Discrete Mathematical Models

Lecture 21

Kane Townsend

Semester 2, 2024

C3: Markov Processes (cont.)

Markov Process Definition (Clearer)

A (discrete) **Markov process** is a system that has:

- a finite number k of states,
- a sequence of time steps $n \in \mathbb{N}^*$,
- probabilities of moving from a state to another state (including itself) that depends only on your current state.

Hence, probabilities of being in a particular state at time $n \geq 1$ depend on

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A **transition diagram** is a complete weighted directed graph with k vertices representing the states of the system and the edge from the j-th vertex to the i-th vertex labelled with the probability T_{ij}.

Example 2

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As a table: Probabilities of weather tomorrow are:

Given that the weather today is:	$\curvearrowright$	fine	cloudy	rain
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	cloudy	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
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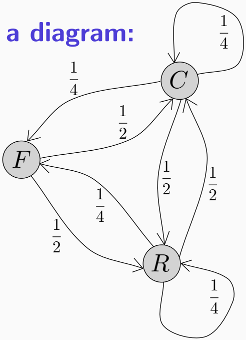

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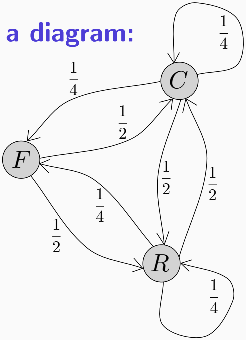

As a matrix:

$$T = \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix}$$

Note: T is a stochastic matrix but not positive. However, since a power of T is positive, Perron-Frobenius still applies and we have a unique steady state vector and our guessing method still works.

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- Then, according to the Markov process theorem:

$$\begin{aligned} \mathrm{x}_{n+1} &= T\mathrm{x}_n \\ &= \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (1/4)y + (1/4)z \\ (1/2)x + (1/4)y + (1/2)z \\ (1/2)x + (1/2)y + (1/4)z \end{bmatrix} \end{aligned}$$

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Perhaps decimals would be more illuminating?

Days 1 through 10 in Oz

Computer calculations give:

$$x_1 = \begin{bmatrix} 0 \\ .5 \\ .5 \end{bmatrix}, \quad x_2 = \begin{bmatrix} .250 \\ .375 \\ .375 \end{bmatrix}, \quad x_3 = \begin{bmatrix} .18750 \\ .40625 \\ .40625 \end{bmatrix}, \quad x_4 = \begin{bmatrix} .2031250 \\ .3984375 \\ .3984375 \end{bmatrix},$$

$$x_5 = \begin{bmatrix} .199218750 \\ .400390625 \\ .400390625 \end{bmatrix}, \quad x_6 = \begin{bmatrix} .1999511719 \\ .4000244141 \\ .4000244141 \end{bmatrix}, \quad x_7 = \begin{bmatrix} .19995511719 \\ .4000244141 \\ .4000244141 \end{bmatrix},$$

$$x_8 = \begin{bmatrix} .2000122070 \\ .3999938965 \\ .3999938965 \end{bmatrix}, \quad x_9 = \begin{bmatrix} .1999969438 \\ .4000015260 \\ .4000015260 \end{bmatrix}, \quad x_{10} = \begin{bmatrix} .2000007629 \\ .3999996185 \\ .3999996185 \end{bmatrix}.$$

A steady state for the weather in Oz

These values seem to be converging to a long-term steady state of

$$v = \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix},$$

i.e. a probability of 0.2 of fine weather, a probability of 0.4 of cloudy weather and a probability of 0.4 of rainy weather.

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To check that this v really is a normalised steady state vector, we also calculate

$$Tv = \begin{bmatrix} 0 & 0.25 & 0.25 \\ 0.50 & 0.25 & 0.50 \\ 0.50 & 0.50 & 0.25 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.1 + 0.1 \\ 0.1 + 0.1 + 0.2 \\ 0.1 + 0.2 + 0.1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}.$$

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Therefore $$Tv = v. \qquad \checkmark$$

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A shortcut

A shortcut to this process is to take the augmented matrix $[T - I|0]$ as below (this is just representing $(T - I)v = 0$ in a more succinct way),

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and solve this new system to directly obtain the unique solution for v.

Solving by Computer (using Reshish)

The system of equations

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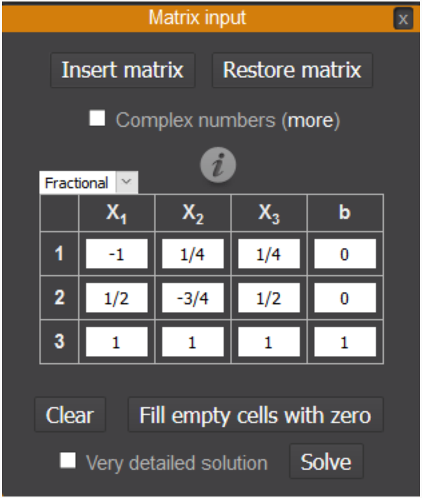

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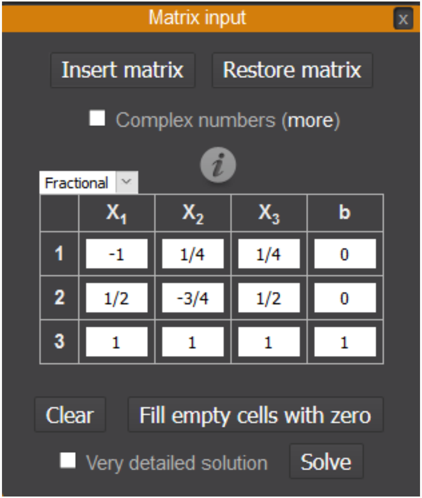
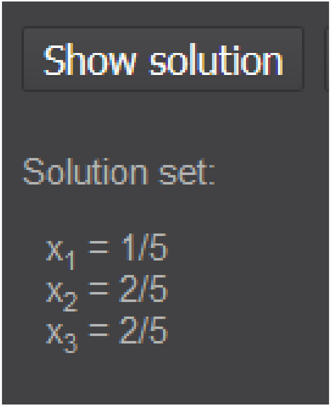


Here is how Reshish responds:



Back to the first example

We have seen that to find the steady state vector v for a Markov process with transition matrix T we need to solve the linear system that results from replacing the last equation in

$$(T - I)v = 0$$

by the equation that says that v is a probability vector.

For Cathy's employment process we had

$$T = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

and, by a 'guess and check' method, we discovered that

$$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}.$$

Solution by matrix inverse

Because T is 2×2, and we have a formula for the inverse of a 2×2 matrix, we can find Cathy's steady state vector directly, without Gaussian elimination or computer. There are three steps:

1. Write out the matrix equation $(T - I)v = 0$:

$$\begin{aligned} \left(\begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \textit{i.e.} \begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

2. Replace the second equation by $x + y = 1$:

$$\begin{bmatrix} -0.2 & 0.6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution by matrix inverse (conclusion)

3. Solve this system using matrix inverse:

$$
\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -0.2 & 0.6 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{-0.2 - 0.6} \begin{bmatrix} 1 & -0.6 \\ -1 & -0.2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{-0.8} \begin{bmatrix} -0.6 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 6/8 \\ 2/8 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}
\end{aligned}
$$

Example 3

New Example — Colours of flowers

A species of flower (carnations say) has three colour varieties. The relevant genetics are as shown in the table:

Colour	Genotype
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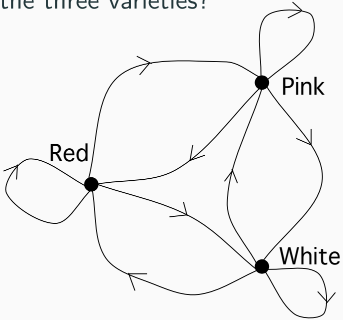
First, we need the transition probabilities. Can you work them out?



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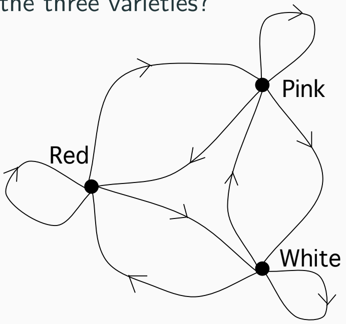

The transition matrix is

$$T = \begin{array}{r} \\ \text{Red} \\ \text{Pink} \\ \text{White} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Red} & \text{Pink} & \text{White} \end{array} \\ \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{bmatrix} \end{array} .$$

Finding the steady state

(a) $[T - I|0]$ is

$$\left[\begin{array}{ccc|c} -0.5 & 0.25 & 0 & 0 \\ 0.5 & -0.5 & 0.5 & 0 \\ 0 & 0.25 & -0.5 & 0 \end{array}\right]$$

(b) Replacing the bottom row with all 1's gives

$$\left[\begin{array}{ccc|c} -0.5 & 0.25 & 0 & 0 \\ 0.5 & -0.5 & 0.5 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

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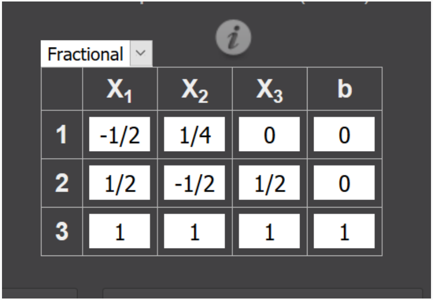

Fractional

	X_1	X_2	X_3	b
1	-1/2	1/4	0	0
2	1/2	-1/2	1/2	0
3	1	1	1	1

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1	-1/2	1/4	0	0
2	1/2	-1/2	1/2	0
3	1	1	1	1

Hence there is a unique steady state vector of

$$v = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

Finding the steady state by computer

We can solve the system using the computer. For example, using Reshish:

Fractional

	X_1	X_2	X_3	b
1	-1/2	1/4	0	0
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Hence there is a unique steady state vector of

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So the species has a steady state in which 25% of the flowers are coloured red, 50% pink, and 25% white.

Checking the answer

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Let's check $Tv = v$:

$$
\begin{aligned}
Tv &= \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix} \\
&= \begin{bmatrix} 1/8 + 1/8 \\ 1/8 + 1/4 + 1/8 \\ 1/8 + 1/8 \end{bmatrix} \\
&= \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}
\end{aligned}
$$

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$$= \begin{bmatrix} 1/8 + 1/8 \\ 1/8 + 1/4 + 1/8 \\ 1/8 + 1/8 \end{bmatrix}$$
$$= \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

So yes, $Tv = v$. ✓

Will a Markov process always get to a steady state?

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Not necessarily!

Example 4

Example: chemical compounds in transition

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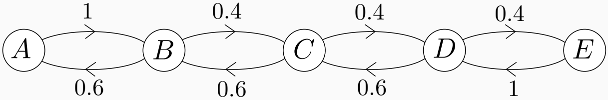

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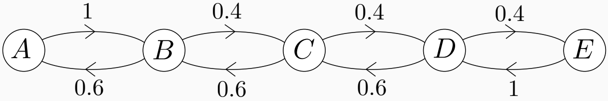

The transition matrix for this Markov Process is

$$T = \begin{bmatrix} 0 & 0.6 & 0 & 0 & 0 \\ 1 & 0 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 1 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}$$

A beaker full of chemical

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- To do a thorough analysis of all possible behaviours of this Markov Process, you need to study '*eigenvalues and eigenvectors*' – a reason to take a course or read a book on Linear Algebra.
- But let's see what we can figure out without those tools.

Chemical example — investigating with a computer

Suppose the beaker only contains form 'A' to start with, *i.e.* $\mathrm{x}_0 = [1, 0, 0, 0, 0]'$. Then by computer to 6dp we find:

$$
\begin{aligned}
\mathrm{x}_{100} &= T^{100}\mathrm{x}_0 \\
&= [0.415383\,,\ 0.000000\,,\ 0.461538\,,\ 0.000000\,,\ 0.123077]^t \\
\mathrm{x}_{101} &= T\mathrm{x}_{100} \\
&= [0.000000\,,\ 0.692308\,,\ 0.000000\,,\ 0.307692\,,\ 0.000000]^t
\end{aligned}
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\end{aligned}
$$

and continuing in the same manner

$$
\begin{aligned}
x_{102} &= [0.415383\,,\ 0.000000\,,\ 0.461538\,,\ 0.000000\,,\ 0.123077]^t \\
x_{103} &= [0.000000\,,\ 0.692308\,,\ 0.000000\,,\ 0.307692\,,\ 0.000000]^t \\
x_{104} &= [0.415383\,,\ 0.000000\,,\ 0.461538\,,\ 0.000000\,,\ 0.123077]^t \\
x_{105} &= [0.000000\,,\ 0.692308\,,\ 0.000000\,,\ 0.307692\,,\ 0.000000]^t \\
&\ \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}
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x_{105} &= [0.000000\,,\ 0.692308\,,\ 0.000000\,,\ 0.307692\,,\ 0.000000]^t \\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

It appears to alternate!

However starting with a beaker half full of A and half of B, *i.e.* $x_0 = [0.5, 0.5, 0, 0, 0]^t$, and again using formulae

$$x_n = T^n x_0 \quad \text{and} \quad x_{n+1} = T x_n$$

repeatedly we get

$$\begin{aligned}
x_{100} &= [0.207692\,,\ 0.346154\,,\ 0.230769\,,\ 0.153846\,,\ 0.061539]^t \\
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This looks like a steady state!

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\vdots & \qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

This looks like a steady state!

So this Markov Process is different to those we used to model employment, weather in Oz, and flower-colours because

eventual behaviour depends on where you start!

Steady state(s) for a beaker of chemical?

We can solve for the steady state to find out if it is unique.

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$$Tv = v$$

for $v = [x_1, x_2, x_3, x_4, x_5]^t$ subject to additional constraint

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1.$$

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(a) First construct $[T - I|0]$.

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$$x_1 + x_2 + x_3 + x_4 + x_5 = 1.$$

We use the 'shortcut' method:

(a) First construct $[T - I|0]$.
(b) Then replace the last row with all 1's.
(c) Then solve by computer (Gaussian elimination).

Steady state(s) for a beaker of chemical?

(a) $[T - I \mid 0]$ is

$$\left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & -1 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0.4 & -1 & 0 \end{array}\right]$$

(b) Replace the last row with all 1's

$$\left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & -1 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Steady state for a beaker of chemical - by computer

Decimal

	X_1	X_2	X_3	X_4	X_5	b
1	-1	0.6	0	0	0	0
2	1	-1	0.6	0	0	0
3	0	0.4	-1	0.4	0	0
4	0	0	0.4	-1	1	0
5	1	1	1	1	1	1

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3	0	0.4	-1	0.4	0	0
4	0	0	0.4	-1	1	0
5	1	1	1	1	1	1

This confirms there is a unique normalised steady state solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 27/130 \\ 45/130 \\ 15/65 \\ 10/65 \\ 4/65 \end{bmatrix} = \begin{bmatrix} 0.2077 \\ 0.3462 \\ 0.2308 \\ 0.1538 \\ 0.0615 \end{bmatrix}.$$

Steady state for a beaker of chemical - by computer

Decimal

	X_1	X_2	X_3	X_4	X_5	b
1	-1	0.6	0	0	0	0
2	1	-1	0.6	0	0	0
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So the steady-state proportions of the five forms of the chemical are:

A: 20.77%, B: 34.62%,
C: 23.08%, D: 15.38%,
E: 6.15%.

A steady state for a beaker of chemical - conclusion

We found that **provided** the beaker reaches a steady-state, then proportions of the various forms of the chemical remain stable at

$$A : 20.77\%, \quad B : 34.62\%, \quad C : 23.08\%, \quad D : 15.38\%, \quad E : 6.15\%.$$

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END OF SECTION C3