $S((x, y)) > 0$ then y has level $n + 1$ provided it has not already been assigned a lower level.

The **label** on a vertex v of level $n \geq 1$ has the form uk, where u is a vertex of level $n - 1$ and $(u, v) \in E(D)$ is an edge on the path for a potential incremental flow through v with flow value k.
The algorithm assigns labels in ascending order of levels, and in alphabetical order within levels.

The spare capacity function S
For vertices u,v of D, where D has capacity and flow functions C, F:

$$S((\mathrm{u,v})) = \begin{cases} C((\mathrm{u,v})) - F((\mathrm{u,v})) & \text{if (u,v)} \in E(D) \\ F((\mathrm{v,u})) & \text{if (v,u)} \in E(D) \\ 0 & \text{otherwise.} \end{cases}$$

When (v,u) $\in E(D)$, $S((\mathrm{u,v}))$ is called a **virtual capacity**.

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If necessary, you may use this space for extra working.