Solution to question number 2:

(1) Given pde

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}$$

initial conditions:

$\frac{\partial u}{\partial t} = 0 \qquad u(x,0) = x(L-x)$

Boundary Conditions

$u(0,t) = u(L,t) = 0$

$u_{xx}(0,t) = u_{xx}(L,t) = 0$

Consider $u(x,t) = F(x)\, G(t)$.

so the pde can be written as

$FG'' = -c^2 F^{IV} G$

$\Rightarrow \frac{G''}{-c^2 G} = \frac{F^{IV}}{F} = k^4$; k is a constant

now if $k=0$ then:

$\frac{F^{IV}}{F} = 0$

$\Rightarrow F^{IV} = ax^3 + bx^2 + cx + d$

From the boundary conditions

$F(0)\, G(t) = F(L)\, G(t) = 0$

So, $F(0) = F(L) = 0$ because if

$G(t) = 0$ then $u(x,t) = 0$, which is not our interest now, $F''(0)\,G(t) = F''(L)\,G(t) = 0$

as $U_{xx}(0,t) = U_{xx}(L,t) = 0$

~~Therefore we have~~

So, $F''(0) = F''(L) = 0$ because $G(t) \neq 0$

now,

$$F'(x) = 3ax^2 + 2bx + C$$

$$\Rightarrow F''(x) = 6ax + 2b$$

$$\Rightarrow F''(0) = 0 = 2b$$

$$\Rightarrow b = 0$$

again $F(L) = aL^3 + cL = 0$

$$F''(L) = 6aL = 0$$

so $a = 0$

This will also make $c = 0$

so finally $F(x) = 0$ [In which we are not interested]

now if $k \neq 0$ then

$$\frac{F^{IV}}{F} = k^4$$

$$\Rightarrow F(x) = A\cos(kx) + B\sin(kx) + Ce^{kx} + De^{-kx}$$

$$= A\cos(kx) + B\sin(kx) + \frac{E+H}{2}e^{kx} + \frac{E-H}{2}e^{-kx}$$

$$\cancel{= A\cos(kx) + B\sin(kx) + C\cosh(kx) + D\sinh}$$

$$= A\cos(kx) + B\sin(kx) + E\cosh(kx) + H\sinh(kx)$$

~~now $F'(x) = -AK\sin(kx) + BK\cos(kx) + EK\sinh(kx) + DK\cosh(kx)$~~

$$F'(x) = -Ak\sin(kx) + Bk\cos(kx) + Ek\sinh(kx) + Hk\cosh(kx)$$

$$F''(x) = -Ak^2\cos(kx) - Bk^2\sin(kx) + Ek^2\cosh(kx) + Hk^2 \cancel{\cos}\sinh(kx)$$

Then $F(0) = A + E = 0$

$$F''(0) = -Ak^2 + Ek^2 = 0$$

So $A = E = 0$

again $F(L) = B\sin(kL) + H\sinh(kL) = 0$

$$F''(L) = -Bk^2\sin(kL) + Hk^2\sinh(kL) = 0$$

Also $H = 0$

now, $F(L) = B\sin(kL) = 0$

$$= \sin(kL) = 0 \quad [B \neq 0]$$

So $k = \frac{n\pi}{L}$; $n = 1, 2, 3, \ldots$

$$F_n(x) = B\sin\left(\frac{n\pi x}{L}\right) \; ; \; n = 1, 2, 3, \ldots$$

now $G'' + c^2k^4 G = 0$

$$\Rightarrow G_n(t) = a_n\cos(ck^2t) + b_n\sin(ck^2t) \; ; \; k = \frac{n\pi}{L}$$

now, $U_n(x,t) = F_n(x)\, G_n(t)$

$$= \sin\left(\frac{n\pi x}{L}\right)\left(Ba_n\cos(ck^2t) + Bb_n\sin(kct)\right)$$

where $k = \frac{n\pi}{L}$

now, $u(x,t) = \sum_{n=1}^{\infty} U_n(x,t)$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(Ba_n\cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right) + Bb_n\sin\left(c\left(\frac{n\pi}{L}\right)^2 t\right)\right)$$

now

$$\frac{\partial u}{\partial t}\Big|_{t=0} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(-Ban\, c\left(\frac{n\pi}{L}\right)^2 \sin\left(c\left(\frac{n\pi}{L}\right)^2 t\right) + Bbn\, c\left(\frac{n\pi}{L}\right)^2 \cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right)\right)\Big|_{t=0}$$

$$\Rightarrow 0 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) Bbn\, c\left(\frac{n\pi}{L}\right)^2$$

$$\Rightarrow Bbn = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) Ban \cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) Ban = f(x) = x(L-x)$$

Here,

$$Ban = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L Lx \sin\left(\frac{n\pi x}{L}\right)dx - \frac{2}{L}\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= I_1 - I_2$$

now

$$I_1 = \left[\frac{-x\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} + \frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}\right]_0^L$$

$$= 2L^2\left[-\frac{\cos(n\pi)}{n\pi}\right]$$

and

$$I_2 = \frac{2}{L}\left[\frac{-x^2\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} + \frac{2x\sin\left(\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}} + \frac{2\cos\left(\frac{n\pi x}{L}\right)}{\frac{n^3\pi^3}{L^3}}\right]_0^L$$

$$= \frac{2}{L}\left[-\frac{L^3}{n\pi}\cos(n\pi) + \frac{2L^3}{n^2\pi^2}\sin(n\pi) + \frac{2L^3}{n^3\pi^3}\cos(n\pi) - \frac{2L^3}{n^3\pi^3}\right]$$

$$= \frac{2}{L}\left[-\frac{L^3}{n\pi}\cos(n\pi) + \frac{2L^3}{n^3\pi^3}\cos(n\pi) - \frac{2L^3}{n^3\pi^3}\right]$$

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$$B_{en} = \frac{2}{L}\left[\frac{2L^3}{n^3\pi^3} - \frac{2L^3}{n^3\pi^3}\cos(n\pi)\right]$$

$$= \frac{4L^2}{n^3\pi^3} \quad \frac{4L^2}{n^3\pi^3}\cos(n\pi)$$

At the end we can write

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{4L^2}{n^3\pi^3} - \frac{4L^2}{n^3\pi^3}\cos(n\pi)\right] \times \cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right)$$

(II)

~~Solution to question no 2~~

$$\frac{\partial^2 u}{\partial t^2} = -c^2\frac{\partial^4 u}{\partial x^4} \qquad \text{(Given)}$$

$\frac{\partial u}{\partial t} = 0$, $u(x,0) = x(1-x)$ [Initial condition]

$u(0,t) = u(L,t) = 0$ [Boundery Condition]

$U_{xx}(0,t) = U_{xx}(L,t) = 0$ [Boundery condition]

Consider $\frac{\partial u(t)}{\partial t} = v(t)$

$$\Rightarrow \frac{\partial^2 u(t)}{\partial t^2} = \frac{\partial v(t)}{\partial t}$$

$$\Rightarrow \frac{\partial \dot{v}(t)}{\partial t} = -c^2\frac{\partial^4 u(t)}{\partial x^4}$$

So, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \Rightarrow v(0) = 0 = v_0$

now,

$$F(u(t)) = -c^2 \frac{\partial^2 u(t)}{\partial x^4}$$

$$u(x,0) = u_0 = x(1-x)$$

also $v_0 = 0$

(iii) $X = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\frac{dX}{dt} = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = \begin{pmatrix} v \\ -c^2 \frac{\partial^4 u}{\partial x^4} \end{pmatrix}$$

$$\Rightarrow \frac{X(n) - X(n-1)}{\tau} = \begin{pmatrix} \frac{u(n) - u(n-1)}{\tau} \\ \frac{v(n) - v(n-1)}{\tau} \end{pmatrix} = \begin{pmatrix} v(n-1) \\ -c^2 \frac{\partial^2 u(n-1)}{\partial x^4} \end{pmatrix}, \quad \tau = \frac{1}{3}$$

So,
$$u(n) - u(n-1) = \tau v(n-1)$$
$$v(n) - v(n-1) = -\tau c^2 \left(\frac{\partial^4 u(n-1)}{\partial x^4}\right)$$

Hence,
$$u(n) = u(n-1) + \tau v(n-1)$$
$$v(n) = v(n-1) - \tau c^2 \frac{\partial^4 u(n-1)}{\partial x^4}$$

In matrix $\rightarrow$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(n) \\ v(n) \end{pmatrix} = \begin{pmatrix} u(n-1) + \tau v(n-1) \\ v(n-1) - \tau c^2 \frac{\partial^4 u(n-1)}{\partial x^4} \end{pmatrix}$$

$$A_\tau = 1$$
$$u_\tau = u(n)$$
$$v_\tau = v(n)$$

$$f_\tau = u(n-1) + \tau v(n-1)$$
$$g_\tau = v(n-1) - \tau c^2 \frac{\partial^4 u(n-1)}{\partial x^4}$$

Solution to problem 3

V = feasible region

= space of functions ω on $\bar{\omega} = \omega \cup \partial\omega$ such that ω, $\nabla\omega$ and $\Delta\omega$ are intregrable

Given,
$$J(v) = \frac{1}{2}\int_\Omega \left(\frac{d^2v}{dx^2}\right)^2 dx - \int_\Omega fv\,dx$$

we have to find $u \in V$ such that
$$J(u) = \min_{v\in V} J(v)$$

It has a unique solution $u \in V$ with certain condition.. Now we need a u such that $u \in V$ and $\langle J'(u), v\rangle = 0 \;\; \forall v \in V$

when $\lambda > 0$
$$\frac{J(u+\lambda v) - J(u)}{\lambda} = \frac{1}{\lambda}\left[\frac{1}{2}\int_\Omega \left(\frac{d^2(u+\lambda v)}{dx^2}\right)^2 du - \int_\Omega f(u+\lambda v)\,dx - \frac{1}{2}\int_\Omega \left(\frac{d^2u}{dx^2}\right)^2 dx + \int_\Omega fu\,dx\right]$$

$$= \frac{1}{\lambda}\left[\frac{1}{2}\int_\Omega \left(\frac{d^2u}{dx^2} + \lambda\frac{d^2v}{dx^2}\right)^2 dx - \int_\Omega fu\,dx - \lambda\int_\Omega fv\,dx - \frac{1}{2}\int_\Omega \left(\frac{d^2u}{dx^2}\right)^2 dx + \int_\Omega fu\,dx\right]$$

$$= \frac{1}{\lambda}\left[\frac{1}{2}\left[\int_\Omega \left(\frac{d^2u}{dx^2}\right)^2 dx + 2\lambda\int_\Omega \Delta u \cdot \Delta v\, dx + \lambda^2 \int_\Omega \left(\frac{d^2v}{dx^2}\right)^2 dx\right] - \lambda\int_\Omega f v\, dx - \frac{1}{2}\int_\Omega \left(\frac{d^2u}{dx^2}\right)^2\right]$$

$$= \int_\Omega \Delta u \cdot \Delta v\, dx + \frac{\lambda}{2}\int_\Omega \left(\frac{d^2v}{dx^2}\right)^2 dx - \int_\Omega f v\, dx$$

So, $$\lim_{\lambda \to 0} \frac{J(u+\lambda v)}{\lambda} = \int_\Omega \Delta u \cdot \Delta v\, dx - \int_\Omega f v\, dx = 0$$

$$\Rightarrow \int_\Omega \Delta u \cdot \Delta v\, dx = \int_\Omega f v\, dx \quad \forall v \in V \quad \ldots (1)$$

$\Delta^2 u = f$ on Ω

$u = 0$ on $\partial\Omega$

$\Delta u = 0$ on $\partial\Omega$

now multiply (1) with v and doing integration

$$\int_\Omega \Delta u\, \Delta v\, dx - \int_{\partial\Omega} \Delta u\, \nabla v\, dx + \int_{\partial\Omega} \Delta u\, v\, dx = \int_\Omega f v\, dx$$

$$\Rightarrow \int_\Omega \Delta u \cdot \Delta v\, dx = \int_\Omega f v\, dx \quad \left[\because \begin{array}{l} u = 0 \\ \Delta u = 0 \text{ on } \partial\Omega \end{array}\right]$$

(iii)

$$u(x) = \frac{f_0}{24}\left(x^4 - 2Lx^3 + L^3x\right)$$

$$u'(x) = \frac{f_0}{24}\left(4x^3 - 6Lx^2 + L^3\right)$$

$$u''(x) = \frac{f_0}{24}\left(12x^2 - 12Lx\right)$$

$$u'''(x) = \frac{f_0}{24}\left(24x - 12L\right)$$

$$u^{IV}(x) = f_0$$

$$u(0) = \frac{f_0}{24}\left(0 - 0 + 0\right) = 0$$

again $u(L) = \frac{f_0}{24}\left(L^4 - 2L^4 + L^4\right) = 0$

$$u''(0) = \frac{f_0}{24}\left(0 - 0\right) = 0$$

$$u''(L) = \frac{f_0}{24}\left(12L^2 - 12L^2\right) = 0$$

So $u(x)$ satisfies the boundary value problem

$$u^{IV}(x) = f(x)$$

where, $u(0) = u(L) = 0$ and

$$u''(0) = u''(L) = 0$$