Chapter 2 Prerequisite Topics

Rules of Exponents

The following table summarizes the laws of exponents and equivalent forms of exponent expressions commonly used in this chapter. Let m and n be any real numbers and let x, y and z take on any values for which the expression is defined.

Laws of Exponents	Examples
Products: $x^m \cdot x^n = x^{m+n}$	$x^5 \cdot x^7 = x^{5+7} = x^{12}$ $x^{-3} \cdot x^{-4} = x^{-3+(-4)} = x^{-7} = \dfrac{1}{x^7}$ $x^{-\frac{1}{2}} \cdot x^{\frac{2}{3}} = x^{-\frac{1}{2}+\frac{2}{3}} = x^{\frac{1}{6}} = \sqrt[6]{x}$
Quotients: $\dfrac{x^m}{x^n} = x^{m-n}$	$\frac{x^5}{x^7} = x^{5-7} = x^{-2} = \frac{1}{x^2}$ $\dfrac{x^{-3}}{x^{-4}} = x^{-3-(-4)} = x^1 = x$ $\dfrac{x^{\frac{2}{3}}}{x^{-\frac{1}{2}}} = x^{\frac{2}{3}-(-\frac{1}{2})} = x^{\frac{7}{6}} = \sqrt[6]{x^7} = x\sqrt[6]{x}$
Power raised to a power: $(x^m)^n = x^{m \cdot n}$	$(x^5)^7 = x^{5 \cdot 7} = x^{35}$ $x^{-3} \cdot x^4 = x^{-3 \cdot 4} = x^{-12} = \dfrac{1}{x^{12}}$ $x^{-\frac{1}{2}} \cdot x^{\frac{2}{3}} = x^{-\frac{1}{2} \cdot \frac{2}{3}} = x^{-\frac{1}{3}} = \dfrac{1}{\sqrt[3]{x}}$
Product and quotient raised to a power: $(xy)^m = x^m y^m$ and $\left(\dfrac{x}{y}\right)^m = \dfrac{x^m}{y^m}$	$(xyz)^7 = x^7 y^7 z^7$ $\left(\dfrac{x}{y}\right)^{-4} = \dfrac{x^{-4}}{y^{-4}} = \dfrac{y^4}{x^4}$

Factoring and Simplifying Complex Fractions

The following examples demonstrate an efficient factoring technique that can be used to create the various equivalent expressions often needed to complete problems that arise in Calculus. The ability to move flexibly and efficiently among different representations of an expression is an important skill to have.

Example 1: Factoring out the lowest power of the common factor among the terms
Factor completely to write an equivalent expression:

1. $x^{\frac{7}{3}} - 5x^{\frac{2}{3}}$

   Solution:

$$x^{\frac{7}{3}} - 4x^{\frac{2}{3}} = x^{\frac{2}{3}}(x^{\frac{5}{3}} - 4)$$

Note: $\sqrt[3]{x^2}(\sqrt[3]{x^5}-4)$ is also equivalent to this expression.

2. $\dfrac{1}{2}x(x-3)^{-\frac{2}{5}}+(x-3)^{\frac{3}{5}}$

$$\begin{aligned}
\frac{1}{2}x(x-3)^{-\frac{2}{5}}+(x-3)^{\frac{3}{5}} &= \frac{1}{2}(x-3)^{-\frac{2}{5}}\big(1+2(x-3)\big)\\
&= \frac{1}{2}(x-3)^{-\frac{2}{5}}(1+2x-6)\\
&= \frac{1}{2}(x-3)^{-\frac{2}{5}}(2x-5)\\
&= \frac{2x-5}{2(x-3)^{\frac{2}{5}}}
\end{aligned}$$

Note: $\dfrac{2x-5}{2\sqrt[5]{(x-3)^2}}$ is also equivalent to this expression.

Example 2: Simplifying complex fractions Factoring out the lowest power of the common factor can also be used to simplify complex fractions.

$$\begin{aligned}
\frac{\frac{2}{3}x(x-2)^{-\frac{1}{3}}+(x-2)^{\frac{2}{3}}}{x^2} &= \frac{\frac{1}{3}(x-2)^{-\frac{1}{3}}\big(2x+3(x-2)\big)}{x^2}\\
&= \frac{2x+3x-6}{3x^2(x-2)^{\frac{1}{3}}}\\
&= \frac{5x-6}{3x^2\sqrt[3]{x-2}}
\end{aligned}$$

Function Composition

Function composition refers to combining functions in a way that the output from one function becomes the input for the next function. In other words, the range (y-values) of one function become the domain (x-values) of the next function. We deonte this as $(f\circ g)(x)=f(g(x))$, where the output of $g(x)$ becomes the input of $f(x)$

Example 3: Composition of two functions
Given $f(x)=\dfrac{1}{x^2}$ and $g(x)=\sqrt{x+4}$, find $(f\circ g)(x)$ and $(g\circ f)(x)$.

To find $(f\circ g)(x)=f(g(x))$, we substitute the function $g(x)$ into the function $f(x)$. Thus,

$$f(g(x))=f\big(\sqrt{x+4}\big)=\frac{1}{(\sqrt{x+4})^2}=\frac{1}{x+4}.$$

For $(g\circ f)(x)=g(f(x))$, we substitute the function $f(x)$ into the function $g(x)$. Thus,

$$g(f(x)) = g\Big(\frac{1}{x^2}\Big) = \sqrt{\frac{1}{x^2}+4} = \sqrt{\frac{1+4x^2}{x^2}}.$$

Example 4: Composition of three functions
Given $f(x) = x^2$, $g(x) = \sqrt{4-x}$ and $h(x) = 3x-5$, find $(f \circ g \circ h)(x)$ and $(g \circ f \circ h)(x)$.

To find $(f \circ g \circ h)(x)$ we must start with the inside and work our way out.

$$\begin{aligned}
(f \circ g \circ h)(x) &= f(g(h(x))) \\
&= f(g(3x-5)) \\
&= f\Big(\sqrt{4-(3x-5)}\Big) = f(\sqrt{9-3x}) \\
&= \big(\sqrt{9-3x}\big)^2 = 9-3x
\end{aligned}$$

For $(g \circ f \circ h)(x)$, we have

$$\begin{aligned}
(g \circ f \circ h)(x) &= g(f(h(x))) \\
&= g(f(3x-5)) \\
&= g((3x-5)^2) = g(9x^2-30x+25) \\
&= \sqrt{4-(9x^2-30x+25)} = \sqrt{30x-9x^2-21}
\end{aligned}$$

In this chapter we will also need to decompose a given function into two or more, less complex functions. For any one function there is often more than one way to write the decomposition. The following examples demonstrate this.

Example 5: Decomposing a function Given $F(x) = \sin(3x^2+5)$, find $f(x)$ and $g(x)$ so that $F(x) = f(g(x))$.

One solution is $f(x) = \sin x$ and let $g(x) = 3x^2+5$.

Another possible solution is $f(x) = \sin(x+5)$ and $g(x) = 3x^2$.

Exercises

Exponents

Simplify each expression. Write your answer so that all exponents are positive.

1. $(5x^4y^5)(2x^2y^3)^4$ Solution: $80x^{12}y^{17}$
2. $\left(\dfrac{4a^{3/2}b^3}{a^2b^{-1/2}}\right)^{-2}$ Solution: $\dfrac{a}{16b^7}$
3. $\dfrac{\left(-2x^{-3}y^7z^5\right)^{-4}}{\left(x^3y^{-2}z^5\right)^3}$ Solution: $\dfrac{-x^3}{16y^{22}z^{35}}$

4. $\sqrt[4]{x^8y^{16}z^{21}}$ Solution: $x^2y^4z^5\sqrt[4]{z} = x^2y^4z^{5/4}$

Factor to write equivalent expressions.

1. $\dfrac{5}{3}x^{\frac{2}{3}} - \dfrac{5}{3}x^{-\frac{1}{3}}$
   Solution: $\dfrac{5}{3}x^{\frac{2}{3}} - \dfrac{5}{3}x^{-\frac{1}{3}} = \dfrac{5}{3}x^{-\frac{1}{3}}(x-1) = \dfrac{5(x-1)}{3x^{\frac{1}{3}}} = \dfrac{5(x-1)}{3\sqrt[3]{x}}$

2. $\dfrac{\frac{1}{2}x^{-\frac{1}{2}}(x+4) - 3x^{\frac{1}{2}}}{(x+4)^2}$
   Solution: $\dfrac{\frac{1}{2}x^{-\frac{1}{2}}(x+4) - 3x^{\frac{1}{2}}}{(x+4)^2} = \dfrac{\frac{1}{2}x^{-\frac{1}{2}}\big((x+4) - 6x\big)}{(x+4)^2} = \dfrac{-5x+4}{2x^{\frac{1}{2}}(x+4)^2} = \dfrac{-5x+4}{2\sqrt{x}(x+4)^2}$

3. $6x(3x^2+2)^4(x^2-5)^2 + 24x(3x^2+2)^3(x^2-5)^3$

   Solution: $6x(3x^2+2)^4(x^2-5)^2 + 24x(3x^2+2)^3(x^2-5)^3 = 6x(3x^2+2)^3(x^2-5)^2\big((3x^2+2) + 4(x^2-5)\big) = 6x(3x^2+2)^3(x^2-5)^2(3x^2+2+4x^2-20) = 6x(3x^2+2)^3(x^2-5)^2(7x^2-18)$

Function Composition:

1. If $f(x) = x^2 + 2x$ and $g(x) = x - 4$ find
   a. $(f \circ g)(6)$ Solution: 8
   b. $(g \circ f)(6)$ Solution: 44
   c. $(f \circ g)(x)$ Solution: $x^2 - 6x + 8$
   d. $(g \circ f)(x)$ Solution: $x^2 + 2x - 4$
2. If $f(x) = \dfrac{1}{x-5}$ and $g(x) = \sqrt{x-2}$ find
   a. $f(g(6))$ Solution: $-\frac{1}{3}$
   b. $g(f(6))$ Solution: Not defined
   c. $f(g(x))$ Solution: $\dfrac{1}{\sqrt{x-2}-5}$
   d. $g(f(x))$ Solution: $\sqrt{\dfrac{1}{x-5} - 2}$
3. $F(x) = f(g(x))$ identify $f(x)$ and $g(x)$.
   a. $F(x) = \frac{5}{x+4}$ Possible solution: $f(x) = \frac{5}{x}$ and $g(x) = x + 4$
   b. $F(x) = |4 - x^2|$ Possible solution: $f(x) = |x|$ and $g(x) = 4 - x^2$
   c. $F(x) = \sqrt{x+h-5}$ Possible solution: $f(x) = \sqrt{x+h}$ and $g(x) = x - 5$
4. $F(x) = f(g(h(x)))$ identify $f(x), g(x)$ and $h(x)$.
   a. $F(x) = \sqrt[3]{(2x+1)^2}$ Possible solution: $f(x) = \sqrt[3]{x}$, $g(x) = x^2$, and $h(x) = 2x + 1$
   b. $F(x) = 2\sqrt[3]{x^2} + 1$ Possible solution: $f(x) = 2x + 1$, $g(x) = \sqrt[3]{x}$, and $h(x) = x^2$