Heaps & Heap Sort
Priority Queues

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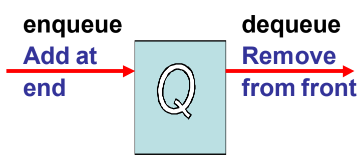
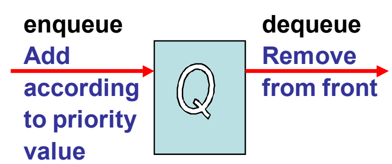
Priority Queues

A queue that is ordered according to some priority value

Standard Queue



Priority Queue



Applications of Priority Queues

Line-up of Incoming Planes at Airport

Possible Criteria for Priority?

Operating Systems Priority Queues?

Several criteria could be mapped to a priority status

Max-Priority Queue Operations

Insert(S, x) – Inserts element x into set S, according to its priority

Maximum(S) – Returns, but does not remove, the element of S with the largest key

Extract-Max(S) – Returns, and also removes the element of S with the largest key

Increase-Key(S, x, k) – Increases the value of element x's key to the new value k

Possible Implementations?

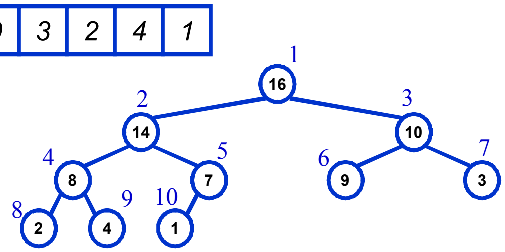
?

Binary Heaps

- The (binary) heap data structure is an array object that can be viewed as a complete binary tree
  - Each node of the tree corresponds to an element of the array that stores the value in the node.
  - The tree is completely filled on all levels except possibly the lowest, where it is filled from the left up to a point.

	0	1	2	3	4	5	6	7	8	9	10
A =	*10*	*16*	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

No. of elements

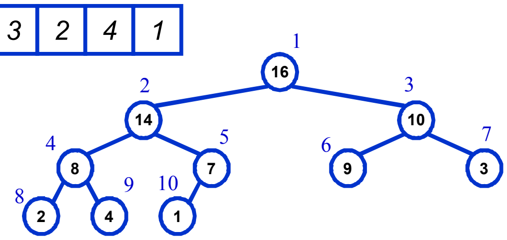


Binary Heaps

- To represent a complete binary tree as an array:
  - The root node is A[1]
  - Node *i* is A[*i*]
  - The parent of node *i* is A[*i*/2]
  - The left child of node *i* is A[2*i*]
  - The right child of node *i* is A[2*i* + 1]

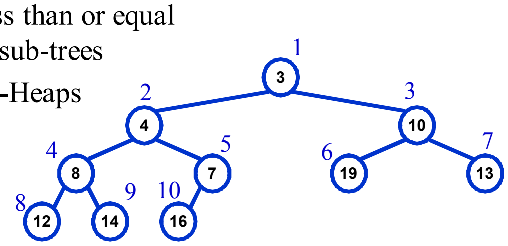
```
Parent(i)
    return floor(i/2)
Right(i)
    return 2i+1
Left(i)
    return 2i
```

	0	1	2	3	4	5	6	7	8	9	10
A =	10	16	14	10	8	7	9	3	2	4	1

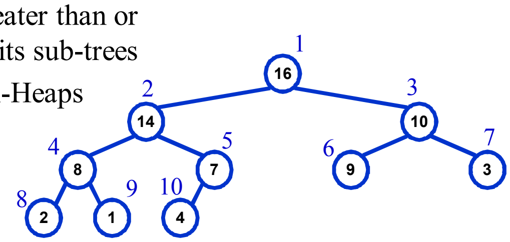


Types of Binary Heaps

- **Min-Heaps:**
  - The element in the root is less than or equal to all elements in both of its sub-trees
  - Both of its sub-trees are Min-Heaps



- **Max-Heaps:**
  - The element in the root is greater than or equal to all elements in both its sub-trees
  - Both of its sub-trees are Max-Heaps

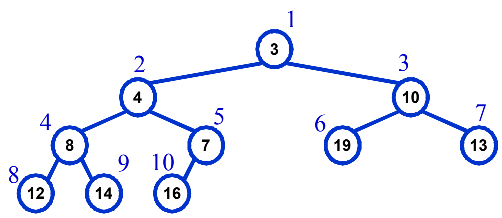


The Min-Heap Property

- Min-Heaps satisfy the *heap property*:

  A[*Parent*(i)] $\leq$ A[i]   for all nodes $i > 1$

  - The value of a node is at least the value of its parent
  - The smallest element in a min-heap is stored at the root
  - Where is the largest element ???
    - Ans: At one of the leaves [leaf indices are $n/2+1$ to n]



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The Max-Heap Property

- Max-Heaps satisfy the *heap property*:

  A[*Parent*(i)] $\geq$ A[i]   for all nodes $i > 1$

  - The value of a node is at most the value of its parent
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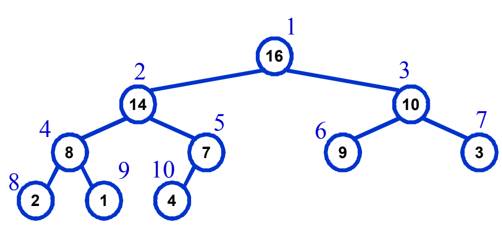

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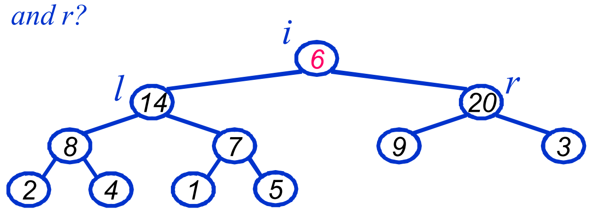
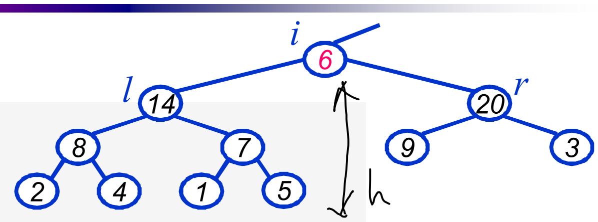
Max-Heap Operations: Max-Heapify()

- **Max-Heapify()**: maintain the max-heap property
  - Given: a node *i* in the heap with children *l* and *r*
    : two subtrees rooted at *l* and *r*, assumed to be heaps
  - Problem: The subtree rooted at *i* may violate the heap property (*How?*)
  - Action: let the value of the parent node "float down" so subtree at *i* satisfies the heap property
    - *What do you suppose will be the basic operation between i, l, and r?*



Max-Heap Operations: Max-Heapify()



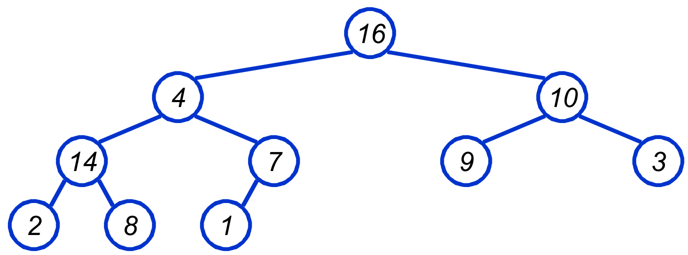
MAX-HEAPIFY(A, i)

$l \leftarrow$ LEFT(i)
$r \leftarrow$ RIGHT(i)
if $l \leq$ *heap-size*$[A]$ and $A[l] > A[i]$
  then *largest* $\leftarrow l$
  else *largest* $\leftarrow i$
if $r \leq$ *heap-size*$[A]$ and $A[r] > A[$*largest*$]$
  then *largest* $\leftarrow r$
if *largest* $\neq i$
  then exchange $A[i] \leftrightarrow A[$*largest*$]$
    MAX-HEAPIFY$(A,$ *largest*$)$

$h = \log_2 n$

Time $= O(\log_2 n)$

Heapify() Example



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A =	*16*	*4*	*10*	*14*	*7*	*9*	*3*	*2*	*8*	*1*

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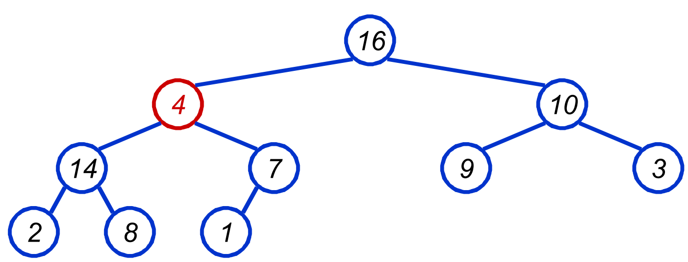

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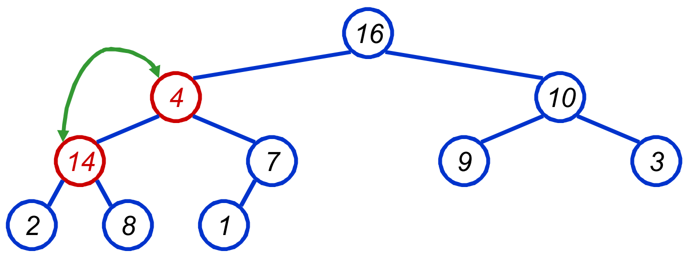

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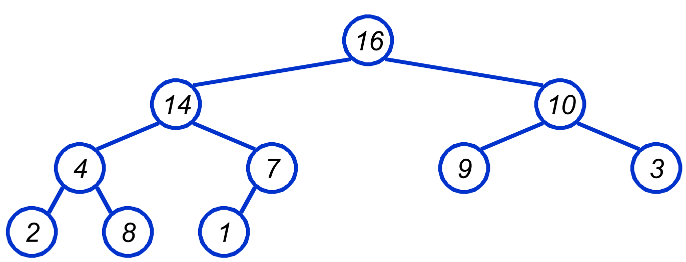

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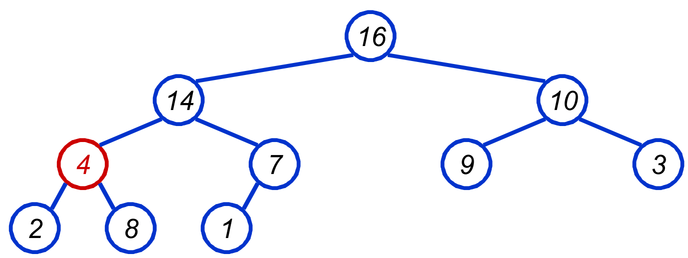
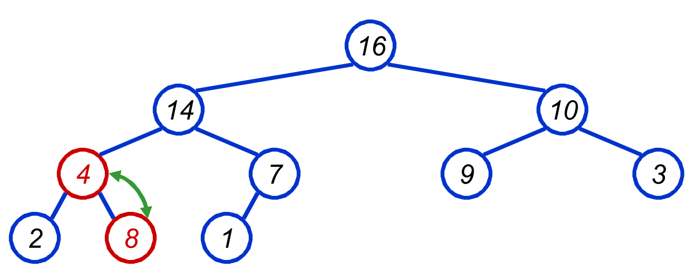
Heapify() Example



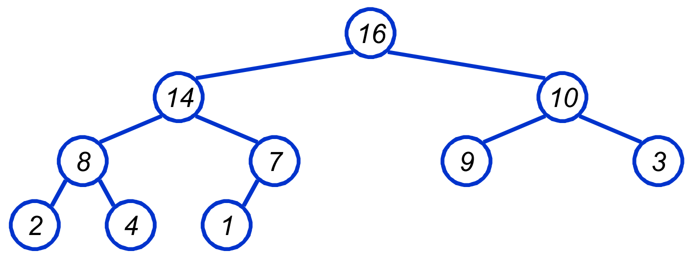
	1	2	3	4	5	6	7	8	9	10
A =	16	14	10	4	7	9	3	2	8	1

Heapify() Example



	1	2	3	4	5	6	7	8	9	10
A =	16	14	10	4	7	9	3	2	8	1

Heapify() Example



	1	2	3	4	5	6	7	8	9	10
A =	*16*	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

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Heapify() Example

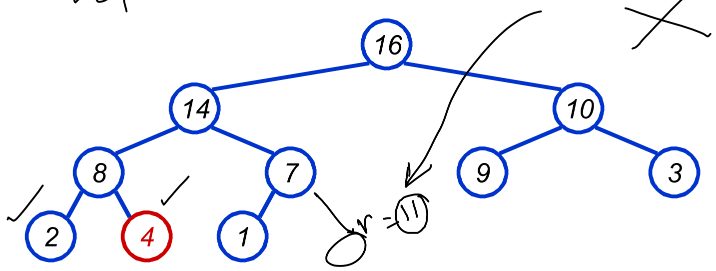
$l < heapsize$

$l = 8$

$heapsize = 10$

$r = 9$ $r < heapsize$

$r > heapsize$



	1	2	3	4	5	6	7	8	9	10
A =	*16*	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

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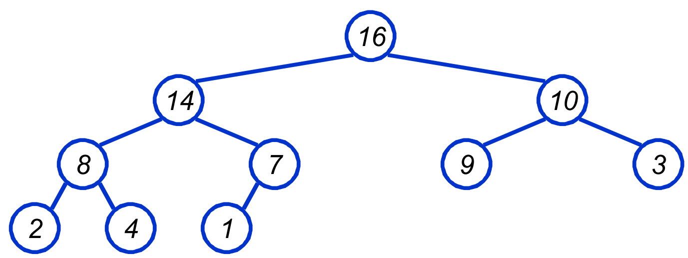
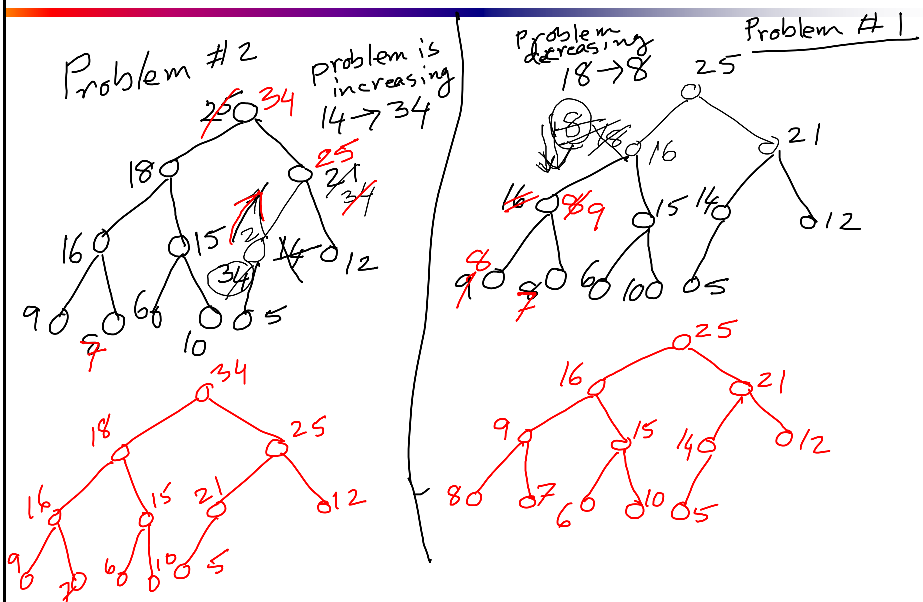
Heapify() Example



	1	2	3	4	5	6	7	8	9	10
A =	*16*	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

Heapify() Example

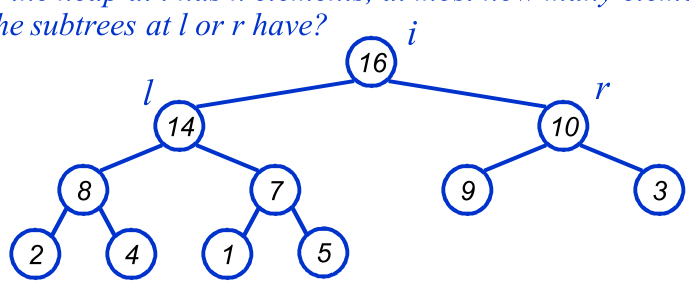


Analyzing Heapify(): Informal

- *Aside from the recursive call, what is the running time of* **`Heapify()`**?
- *How many times can* **`Heapify()`** *recursively call itself?*
- *What is the worst-case running time of* **`Heapify()`** *on a heap of size n?*

Analyzing Heapify(): Formal

- Fixing up relationships among the elements $A[i]$, $A[l]$, and $A[r]$ takes $\Theta(1)$ time
- *If the heap at i has n elements, at most how many elements can the subtrees at l or r have?*



- Answer: $2n/3$ (worst case: bottom row half full)
- So time taken by **`Heapify()`** is given by

  $T(n) \le T(2n/3) + \Theta(1)$

Analyzing Heapify(): Formal

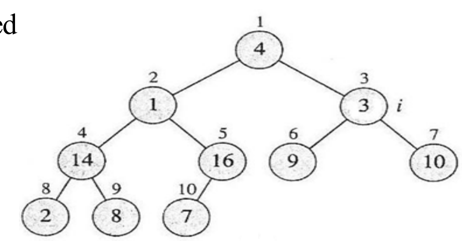
- So we have

  $T(n) \leq T(2n/3) + \Theta(1)$
- Solving the recurrence, we have

  $T(n) = O(\log n)$
- Thus, **`Heapify()`** takes $O(h)$ time for a node at height h.

Heap Operations: BuildHeap()

- We can build a heap in a bottom-up manner by running **`Heapify()`** on successive subarrays
  - Fact: for array of length n, all elements in the range $A[\lfloor n/2 \rfloor + 1 \ldots n]$ are heaps (*Why?*)
  - So
    - Walk backwards through the array from $n/2$ to 1, calling **`Heapify()`** on each node.
    - Order of processing guarantees that the children of node i are heaps when i is processed



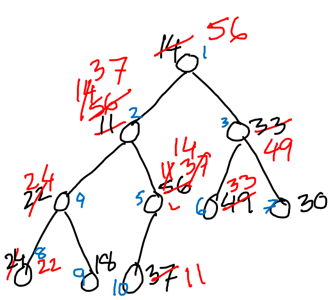
Heap Operations: BuildHeap()

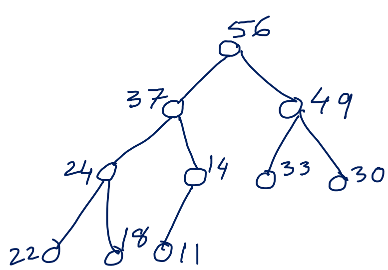
Converts an unorganized array A into a max-heap.

```
BUILD-MAX-HEAP(A)
  heap-size[A] ← length[A]
  for i ← ⌊length[A]/2⌋ downto 1
      do MAX-HEAPIFY(A, i)
```

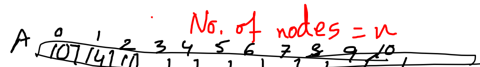

Max BuildHeap() Example

- Work through example
  A = {14, 11, 33, 22, 56, 49, 30, 24, 18, 37}





No. of nodes = n

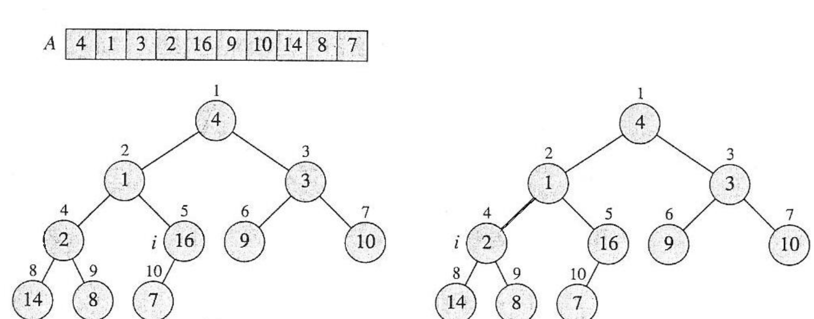


No. of Leaves = n/2 if n is even, n/2+1 if n is odd

No. of internal nodes = n/2

BuildHeap() Example

- Work through example
  A = {4, 1, 3, 2, 16, 9, 10, 14, 8, 7}



(a) (b)

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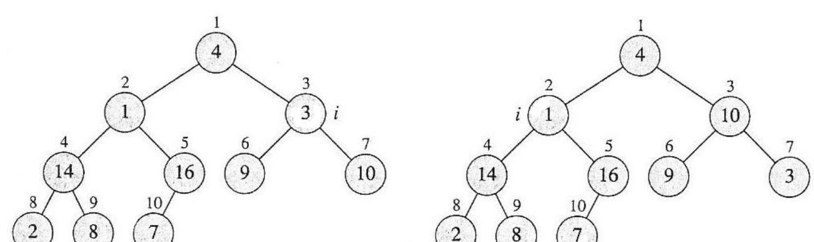
BuildHeap() Example



(c) (d)

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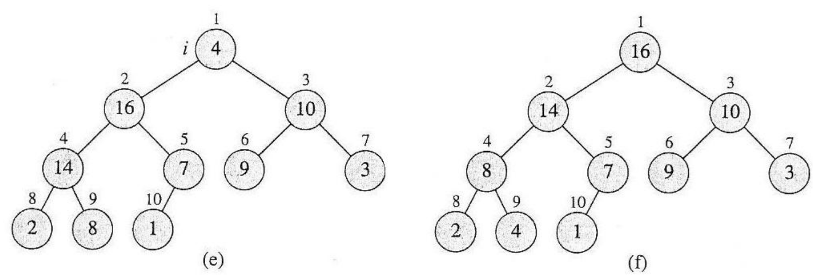
BuildHeap() Example



Analyzing BuildHeap()

- Each call to **`Heapify()`** takes $O(\log n)$ time
- There are $O(n)$ such calls (specifically, $\lfloor n/2 \rfloor$)
- Thus the running time is $O(n \log n)$
  - *Is this a correct asymptotic upper bound?*
  - *Is this an asymptotically tight bound?*
- A tighter bound is $O(n)$
  - *How can this be? Is there a flaw in the above reasoning?*

Analyzing BuildHeap(): Tight

- To **`Heapify()`** a subtree takes $O(h)$ time, where h is the height of the subtree
  - $h = O(\log m)$, m = # nodes in the subtree
  - The height of most subtrees is small
- Fact: an n-element heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height h
- Prove that **`BuildHeap()`** takes $O(n)$ time

Heapsort

- Given **`BuildHeap()`**, a sorting algorithm can easily be constructed:
  - Maximum element is at A[1]
  - Discard by swapping with element at A[n]
    - Decrement heap_size[A]
    - A[n] now contains correct value
  - Restore heap property at A[1] by calling **`Heapify()`**
  - Repeat, always swapping A[1] for A[heap_size(A)]

Heapsort

```
Heapsort(A)
{
        BuildHeap(A);
        for (i = length(A) downto 2)
        {
                Swap(A[1], A[i]);
                heap_size(A) = heap_size(A) – 1;
                Heapify(A, 1);
        }
}
```

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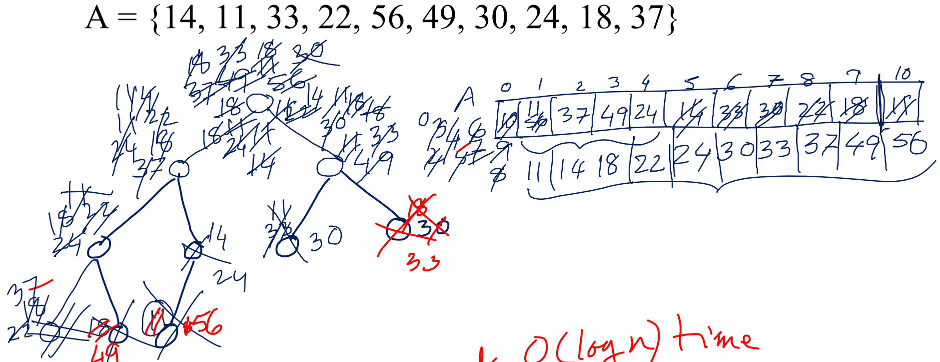
Heapsort

- Work through example
  A = {14, 11, 33, 22, 56, 49, 30, 24, 18, 37}



needs O(log n) time

Call Heapify() ←

No. of calls O(n)

Total time : O(nlogn)

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Analyzing Heapsort

- The call to **`BuildHeap()`** takes $O(n)$ time
- Each of the $n - 1$ calls to **`Heapify()`** takes $O(\log n)$ time
- Thus the total time taken by **`HeapSort()`**
  $= O(n) + (n - 1)\, O(\log n)$
  $= O(n) + O(n \log n)$
  $= O(n \log n)$

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Priority Queues

- Heapsort is a nice algorithm, but in practice Quicksort (coming up) usually wins
- The heap data structure is incredibly useful for implementing *priority queues*
  - A data structure for maintaining a set S of elements, each with an associated value or *key*
  - Supports the operations **`Insert()`**, **`Maximum()`**, and **`ExtractMax()`**

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Priority Queue Operations

Insert(*S, x*) – Inserts element *x* into set *S*, according to its priority

Maximum(*S*) – Returns, but does not remove, the element of *S* with the largest key

Extract-Max(*S*) – Removes and returns the element of *S* with the largest key

Increase-Key(*S, x, k*) – Increases the value of element *x*'s key to the new value *k*

- *How could we implement these operations using a heap?*

Priority Queue Operations

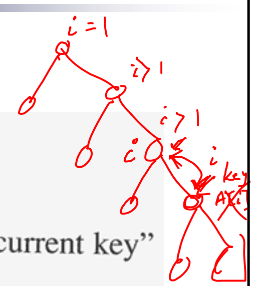
```
HEAP-MAXIMUM(A)
    return A[1]
```

Time = O(1)

```
HEAP-EXTRACT-MAX(A)
    if heap-size[A] < 1
        then error "heap underflow"
    max ← A[1]
    A[1] ← A[heap-size[A]]
    heap-size[A] ← heap-size[A] − 1
    MAX-HEAPIFY(A, 1)
    return max
```

Time = O(log n)

Priority Queue Operations




```
HEAP-INCREASE-KEY(A, i, key)
1  if key < A[i]
2     then error "new key is smaller than current key"
3  A[i] ← key
4  while i > 1 and A[PARENT(i)] < A[i]
5       do exchange A[i] ↔ A[PARENT(i)]
6          i ← PARENT(i)
```

Time = $O(\log_2 n)$

Priority Queue Operations

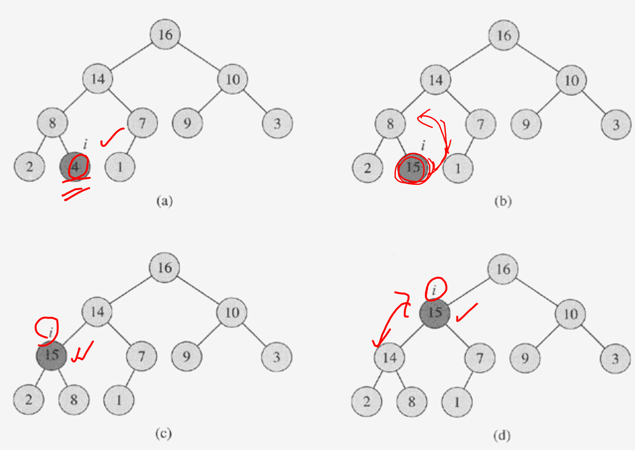
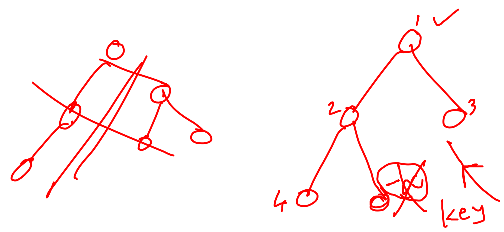


Figure 6.5 The operation of HEAP-INCREASE-KEY. **(a)** The max-heap of Figure 6.4(a) with a node whose index is i heavily shaded. **(b)** This node has its key increased to 15. **(c)** After one iteration of the **while** loop of lines 4–6, the node and its parent have exchanged keys, and the index i moves up to the parent. **(d)** The max-heap after one more iteration of the **while** loop. At this point, $A[\text{PARENT}(i)] \geq A[i]$. The max-heap property now holds and the procedure terminates.

Priority Queue Operations

```
MAX-HEAP-INSERT(A, key)
  heap-size[A] ← heap-size[A] + 1
  A[heap-size[A]] ← −∞
  HEAP-INCREASE-KEY(A, heap-size[A], key)
```

Time = $O(\log n)$



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