Discrete Mathematical Models

Lecture 6

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Section A2: Sets (Continued)

Russell's Paradox

Russell's paradox

Most sets are not members of themselves.

Example: If $P = \{\text{pigeon, parrot}\}$ then $P \notin P$
(since $\{\text{pigeon, parrot}\} \neq \text{pigeon}$, and $\{\text{pigeon, parrot}\} \neq \text{parrot}$.)

But some sets **are** members of themselves.

Examples:

- The set of all sets.
- The set of all things that are not birds.

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(since $\{\text{pigeon, parrot}\} \neq$ pigeon, and $\{\text{pigeon, parrot}\} \neq$ parrot.)

But some sets **are** members of themselves.

Examples:

- The set of all sets.
- The set of all things that are not birds.

We shall call a set **regular** if it is not a member of itself.

We let $\mathcal{R}$ denote the set of all regular sets.

Question: **Is $\mathcal{R}$ itself regular?**

$$
\begin{array}{ll}
 & \mathcal{R} \text{ not regular} \\
\Longrightarrow & \mathcal{R} \text{ is a member of itself} \\
\Longrightarrow & \mathcal{R} \in \mathcal{R} \\
\Longrightarrow & \mathcal{R} \in \text{the set of all regular sets} \\
\Longrightarrow & \mathcal{R} \text{ is regular}
\end{array}
$$

$$
\begin{array}{ll}
 & \mathcal{R} \text{ is regular} \\
\Longrightarrow & \mathcal{R} \text{ is not a member of itself} \\
\Longrightarrow & \mathcal{R} \notin \mathcal{R} \\
\Longrightarrow & \mathcal{R} \notin \text{the set of all regular sets} \\
\Longrightarrow & \mathcal{R} \text{ not regular}
\end{array}
$$

A contradiction either way! (paradox)... Naive set theory fails!

Axiomatic Set Theory

The paradox on the previous slide was discovered by Bertrand Russell in 1901, and is known as **Russell's Paradox**.

To avoid problems such as Russell's paradox we need to move to a more formal approach. The most classical is an axiomatic system called ZFC (Zermelo-Fraenkel with choice). We will not study it in this course, but will assume some of its axioms. In particular:

Axiom 1: A set T can only be defined as a subset of a known set U. That is, the definition must have the form:

$$T \quad = \quad \{x \in U \mid p(x)\}$$ where the domain of predicate p includes all elements of U.

Example

The phrase "the set of all birds" does not define a set in ZFC. But, if we have managed to define A as the set of all animals, then "the sets of all animals that are birds" does define a set in ZFC as we can define:

A: set of all animals.

$B = \{b \in A \mid b \text{ is a bird}\} \subseteq A$.

Russell's paradox removed

In ZFC we can still say that a set S is **regular** if and only if it satisfies the condition (predicate) $S \notin S$.

However we cannot define $\mathcal{R}$ as the set of *all* regular sets.

Instead, for any *known set of sets* $\mathcal{U}$ we can define $\mathcal{R}_{\mathcal{U}}$ by

$$\boxed{\mathcal{R}_{\mathcal{U}} = \{S \in \mathcal{U} \mid S \text{ is regular}\} = \{S \in \mathcal{U} \mid S \notin S\}}$$

So we again ask: **For any set of sets $\mathcal{U}$, is $\mathcal{R}_{\mathcal{U}}$ itself regular?**

Let $\mathcal{U}$ be a known set of sets.

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Let $\mathcal{U}$ be a known set of sets. Then

$$\begin{aligned}
& \mathcal{R}_{\mathcal{U}} \text{ not regular} \\
\implies \quad & \mathcal{R}_{\mathcal{U}} \text{ is a member of itself} \\
\implies \quad & \mathcal{R}_{\mathcal{U}} \in \mathcal{R}_{\mathcal{U}} \\
\implies \quad & \mathcal{R}_{\mathcal{U}} \in \{S \in \mathcal{U} \mid S \text{ is regular } \} \\
\implies \quad & \mathcal{R}_{\mathcal{U}} \text{ is regular} \qquad \text{Contradiction!}
\end{aligned}$$

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$$

$$
\begin{aligned}
& \mathcal{R}_{\mathcal{U}} \text{ regular} \\
\implies \quad & \mathcal{R}_{\mathcal{U}} \text{ is not a member of itself} \\
\implies \quad & \mathcal{R}_{\mathcal{U}} \notin \mathcal{R}_{\mathcal{U}} \\
\implies \quad & \mathcal{R}_{\mathcal{U}} \notin \{ S \in \mathcal{U} \mid S \text{ is regular } \} \\
\implies \quad & (\mathcal{R}_{\mathcal{U}} \notin \mathcal{U}) \vee (\mathcal{R}_{\mathcal{U}} \text{ is not regular})
\end{aligned}
$$

So we again ask: **For any set of sets $\mathcal{U}$, is $\mathcal{R}_{\mathcal{U}}$ itself regular?**

Let $\mathcal{U}$ be a known set of sets. Then

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Since

$$\begin{array}{ll} & \mathcal{R}_{\mathcal{U}} \text{ regular} \\ \Longrightarrow & \mathcal{R}_{\mathcal{U}} \text{ is not a member of itself} \\ \Longrightarrow & \mathcal{R}_{\mathcal{U}} \notin \mathcal{R}_{\mathcal{U}} \\ \Longrightarrow & \mathcal{R}_{\mathcal{U}} \notin \{S \in \mathcal{U} \mid S \text{ is regular }\} \\ \Longrightarrow & (\mathcal{R}_{\mathcal{U}} \notin \mathcal{U}) \vee (\mathcal{R}_{\mathcal{U}} \text{ is not regular}) \end{array}$$

$\Big((\mathcal{R}_{\mathcal{U}} \text{ regular}) \Longrightarrow (\mathcal{R}_{\mathcal{U}} \text{ is not regular})\Big)$ is contradictory, we conclude that $\mathcal{R}_{\mathcal{U}}$ is regular but not in $\mathcal{U}$. There is no overall contradiction.

Section A3: Relations and functions

Relations

Relations

Let A, B be non-empty sets. Any subset of $A \times B$ is called a **relation from** A **to** B. A relation from A to A is called a **relation on** A.

Given a relation R from A to B and an element $(a, b) \in A \times B$, we usually write $a \; R \; b$ instead of $(a, b) \in R$ and we usually write $a \not R b$ instead of $(a, b) \notin R$.

An example will help us see why relations are important, and why these choices of notation are made.

Example

Let

$$B = \{\text{Breakfast}, \text{Lunch}, \text{Dinner}, \text{Snack}\}$$

$$L = \{\text{am}, \text{pm}\}$$

$$\sim = \{(\text{Breakfast}, \text{am}), (\text{Snack}, \text{am}), (\text{Lunch}, \text{pm}), (\text{Dinner}, \text{pm}), (\text{Snack}, \text{pm})\}$$

Then $\sim$ is a relation from B to L, and we may write

$$(\text{Breakfast}, \text{am}) \in \sim$$

or

$$\text{Breakfast} \sim \text{am}$$

Example

We define

$$< \; = \; \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b - a \in \mathbb{N}\}.$$

We have defined a relation $<$ on $\mathbb{Z}$.

We could write $(5, 12) \in <$, but we usually prefer $5 < 12$.

Another way to write the same definition: For all integers a and b,

$$((a, b) \in <) \Leftrightarrow (b - a \in \mathbb{N}).$$

Another way to write the same definition: For all integers a and b,

$$(a < b) \Leftrightarrow (b - a \in \mathbb{N}).$$

Another example

Q: Define the familiar relation $\geq$ on $\mathbb{Z}$

A: We define

$$\geq = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b \in \mathbb{Z}_{\geq 0}\}.$$

A: For all integers a and b,

$$(a \geq b) \Leftrightarrow (a - b \in \mathbb{Z}_{\geq 0}).$$

Inverse relation

The **inverse** R^{-1} of a relation $R \subseteq A \times B$ is the relation $R^{-1} \subseteq B \times A$ defined by

$$R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}.$$

Thus $\qquad b \, R^{-1} \, a \iff a \, R \, b.$

Example: Let A be a set of canned soup suppliers, let B be a set of supermarkets, and let $R \subseteq A \times B$ defined by $a \, R \, b \iff a$ sells to b. Then

$$\begin{aligned} b \, R^{-1} \, a \quad &\iff \quad a \, R \, b \\ &\iff \quad a \text{ sells to } b \\ &\iff \quad b \text{ buys from } a. \end{aligned}$$

Diagram of a relation

For small sets, relations can be expressed with **arrow diagrams**.

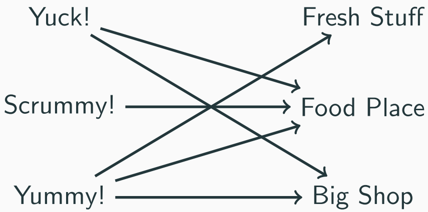
Example: Let

$$\begin{aligned}
A &= \{\text{Yummy!}, \text{Scrummy!}, \text{Yuck!}\}\\
B &= \{\text{Big Shop}, \text{Food Place}, \text{Fresh Stuff}\}\\
R &= \{(\text{Yummy!}, \text{Big Shop}), (\text{Yummy!}, \text{Food Place}),\\
&\quad (\text{Yummy!}, \text{Fresh Stuff}), (\text{Scrummy!}, \text{Food Place}),\\
&\quad (\text{Yuck!}, \text{Food Place}), (\text{Yuck!}, \text{Big Shop})\}
\end{aligned}$$

Here A is a set of brands, and B is a set of vendors. Perhaps aRb means "a sells to b."
Q: How would you represent R in a diagram?

Example

Directed arrows from elements in set A to elements in set B can be used to show which elements are related.



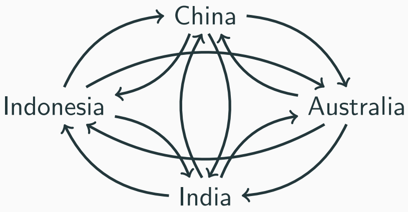
Q: The relation R^{-1} represents which brands are stocked by each vendor. How would you change the diagram above to represent the inverse relation R^{-1}?

A: Change the direction of the arrows.

Directed Graphs

A **directed graph** or **digraph** is a set A of **vertices** together with a subset $R \subseteq A \times A$ of **directed edges**. If $(x, y) \in R$ we say 'there is a directed edge from x to y'. When A is small, a digraph can be drawn with the vertices as points and directed edges as arrows.

Example: A a set of countries. $\quad a\,R\,b$ means a exports to b.



Each of the four countries exports to all the other three.

Example

Friends Ami, Bo, Chi and Di took photos of themselves visiting Parliament House.

- Ami took photos of Bo and Chi.
- Bo took photos of each of the others.
- Chi didn't take any photos.
- Di just took a selfie.

Draw a digraph for the relation P on $\{\text{Ami,Bo,Chi,Di}\}$ given by

$$x \, P \, y \iff x \text{ photographed } y.$$

$$\begin{tikzcd}
\text{Ami} \arrow[r, bend left] \arrow[dr] & \text{Bo} \arrow[l, bend left] \arrow[r] \arrow[d] & \text{Di} \arrow[loop right] \\
 & \text{Chi} &
\end{tikzcd}$$

(You can position the vertices how you like, of course.)

Functions

Functions

Let A, B be sets. A relation f from A to B is called a **function from A to B** when

$$\forall a \in A \; \exists! b \in B \; (a, b) \in f$$

The set A is called the **domain** of the function; the set B is called the **codomain** of the function. The **range** of f is the set

$$\{b \in B \mid \exists a \in A \; (a, b) \in f\}.$$

We write $f : A \to B$ to say that f is a function from A to B. Even though a function is a relation, we usually write $f(a) = b$ instead of $(a, b) \in f$ or afb.

Functions as machines

You may like to think of a function as an abstract machine (this is an example of a 'model'). It takes inputs. When given an input, the machine runs and produces an output according to some process. Using this model, we may interpret the function $f : A \to B$ as follows:

- the domain A is the set of allowed inputs to the function f
- the codomain B is the set from which outputs are selected by f
- the range is the subset of the codomain comprising the elements that are actually selected as outputs of the function.

About functions

We use the language of inputs and outputs to observe that:

- every input is assigned exactly one output—this is exactly the criteria under which a relation is a function;
- in general, the same output may be assigned to any number of inputs
- in general, some elements of the codomain may not be selected as the output for any input—that is, the range and the codomain may be different.

Example: Consider the function $s : \mathbb{Z} \to \mathbb{Z}$ that squares each input. Then $s(-3) = 9 = s(3)$, and -2 is in the codomain but not the range.

How to specify a function

To describe a function f you must:

- Describe the domain A
- Describe the codomain B
- Specify enough about "how the machine works" that for all $a \in A$, the predicate

  $$p_a(b) : f(a) = b$$

  is true for exactly one $b \in B$. It does not have to be easy to describe how to get $f(a)$ from a (in particular, the description may be very complicated), but the description must effectively select one output for each a.

Equal functions

Function f and g are **equal** if they have the same domain, the same codomain, and $f(x) = g(x)$ for all x in the domain.

Q: Are the following functions equal?
$f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(z) = z^2$
$g : \mathbb{Z} \to \mathbb{N}$ defined by $f(z) = z^2$

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A: No, because they have different codomains.

Another example

Q: Are the following functions equal?

$f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(z) = \dfrac{z^2 + 2z + 1}{(z + 1)}$

$g : \mathbb{Z} \to \mathbb{Z}$ defined by $f(z) = z + 1$

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A: A TRICK QUESTION! The first function is not well-defined because the rule does not make sense when $z = -1$. *I refuse to answer your ill-posed problem!*

Q: Are the following functions equal?

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A: Yes, because $\frac{z^2+2z+1}{(z+1)} = \frac{(z+1)^2}{(z+1)} = z + 1$.

Examples

Here are several ways to define the same function:

- Let f be the function from $\mathbb{Z}$ to $\mathbb{Z}$ that squares its input.
- Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by the rule $f(z) = z^2$.
- Let $f = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b = a^2\}$.
- Consider the function $\mathbb{Z} \to \mathbb{Z}$ given by $z \overset{f}{\mapsto} z^2$
- Let $f : \mathbb{Z} \to \mathbb{Z}$ be $z \mapsto z^2$

In the above, the arrow $\to$ is read 'to' and the arrow $\mapsto$ is read 'maps to'.

Another example

Here is a function that takes an input from a Cartesian product. This is a sneaky way to take two inputs, yet still fit the definition of a function.

Let $A = \{\text{cat}, \text{dog}, \text{chicken}\}$ and $B = \{+, -\}$ and $C = \{1, 2, 3\}$ and let $R : A \times B \to C$ be defined by the rule:

$$R((\text{pet}, \text{test})) = \begin{cases} 1, \text{if test is } + \\ 2, \text{if test is } - \text{ and pet } \neq \text{ cat} \\ 3, \text{if test is } - \text{ and pet } = \text{ cat.} \end{cases}$$

In a mild abuse of notation, for such functions we tend to write $R(\text{cat}, +)$ instead of $R((\text{cat}, +))$.

Example

A function can be well-defined, but slow to compute.

Let $f : \mathbb{N} \to \mathbb{N}$ be defined by the rule

$$f(z) = \begin{cases} 1, \text{if } z = 1 \\ \text{the smallest prime that divides } z, \text{if } z \neq 1 \end{cases}$$

We do not know 'fast' ways to compute $f(z)$ when z is large.

This example illustrates that the 'rule of the function' (a description of the process by which the output is determined by the input) does not have to be easily expressible in a single simple formula. As long as each element of the domain is related to exactly one element of the codomain, a function has been defined.

Example

A function can be well-defined, but difficult to compute.

Let $\cos : \mathbb{R} \to \mathbb{R}$ be the function such that θ maps to the x-coordinate of the point reached by starting at the point $(1, 0)$ in the Euclidean plane and travelling counter-clockwise θ units around the unit circle centred at $(0, 0)$.

Let $\sin : \mathbb{R} \to \mathbb{R}$ be the function such that θ maps to the y-coordinate of the point reached by starting at the point $(1, 0)$ in the Euclidean plane and travelling counter-clockwise θ units around the unit circle centred at $(0, 0)$.

A very difficult question (without a computer):
Evaluate $\cos(2.3)$ and $\sin(-1.45)$.