2.6 Implicit Differentiation

In the previous sections we learned to find the derivative, $\frac{dy}{dx}$, or y', when y is given *explicitly* as a function of x. That is, if we know $y = f(x)$ for some function f, we can find y'. For example, given $y = 3x^2 - 7$, we can easily find $y' = 6x$. (Here we explicitly state how x and y are related. Knowing x, we can directly find y.)

Sometimes the relationship between y and x is not explicit; rather, it is *implicit*. For instance, we might know that $x^2 - y = 4$. This equality defines a relationship between x and y; if we know x, we could figure out y. Can we still find y'? In this case, sure; we solve for y to get $y = x^2 - 4$ (hence we now know y explicitly) and then differentiate to get $y' = 2x$.

Sometimes the *implicit* relationship between x and y is complicated. Suppose we are given $\sin(y) + y^3 = 6 - x^3$. A graph of this ~~implicit~~ equation is given in Figure 2.19. In this case there is absolutely no way to solve for y in terms of elementary functions. The surprising thing is, however, that we can still find y' via a process known as **implicit differentiation**.

Implicit differentiation is a technique based on the Chain Rule that is used to find a derivative when the relationship between the variables is given implicitly rather than explicitly (solved for one variable in terms of the other).

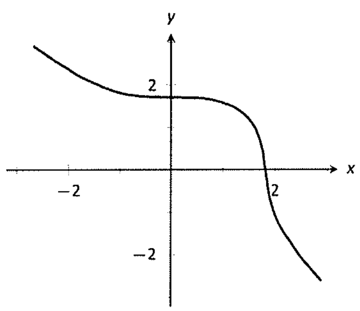

Figure 2.19: A graph of the ~~implicit function~~ $\sin(y) + y^3 = 6 - x^3$. equation

rethink equation vs function

We begin by reviewing the Chain Rule. Let f and g be functions of x. Then

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x).$$

Suppose now that $y = g(x)$. We can rewrite the above as

$$\frac{d}{dx}\Big(f(y)\Big) = f'(y) \cdot y', \quad \text{or} \quad \frac{d}{dx}\Big(f(y)\Big) = f'(y) \cdot \frac{dy}{dx}. \tag{2.1}$$

These equations look strange; the key concept to learn here is that we can find y' even if we don't exactly know how y and x relate.

Watch the video:
Implicit Differentiation — More Examples at
`https://youtu.be/Wn4aVYW4kFk`

Change video to ~~Patrick JMT~~ ~~Implicit Diff~~ ~~for Calc - More examples #1~~

We demonstrate this process in the following example.

Notes:

Khan Academy Implicit Differentiation

see tex file for address

Example 2.6.1 Using Implicit Differentiation

Find y' given that $\sin(y) + y^3 = 6 - x^3$.

SOLUTION We start by taking the derivative of both sides (thus maintaining the equality). We have:

$$\frac{d}{dx}\Big(\sin(y) + y^3\Big) = \frac{d}{dx}\Big(6 - x^3\Big).$$

The right hand side is easy; it returns $-3x^2$.

The left hand side requires more consideration. We take the derivative term–by–term. Using the technique derived from Equation 2.1 above, we can see that

$$\frac{d}{dx}\Big(\sin y\Big) = \cos y \cdot y'.$$

We apply the same process to the y^3 term.

$$\frac{d}{dx}\Big(y^3\Big) = \frac{d}{dx}\Big((y)^3\Big) = 3(y)^2 \cdot y'.$$

Putting this together with the right hand side, we have

$$\cos(y)y' + 3y^2y' = -3x^2.$$

Now solve for y'.

$$\begin{aligned} \cos(y)y' + 3y^2y' &= -3x^2. \\ \big(\cos y + 3y^2\big)y' &= -3x^2 \\ y' &= \frac{-3x^2}{\cos y + 3y^2} \end{aligned}$$

This equation for y' probably seems unusual for it contains both x and y terms. How is it to be used? We'll address that next.

Implicit functions ~~equations~~ are generally harder to deal with than explicit functions. With an explicit function, given an x value, we have an explicit ~~formula for com-~~ function puting the corresponding y value. With an implicit ~~equation~~, one often has to find x and y values *at the same time* that satisfy the equation. It is much easier to demonstrate that a given point satisfies the equation than to actually find such a point.

For instance, we can affirm easily that the point $(\sqrt[3]{6}, 0)$ lies on the graph of the ~~implicit~~ equation $\sin y + y^3 = 6 - x^3$. Plugging in 0 for y, we see the left hand side is 0. Setting $x = \sqrt[3]{6}$, we see the right hand side is also 0; the equation is satisfied. The following example finds an equation of the tangent line to this equation at this point.

Notes:

Example 2.6.2 Using Implicit Differentiation to find a tangent line
Find the equation of the line tangent to the implicitly defined curve $\sin y + y^3 = 6 - x^3$ at the point $(\sqrt[3]{6}, 0)$.

SOLUTION In Example 2.6.1 we found that

$$y' = \frac{-3x^2}{\cos y + 3y^2}.$$

We find the slope of the tangent line at the point $(\sqrt[3]{6}, 0)$ by substituting $\sqrt[3]{6}$ for x and 0 for y. Thus at the point $(\sqrt[3]{6}, 0)$, we have the slope as

$$y' = \frac{-3(\sqrt[3]{6})^2}{\cos 0 + 3 \cdot 0^2} = \frac{-3\sqrt[3]{36}}{1} \approx -9.91.$$

Therefore an equation of the tangent line to the implicitly defined curve $\sin y + y^3 = 6 - x^3$ at the point $(\sqrt[3]{6}, 0)$ is

$$y = -3\sqrt[3]{36}(x - \sqrt[3]{6}) + 0 \approx -9.91x + 18.$$

The curve and this tangent line are shown in Figure 2.20.

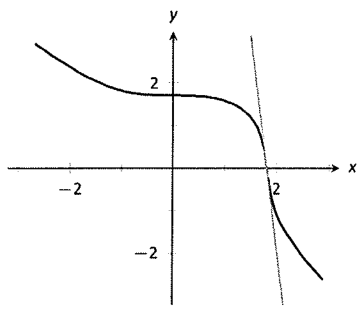

Figure 2.20: The function $\sin y + y^3 = 6 - x^3$ and its tangent line at the point $(\sqrt[3]{6}, 0)$.

This suggests a general method for implicit differentiation. For the steps below assume y is a function of x.

1. Take the derivative of each term in the equation. Treat the x terms like normal. When taking the derivatives of y terms, the usual rules apply except that, because of the Chain Rule, we need to multiply each term by y'.
2. Get all the y' terms on one side of the equal sign and put the remaining terms on the other side.
3. Factor out y'; solve for y' by dividing.

Practical Note: When working by hand, it may be beneficial to use the symbol $\frac{dy}{dx}$ instead of y', as the latter can be easily confused for y or y^1.

Example 2.6.3 Using Implicit Differentiation
Given the implicitly defined function $y^3 + x^2y^4 = 1 + 2x$, find y'.

SOLUTION We will take the implicit derivatives term by term. Using the Chain Rule the derivative of y^3 is $3y^2y'$.

Notes:

The second term, x^2y^4 is a little more work. It requires the Product Rule as it is the product of two functions of x: x^2 and y^4. We see that $\frac{d}{dx}(x^2y^4)$ is

$$\begin{gathered} x^2 \cdot \frac{d}{dx}(y^4) + \frac{d}{dx}(x^2) \cdot y^4 \\ x^2 \cdot (4y^3y') + 2x \cdot y^4 \end{gathered}$$

check prime

The first part of this expression requires a y' because we are taking the derivative of a y term. The second part does not require it because we are taking the derivative of x^2.

The derivative of the right hand side of the equation is found to be 2. In all, we get:

$$3y^2y' + 4x^2y^3y' + 2xy^4 = 2.$$

Move terms around so that the left side consists only of the y' terms and the right side consists of all the other terms:

$$3y^2y' + 4x^2y^3y' = 2 - 2xy^4.$$

Factor out y' from the left side and solve to get

$$y' = \frac{2 - 2xy^4}{3y^2 + 4x^2y^3}.$$

To confirm the validity of our work, let's find the equation of a tangent line to this curve at a point. It is easy to confirm that the point $(0,1)$ lies on the graph of this curve. At this point, $y' = 2/3$. So the equation of the tangent line is $y = 2/3(x - 0) + 1$. The equation and its tangent line are graphed in Figure 2.21.

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Notice how our curve looks much different than other functions we have worked with up to this point. Such curves are important in many areas of mathematics, so developing tools to deal with them is also important.

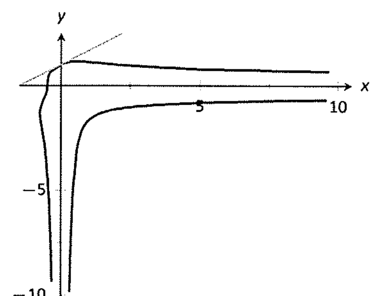

Figure 2.21: A graph of the ~~implicitly defined curve~~ equation $y^3 + x^2y^4 = 1 + 2x$ along with its tangent line at the point $(0,1)$.

Example 2.6.4 Using Implicit Differentiation

Given the implicitly defined curve $\sin(x^2y^2) + y^3 = x + y$, find y'.

Solution Differentiating term by term, we find the most difficulty in the first term. It requires both the Chain and Product Rules.

$$\begin{aligned} \frac{d}{dx}\Big(\sin(x^2y^2)\Big) &= \cos(x^2y^2) \cdot \frac{d}{dx}\Big(x^2y^2\Big) \\ &= \cos(x^2y^2) \cdot \big(x^2(2yy') + 2xy^2\big) \\ &= 2(x^2yy' + xy^2)\cos(x^2y^2). \end{aligned}$$

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Notes:

We leave the derivatives of the other terms to the reader. After taking the derivatives of both sides, we have

$$2(x^2yy' + xy^2)\cos(x^2y^2) + 3y^2y' = 1 + y'.$$

We now have to be careful to properly solve for y', particularly because of the product on the left. It is best to multiply out the product. Doing this, we get

$$2x^2y\cos(x^2y^2)y' + 2xy^2\cos(x^2y^2) + 3y^2y' = 1 + y'.$$

From here we can safely move around terms to get the following:

$$2x^2y\cos(x^2y^2)y' + 3y^2y' - y' = 1 - 2xy^2\cos(x^2y^2).$$

Then we can solve for y' to get

$$y' = \frac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2) + 3y^2 - 1}.$$

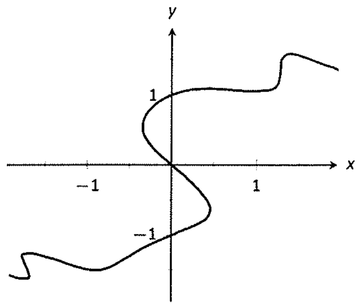
A graph of this implicit equation is given in Figure 2.22(a). It is easy to verify that the points $(0,0)$, $(0,1)$ and $(0,-1)$ all lie on the graph. We can find the slopes of the tangent lines at each of these points using our formula for y'.

At $(0,0)$, the slope is -1.

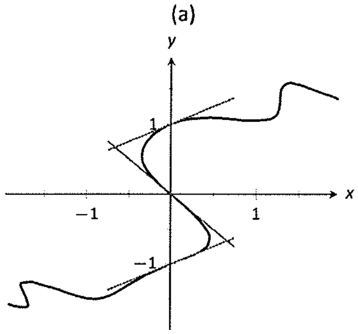
At $(0,1)$, the slope is $1/2$.

At $(0,-1)$, the slope is also $1/2$.

The tangent lines have been added to the graph of the function in Figure 2.22(b).



(a)



(b)

Figure 2.22: A graph of the ~~implicitly defined curve~~ equation $\sin(x^2y^2) + y^3 = x + y$ and certain tangent lines.

Quite a few "famous" curves have equations that are given implicitly. We can use implicit differentiation to find the slope at various points on those curves. We investigate two such curves in the next examples.

Example 2.6.5 Finding slopes of tangent lines to a circle

Find the slope of the tangent line to the circle $x^2+y^2 = 1$ at the point $(1/2, \sqrt{3}/2)$.

SOLUTION Taking derivatives, we get $2x+2yy' = 0$. Solving for y' gives:

$$y' = \frac{-x}{y}.$$

This is a clever formula. Recall that the slope of the line through the origin and the point (x, y) on the circle will be y/x. We have found that the slope of the tangent line to the circle at that point is the opposite reciprocal of y/x, namely, $-x/y$. Hence these two lines are always perpendicular.

Notes:

At the point $(1/2, \sqrt{3}/2)$, we have the tangent line's slope as

$$y' = \frac{-1/2}{\sqrt{3}/2} = \frac{-1}{\sqrt{3}} \approx -0.577.$$

A graph of the circle and its tangent line at $(1/2, \sqrt{3}/2)$ is given in Figure 2.23, along with a thin dashed line from the origin that is perpendicular to the tangent line. (It turns out that all normal lines to a circle pass through the center of the circle.)

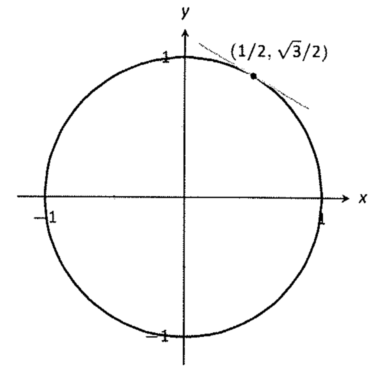


Figure 2.23: The unit circle with its tangent line at $(1/2, \sqrt{3}/2)$.

This section has shown how to find the derivatives of implicitly defined functions curves, whose graphs include a wide variety of interesting and unusual shapes. Implicit differentiation can also be used to further our understanding of "regular" differentiation.

Implicit Differentiation and the Second Derivative

We can use implicit differentiation to find higher order derivatives. In theory, this is simple: first find $\frac{dy}{dx}$, then take its derivative with respect to x. In practice, it is not hard, but it often requires a bit of algebra. We demonstrate this in an example.

Example 2.6.6 **Finding the second derivative**

Given $x^2 + y^2 = 1$, find $\dfrac{d^2y}{dx^2} = y''$.

SOLUTION We found that $y' = \frac{dy}{dx} = -x/y$ in Example 2.6.5. To find y'',

Notes:

we apply implicit differentiation to y'.

$$\begin{aligned}
y'' &= \frac{d}{dx}(y') \\
&= \frac{d}{dx}\left(-\frac{x}{y}\right) && \text{now use the Quotient Rule} \\
&= -\frac{y(1) - x(y')}{y^2} && \text{replace } y' \text{ with } -x/y \\
&= -\frac{y - x(-x/y)}{y^2} \\
&= -\frac{y + x^2/y}{y^2} \\
&= -\frac{y + x^2/y}{y^2} \cdot \frac{y}{y} \\
&= -\frac{y^2 + x^2}{y^3}, && \text{since we were given } x^2 + y^2 = 1 \\
&= -\frac{1}{y^3}.
\end{aligned}$$

We can see that $y'' > 0$ when $y < 0$ and $y'' < 0$ when $y > 0$. In section 3.4, we will see how this relates to the shape of the graph.

Implicit differentiation proves to be useful as it allows us to find the instantaneous rates of change of a variety of functions.

In this chapter we have defined the derivative, given rules to facilitate its computation, and given the derivatives of a number of standard functions. We restate the most important of these in the following theorem, intended to be a reference for further work.

Notes:

Theorem 21 Glossary of Derivatives of Elementary Functions
Let u and v be differentiable functions, and let c and n be real numbers, $n \neq 0$.

1. $\frac{d}{dx}\big(cu\big) = cu'$	2. $\frac{d}{dx}\big(u \pm v\big) = u' \pm v'$
3. $\frac{d}{dx}\big(u \cdot v\big) = uv' + u'v$	4. $\frac{d}{dx}\big(\frac{u}{v}\big) = \frac{u'v - uv'}{v^2}$
5. $\frac{d}{dx}\big(u(v)\big) = u'(v)v'$	6. $\frac{d}{dx}\big(e^x\big) = e^x$
7. $\frac{d}{dx}\big(c\big) = 0$	8. $\frac{d}{dx}\big(x^n\big) = nx^{n-1}$
9. $\frac{d}{dx}\big(\sin x\big) = \cos x$	10. $\frac{d}{dx}\big(\cos x\big) = -\sin x$
11. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$	12. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$
13. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$	14. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$

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Notes: