Discrete Mathematical Models

Lecture 4

Kane Townsend

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Section A2: Sets

References

The ideas about sets that we cover in the next few lectures are covered in the following sections in our optional text:

3ed: Chapter 5

4ed: Chapter 6

5ed: Sections 1.2-1.3 and Chapter 6

Sets and elements

A **set** is a collection of **elements**.

This is an intuitive statement, but cannot be considered a definition because we do not give a precise definition of "collection" or "element" (other than saying they are the things that are in sets). Even so, this is where we start.

The notation $a \in S$ is read "a is an element of S".

The notation $a \notin S$ is read "a is not an element of S".

Axiom of extensionality

A set is determined by what its elements are. No importance is placed on the order in which elements are considered, or how many times an element appears in the set. Membership is the only things that matters.

Methods for describing a set

To effectively communicate which set you are talking about, you may:

- Describe S with language that communicates the precise nature of the set
- Use set-roster notation
- Use set-builder notation

Some important sets described with language

We write $\emptyset$ for the **emptyset**. It is the set with no elements

Let $\mathbb{Z}_{\geq 0}$ denote the set of **non-negative integers**

Let $\mathbb{N}$ denote the set of **positive integers** (sometimes called the natural numbers). Note that $0 \notin \mathbb{N}$ but $0 \in \mathbb{Z}_{\geq 0}$.

Let $\mathbb{Z}$ denote the set of **integers**.

Let $\mathbb{Q}$ denote the set of **rational numbers**.

Let $\mathbb{R}$ denote the set of **real numbers**.

Predicates with domain $\emptyset$

Let $p(x)$ be a predicate with x taking values from the domain $\emptyset$.

Are the following statements true or false?

- $\forall x\ p(x)$
- $\exists x\ p(x)$

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- $\exists x \, p(x)$

Note that for any predicate $p(x)$ with x taking values from domain D that is not the $\emptyset$ we have

$$\forall x \, p(x) \rightarrow \exists x \, p(x).$$

Meanwhile, for any predicate $p(x)$ with x taking values from domain D (could be empty) we have

$$\exists ! x \, p(x) \rightarrow \exists x \, p(x).$$

Specifying membership with precision

Which, if any, of the following sentences define sets:

A. Let E denote the set of Australian species that are endangered.

B. Let E denote the set of Australian species that are *officially* endangered.

C. Let E denote the set of species that are currently listed under Section 178 of the Environment Protection and Biodiversity Conservation Act 1999 (EPBC Act).

Specifying membership with precision

Which, if any, of the following sentences define sets:

A. Let P denote the set of all (computer) programs.

B. Let P denote the set of all programs written in the language C.

C. Let P denote the set of all *correct* programs written in the language C.

E. Let P denote the set of programs written in the language C that terminate in finite time.

F. Let P denote the set of programs written in the language C that accept no input from the user and will run to completion without a run-time error in finite time.

Set-roster notation

A set S may be specified using **set-roster notation** by:

- writing all of its elements, with elements separated by commas, and the entire collection enclosed by braces;
- writing some elements and ellipses (ellipses are ... that read "and so on"), enclosing the entire description between braces. The elements written must establish enough of a pattern, in a way that is obvious to the reader, that it becomes clear how the pattern continues and whether or not the pattern ends at some point.

Examples

Describe each of the following sets in words.

$S = \{\text{Helium, Argon, Neon, Krypton, Xenon, Radon}\}$

$O = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$

$T = \{2, 3, 5, 7, \ldots\}$

$U = \{3, 5, 7, \ldots, 19\}$

Examples

Describe each of the following sets in words.

$S = \{$Helium, Argon, Neon, Krypton, Xenon, Radon$\}$
The noble gases.

$O = \{\dots, -3, -1, 1, 3, 5, \dots\}$.
The odd integers.

$T = \{2, 3, 5, 7, \dots\}$.
The prime numbers.

$U = \{3, 5, 7, \dots, 19\}$.
This could be "The odd primes less than 20" or "The odd integers between 2 and 20." As soon as we identify two reasonable interpretations, we know this description needs improvement to be effective.

Axiom of extensionality again

Recall: A set is determined by what its elements are. No importance is placed on the order in which elements are considered, or how many times an element appears in the set.

So $\{a, b, c\} = \{c, b, a\} = \{a, a, a, a, b, b, c\}$

Set-builder notation

Recall that a predicate is a sentence that involves at least one variable, and the domain of the variable must be specified.

To describe a set using **set-builder notation** we use a predicate $p(x)$, a predicate domain D, and we write

$$\{x \in D \mid p(x)\}$$

for "the set of all x in D for which $p(x)$ is true.".

Note: The use of $|$ to separate the specification of the domain from the predicate is not universal. Other commonly used symbols include ':' and ';'. In all cases, the symbol may be read as "such that" or "for which".

Universe of discourse

In any mathematical context, there is usually a type of object you are thinking about. We will usually consider sets that are subsets of a set U, called the **universe of discourse**, or **universal set**. This often forms the domain of the predicates we use to describe sets in a given situation.

For example, in one context the universe of discourse may be the set of positive integers, while in another it may by the set of connected graphs (we will find out about these later).

The importance of having a universe of discourse will be explored later.

An Example

Example:

$$S = \{\underbrace{x \in \mathbb{Z}}_{\text{domain}} \mid \underbrace{x^2 + 2x - 15 = 0}_{\text{predicate}}\}.$$

The expression may be read aloud as:

- "S is equal to the set of all x from the set of integers such that $x^2 + 2x - 15 = 0$."
- "S is equal to the set of all integers x such that $x^2 + 2x - 15 = 0$."
- "S is equal to the set of all integers x for which $x^2 + 2x - 15 = 0$."

Easily defined sets are not always easy

Example: Let $T = \{p \in \mathbb{Z}^+ \mid p \text{ and } p+2 \text{ are both prime}\}$.

I can, for example, tell you that $3, 11, 17 \in T$ and $6, 7 \notin T$.

If you give me any positive integer n, and leave me alone long enough, I can tell you whether or not $n \in T$.

The statement:

$$\text{The set } T \text{ has infinitely many elements}$$

is known as the Twin Prime Conjecture. It has been studied since 1849. At the time of writing, no one knows whether the statement is true or false.

Subsets

Let A and B be sets. We say that A is a **subset** of B, or A is contained in B, and we write $A \subseteq B$, when every element of A is also an element of B. Symbolically,

$$A \subseteq B \Leftrightarrow \forall x \, (x \in A \to x \in B).$$

You may say that $\subseteq$ is the set theoretic analogue of IMPLIES in logic.

Understanding $A \not\subseteq B$

We often indicate that the negation of a statement is true by placing a diagonal slash through the key symbol in the statement. Now

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$$\begin{aligned} A \not\subseteq B \Leftrightarrow &\quad \neg(A \subseteq B) \\ \Leftrightarrow &\quad \neg\forall x\ (x \in A \to x \in B) \\ \Leftrightarrow &\quad \exists x\ \neg(x \in A \to x \in B) \\ \Leftrightarrow &\quad \exists x\ x \in A \wedge (\neg(x \in B)) \\ \Leftrightarrow &\quad \exists x\ x \in A \wedge x \notin B. \end{aligned}$$

Understanding $A \not\subseteq B$

We often indicate that the negation of a statement is true by placing a diagonal slash through the key symbol in the statement. Now

$$\begin{aligned} A \not\subseteq B \Leftrightarrow &\quad \neg(A \subseteq B) \\ \Leftrightarrow &\quad \neg\forall x\, (x \in A \rightarrow x \in B) \\ \Leftrightarrow &\quad \exists x\, \neg(x \in A \rightarrow x \in B) \\ \Leftrightarrow &\quad \exists x\; x \in A \wedge (\neg(x \in B)) \\ \Leftrightarrow &\quad \exists x\; x \in A \wedge x \notin B. \end{aligned}$$

We can interpret this intuitively: $A \not\subseteq B$ means that there is at least one element in A that is not in B.

Proper containment

We say that A is a **proper subset** of B, or A is properly contained in B, and write $A \subsetneq B$, when every element of A is in B but there is at least one element of B that is not in A. Symbolically:

$$A \subsetneq B \Leftrightarrow (A \subseteq B) \wedge (B \not\subseteq A)$$

You are well acquainted with the following proper containments:

$$\mathbb{N} \subsetneq \mathbb{Z}_{\geq 0} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}.$$

BEWARE: Some mathematicians use $\subset$ for $\subsetneq$. I don't do this because others mathematicians use $\subset$ for $\subseteq$.

Set equality

Let A and B be sets. We say that A **equals** B, written $A = B$, when $A \subseteq B$ and $B \subseteq A$. Symbolically,

$$\begin{aligned} A = B &\Leftrightarrow (A \subseteq B) \wedge (B \subseteq A) \\ &\Leftrightarrow \forall x \, (x \in A \leftrightarrow x \in B) \\ &\Leftrightarrow \forall x \, (x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A) \end{aligned}$$

You may say that $=$ is the set theoretic analogue of IFF in logic.

An Example

A set may be described in more than one way. Some ways say more about the set, perhaps by saying why it is interesting, while others make it more obvious which things are elements of the set. For example:

$$\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} = \{-5, 3\}$$

The first description tells me why the set is interesting in a given context, while the second description gives me a particularly clear understanding of the which integers are in the set and which integers are not in the set.

Set equality as a type of problem

Many problems in mathematics (and computer science) can be framed as: we know one way to describe a particular set because we know what makes the set interesting, but we want to know another way to describe the same set that makes membership easier to recognise or understand. Thus we wish to show a set equality.

For example: The problem of solving the equation $x^2 + 2x - 15 = 0$ over the domain of integers is essentially the following:
Show that

$$\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} = \{-5, 3\}?$$

Proof

To show that $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} = \{-5, 3\}$ we will show:

- $\{-5, 3\} \subseteq \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$; and
- $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \subseteq \{-5, 3\}$.

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First we show that $\{-5, 3\} \subseteq \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Let $x \in \{-5, 3\}$. We consider cases.

Case $x = -5$: Since $(-5)^2 + 2 \times (-5) - 15 = 25 - 10 - 15 = 0$,
$-5 \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Case $x = 3$: Since $3^2 + 2 \times 3 - 15 = 9 + 6 - 15 = 0$,
$3 \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

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Let $x \in \{-5, 3\}$. We consider cases.

Case $x = -5$: Since $(-5)^2 + 2 \times (-5) - 15 = 25 - 10 - 15 = 0$, $-5 \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Case $x = 3$: Since $3^2 + 2 \times 3 - 15 = 9 + 6 - 15 = 0$, $3 \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

In all cases, $x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$. Hence $\{-5, 3\} \subseteq \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Proof (cont.)

Now we show that $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \subseteq \{-5, 3\}$.

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Let $x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Proof (cont.)

Now we show that $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \subseteq \{-5, 3\}$.

Let $x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Since $x^2 + 2x - 15 = 0$, we have that $(x + 5)(x - 3) = 0$. If a product of two real numbers is zero, at least one of the numbers is zero. We consider cases.

Proof (cont.)

Now we show that $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \subseteq \{-5, 3\}$.

Let $x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Since $x^2 + 2x - 15 = 0$, we have that $(x + 5)(x - 3) = 0$. If a product of two real numbers is zero, at least one of the numbers is zero. We consider cases.

Case $x + 5 = 0$: Then $x = -5$ and $x \in \{-5, 3\}$.

Case $x + 5 \neq 0$: Then $x - 3 = 0$. Hence $x = 3$ and $x \in \{-5, 3\}$.

In all cases, $x \in \{-5, 3\}$. Hence $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \subseteq \{-5, 3\}$.$\square$

Alternate proof

Let x be an integer. Then

$$
\begin{aligned}
& x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \\
\Leftrightarrow & x^2 + 2x - 15 = 0 \\
\Leftrightarrow & (x+5)(x-3) = 0 \\
\Leftrightarrow & (x+5=0) \vee (x-3=0) \\
& \text{(because a product of real numbers is zero iff} \\
& \text{at least one of the numbers is zero)} \\
\Leftrightarrow & (x=-5) \vee (x=3) \\
\Leftrightarrow & x \in \{-5, 3\}. \qquad \square
\end{aligned}
$$

Example 6.1.2 on p.378 of our optional text.

Let

$$A = \{m \in \mathbb{Z} \mid \exists r \in \mathbb{Z}\ m = 6r + 12\}\}$$
$$B = \{n \in \mathbb{Z} \mid \exists s \in \mathbb{Z}\ n = 3s\}$$

Prove that $A \subsetneq B$.