Discrete Mathematical Models

Lecture 2

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Section A1: Logic (continued)

Logical equivalence

When two statement forms f, g have identical truth tables we say they are **logically equivalent** and write $f \equiv g$.

(Alt. Def.: When f, g are statement forms and $f \leftrightarrow g$ is a tautology, we say that f and g are **logically equivalent** and write $f \equiv g$.)

Knowing some logical equivalences allows us to replace complicated statement forms by simpler, logically equivalent, statement forms.

Some well-known logical equivalences are shown on the next frame.

From page 49 of your optional text

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1.	*Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2.	*Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3.	*Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4.	*Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5.	*Negation laws:*	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6.	*Double negative law:*	${\sim}({\sim}p) \equiv p$	
7.	*Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8.	*Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9.	*De Morgan's laws:*	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10.	*Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11.	*Negations of* $\mathbf{t}$ *and* $\mathbf{c}$:	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

Note: The text uses $\sim$ for negation instead of $\neg$.

Example: An equivalent form of XOR

Q: Justify the earlier claim that $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$.

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We compute a truth table:

p	q	$p \oplus q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
T	T	F	T	T	F	F
T	F	T	T	F	T	T
F	T	T	T	F	T	T
F	F	F	F	F	T	F

Because the entries in column 3 and column 7 agree in every row, the truth table above establishes that $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$.

Example: Another equivalent form for XOR

Q: Show that $p \oplus q \equiv (p \land \neg q) \lor (\neg p \land q)$

Example: Another equivalent form for XOR

Q: Show that $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

We compute a truth table:

p	q	$p \oplus q$	$\neg p$	$\neg q$	$p \wedge \neg q$	$\neg p \wedge q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
T	T	F	F	F	F	F	F
T	F	T	F	T	T	F	T
F	T	T	T	F	F	T	T
F	F	F	T	T	F	F	F

Because the entries in column 3 and column 8 agree in every row, the truth table above establishes that $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$.

A logical connective: IMPLIES

Recall that the logical connective "implies" is defined by the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

This definition is not as intuitive as the definitions of the other logical connectives.

A scenario

A scenario (from p. 54 of our optional text) to explain why our definition of implies is reasonable

Suppose you go to interview for a job at a store and the owner of the store makes you the following promise:

> If you show up for work Monday morning, then you will get the job.

In which of the following situations has the store owner been proven a liar?

A. You show up for work Monday morning and you get the job.

B. You show up for work Monday morning and you don't get the job.

C. You do not show up for work Monday morning and you get the job.

D. You do not show up for work Monday morning and you don't get the job.

A scenario (cont.)

The store owner did not say anything about what will happen if you don't show up for work Monday morning. Therefore, the only situation in which the owner has been proven a liar is (B).

With
p: You show up for work Monday morning
q: You get the job,
the store owner's promise is represented by $p \to q$.

The store owner is proven a liar only if the statement they made has been demonstrated to be false. It therefore seems reasonable to say that $p \to q$ is false only in the situation (B).

A mathematical scenario

Given the quadratic equation $ax^2 + bx + c = 0$, recall the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We have the following true statement:

If $b^2 - 4ac > 0$, then the equation $ax^2 + bx + c = 0$ has a real solution.

This can be converted into the statement form $p \to q$ with the replacements
p: $b^2 - 4ac > 0$
q: $ax^2 + bx + c = 0$ has a real solution.

We want $p \to q$ to be true when p is false since our statement is true.

Another true statement: $ax^2 + bx + c = 0$ has two distinct real solutions if and only if $b^2 - 4ac > 0$.

More vocabulary associated to $p \to q$

The statement $p \to q$ is read aloud in each of the following ways:

- p implies q
- if p then q
- p only if q
- q if p

Please note the potential for confusion.

More vocabulary associated to $p \to q$

It is helpful to have words to describe the relationships between certain implications. We have

- The **converse** of $p \to q$ is $q \to p$
- The **inverse** of $p \to q$ is $\neg p \to \neg q$
- The **contrapositive** of $p \to q$ is $\neg q \to \neg p$

Example: Consider the statement:

If you work hard, then you will pass the exam.

With

$$p : \text{You work hard}$$
$$q : \text{You pass the exam,}$$

the statement is $p \to q$.
The contrapositive is:

If you fail the exam, then you have not worked hard.

The converse is:

If you pass the exam, then you have worked hard.

The inverse is:

If you don't work hard, then you will fail the exam.

More vocabulary associated to $p \to q$

Which, if any, of the following statements concerning $p \to q$ are true:

A. The converse of the inverse is the contrapositive.
B. The inverse of the converse is the contrapositive.
C. The inverse of the contrapositive is the converse.
D. The converse of the contrapositive is the inverse.
E. The statement form is logically equivalent to its contrapositive.
F. The inverse is logically equivalent to the converse.
G. The statement is not logically equivalent to its converse.

More vocabulary associated to $p \to q$

Which, if any, of the following statements concerning $p \to q$ are true:

A. The converse of the inverse is the contrapositive.
B. The inverse of the converse is the contrapositive.
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D. The converse of the contrapositive is the inverse.
E. The statement form is logically equivalent to its contrapositive.
F. The inverse is logically equivalent to the converse.
G. The statement is not logically equivalent to its converse.

Answer: All of the above are true. Statements E, F, and G are particularly important.

Necessary and sufficient conditions

p is a **sufficient condition** for q means $p \to q$.

p is a **necessary condition** for q means $\neg p \to \neg q$.

Since $(\neg p \to \neg q) \equiv (q \to p)$, we can also say:

p is a **necessary condition** for q means $q \to p$;

p is a **necessary and sufficient condition** for q means $p \leftrightarrow q$.

In view of the above terminology, statements of the form $p \to q$ and $p \leftrightarrow q$ are often referred to as **conditional statements**.

Examples

- Being older than 16 is a necessary condition for being allowed to drive. It is not sufficient.
- Having a drivers license is a necessary and sufficient condition for being allowed to drive.
- Being a taxi driver is a sufficient condition for being allowed to drive. It is not necessary.

Negating compound statements

For statement variables p and q, the following logical equivalences hold:

$$\neg(\neg p) \equiv p \tag{1}$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q \tag{2}$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q \tag{3}$$
$$\neg(p \oplus q) \equiv (p \wedge q) \vee (\neg p \wedge \neg q) \tag{4}$$
$$\neg(p \to q) \equiv \quad p \wedge \neg q \tag{5}$$
$$\neg(p \leftrightarrow q) \equiv \quad (p \wedge \neg q) \vee (\neg p \wedge q) \tag{6}$$

RECALL: Each of these logical equivalences may be confirmed by demonstrating that the truth tables for the left-hand side and right-hand side match in every row.

Simplification using logical equivalences

The right-hand side of each logical equivalence on the previous slide has the pleasant property that $\neg$ only appears in front of statement variables. Using these rules, we can arrange to replace any compound statement by a logically equivalent statement in which $\neg$ only appears in front of statement variables.

Example: Consider the compound statement

$$\begin{aligned} &\neg(\neg p \wedge q) \wedge (p \vee q) \\ \equiv&(\neg(\neg p) \vee \neg q) \wedge (p \vee q) && \text{Using (2)} \\ \equiv&(p \vee \neg q) \wedge (p \vee q) && \text{Using (1)} \end{aligned}$$

Using the logical equivalences on slide #38 and #53 together (there is overlap), we can simplify further.

Example:

$$
\begin{aligned}
& \neg(\neg p \wedge q) \wedge (p \vee q) & \\
\equiv & (\neg(\neg p) \vee \neg q) \wedge (p \vee q) & \text{Using (2)} \\
\equiv & (p \vee \neg q) \wedge (p \vee q) & \text{Using (1)} \\
\equiv & p \vee (\neg q \wedge q) & \text{Using a Distributive law} \\
\equiv & p \vee (q \wedge \neg q) & \text{Using a Commutative law} \\
\equiv & p \vee c & \text{Using a Negation law} \\
\equiv & p & \text{Using an Identity law}
\end{aligned}
$$

In the above, c represent a contradiction.

Functional completeness

A question

So far, we have defined a number of logical connectives. Do we need to keep defining new connectives, or do we have all of the logical connectives we need?

That is, suppose I present you with a truth table but I do not label the right-hand column with a compound statement. Can you construct a compound statement for which the given truth table is correct?

Said another way: Is every compound statement form logically equivalent to a compound statement form made using only statement variables, parentheses, and the logical connectives we have already defined?

An example

Q: Find a compound statement to replace ?

p	q	r	?
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

In the truth table table, ? can be replaced by:

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T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

In the truth table table, ? can be replaced by:

$$(p \land q \land \neg r) \lor (\neg p \land q \land r) \lor (\neg p \land \neg q \land r)$$

DNF

Suppose that $p_1, p_2, \ldots, p_n$ are statements. A compound statement of the form $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ is said to be a **conjuction** of the statements $p_1, p_2, \ldots, p_n$. A compound statement of the form $p_1 \vee p_2 \vee \cdots \vee p_n$ is said to be a **disjunction** of the statements $p_1, p_2, \ldots, p_n$.

A compound statement is in **disjunctive normal form** it is a disjunction of conjunctions, and in each of the conjunctions each statement variable or its negation appears but not both.

Example:

$$(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$$

is in disjunctive normal form.

A theorem

Theorem: Every statement form that is not a contradiction is logically equivalent to a disjunctive normal form.

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Using a conjunction in which each statement variable or its negation appears but not both, we can make a statement form with a truth table in which there is exactly one T.

A theorem

Theorem: Every statement form that is not a contradiction is logically equivalent to a disjunctive normal form.

Proof: Consider the truth table of some arbitrary statement form that is not a contradiction.

Using a conjunction in which each statement variable or its negation appears but not both, we can make a statement form with a truth table in which there is exactly one T.

By intentionally choosing between the statement variables and their negations, we can make that T be in any row of the truth table. To make a statement form to match any particular truth table we note the rows in which T's appear, we use a conjunction for each such row and we combine these conjunctions into a disjunction. $\square$

Functional completeness

A set of logical connectives A is **functionally complete** if any statement form is logically equivalent to a statement form that is made using only statement variables, parentheses, and connectives from A.

Corollary: The set $\{\wedge, \vee, \neg\}$ is a functionally complete set of logical connectives.

Proof.
Consider the truth table of an arbitrary statement form X. If X is not a contradiction, then by the previous frame we can present the statement form in disjunctive normal form. If X is a contradiction, then then the contradiction $p \wedge \neg p$, where p is a statement variable of X is logically equivalent to the statement form X. $\square$

Another example

Claim: The set $\{\wedge, \neg\}$ is functionally complete.

Q: How can we show this?

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Claim: The set $\{\wedge, \neg\}$ is functionally complete.

Q: How can we show this?

Plan: Show that $p \vee q \equiv \neg(\neg p \wedge \neg q)$ and then appeal to the fact that $\{\wedge, \vee, \neg\}$ is functionally complete.

Suggested activities

- Ensure you know how to watch videos and download frames from the ECHO360 system. Do this by watching Lecture 0 and downloading the frames for this week.
- Master the vocabulary introduced in this lecture and practice producing a disjunctive normal forms for random truth tables with different numbers of variables that do not representing contradictions.
- Find the ebook of our optional text in the ANU library catalogue and see what you think
- Look at the materials, including the practice problems, in the Week 1 section on Wattle.