Is the set of odd numbers with binary operations $(+)$ i.e. $(O,+)$ an abelian group? If not explain with necessary notation.

Soln:

let O be the set of odd numbers. $O = \{n \in \mathbb{Z} .$ where $n = 2k+1\}$.

Let's check against the axioms for an abelian group check if it's really an abelian group.

1. Closure: for any odd numers $a, b \in O$, $a+b \in O$ must hold.
   but odd + odd = even so $a+b \notin O$. so it breaks.

2. Associativity: Addition of odd Integers must behave like $(a+b)+c = a+(b+c)$
   this holds since it holds for $n \in \mathbb{Z}$ and $O \in \mathbb{Z}$

3. Identity element: The additive identity is 0. but 0 is even.
   so there is no identity element.

4. Inverse element: The additive invese of odd numbers exist. but due to the lack of identity element, it can't satisfy group axiom.

5. Commutivity: It will hold as $n \in \mathbb{Z}$ and $O \in \mathbb{Z}$ and it holds for $n \in \mathbb{Z}$.

As the result of the analysis, we can safely say $(O,+)$ is not an abelian group. It fails in axiom closure and Identity element.