CHAPTER - 9

RANDOM VARIABLE

9.1. Introduction

It is not rare that the results of random experiments are expressed in terms of numerical values. For example, number of heads in case of tossing two coins, number of defective items among three items produced by a machine, number of boys in case of three children family, etc. The number of heads, number of defective items, and number of children are expressed in numerical quantities. The specific values of these quantities depend on the outcome of the random experiments and cannot be predicted before the experiment. These variable quantities, which vary from outcomes to outcomes, are known as random variables. Random variables and their distributions called, probability distributions, play an important role in business decision-making. The expected values of these random variables, also play important role in business decision making. In this chapter, we shall discuss random variable and its expectation.

9.2. Random Variable

Definition. A variable whose values are determined by the outcomes of a random experiment is called random variable.

Definition . A random variable is a function that assigns a real number to each sample point in a sample space.

We shall use capital letters, such as X, Y, Z etc. to denote random variables and the corresponding small letters x, y, z to denote any specific value of a random variable. Like variable, there are two types of random variable. They are

i). Discrete random variable and
ii) Continuous random variable.

9.3. Discrete Random Variable

Definition. A discrete random variable is a function that assigns a numerical value to each sample point in a discrete sample space.

That is, a variable, which takes different discrete values with specified probabilities, is called discrete random variable. Now we shall cite one example.

Example 9.3.1. Toss a fair coin twice. Construct the sample space and define a random variable.

Solution. Let H and T denote the head and tail of the coin respectively. The sample space of the experiment is S = { TT, TH, HT, HH }

Let X denotes the number of heads. Then X can take values 0, 1 and 2, which are the quantities determined by the outcomes of the random experiment. Here P [H] = P [T] = 1/2, since the coins are fair .Let p(x) be the probability that the random variable X takes value x. Here X, number of heads is a random variable. The sample points of the experiment, number of heads and the probabilities of the number of heads are presented in a tabular form given below:

Sample point	Number of heads	Probability of X
TT	0	1/4
TH	1	1/4
HT	1	1/4
HH	2	1/4
Total		1

It is seen that the number of heads equals to 1 has probability 1/2. The different values of X with the probabilities will be as follows:

Values of X : x	0	1	2	Total
p(x)	1/4	1/2	1/4	1

Here it is seen that the value of p(x) is greater than 0 and the sum of p(x) is 1. Here p(x) is the probability that X takes value x. That is P [X = x] = p (x).

It is seen that (i) p (x) > 0 and (ii) Σ p (x) = 1.

The set of ordered pairs (x, p (x)) is called probability function or probability mass function. It is also called probability distribution.

Usually discrete random variable takes integer value. But sometimes it can take fractional isolated values too.

9.3.1. Probability function, probability mass function or probability distribution. The probability function of a discrete random variable can be shown in a table or by a formula.

A table or a formula by which the different values of a random variable with their associated probabilities are shown is called discrete probability distribution. Table 9.1 is an example of a probability distribution.

Let X be a random variable which can take values x_1, x_2, ..., x_n with associate probabilities $p(x_1)$, $p(x_2)$, ..., $p(x_n)$, then the probability function of X can be defined by the following table:

Table 9.1. Discrete probability distribution

Variable X : x	x_1	x_2	....	x_n
p(x)	$p(x_1)$	$p(x_2)$	....	$p(x_n)$

Probability function of a random variable X can also be expressed by a formula. Suppose X is a discrete random variable, then the probability function of X may be defined by the formula

$$p(x) = \binom{4}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{4-x} \quad ; x = 1, 2, 3, 4.$$

Definition. The set of ordered pairs (x, p(x)) is called probability function, probability mass function or probability distribution of the discrete random variable X, if for each value of x

i) $p(x) \geq 0$,
ii) $\Sigma p(x) = 1$ and
iii) $P[X = x] = p(x)$.

Example 9.3.2. Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified as defective D, or non-defective N. Let X denotes the number of defective items. Suppose the probability of a defective item is 0.02. (i) Find the probability function of X. Also compute, (ii) P [X > 2], (iii) P [X ≥ 2] and (iv) P [X = 2].

Solution. (i) The sample space of the experiment is

S = { NNN, NND, NDN, DNN, NDD, DND, DDN, DDD }

Here P [D] = 0.02, then P [N] = 1 – P[D] = 1 – 0.02 = 0.98 .

It is easily seen that the possible values of X are 0, 1, 2, 3.

P [NNN] = P [X = 0] = P (0) = P [N] P [N] P [N]
= (0.98) (0.98) (0.98) = 0.941192.

P [NND] = P [NDN] = P [DNN] = P [X = 1] = p (1)
=P [N] P [N] P [D] = (0.98) (0.98) (0.02) = 0.19208

P [NDD] = P [DND] = P [DDN] = P [X = 2] = p (2)
= P [N] P [D] P [D] = (0.98) (0.02) (0.02) = 0.000392

P [DDD] = P [X = 3] = p (3) =P [D] P [D] P [D]
= (0.02)(0.02)(0.02) = 0.000008.

The probability function of X is

Values of X : x	0	1	2	3
p(x)	0.941192	0.057624	0.001176	0.000008

Table 9.1. Discrete probability distribution

Variable X : x	x_1	x_2	....	x_n
p(x)	$p(x_1)$	$p(x_2)$	....	$p(x_n)$

Probability function of a random variable X can also be expressed by a formula. Suppose X is a discrete random variable, then the probability function of X may be defined by the formula

$$p(x) = \binom{4}{x}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{4-x} \quad ; x = 1, 2, 3, 4.$$

Definition. The set of ordered pairs (x, p(x)) is called probability function, probability mass function or probability distribution of the discrete random variable X, if for each value of x

i) $p(x) \geq 0$,
ii) $\Sigma p(x) = 1$ and
iii) $P[X = x] = p(x)$.

Example 9.3.2. Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified as defective D, or non-defective N. Let X denotes the number of defective items. Suppose the probability of a defective item is 0.02. (i) Find the probability function of X. Also compute, (ii) $P[X > 2]$, (iii) $P[X \geq 2]$ and (iv) $P[X = 2]$.

Solution. (i) The sample space of the experiment is

S = { NNN, NND, NDN, DNN, NDD, DND, DDN, DDD }

Here $P[D] = 0.02$, then $P[N] = 1 - P[D] = 1 - 0.02 = 0.98$.

It is easily seen that the possible values of X are 0, 1, 2, 3.

$P[NNN] = P[X = 0] = P(0) = P[N]\,P[N]\,P[N]$
$= (0.98)(0.98)(0.98) = 0.941192.$

$P[NND] = P[NDN] = P[DNN] = P[X = 1] = p(1)$
$= P[N]\,P[N]\,P[D] = (0.98)(0.98)(0.02) = 0.19208$

$P[NDD] = P[DND] = P[DDN] = P[X = 2] = p(2)$
$= P[N]\,P[D]\,P[D] = (0.98)(0.02)(0.02) = 0.000392$

$P[DDD] = P[X = 3] = p(3) = P[D]\,P[D]\,P[D]$
$= (0.02)(0.02)(0.02) = 0.000008.$

The probability function of X is

Values of X : x	0	1	2	3
p(x)	0.941192	0.057624	0.001176	0.000008

The probability function of the random variable X can also be defined as

$$P[X=x] = p(x) = \binom{4}{x}(0.02)^x(0.98)^{3-x};\ x = 0, 1, 2, 3.$$

(ii) $P[X > 2] = P[X = 3] = P(3) = 0.000008.$

(iii) $P[X \geq 2] = P[X = 2] + P[X = 3] = p(2) + P(3)$

$= 0.001176 + 0.000008 = 0.001184.$

(iv) $P[X = 2] = P(2) = 0.001176.$

Example 9.3.3. Suppose the different sizes of shoes sold by a departmental store in the last month are as follows:

Sizes of Shoes	5	5.5	6	6.5	7	7.5	8	8.5	9
Number of pairs sold	25	30	95	105	155	210	170	125	85

(i) Find the probability function of sizes of the shoes sold by the departmental store. Compute the probability of the sizes of the shoes sold by the shop (ii) more than 7, (iii) less then 6, (iv) between 6 and 8.

Solution. (i) Here the sizes of the shoes may be considered as a random variable X. The possible values of X are 5, 5.5, 6, 6.5, 7, 7.5, 8, 8.5, 9 and their relative frequencies are 0.025, 0.030, 0.095, 0.105, 0.155, 0.210, 0.170, 0.125, and 0.085. The relative frequencies for the different sizes of the shoes are considered as their respective probabilities. Then the possible values of X and their probabilities constitute the probability distribution of X and is given by the following table:

Table 9.3. Probability distribution of the sizes of the shoes

Sizes of Shoes X : x	5	5.5	6	6.5	7	7.5	8	8.5	9
p(x)	0.025	0.030	0.095	0.105	0.155	0.210	0.170	0.125	0.085

(ii) $P[X > 7] = P[X = 7.5] + P[X = 8] + P[X = 8.5] + P[x = 9]$

$= p(7.5) + p(8) + p(8.5 + p(9)$

$= 0.210 + 0.170 + 0.125 + 0.085 = 0.615.$

(iii) $P[X < 7] = P[X = 6.5] + P[X = 6.] + P[X = 5.5] + P[X = 5]$

$= p(6.5) + p(6) + p(5.5) + p(5)$

$= 0.105 + 0.095 + 0.030 + 0.025 = 0.255.$

(iv) $P[6 < X < 8] = P[X = 6.5] + P[X = 7] + P[7.5] = p(6.5) + p(7) + p(7.5)$

$= 0.105 + 0.155 + 0.210 = 0.47.$

Example 9.3.4. Suppose a lot contains 4 items. Let the random variable X denote the number of defective items in the lot. Suppose the probability function of the random variable X is

Values of X : x	0	1	2	3
p(x)	0.50	0.3	0.15	0.05

Find (i) P [X<1], (ii) P [X ≤ 1], (iii) P [1≤ X < 3]

Solution. (i) P [X<1] = P [X=0] = p (0) =0.50.

(ii) P [X ≤ 1] = P[X = 0] + P[X = 1] = p (0) + p (1) = 0.50 + 0.3 = 0.8.

(iii) P [1≤ X <3] = P [X = 1] +P [X=2] = p (1) + p (2) = 0.30 + 0.15 = 0. 45.

Example 9.3.5. A random variable has the following probability function:

Values of X : x	-2	-1	0	1	2	3
p(x)	0.1	K	0.2	2k	0.3	0.1

(i) Find the value of k.

(ii) P [X > 1], (iii) P [-1 < X < 2], (iv) P [X < 1].

Solution. (i) Since $\Sigma p(x) = 1$, we have

0.1 + k + 0.2 +2k +0.3 +0.1 = 1

⇒ 0.7+ 3k = 1

⇒ 3k = 0.3

⇒ k = 0.1

(ii) P [X>1] = P [X=2] + P [X=3] = p (2) + p (3) = 0.3 + 0.1 = 0.4

(iii) P [-1<X<2] = P [X= 0] + P [X=1] = p (0) + p (1) = 0.2 +0.2 = 0.4

(iv) P [X<1] = P [X= 0] +P [X = -1] +P [X = -2]

= p (0) + p (-1) + p(-2) = 0.2 +0.1 +0.1 = 0.4

Example 9.3.6. Consider the sample space with three children family. Suppose the probability of a boy is 4/7. Let X be the number of boys. Compute the probability function of the number of boys. A family with three children is selected at random, what is the probability that the family contains (i) exactly 2 boys, (ii) at most two boys, (iii) at least two boys and (iv) no boys.

Solution. B denotes a boy and G denotes a girl. The sample space of the experiment is

S = {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

Let X denotes the number of boys.

Here, P [B] = 4/7, then P [G] = 1 - 4/7 = 3/7.

P [GGG] = P [X= 0] = p (0) = P [G] P [G] P [G] = (3/7) (3/7) (3/7) = 27/343

Since the events are independent.

P [GGB] = P [GBG] = P [BGG] = P [B] P [G] P [G]

Here $P[GGB] = (3/7)(3/7)(4/7) = 36/343$

So, $P[X = 1] = p(1) = 36/343 + 36/343 + 36/343 = 108/343$.

$P[GBB] = P[BGB] = P[BBG] = P[B]\,P[B]\,P[G]$

Here, $P[GBB] = (3/7)(4/7)(4/7) = 48/343$

$P(X=2) = p(2) = 48/343 + 48/343 + 48/343 = 144/343$.

$p(3) = P[BBB] = P[B]P[B]P[B] = (4/7)(4/7)(4/7) = 64/343$.

The probability function of X is

Values of X : x	0	1	2	3
p(x)	27/343	108/343	144/343	64/343

(i) P [exactly two boys] = $P[X = 2] = p(2) = 108/343$.

(ii) P [at most two boys] = $P[X \le 2] = P[X = 0] + P[X = 1] + P[X = 2]$

$= p(0) + p(1) + p(2) = 27/343 + 108/343 + 144/343 = 279/343$.

Or, $P[X \le 2] = 1 - P[X = 3] = 1 - 64/343 = 279/343$.

(iii) P [at least two boys] = $P[X \ge 2] = P[X = 3] = p(3) = 64/343$.

(iv) P [no boys] = $P[X = 0] = p(0) = 27/343$.

A matched problem

Consider three children in which the probability of a boy is 0.50. Let X be the number of girls. Write probability function of the number of girls. What is the probability that a randomly selected family contains (i) exactly two girls; (ii) at least two girls, (iii) at most two girls and (iv) no girls.

Values of X : x	0	1	2	3
p(x)	1/8	1/8	3/8	1/8

Ans: (i) 3/8; (ii) ½; (iii) 7/8; (iv) 1/8.

9.4. Continuous Random Variable

A random variable obtained from a continuous sample space is called a continuous random variable. In this case a random variable cannot take any isolated value. It can take any value in a certain range. Age of a person is a good example of a continuous variable.

Other examples of continuous variable may be the length of life of a bulb, height of a student, weight of a student etc.

9.4.1. Probability density function. A function f (x) of a continuous random variable X is called a probability density function if it satisfies the following two conditions

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

The probability that a continuous random variable X takes a particular value x is zero. That is $P[X = x] = 0$.

Example 9.4.1. A continuous random variable has the following probability density function

$$f(x) = kx^2, 0 \leq x \leq 1.$$

(i) Find the value of k.

Find the probability of (ii) $P[0.2 \leq X \leq 0.5]$, (iii) $P[X < 0.3]$, (iv) $P[0.25 < X < 0.5]$, (v) $P[X > 0.75]$.

Solution. (i) Since the total probability is one,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 kx^2\,dx = 1$$

$$\Rightarrow \left[k\frac{x^3}{3}\right]_0^1 = 1 \Rightarrow \frac{k}{3} = 1$$

Hence k = 3

(ii) $P[0.2 \leq X \leq 0.5] = \int_{0.2}^{0.5} 3x^2\,dx = \left[3\frac{x^3}{3}\right]_{0.2}^{0.5} = (0.5)^3 - (0.2)^3$

$= 0.125 - 0.008 = 0.117.$

(iii) $P[X < 0.3] = \int_0^{0.3} 3x^2\,dx = \left[3\frac{x^3}{3}\right]_0^{0.3} = (0.3)^3 = 0.027.$

(iv) $P[0.25 < X < 0.5] = \int_{0.25}^{0.5} 3x^2\,dx = \left[3\frac{x^3}{3}\right]_{0.25}^{0.5} = (0.5)^3 - (0.25)^3$

$= 0.125 - 0.016 = 0.109.$

(v) $P[X > 0.75] = \int_{0.75}^{1} 3x^2\,dx = \left[3\frac{x^3}{3}\right]_{0.75}^{1} = (1)^3 - (0.75)^3 = 1 - 0.423 = 0.578.$

Example 9.4.2. Suppose that in a certain region of a country the daily rainfall (in inches) is a continuous random variable X with probability density function f(x) given by

$$f(x) = \frac{3}{4}(2x - x^2),\ 0 < x < 2.$$

Find the probability that a at a given day in this region the rainfall is (i) not more than 1 inch, (ii) more than 1.5 inches, (iii) between 0.5 and 1.5 inches (iv) equal to one inch and (v) less than one inch.

Solution. (i) $P[X \leq 1] = \int_0^1 \frac{3}{4}(2x - x^2)\,dx$

$$= \frac{3}{4}\left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{3}{4}\left(1 - \frac{1}{3}\right) = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}.$$

(ii) $P[X > 1.5] = \int_{3/2}^{2} \frac{3}{4}(2x - x^2)\,dx = \frac{3}{4}\left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_{3/2}^{2}$

$$= \frac{3}{4}\left(4 - \frac{8}{3} - \frac{9}{4} + \frac{27}{24}\right) = \frac{3}{4}\left(\frac{96 - 64 - 54 + 27}{24}\right) = \frac{3}{4} \times \frac{5}{24} = \frac{5}{32}.$$

(iii) $P[0.5 < X < 1.5] = \int_{1/2}^{3/2} \frac{3}{4}(2x - x^2)\,dx = \frac{3}{4}\left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/2}^{2/3}$

$$= \frac{3}{4}\left(\frac{4}{9} - \frac{8}{81} - \frac{1}{4} + \frac{1}{24}\right) = \frac{3}{4}\left(\frac{96 - 64 - 54 + 27}{24}\right) = \frac{3}{4} \times \frac{89}{648} = \frac{89}{864}.$$

(iv) P [X = 1] = 0, since the probability that a continuous variable takes a particular value is zero.

(v) It is the same as (i). Hence

$$P[X < 1] = \int_0^1 \frac{3}{4}(2x - x^2)\,dx$$

$$= \frac{3}{4}\left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{3}{4}\left(1 - \frac{1}{3}\right) = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}.$$

9.5. Mean and Variance of a Random Variable

The probability distribution provides a model for the theoretical frequency distribution of a random variable and hence must possess a mean, median, mode, variance, standard deviation and other descriptive measures associated with the theoretical population, which it represents. The mean of a random variable is called expected value or mathematical expectation of the random variable.

9.5.1. Mathematical expectation or expected value or mean of a discrete random variable

Definition. Let X be a discrete random variable which can take values $x_1, x_2, \ldots\ldots, x_n$ with associate probabilities $p(x_1), p(x_2), \ldots\ldots, p(x_n)$, then mathematical expectation or mean of X, denoted by E[X] is defined by

$$E[X] = x_1 p(x_1) + x_2 p(x_2) + \ldots\ldots + x_n p(x_n) = \sum_{i=1}^{n} x_i p(x_i).$$

In other words, each value of the random variable is multiplied by the probability of the occurrence of the value and then all these products are summed up.

Actually mathematical expectation of a random variable is the mean of the population. It is denoted by a Greek symbol μ (mu).

9.5.2. Properties of expectation. The following are the important properties of an expected value of a random variable:

1. The expected value of a constant c is constant, i,e, E(c) = c for every constant c
2. The expected value of the product of a constant c and a random variable X is equal to c times the expected value of the random variable, i.e. E(cX) = c E(X)
3. The expected value of a linear function of a random variable X is same as the linear function of its expectation i,e, E(a + bX) = a + b E(X)
4. The variance of the linear function of a random variable X is equal to the constant squared times the variance of the random variable X, i,e, $Var(a + bX) = b^2 Var(X)$

Example 9.5.1. A fair coin is tossed twice. Then the number of heads is a random variable that takes values 0, 1 and 2 with the following probability function:

Values of X : x	0	1	2	Total
p(x)	1/4	1/2	1/4	1

Find the expected number of heads.

Solution. The expected number of heads is

$$E[X] = x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3)$$
$$= 0 \times 0.25 + 1 \times 0.5 + 2 \times 0.25 = 0.5 + 0.5 = 1.$$

This means on an average, we can expect one head if we toss one fair coin twice.

Example 9.5.2. Suppose a lot contains 4 items. Let the random variable X denotes the number of defective items in the lot. Suppose the probability function of the random variable X is

Values of X : x	0	1	2	3
p(x)	0.50	0.3	0.15	0.05

Find the expected number of defective items in the lot.

Solution. The expected number of defective items in the lot is

$$E[X] = x_1\, p(x_1) + x_2\, p(x_2) + \ldots\ldots\ldots + x_n\, p(x_n)$$
$$= 0 \times 0.5 + 1 \times 0.3 + 2 \times 0.15 + 3 \times 0.05$$
$$= 0.3 + 0.3 + 0.15 = 0.75.$$

This means that on an average 0.75 defective items can expected from the lot.

9.5.3. Variance of a discrete random variable. If X is a discrete random variable with mean μ which can take values $x_1, x_2, \ldots\ldots, x_n$ with associate probabilities $p(x_1), p(x_2), \ldots\ldots, p(x_n)$, then the variance of X denoted by σ^2. is defined as

$$\sigma^2 = E[X - E(X)]^2 = E[X - \mu]^2 = E[X^2] - [E(X)]^2$$

$$\text{Here, } E[X^2] = x_1^2\, p(x_1) + x_2^2\, p(x_2) + \ldots\ldots\ldots + x_n^2\, p(x_n)$$

Standard deviation: The positive square root of variance is called standard deviation and it is denoted by σ.

Example 9.5.3. A company introduces a new product in the market and expects to make a profit of Tk. 2.5 lakh during the first year if the demand is good; Tk.1.5 if the demand is moderate; and a loss of Tk. 1 lakh if the demand is poor. Market research studies indicate that the probabilities for the demand to be good, moderate and poor are 0.2, 0.5 and 0.3 respectively. Find the company's expected profit and the standard deviation.

Solution. Let X be a random variable representing the profit in three types of demand. Thus X may assume the values:

x_1 = Tk. 2.5 lakh when demand is good,

x_2 = Tk. 1.5 lakh when demand is moderate, and

x_3 = Tk. 1 lakh when demand is poor

The probability distribution of X is given by

Values of X : x	−1	1.5	2.5
p(x)	0.3	0.5	0.2

Hence, the expected profit is given by

$$E(X) = (-1) \times 0.3 + 1.5 \times 0.5 + 2.5 \times 0.2 = \text{Tk. } 0.95 \text{ lakh.}$$

On an average, the company can expect a profit of Tk. 0.95 lakh.

$$E(X^2) = x_1^2\, p(x_1) + x_2^2\, p(x_2) + x_3^2\, p(x_3)$$

$$= (-1)^2 \times 0.3 + (1.5)^2 \times 0.5 + (2.5)^2 \times 0.2 = 2.675 \text{ lakh.},$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 2.675 - (0.95)^2$$

$$= 2.675 - 0.9025 = 1.7725.$$

$$\text{S.D.}(X) = \sqrt{1.7725} = \text{Tk.} 1.331 \text{ lakh.}$$

Example 9.5.4. A bakery has the following probability function of daily demand for marriage day cake:

No. of cakes demanded $X : x$	1	2	3	4	5	6	7	8
Probability p(x)	0.02	0.07	0.09	0.12	k	0.2	0.18	0.02

(i) Find the value of k.

(ii) Find the expected number of marriage day cakes demanded per day.

Solution. Since $\Sigma\, p(x) = 1$.

We have, $0.02 + 0.07 + 0.09 + 0.12 + k + 0.2 + 0.18 + 0.02 = 1$

$$0.7 + k = 1 \Rightarrow k = 1 - 0.7 = 0.3.$$

The expected number of marriage day cakes is

$$E[X] = x_1\, p(x_1) + x_2\, p(x_2) + \ldots\ldots\ldots\ldots + x_n\, p(x_n)$$

$$= 1\times0.02 + 2\times0.07 + 3\times0.09 + 4\times0.12 + 5\times0.3 + 6\times0.2 + 7\times0.18 + 8\times0.02$$

$$= 0.02 + 0.14 + 0.27 + 0.48 + 1.50 + 1.20 + 1.26 + 0.16 = 5.03 \approx 5.$$

That is on an average, the expected number of cakes demanded per day is 5.

Some Matched Problems

Example 9.5.5. The following table shows the probability distribution of a number of long-distance calls made in a month by the residents of urban households in an area:

No. of calls	0	1	2	3	4	5
Probability	0.05	0.21	0.56	0.06	0.08	0.04

If X denotes the number of calls, find the expected value and the variance of X.

Solution. Mean = Expected value of the long – distance call is

$$\mu = 0\times(0.05)+1\times(0.21)+2\times(0.56)+3\times(0.06)+4\times(0.08)+5\times(0.04)$$

$$= 0 + 0.21 + 1.12 + 0.18 + 0.32 + 0.20 = 2.03$$

$$\text{Varianc} = \sigma^2 = E[X^2] - (E[X])^2$$

$$E[X^2] = 0\times(0.05)+1\times(0.21)+4\times(0.56)+9\times(0.06)+16\times(0.08)+25\times(0.04)$$

$$= 0 + 0.21 + 2.24 + 0.54 + 1.28 + 1.00 = 5.27$$

$$\sigma^2 = E[X^2] - (E[X])^2 = 5.27 - (2.03)^2 = 1.15.$$

Example 9.5.6. Bangladesh Bank has six tellers available to serve customers. The number of tellers busy with customers at peak time, say, 2:00 p.m. varies from day to day. So it is a random variable denoted as X. It is known from the past records that the probability distribution of X is as follows:

Values of X : x	0	1	2	3	4	5	6
p(x)	0.03	0.05	0.08	0.15	0.21	0.26	0.22

a) Find the mean number of tellers busy with the customers at 2:00 p.m.

b) Also find the variance and standard deviation of the number of tellers busy with the customers at 2:00 p.m.

Solution. (a) Mean $= \mu = E[X] = \Sigma x p(x)$

$$= 0(0.03)+1(0.05)+2(0.08)+3(0.15)+4(0.21)+5(0.26)+6(0.22)$$

$$= 0 + 0.05 + 0.16 + 0.45 + 0.84 + 1.30 + 1.32 = 4.12$$

Hence the mean number of tellers busy with the customer is approximately 4.

(b) Variance $= \sigma^2 = E[X^2] - (E[X])^2$

$$E[X^2] = 0\times(0.03)+1\times(0.05)+4\times(0.08)+9\times(0.15)+16\times(0.21)+25\times(0.26)+36\times(0.22)$$

$$= 0 + 0.05 + 0.32 + 0.54 + 1.35 + 3.36 + 6.50 + 7.92 = 20.04$$

$$\sigma^2 = E[X^2] - (E[X])^2 = 20.04 - (4.12)^2 = 3.07$$

Standard deviation $= \sigma = 1.75.$

Example 9.5.7. Suppose you are interested in insuring a car stereo system for Tk. 500 against theft. An insurance company charges a premium of Tk.60 for coverage for 1 year, claiming an empirically determined probability 0.1 that the stereo will be stolen some time during the year. What is your expected return from the insurance company if you take out this insurance?

Solution. This is actually a game of chance in which your stake is Tk.60. You have a 0.1 chance of receiving Tk.440 from the insurance company (Tk.500 minus your stake Tk.60) and a 0.9 chance of losing your stake of Tk.60. What is the expected value of the game? Here X be the amount payoff. The probability distribution of X is

Value of X : x	440	– 60
p(x)	0.1	0.9

The expected value of the game is

$$E(X) = (440)(0.1) + (-60)(0.9) = 44 - 54 = -10.$$

This means that if you insure with this company over many years and the circumstances remain the same, you would have an average net loss to the insurance company of Tk. 10 per year.

Matched problem

Insurance. The annual premium for a Tk. 5000 insurance policy against the theft of a painting is Tk.150. If the probability that the painting will be stolen during the year is .01, what is your expected return from the insurance company if you take out this insurance?

Pay off table

X : x	4850	–150
p(x)	0.01	0.99 ; E(X)= Tk. 100

9.5.4. Mean and variance of a continuous random variable. Suppose X is a continuous random variable with probability density function f(x), then the expected value or mean of X is defined as

$$\mu = E[X] = \int_{-\infty}^{\infty} f(x)\,dx\,; \quad -\infty < x < \infty$$

$$\text{Variance} = E[X - \mu]^2 = E[X^2] - \mu^2.$$

$$\text{Here } E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx\,.$$

Example 9.5.8. Suppose that in a certain region of a country the daily rainfall (in inches) is a continuous random variable X with probability density function f(x) given by $f(x) = \frac{3}{4}(2x - x^2),\ 0 < x < 2$. Find the expected daily rainfall (in inches) in that region. Also find variance and standard deviation.

Solution. The expected daily rainfall is

$$\mu = E[X] = \int_0^2 x\, f(x)\,dx$$

$$= \int_0^2 x \frac{3}{4}(2x - x^2)\,dx = \frac{3}{4}\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{3}{4} \times \frac{4}{3} = 1.$$

This means, on an average the daily rainfall of that region is one inch.

Variance $= \sigma^2 = E[X - \mu]^2 = E[X^2] - \mu^2$

$$E[X^2] = \frac{3}{4}\int_0^2 x^2(2x - x^2)\,dx$$

$$= \frac{3}{4}\left[2\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 = \frac{3}{4}\left(\frac{32}{4} - \frac{32}{5}\right) = \frac{3}{4} \times \frac{32}{20} = \frac{6}{5} = 1.2$$

Variance $= 1.2 - 1 = 0.2.$

Standard deviation $= \sigma = \sqrt{0.2} = 0.45$ inches.

Matched problem.

Let X be a random variable with probability function $f(x) = 2x;\ 0 < x < 1$. Find mean, variance and standard deviation of X.

Ans. Mean=2/3, variance=5/9 and $\sigma = 0.745$.

Questions

1. What is a random variable? Define a discrete random variable with examples.
2. Define a discrete random variable. Explain its probability function with an example.
3. Define a continuous random variable with example. Distinguish between probability function and probability density function.
4 Define discrete and continuous random variable. Cite one example for each.
5. Define a random variable. What is the mathematical expectation of a random variable?
6. Define variance and standard deviation of a random variable.

Exercise

7. The probability function of a discrete random variable X is as follows:

X:x	-5	-3	-1	1	3	5
p(x)	a	2a	3a	4a	5a	6a

(i) Find the value of a, (ii) Find P [X = 1], (iii) P[-1 < X < 3], (iv) P[- 1 ≤ X ≤ 3], (v) P[X > - 1], (vi) P[X ≥ 4], (vii) P[X ≤ 1].

8. Let X be a discrete random variable with the following probability function

Values of X : x	0	1	2	3	4
p(x)	0.12	0.18	k	0.30	0.16

(i) Find the value of k, Compute (ii) P [X > 3], (iii) P [1 < X < 4], (iv) P [X < 1].

9. Let X be a random variable with probability function given below;

x	0	1	2	3
p(x)	1/6	1/2	3/10	1/30

Find P [X ≤ 1]; P [X < 1] and P [0, X, 2]. Ans. 2/3, 5/6, 1/2.

10. A continuous random variable X has the following probability distribution: $f(x) = kx^2$; $0 \leq x \leq 1$. (i) Find the value of k, (ii) Calculate the probability that X lies between 0. 2 and 0.5, (iii) X less than 0.3, (iv) 1/4 < X < 1/2. Ans. (i) 0.3, (ii) 0.117, (iii) 0.027

11. Find mean, variance and standard deviation of the following probability function:

Values of X : x	0	1	2	3
p(x)	1/8	3/8	3/8	1/8

Ans. 1.5; 0.74; 0.86.

12. A random variable X can assume five values: 0, 1, 2, 3, 4, A portion of the probability distribution is shown here:

Values of X : x	0	1	2	3	4
p(x)	.1	.3	.3	?	.1

(i) Find p (3).
(ii) Calculate mean, variance and standard deviation
(iii) What is the probability that X is greater than 2?
(iv) What is the probability that X is 3 or less?

Ans. (i) 0.2; (ii) $\mu = 1.9$; $\sigma^2 = 1.29$; $\sigma = 1.136$; (iii) 0.3; (iv) 0.9

Application

13. A company has five applicants for two positions: two women and three men. Suppose that the five applicants are equally qualified and that no preference is given for choosing either gender. Let X be the number of women chosen to fill the two positions (i) Find the probability function of X. (ii) What is the probability that exactly 1 woman will be chosen? (iii) Also find the probability that at least one woman will be chosen.

Ans. p(0)= 3/10; p(1)=6/10; p(2)= 1/10. (ii) 6/10; (iii) 7/10.