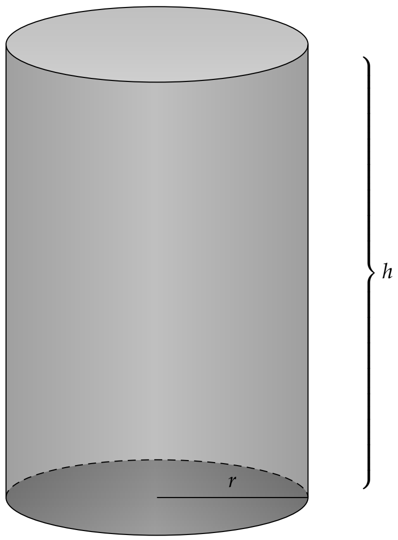
4.2 Additional Example

Example 4.2.4 Optimization: Minimizing Surface Area
Design a closed cylindrical can of volume 8 ft^3 so that it uses the least amount of metal. In other words, minimize the surface area of the can.



Solution:

Following the strategy of Key Idea 9 make a sketch (see Figure #.#) and identify the quantity to be minimized, surface area of the cylinder. The formula for the surface area is our fundamental equation since it relates all of our relevant quantities.

$$A = \underbrace{\pi r^2}_{\text{Top}} + \underbrace{\pi r^2}_{\text{Bottom}} + \underbrace{2\pi rh}_{\text{Side}} = 2\pi r^2 + 2\pi rh.$$

Our surface area is now defined in terms of two variables. To reduce this to a single variable we use the volume of a can, $V = \pi r^2 h$. Since the can must have $V = 8$ ft^3, we set $\pi r^2 h = 8$. Thus $h = \dfrac{8}{\pi r^2}$ and

$$A(r) = 2\pi r^2 + 2\pi r \tfrac{8}{\pi r^2} = 2\pi r^2 + \tfrac{16}{r}$$

Next we find the critical values of $A(r)$. We compute $A'(r)$

$$A'(r) = 4\pi r - \tfrac{16}{r^2} = \tfrac{4\pi r^3 - 16}{r^2}$$

and solve $A'(r) = 0$

$$r^3 = \frac{4}{\pi} \quad \text{or} \quad r = \left(\frac{4}{\pi}\right)^{\frac{1}{3}} \approx 1.08 \text{ ft.}$$

Looking back at $A(r)$ we notice that r is not restricted to a closed interval. The radius can take on any positive value making the interval of optimization $(0, \infty)$. Since we do not have endpoints to test in $A(r)$ we consider what happens to $A(r)$ as r approaches the enpoints of $(0, \infty)$. We see that

$$A(r) \to \infty \text{ as } r \to \infty \text{ (because of the } r^2 \text{ term) and}$$

$$A(r) \to \infty \text{ as } r \to 0 \text{ (because of the } \tfrac{16}{r} \text{ term)}$$

Thus, the surface area must minimized at the critical value, $r = \left(\frac{4}{\pi}\right)^{\frac{1}{3}} \approx$. Finally, we determine the height of the cylinder.

$$h = \tfrac{8}{\pi r^2} = \tfrac{8}{\pi} r^{-2} = \tfrac{8}{\pi}\big((\tfrac{4}{\pi})^{\frac{1}{3}}\big)^{-2} = 2(\tfrac{4}{\pi})(\tfrac{4}{\pi})^{-\frac{2}{3}} = 2(\tfrac{4}{\pi})^{\frac{1}{3}} \approx 2.17 \text{ ft.}$$

Notice that the height is twice the length of the radius. This means that the surface area is minimized when the can is as tall as it is wide.