IT24624

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Assignment - 03

Prove that the set of rational numbers $\mathbb{Q}$, equipped with two binary operations of addition and multiplication, from a field.

↳ we take the rational numbers $\mathbb{Q}$ to be the set of equivalence classes of ordered pairs (a, b) with $a, b \in \mathbb{Z}$ and $b \neq 0$, where $(a, b) \sim (a', b')$ if $ab' = a'b$. we identify the class of (a, b) with the usual fraction a/b. Define addition and multiplication in the usual way:

$$a/b + c/d = \frac{ad+bc}{bd}, \quad a/b \cdot c/d = ac/bd$$

for, $b \neq 0$, $d \neq 0$. Below we show these operations make $\mathbb{Q}$ a field.

1. The operation are well-defined:

we must check that if $a/b = a'/b'$ and $c/d = \frac{c'}{d'}$

then,

$$\frac{ad+bc}{bd} = \frac{a'd'+b'c'}{b'd'} \quad \text{and} \quad \frac{ac}{bd} = \frac{a'c'}{b'd'}.$$

From, $a/b = \frac{a'}{b'}$ and $c/d = \frac{c'}{d'}$ we have $ab' = a'b$ and $cd' = c'd$. Compute,

$$(ad+bc)b'd' = (ab')(dd') + (bc)(b'd') = (a'b)(dd') + (bc)(b'd')$$

and similarly expand the right-hand numerator times $bdb'd'$. Rearranging and using $ab' = a'b$, $cd' = c'd$, shows both cross-products are equal, therefore the sums (and similarly the products) represent the same equivalence class. So addition and multiplication are well-defined.

2. $(\mathbb{Q}, +)$ is an abelian group:

Take any $a/b, c/d, e/f \in \mathbb{Q}$

- Closure: $a/b + c/d = \frac{ad+bc}{bd}$ is a rational number since $bd \neq 0$

- Associativity: Follows from associativity of integer addition:

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f} = \frac{ad+bc}{bd} + \frac{e}{f} = \frac{(ad+bc)f + e(bd)}{(bd)f}$$

and a similar expansion for $\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$; both give the same numerator by associativity/commutativity of integer operations.

- Identity: $0 = \frac{0}{1}$ satisfies $\frac{a}{b}+0 = \frac{a}{b}$.

- Inverse: additive inverse of $\frac{a}{b}$ is $-\frac{a}{b} = \frac{-a}{b}$ because $\frac{a}{b} + \frac{-a}{b} = \frac{0}{b} = 0$.

- Commutativity: $\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd} = \frac{bc+ad}{db}$

$$= \frac{c}{d}+\frac{a}{b}.$$

Thus $(\mathbb{Q},+)$ is an abelian group.

3. <u>Multiplication on $\mathbb{Q}\setminus\{0\}$ is an abelian group</u>

<u>(except we first show ring axioms):</u>

- <u>Closure:</u> product $a/b \cdot c/d = ac/bd$ is rational since $bd \neq 0$.

- <u>Associativity and Commutativity:</u> follow from associativity and commutativity of integer multiplication:

$$\left(a/b \cdot \frac{c}{d}\right) \cdot e/f = \frac{ac}{bd} \cdot e/f = \frac{(ac)e}{(bd)f} = \frac{a(ce)}{b(df)} = \frac{a}{b} \cdot \frac{ce}{df}$$

$$= \frac{a}{b} \cdot \left(\frac{c}{d} \cdot e/f\right)$$

- Multiplicative identity: $1 = 1/1$ satisfies $a/b \cdot 1 = a/b$.

- Distributivity: For addition and mutiplication

$$a/b \cdot (c/d + e/f) = a/b \cdot \frac{cf + ed}{df} = \frac{a(cf+ed)}{bdf} = \frac{acf + aed}{bdf}$$

$$= \frac{ac}{bd} + \frac{ae}{bf} = a/b \cdot c/d + a/b \cdot e/f.$$

using integer distributivity.

So, $\mathbb{Q}$ is a commutative ring with unity 1.

4. Multiplicative inverses exist for nonzero rationals:

Take a rational nonzero a/b (so $a \neq 0$, $b \neq 0$). Its multiplicative inverse is b/a because

$$a/b \cdot b/a = ab/ab = 1/1 = 1.$$

We also must check this inverse is well-defined: if $a/b = \frac{a'}{b'}$ and $a \neq 0$, then $ab' = a'b$.

Multiplying both sides by $1/(aa')$ is informal but the correct check is: $b/a = \frac{b'}{a'}$ if and only if $ba' = b'a$; but from $ab' = a'b$ we get exactly $ba' = b'a$, so inverses agree for different representatives.

5. Nontrividlity: $0 \neq 1$

Clearly $0/1 \neq 1/1$ because if $0.1 = 1.1$ the $0=1$, Contradicting the integer's properties. So the field is not the zero ring.