Discrete Mathematical Models

Lecture 3.5

Kane Townsend

Semester 2, 2024

Section A1: Logic (continued)

How to prove things

Putting our logic to work.

How to start

Before trying to prove a statement, you should clearly identify the logical structure of the statement. Doing so allows you to understand the choices you have in choosing a logical structure for your proof.

Let's understand the logical structures that can be used to prove statements with various logical structures.

Proving $\forall$

To prove a statement of the form $\forall x \; p(x)$, one may follow this plan:

Let x be a (fixed but arbitrary) element of the predicate domain. Argue that $p(x)$ is true.

Proving $\forall$

To prove a statement of the form $\forall x\ p(x)$, one may follow this plan:

Let x be a (fixed but arbitrary) element of the predicate domain. Argue that $p(x)$ is true.

Example
Prove the following statement: Whenever x is an integer, $6x^2 + 4$ is even.

Working the example

We need some definitions:

Defn: An integer x is **even** if there exists an integer k such that $x = 2k$.

Defn: An integer x is **odd** if there exists an integer k such that $x = 2k + 1$.

Theorem: (The even/odd theorem)
Every integer is either even or odd; no integer is both even and odd.

We note that the theorem could be stated succinctly using our logic notation:

$$\forall x \ \ (x \text{ is even}) \oplus (x \text{ is odd})$$

where the domain of quantification is understood to be the integers.

Back to our example

Prove the following statement: Whenever x is an integer, $6x^2 + 4$ is even.

Proof: Let x be an integer.

$\vdots$

Hence $6x^2 + 4$ is even. $\square$

What we have written above is the **logical structure of a proof**.

Back to our example

Prove the following statement: Whenever x is an integer, $6x^2 + 4$ is even.

Proof: Let x be an integer. We note that the integers are closed under multiplication and addition.

Since x is an integer and the integers are closed under multiplication, x^2 is an integer.
Since x^2 is an integer and the integers are closed under multiplication, $3x^2$ is an integer.
Since $3x^2$ is an integer and the integers are closed under addition, $3x^2 + 2$ is an integer.
Since $3x^2 + 2$ is an integer, $2(3x^2 + 2)$ is even.
But $2(3x^2 + 2) = 6x^2 + 4$.

Hence $6x^2 + 4$ is even. $\square$

Proving $\exists$

To prove a statement of the form $\exists x \; p(x)$, one may identify a particular element of the predicate domain and establish that $p(x)$ is true. Please note, it is not enough to simple state which element x of the domain has the required property, you should explain how you know that the particular element you identified has the required property (how you know that $p(x)$ is true). This is called **exhibiting an example**.

An example

Prove the following statement: The equation $x^2 - 6x + 8 = 0$ has an integer solution.

An example

Prove the following statement: The equation $x^2 - 6x + 8 = 0$ has an integer solution.

Proof: When $x = 2$, the left-hand side of the equation evaluates to $2^2 - 6(2) + 8 = 4 - 12 + 8 = 0$. Hence 2 is an integer solution to the equation. $\square$

Disproving $\forall$

To disprove a statement of the form $\forall x \; p(x)$, one should prove the statement $\exists x \; \neg p(x)$. (This is called providing a **counterexample**)

An example

Prove or disprove the following statement: For every integer x, $(x-1)^2 + (x-1)$ is positive.

An example

Prove or disprove the following statement: For every integer x, $(x-1)^2 + (x-1)$ is positive.

The statement is false because $x = 0$ is a counterexample. When $x = 0$,

$$(x-1)^2 + (x-1) = (0-1)^2 + (0-1) = (-1)^2 + (-1) = 1 - 1 = 0$$

and 0 is not positive.

Disproving $\exists$

To disprove a statement of the form $\exists x \; p(x)$, one should prove the statement $\forall x \; \neg p(x)$.

An example

Prove or disprove the following statement:

$$\exists y \ \ \forall x \ \ (y \leq x),$$

where the quantification is over the set of integers.

An example

Prove or disprove the following statement:

$$\exists y \;\; \forall x \;\; (y \leq x),$$

where the quantification is over the set of integers.

The statement is false.

(To show that the statement is false, we must show the following

$$\neg \exists y \;\; \forall x \;\; (y \leq x) \equiv \forall y \;\; \exists x \;\; \neg(y \leq x) \equiv \forall y \;\; \exists x \;\; (y > x))$$

Let y be an integer. Let $x = y - 1$. It is clear that $y > x$.

Proving $\rightarrow$

To prove $p \rightarrow q$ you may:

- Suppose that p is true.
- Deduce by valid and explicit reasoning that q must be true (using the truth of p along the way).

This is called **arguing directly**.

You may also:

- Suppose that $\neg q$ is true.
- Deduce by valid and explicit reasoning that $\neg p$ must be true (using the truth of $\neg q$ along the way).

This is called **arguing via the contrapositive**.

The logical structure I

Let x be an integer. Prove the following statement: For all integers x, if x is even then $x^2 + 2$ is even.

Proof: Let x be an integer. We shall argue directly. Suppose that x is even.

$\vdots$

Hence $x^2 + 2$ is even. $\square$

The logical structure II

Prove the following statement: For all integers x, if $x^2 + 2$ is even, then x is even.

Proof: Let x be an integer. We shall argue via the contrapositive. Suppose that x is not even. By the even/odd theorem, x is odd.

$\vdots$

Hence $x^2 + 2$ is odd. By the even/odd theorem, $x^2 + 2$ is not even. $\square$

Proving $\leftrightarrow$

To prove $p \leftrightarrow q$, you may first prove $p \rightarrow q$ and then prove $q \rightarrow p$

It is possible to accomplish "both directions" of proof simultaneously by arguing with biconditionals throughout your proof, but you must be careful when doing so.

An example

Prove the following statement: For all integers x, x is even if and only if $x^2 + 2$ is even.

The logical structure

Prove the following statement: For all integers x, x is even if and only if $x^2 + 2$ is even.

Proof: Let x be an integer.

For the $\rightarrow$ direction, we argue directly. Suppose first that x is even.

$\vdots$

Hence $x^2 + 2$ is even.

For the $\leftarrow$ direction, we argue via the contrapositive. Now suppose $x^2 + 2$ is not even

$\vdots$

Hence $x^2 + 2$ is not even.

The proof

Prove the following statement: For all integers x, x is even if and only if $x^2 + 2$ is even.

Proof: Let x be an integer.

For the $\rightarrow$ direction, we argue directly. Suppose that x is even. Since x is even, there exists an integer k such that $x = 2k$. Then $x^2 + 2 = (2k)^2 + 2 = 4k^2 + 2 = 2(2k^2 + 1)$. Since k is an integer, so is $2k^2 + 1$. Hence $x^2 + 2$ is even.

For the $\leftarrow$ direction, we argue via the contrapositive. Suppose that x is not even. By the even/odd theorem, x is odd. So there exists an integer k such that $x = 2k + 1$. Now

$$\begin{aligned} x^2 + 2 = (2k+1)^2 + 2 = (4k^2 + 4k + 1) + 2 = 4k^2 + 4k + 2 + 1 \\ = 2(2k^2 + 2k + 1) + 1. \end{aligned}$$

Since k is an integer, so is $2k^2 + 2k + 1$. Hence $x^2 + 2$ is odd. By the even/odd theorem, $x^2 + 2$ is not even. $\square$

Arguing by cases

If the domain of a predicate is partitioned into subsets, you may prove a $\forall$ statement by proving it for each subset.

Example

Prove the following statement: For all integers x, $x^2 + x + 6$ is even.

The logical structure

Prove the following statement: For all integers x, $x^2 + x + 6$ is even.

Proof: Let x be an integer. By the even/odd theorem, every integer is either even or it is odd.

Consider first the case that x is even.
$\vdots$
Hence $x^2 + x + 6$ is even.

Now consider the case that x is odd.
$\vdots$
Hence $x^2 + x + 6$ is even.

In all cases, $x^2 + x + 6$ is even. $\square$

The proof

Prove the following statement: For all integers x, $x^2 + x + 6$ is even.

Proof: Let x be an integer. By the even/odd theorem, every integer is either even or it is odd.

Consider first the case that x is even. Since x is even, there exists an integer k such that $x = 2k$. Then

$$x^2 + x + 6 = (2k)^2 + (2k) + 6 = 4k^2 + 2k + 6 = 2(2k^2 + k + 3)$$

Since k is an integer, so is $2k^2 + k + 3$. Hence $x^2 + x + 6$ is even.

Now consider the case that x is odd. Since x is odd, there exists an integer k such that $x = 2k + 1$. Then

$$\begin{aligned} x^2 + x + 6 = (2k+1)^2 + (2k+1) + 6 &= 4k^2 + 4k + 1 + 2k + 1 + 6 \\ &= 4k^2 + 6k + 8 = 2(2k^2 + 3k + 4). \end{aligned}$$

Since k is an integer, so is $2k^2 + 3k + 4$. Hence $x^2 + x + 6$ is even.

In all cases, $x^2 + x + 6$ is even. $\square$

Proof by contradiction

To prove a statement p, you may disprove $\neg p$. One way to do this is to suppose $\neg p$, and use this fact to deduce a statement we know to be false. Since a true statement cannot imply a false statement, we must have that $\neg p$ is false. This is called a **proof by contradiction**.

An example

Prove the following statement: No integers x and y exist for which $5x + 20y = 4$.

The logical structure

Prove the following statement: No integers x and y exist for which $5x + 20y = 4$.

Proof: We shall use a proof by contradiction. Suppose there exist integers x and y such that $5x + 20y = 4$.

$\vdots$

Since have deduced something we know to be false, our original supposition is impossible. $\square$

The proof

Prove the following statement: No integers x and y exist for which $5x + 20y = 4$.

Proof: We shall use a proof by contradiction. Suppose there exist integers x and y such that $5x + 20y = 4$.

Dividing both sides of the equation by 5 yields $x + 4y = \frac{4}{5}$. Since x and y are integers, $x + 4y$ is an integer. Since $x + 4y$ is an integer and $x + 4y = \frac{4}{5}$, we deduce that $\frac{4}{5}$ is an integer; this is, of course, false.

Since have deduced something we know to be false, our original supposition is impossible. $\square$

Some advice

1. Before starting a proof, clearly identify the logical structure of the statement to be proved.
2. Consider your options for a logical structure that will prove the statement.
3. Write down the logical structure of your argument so that the reader knows what is going on.
4. When deciding between a direct argument and an argument via the contrapositive, try whichever direction appears to allow you to make the strongest supposition first. The same advice applies when considering a proof by contradiction or one of the other methods.

How to prove things I

- To prove a statement of the form $\forall x\ p(x)$, one may follow this plan: *Let x be a (fixed but arbitrary) element of the predicate domain. Argue that $p(x)$ is true.*

- To disprove a statement of the form $\forall x\ p(x)$, one should prove the statement $\exists x\ \neg p(x)$. (This is called providing a **counterexample**)

- To prove a statement of the form $\exists x\ p(x)$, one may identify a particular element of the predicate domain and establish that $p(x)$ is true.

- To disprove a statement of the form $\exists x\ p(x)$, one should prove the statement $\forall x\ \neg p(x)$.

How to prove things II

To prove $\forall x\ p(x) \rightarrow q(x)$ you may:

- Let x be an arbitrary element of the domain.
- Suppose that $p(x)$ is true.
- Deduce by valid reasoning that $q(x)$ must be true (using the truth of $p(x)$ along the way).

This is called **arguing directly**.

You may also:

- Let x be an arbitrary element of the domain.
- Suppose that $\neg q(x)$ is true.
- Deduce by valid reasoning that $\neg p(x)$ must be true (using the truth of $\neg q(x)$ along the way).

This is called **arguing via the contrapositive**.

How to prove things III

Some advice:

1. Before starting a proof, clearly identify the logical structure of the statement to be proved.
2. Write down the logical structure of your argument so that the reader knows what is going on.
3. When deciding between a direct argument and an argument via the contrapositive, try whichever direction appears to allow you to make the strongest supposition first.

Suggested activities

- Ensure you know how to watch videos and download frames from the ECHO360 system. Do this by watching Lecture 0.
- Look at the materials, including the practice problems, in the Week 1 section on Wattle.
- The workshop quiz in Week 2 will be three questions based on the material for Week 1 lectures.