□ Prove that a set of rational numbers $\mathbb{Q}$, equiped with two binary operations of addition and multiplication form a field.

Soln: A set F with two binary operation + and · is a field if the following conditions hold.

1. $(F, +)$ is an abelian Group.

2. $F \setminus \{0\}, \cdot\}$ is an abelian group.

3. Distributivity: $x \cdot (y+z) = x \cdot y + x \cdot z$ for al $x, y, z \in F$.

4. Finally $0 \neq 1$ must hold.

Let's us verify this properties for $\mathbb{Q}$:

Every rational number can be written as a/b with $a \in \mathbb{Z}$, $b \in \mathbb{Z} / \{0\}$.

Case 1: $(\mathbb{Q}, +)$ is an abelian Group.

- Closure under addition: if $x = a/b$ and $y = c/d$

  then, $x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

  - Here $ad + bc$ is an integer with $bd \neq 0$. Thus $x+y \in \mathbb{Q}$.

- Associativity: addition of rationals is associative because it follow from associativity of Integers addition.

- Additive Identity: 0 satisfies $x + 0 = x$ and $0 + x = x$ for x is rati

- Additive Inverse: for $x = a/b$, the additive inverse is $-x = -a/b$ which is rational and satisfies $x + (-x) = 0$.

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• Commutivity:

$$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$$

Hence $(Q, +)$ is an abelian group.

Case 2: $(Q \setminus \{0\})$ is an abblian Group:

• Closure under multiplication:

with $x = a/b$, $y = c/d$ $\quad x \cdot y = \frac{ac}{bd}$

Here ac and bd are intergers with bd ≠ 0 so the product is in Q. if neither x nor y is zero, then ac ≠ 0 so the product is non zero.

• Associativity: multiplication of rationals is associative.
• Multiplicative Identity: 1 satisfies $x \cdot 1 = 1 \cdot x = x \ \forall x \in Q$
• multipliative Inverse: For a non zero element $x = a/b$ with $a \neq 0$, the inverse is b/a which $\in Q$ and $\frac{a}{b} \cdot \frac{b}{a} = 1$.

• Commutativity: Intergers und multiplication is commutative

Thus $(Q \setminus \{0\})$ is an abelian group.

Case 3: Distributability: For rationals $x = \frac{a}{b}$, $y = \frac{c}{d}$, $z = \frac{e}{f}$

$$x(y+z) = \frac{a}{b} \cdot \frac{cf+de}{df} = \frac{acf+ade}{bdf} = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}.$$

$$= x \cdot y + x \cdot z.$$

~~Thus Q are~~ Thus Q is distributive.

Case 4: $0 \neq 1$.

In $\mathbb{Q}$, 0 is $0/1$ and 1 is $1/1$. These are different rationals, so $0 \neq 1$. This prevents the degenerate one element ring.

All field axioms hold for $\mathbb{Q}$: $(\mathbb{Q}, +)$ is an abelian Group, $(\mathbb{Q} \setminus \{0\})$ is an abelian group, multiplication distributes over addition and $0 \neq 1$. Therefore $\mathbb{Q}$ under addition and multiplication is a field.