APEX Section 5.3 Changes

Note: throughout, a positive line number refers to a line that far from the top of the page, a negative line number refers to a line that far from the bottom of the page.

Text

1. There should be a paragraph break between lines 14 and 15 on page 211.

2. Mimicking what the text does for integral notation on page 190, we should insert the following after line 4 on page 212:

$$\sum_{i=1}^{9} a_i$$

   Lets analyze this notation.

3. Insert the following note after Theorem 37:

   Note: In practice we will sometimes need variations on formulas 5, 6, and 7 above. For example, we note that

$$\sum_{i=0}^{n} i = 0 + 1 + 2 + \cdots + n = 0 + \sum_{i=1}^{n} i = 0 + \frac{n(n+1)}{2} = \frac{n(n+1)}{2},$$

   so we see that

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

   Similarly, we find that

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \textit{and}$$
$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

4. The solution to Example 120 (page 213) should include justifications, i.e.

$$\begin{aligned}
\sum_{i=1}^{6}(2i-1) &= \sum_{i=1}^{6} 2i - \sum_{i=1}^{6}(1) \quad \text{(Theorem 37(2))}\\
&= \left(2\sum_{i=1}^{6} i\right) - \sum_{i=1}^{6}(1) \quad \text{(Theorem 37(3))}\\
&= 2\left(\frac{6(6+1)}{2}\right) - 6 \quad \text{(Theorem 37(1,5))}\\
&= 2(21) - 6 = 36
\end{aligned}$$

5. Reindexing Riemann sums.

   (a) The subscripts in Figure 5.17 should each be reduced by 1.

   (b) Beginning with the second paragraph after the subsection heading **Riemann Sums**, change the text to read as follows:

   Figure 5.17 shows a number line of $[0,4]$ *subdivided into 16 equally spaced subintervals. We denote* 0 *as* x_0*; we have marked the values of* x_4*,* x_8*,* x_{12}*, and* x_{16}*. We could mark them all, but the figure would get crowded. While it is easy to figure that* $x_9 = 2.25$*, in general, we want a method of determining the value of* x_i *without consulting the figure. Consider:*

   (c) In the displayed equation immediately following the above text, x_1 should be replaced with x_0 in both instances.

   (d) Line -5 on page 214 should read *So* $x_9 = x_0 + 9(4/16) = 9/4 = 2.25$*.*

   (e) In lines -3 & -2 on page 214, change to read *We could compute* x_{31} *as*

   $$x_{31} = x_0 + 31(4/100) = 124/100 = 1.24.$$

   (f) At the top of page 215, it should read *Given any subdivision of* $[0,4]$*, the first subinterval is* $[x_0, x_1]$*; the second is* $[x_1, x_2]$*; the* i^{th} *subinterval is* $[x_{i-1}, x_i]$*.*
   When using the Left Hand Rule, the height of the i^{th} *rectangle will be* $f(x_{i-1})$*.*
   When using the Right Hand Rule, the height of the i^{th} *rectangle will be* $f(x_i)$*.*
   When using the Midpoint Rule, the height of the i^{th} *rectangle will be* $f\left(\frac{x_{i-1}+x_i}{2}\right)$*.*

   (g) The indexing on the Left Hand, Right Hand, and Midpoint Rules at the top of page 215 should be changed so that the sums read $\sum\limits_{i=0}^{15}$

   (h) I would strongly suggest that we change Example 121 so that it uses the Left Hand Rule instead of the Right Hand Rule. After we change the indexing of the sums, the Right Hand Rule seems to be more complicated in an unhelpful manner. Here is the replacement text, beginning with the paragraph before Example 121 on page 215. Note that this replacement also corrects several typos.

   We use these formulas in the next two examples. The following example lets us practice using the Left Hand Rule and the summation formulas introduced in Theorem 37.

   Example 121 Approximating definite integrals using sums

Approximate $\int_0^4 (4x - x^2)\,dx$ using the Left Hand Rule and summation formulas with 16 and 1000 equally spaced intervals.

Solution *Using the formula derived before, using 16 equally spaced intervals and the Left Hand Rule, we can approximate the definite integral as*

$$\sum_{i=0}^{1} 5 f(x_i)\Delta x.$$

We have $\Delta x = 4/16 = 0.25$*,* $x_i = 0 + i\Delta x = i\Delta x$*, and* $f(x_i) = f(i\Delta x) = 4i\Delta x - i^2\Delta x^2$*. Using the summation formulas, we see:*

$$\begin{aligned}
\int_0^4 (4x - x^2)\,dx &\approx \sum_{i=0}^{15} f(x_i)\Delta x \\
&= \sum_{i=0}^{15} f(i\Delta x)\Delta x \\
&= \sum_{i=0}^{15} (4i\Delta x - i^2\Delta x^2)\Delta x \qquad \textit{(from our work above)} \\
&= \sum_{i=0}^{15} (4i\Delta x^2 - i^2\Delta x^3) \\
&= \sum_{i=0}^{15} 4i\Delta x^2 - \sum_{i=0}^{15} i^2\Delta x^3 \qquad \textit{(Theorem 37(2))} \\
&= (4\Delta x^2)\sum_{i=0}^{15} i - (\Delta x^3)\sum_{i=0}^{15} i^2 \qquad \textit{(Theorem 37(3))} \\
&= 4(1/4)^2\left(\frac{(15)(16)}{2}\right) - (1/4)^3\left(\frac{(15)(16)(31)}{6}\right) \qquad \textit{(Theorem 37(5,6))} \\
&= 30 - \frac{155}{8} = \frac{85}{8} = 10.625
\end{aligned}$$

We were able to sum up the areas of 16 rectangles with very little computation. In Figure 5.18 the function and the 16 rectangles are graphed. While some rectangles over-approximate the area, others under-approximate the area by about the same amount. Thus our approximate area of 10.625 is likely a fairly good approximation.

Notice Equation (5.3); by changing the 15's to 999's and changing the value of Δx *to* $4/1000 = 0.004$*, we can use the equation to sum up the areas of 1000 rectangles. We do so here, skipping from the original summand to the equaivalent of Equation (5.3) to save space.*

$$\begin{aligned}
\int_0^4 (4x - x^2)\,dx &\approx \sum_{i=0}^{999} f(x_i)\Delta x \\
&= (4\Delta x)^2\sum_{i=0}^{999} i - (\Delta x^3)\sum_{i=0}^{999} i^2 \\
&= 4(.004)^2\left(\frac{(999)(1000)}{2}\right) - (0.004)^3\left(\frac{(999)(1000)(1999)}{6}\right) \\
&= 10.666656
\end{aligned}$$

Using many, many rectangles, we likely have a good approximation of $\int_0^4 (4x - x^2)\,dx$. *That is,*

$$\int_0^4 (4x - x^2)\,dx \approx 10.666656.$$

Note that we also have to change Figure 5.18 for this to work.

6. Typo: on line 15 of page 219: *When the partition size is small* should be *When* Δx *is small*.

7. On page 222, replace the first sentence under the heading **Limits of Riemann Sums** with the following:

   We have used limits to find the exact value of certain definite integrals.

8. Insert the following after Theorem 38 on page 224:

   Now that we have more tools to work with, we can justify the remaining properties in Theorem 36.

   Theorem 36   Properties of the Definite Integral

   Let f and g be continuous on a closed interval I that contains the values a, b, and c, and let k, m, and M be constants. The following hold:

   (a) $\displaystyle\int_a^a f(x)\,dx = 0$

   (b) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

   (c) $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

   (d) $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

   (e) $\displaystyle\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$

   (a) *To see why this property holds note that for any Riemann sum we have $\Delta x = 0$, from which we see that:*

   $$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty}\sum_{i=0}^{n} f(c_i)\Delta x = 0 \quad \textit{(by Theorem 38(2))}\\ &= \lim_{n\to\infty} 0\\ &= 0\end{aligned}$$

   (b) *Applying Theorem 38(2), we have:*

   $$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=0}^{n} f(c_i)\Delta x.$$

   When we compute $\int_b^a f(x)\,dx$, we can use the same partitions and the same points c_i, so the heights $f(c_i)$ will remain the same. Since we want to start at $x = b$ and finish at

$x = a$, we use $\widetilde{\Delta}x = \frac{a-b}{n} = -\Delta x$. We now have:

$$\begin{aligned}
\int_b^a f(x)\,dx &= \lim_{n\to\infty} \sum_{i=0}^{n} f(c_i)\widetilde{\Delta}x \quad \text{(Theorem 38(2))}\\
&= \lim_{n\to\infty} \sum_{i=0}^{n} f(c_i)(-\Delta x)\\
&= \lim_{n\to\infty} -\left(\sum_{i=0}^{n} f(c_i)\Delta x\right) \quad \text{(using Theorem 37(3))}\\
&= -\lim_{n\to\infty} \sum_{i=0}^{n} f(c_i)\Delta x\\
&= -\int_a^b f(x)\,dx \quad \text{(Theorem 38(2))}
\end{aligned}$$

(c) This property was justified previously.

(d) To see why this property holds, we again use Theorems 37 and 38. In this case we have:

$$\begin{aligned}
\int_a^b (f(x)+g(x))\,dx &= \lim_{n\to\infty} (f(c_i)+g(c_i))\Delta x\\
&= \lim_{n\to\infty} \sum_{i=0}^{n} (f(c_i)\Delta x + g(c_i)\Delta x)\\
&= \lim_{n\to\infty} \left(\sum_{i=0}^{n} f(c_i)\Delta x + \sum_{i=0}^{n} g(c_i)\Delta x\right)\\
&= \lim_{n\to\infty} \sum_{i=0}^{n} f(c_i)\Delta x + \lim_{n\to\infty} \sum_{i=0}^{n} g(c_i)\Delta x\\
&= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx
\end{aligned}$$

(e) The justification of this property is left as an exercise.

9. Insert the following result:

Theorem 39 Further Properties of the Definite Integral Let f be continuous on the interval $[a,b]$ and let k, m, and M be constants. The following hold:

(a) $\displaystyle\int_a^b k\,dx = k(b-a)$.

(b) If $m \leq f(x)$ for all x in $[a,b]$, then $m(b-a) \leq \displaystyle\int_a^b f(x)\,dx$.

(c) If $f(x) \leq M$ for all x in $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx \leq M(b-a)$.

Before justifying these properties, note that for any subdivision of $[a,b]$ we have:

$$\sum_{i=0}^{n} \Delta x = n\frac{b-a}{n} = b-a.$$

To see why (a) holds, let k be a constant. We apply Theorem 38 to see that:

$$\begin{aligned}
\int_a^b k\,dx &= \lim_{n\to\infty}\sum_{i=0}^{n} k\Delta x \\
&= \lim_{n\to\infty} k\left(\sum_{i=0}^{n}\Delta x\right) \qquad \textit{(using Theorem 37)} \\
&= k\left(\lim_{n\to\infty}\sum_{i=0}^{n}\Delta x\right) \\
&= k\left(\lim_{n\to\infty}(b-a)\right) \\
&= k(b-a)
\end{aligned}$$

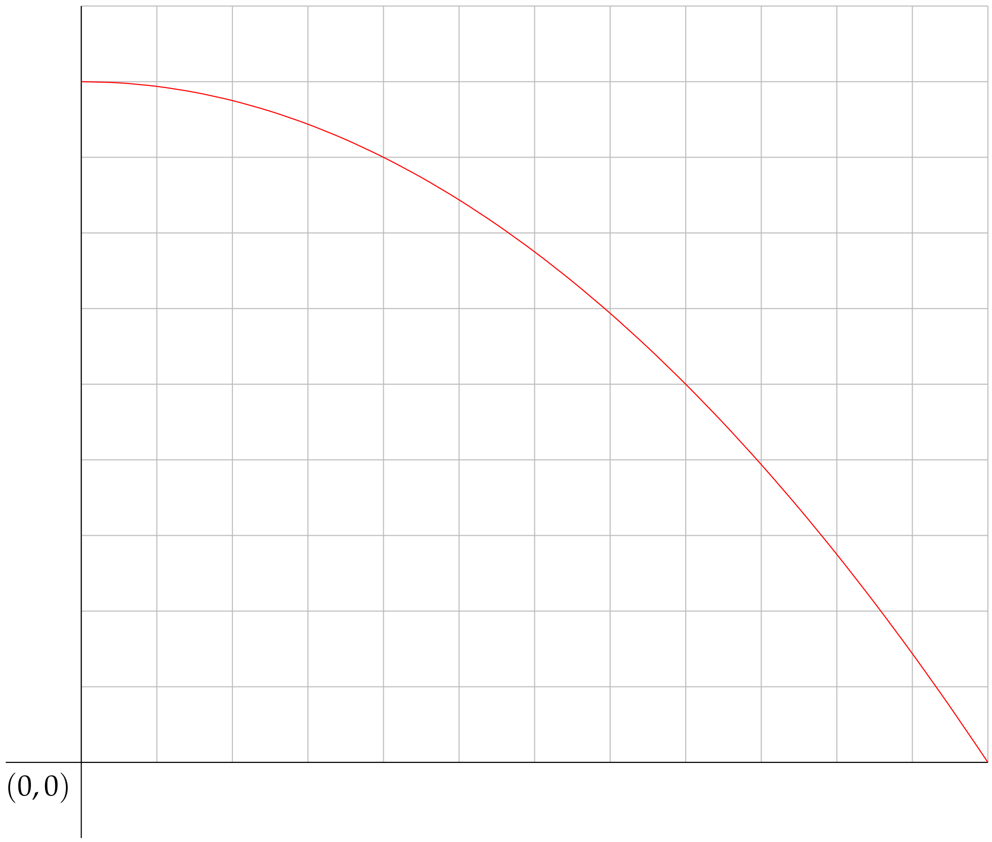
We can now use this property to see why (b) holds. Let f and m be as given. Then we have:

$$\begin{aligned}
m(b-a) &= \int_a^b m\,dx \\
&= \lim_{n\to\infty}\sum_{i=0}^{n} m\Delta x \quad \textit{(Theorem 38)} \\
&\le \lim_{n\to\infty}\sum_{i=0}^{n} f(c_i)\Delta x \\
&= \int_a^b f(x)\,dx \quad \textit{(Theorem 38)}
\end{aligned}$$

Justifying property (c) is similar and is left as an exercise.

Problems

1. Replace current problem 30 with $\int_1^3 \sqrt{10-x^2}\,dx$, 4 rectangles, right hand rule.
2. Add the following problems:
   - Use six rectangles to approximate the area under the given graph of f from $x = 0$ to $x = 12$, using:
     (a) The Left Hand Rule,
     (b) The Right Hand Rule,
     (c) The Midpoint Rule.



- A car accelerates from 0 to 40 mph in 30 seconds. The speedometer reading at each 5 second interval during this time is given in the table below. Estimate how far the car travels during this 30 second period using the velocities at:
  (a) The beginning of each time interval.
  (b) The end of each time interval.

t (sec)	0	5	10	15	20	25	30
v (mph)	0	6	14	23	30	36	40

- Use Theorems 37 and 38 to justify the remaining property in Theorem 36:

$$\int_a^b k \cdot f(x)\,dx = k \int_a^b f(x)\,dx$$

- Use Theorems 37 and 38 to justify the remaining property in Theorem 39: If $f(x) \leq M$ for all x in $[a, b]$, then

$$\int_a^b f(x)\,dx \leq M(b-a).$$