Chapter 2: Derivatives

Section 2.4 The Product and Quotient Rules

All page numbers refer to original APEX text page numbers.

p. 85

Under the Theorem 14 box remove the ! after the bad "product" rule. At the end of this same paragraph after "... it is wrong." Insert, "We can show that this is wrong by considering $f(x) = x^2$ and $g(x) = x^5$.

Using the WRONG rule we get $\frac{d}{dx}[f(x)g(x)] = 2x \cdot 5x^4 = 10x^5$. However, when we simplify the product first and apply the Power Rule, $f \cdot g = x^2 \cdot x^5 = x^7$ and

$$\frac{d}{dx}[f(x)g(x)] = 7x^6 \neq 10x^5.$$

Applying the **real** Product Rule we see that,

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= x^2 \frac{d}{dx}(x^5) + \frac{d}{dx}(x^2) \cdot x^5 \\ &= x^2 \cdot 5x^4 + 2x \cdot x^5 \\ &= 7x^6\end{aligned}$$

Last paragraph, "Produce Rule" should be "Product Rule"

p. 86

"Exampe 50" should be renamed as "Proof"

p. 87

In example at top of page insert a line specifically showing product rule, i.e. $y' = (x^2+3x+1) \cdot \frac{d}{dx}(2x^2-3x+1) + \frac{d}{dx}(x^2+3x+1) \cdot (2x^2-3x+1)$ then align with $= (x^2 + 3x+1)$...

In the paragraph that starts "Recognize the pattern..." the second sentence should be "Each term..." (singular)

p. 88

In the paragraph just under solution at top of page: "This seems significant..." In the () at the end of the paragraph replace "another" with "others" or "other functions".

Theorem 15 write denominator of quotient rule as $[g(x)]^2$

Right after Theorem 15 box Delete the "low dee high minus high dee low" paragraphs and insert the proof:

Proof: Quotient Rule for Differentiation

Let the functions f and g be defined and $g(x) \neq 0$ on an open interval I By the definition of derivative,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}, \text{ where } h \neq 0$$

$$= \lim_{h\to 0}\left[\left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right)\cdot\frac{1}{h}\right]$$

$$= \lim_{h\to 0}\left[\left(\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}\right)\cdot\frac{1}{h}\right]$$

Adding and subtracting the term $f(x)g(x)$ in the numerator does not change the value of the expression and allows us to separate f and g so that

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \lim_{h\to 0}\left[\left(\frac{f(x+h)g(x)\textbf{- f(x)g(x) + f(x)g(x)} - f(x)g(x+h)}{g(x+h)g(x)}\right)\cdot\frac{1}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)g(x) - f(x)g(x)}{hg(x+h)g(x)} + \frac{f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}\right]$$

$$= \lim_{h\to 0}\left[g(x)\frac{f(x+h) - f(x)}{hg(x+h)g(x)} + f(x)\frac{g(x) - g(x+h)}{hg(x+h)g(x)}\right]$$

$$= \lim_{h\to 0}\frac{g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)}$$

$$= \lim_{h\to 0}\frac{\lim\limits_{h\to 0}g(x)\cdot\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h} - \lim\limits_{h\to 0}f(x)\cdot\lim\limits_{h\to 0}\frac{g(x+h)-g(x)}{h}}{\lim\limits_{h\to 0}g(x+h)\cdot\lim\limits_{h\to 0}g(x)}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

p. 89

Example 54 solution insert the details of quotient rule in the first step:

$$\frac{d}{dx}\left(\frac{5x^2}{\sin x}\right) = \frac{\sin x\frac{d}{dx}(5x^2) - 5x^2\frac{d}{dx}(\sin x)}{(\sin x)^2}$$

Similarly for Example 55 insert a second step:

$$= \frac{\cos x\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(\cos x)}{(\cos x)^2}$$

p. 90

Add the following sentence to the paragraph after the Theorem 16 box: The proofs of these derivatives have been presented or left as exercises. To remember... sign in them.

Example 56 SOLUTION

Typo "We found in Example 54 that ~~the~~ $f'(x)$...

Add a line showing product rule details: $= 5x^2\dfrac{d}{dx}(\csc x) - \csc x\dfrac{d}{dx}(5x^2)$

Last line: ~~The Quotient Rule gives...~~ and replace it with:
When we stated the Power Rule in Section 2.3 we claimed that it worked for all $n \in \mathbb{R}$ but only provided the proof for $n \in \mathbb{Z}^+$. The next example uses the Quotient Rule tp provide justification of the Power Rule for $n \in \mathbb{Z}, n \neq 0$.

p. 91

Paragraph after Example 57 solution: ~~This is reminiscent... restriction of $n > 0$~~ and replace with "Thus, for all $n \in \mathbb{Z}, n \neq 0$ we can officially apply the Power Rule: multiply by the power, then subtract 1 from the power."

Cut Theorem 17 box.

p. 92

Last paragraph, last line: replace the word "reduce" with "simplify"

p. 93

First lines should be:
to a form that seems "simple" and easy to interpret. ~~In that example, we saw different espressions for f', including:~~ They are equal; they are all correct. The most appropriate form of f' depends on what we need to do with the function next. For later problems it will be important for us to determine the most appropriate form to use and to move flexibly between the different forms.

Delete line of equivalent f''s.

Section 2.4 Exercises

Is there a typo in #2? Should it be $= \dfrac{2x}{\cos x}$?

Place each of the followling after the stated **current** problem #.

After #6:
In Exercises 7 - 9 use the Quotient Rule to verify these derivatives.

7. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
8. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
9. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

Answers:

7.

$$\begin{aligned}\frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\ &= \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} \\ &= \frac{-[(\sin x)^2 + (\cos x)^2]}{(\sin x)^2} \\ &= \frac{-1}{(\sin x)^2} = -\csc^2 x\end{aligned}$$

8.

$$\begin{aligned}\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) \\ &= \frac{\cos x \cdot 0 - 1 \cdot (-\sin x)}{(\cos x)^2} \\ &= \frac{\sin x}{(\cos x)^2} = \sec x \tan x\end{aligned}$$

8.

$$\begin{aligned}\frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) \\ &= \frac{\sin x \cdot 0 - 1 \cdot (\cos x)}{(\sin x)^2} \\ &= \frac{-\cos x}{(\sin x)^2} = -\csc x \cot x\end{aligned}$$

After #16:

$H(y) = (y^5 - 2y^3)(7y^2 + y - 8)$

$F(y) = \sqrt[3]{y^2}(y^2 + 9y)$

Answers:

$H'(y) = (y^5 - 2y^3)(14y + 1) + (5y^4 - 6y^2)(7y^2 + y - 8)$

$F'(y) = \frac{8}{3}y^{\frac{5}{3}} + 15y^{\frac{2}{3}} = \frac{\sqrt[3]{y^2}(8y + 45)}{3}$

After #17 :

$y = \frac{\sqrt{x}}{x + 4}$

$g(x) = \frac{x}{\sqrt{x} + 4}$

Answers:

$y' = \frac{4 - x}{2\sqrt{x}(x + 4)^2}$

$$g'(x) = \frac{\sqrt{x}+8}{2(\sqrt{x}+4)^2}$$

After #21:

$$f(x) = \frac{x^2 - \sqrt{x}}{x^3}$$

$$y = \left(\frac{1}{x^3} + \frac{5}{x^4}\right)(2x^3 - x^5)$$

$$g(x) = \frac{1}{1+x+x^2+x^3}$$

$$p(x) = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$$

Answers:

$$f'(x) = -\frac{1}{x^2} + \frac{5}{2x^3\sqrt{x}} = \frac{-2x\sqrt{x}+5}{2x^3\sqrt{x}}$$

$$y' = 2x - 5 + \frac{10}{x^2} = \frac{2x^3 - 5x^2 + 10}{x^2}$$

$$g'(x) = -\frac{1+2x+3x^2}{(1+x+x^2+x^3)^2}$$

$$p'(x) = -\frac{1}{x^2} - \frac{2}{x^3} - \frac{3}{x^4} = -\frac{x^2+2x+3}{x^4}$$

After #26

$$f(y) = y(2y^3 - 5y - 1)(6y^2 + 7)$$

$$F(x) = (8x-1)(x^2+4x+7)(x^3-5)$$

Answers:

$$f'(y) = y(2y^3-5y-1)(12y) + y(6y^2-5)(6y^2+7) + 1(2y^3-5y-1)(6y^2+7) = 72y^5$$

$$F'(x) = (8x-1)(x^2+4x+7)(3x^2) + (8x-1)(2x+4)(x^3-5) + (8)(x^2+4x+7)(x^3-5)$$

After #41:

In Exercises # - #+5, f and g are differentiable funcitons such that $f(2) = 3, f'(2) = -1, g(2) = -5$, and $g'(2) = 2$,. Evaluate the expressions.

#. $(f+g)'(2)$

#+1. $(f-g)'(2)$

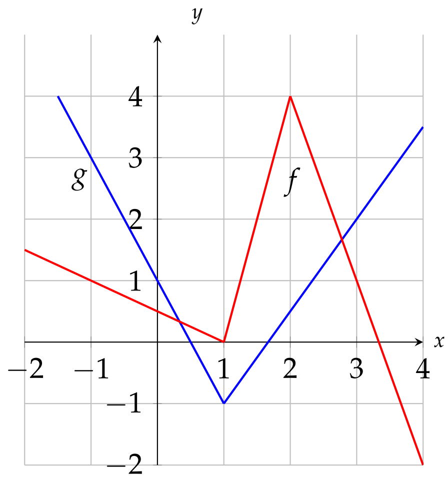
#+2. $(4f)'(2)$

#+3. $(f \cdot g)'(2)$

#+4. $\left(\frac{f}{g}\right)'(2)$

#+5. $\left(\dfrac{g}{f+g}\right)'(2)$

If f and g are functions whose graphs are shown, evaluate the expressions.



You can write this as 4 separately numbered problems, instead of 4 parts of one problem, if it works better that way.
(a) $(fg)'(-1)$ (b) $(f/g)'(-1)$ (c) $(fg)'(3)$ (d) $(g/f)'(3)$

Answers In the solutions it is fine to list the answer w/o the problem. I just wanted to make sure it was clear which answer went with a problem:
#. $(f+g)'(2) = 1$
#+1. $(f-g)'(2) = -3$
#+2. $(4f)'(2) = -4$
#+3. $(f \cdot g)'(2) = 11$
#+4. $\left(\dfrac{f}{g}\right)'(2) = -\dfrac{1}{25}$
#+5. $\left(\dfrac{g}{f+g}\right)'(2) = \dfrac{1}{4}$

Graph problem:
(a) $(fg)'(-1) = -\frac{7}{2}$

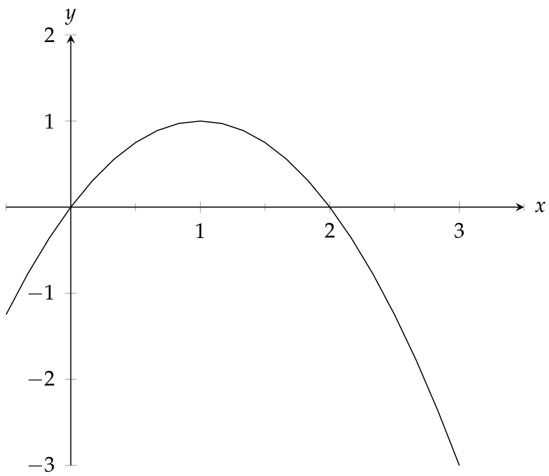
(b) $(f/g)'(-1) = \frac{1}{8}$

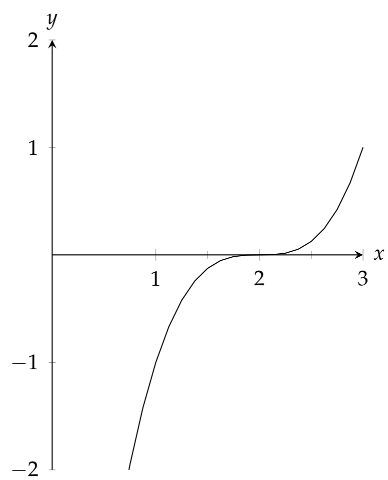
(c) $(fg)'(3) = -\frac{9}{2}$

(d) $(g/f)'(3) = \frac{15}{2}$

Answers

20. Answers vary. Possible solution





Proof: Differentiation Power Rule when n is a positive integer

Let $f(x) = x^n$, where $n \in \mathbb{Z}^+$. By the definition of derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}, \text{ where } h \neq 0\\
&= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}, \text{ use the Binomial Theorem to expand } (x+h)^n\\
&= \lim_{h\to 0}\frac{x^n + \binom{n}{1}hx^{n-1} + \binom{n}{2}h^2x^{n-2} + \ldots + \binom{n}{n-1}h^{n-1}x + \binom{n}{n}h^n) - x^n}{h},\\
&= \lim_{h\to 0}\frac{\binom{n}{1}hx^{n-1} + \binom{n}{2}h^2x^{n-2} + \ldots + \binom{n}{n-1}h^{n-1}x + \binom{n}{n}h^n}{h}\\
&= \lim_{h\to 0}\frac{\binom{n}{1}hx^{n-1} + \binom{n}{2}h^2x^{n-2} + \ldots + \binom{n}{n-1}h^{n-1}x + \binom{n}{n}h^n}{h}\\
&= \lim_{h\to 0}\frac{h[\binom{n}{1}x^{n-1} + \binom{n}{2}hx^{n-2} + \ldots + \binom{n}{n-1}h^{n-2}x + \binom{n}{n}h^{n-1}]}{h}, \text{ we divide out } h\\
&= \lim_{h\to 0}\binom{n}{1}x^{n-1} + \binom{n}{2}hx^{n-2} + \ldots + \binom{n}{n-1}h^{n-2}x + \binom{n}{n}h^{n-1}, \text{ since} \binom{n}{1} = n\\
&= nx^{n-1}
\end{aligned}
$$

Insert proof of Sum Rule

Proof: Sum Rule for Differentiation

Let f and g be differentiable on an open interval I and let c be a real number,

$$
\begin{aligned}
\frac{d}{dx}(f(x)+g(x)) &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)] - [f(x)+g(x)]}{h}, \text{ where } h \neq 0\\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)] + [g(x+h)-g(x)]}{h}\\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x) + g'(x)
\end{aligned}
$$

SOLUTION
Given the differentiation rules we have thus far, our only option for finding $g'(x)$ is to first multiply $g(x)$ out and then apply the sum and power rules.

$$g(x) = x^6 + 3x^4 + 3x^2 + 1$$

thus,

$$g'(x) = 6x^5 + 12x^3 + 6x$$

To differetiate $f(x)$ we will first need to use the Laws of Logarithms to expand f.

$$
\begin{aligned}
f(x) &= \ln\frac{\sqrt{x}}{8}\\
&= \ln x^{\frac{1}{2}} - \ln 8\\
&= \frac{1}{2}\ln x - \ln 8
\end{aligned}
$$

so that,

$$f'(x) = \frac{1}{2} \cdot \frac{1}{x} - 0 = \frac{1}{2x}$$

p. 84: Exercises 2.3 Add the following problems:

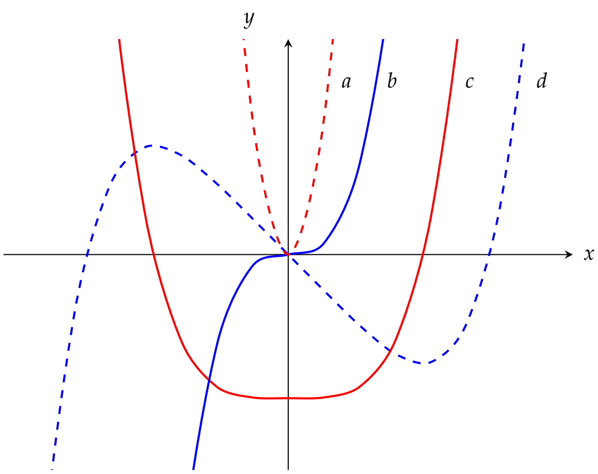
new #26 $h(x) = \frac{x^5 - 2x^3 + x^2}{x^2}$

new #27 $f(x) = \frac{x^2+1}{\sqrt{x}}$

new #28 $g(\theta) = \frac{1-\sin^2\theta}{\cos\theta}$

The current #26 becomes #29 then insert the following:

new#32 The figure shows the graphs of f, f', f'' and f'''. Identify each curve and explain your choices.



The current #27 - 32 become #33 - 38 then insert the following:

new #39. The position of a object is described by $s(t) = t^4 - 4t^2, t \geq 0$, where s is in feet and t is in seconds. Find
(a) the velocity and acceleration functions for the object,
(b) the acceleration after 1.5 seconds, and
(c) the time, in seconds, the object is at rest.

new #20. Sketch the graph of the function f for which $f(0) = 0, f'(0) > 0, f'(1) = 0$, and $f'(3) < 0$.
new #21. Sketch the graph of the function h for which $h(1) = 0, h'(1) > 0, h'(2) = 0$, and $h'(3) > 0$.

Answers

26. $h'(x) = 3x^2 - 2$

27. $f'(x) = \frac{3}{2}\sqrt{x} - \frac{1}{2x\sqrt{x}}$

28. $g'(\theta) = -\cos\theta$

30. $\displaystyle \frac{d}{dx}(c) = \lim_{h\to 0} \frac{c-c}{h} = 0, \;$ where $h \neq 0$

31. a is f, b is f', c is f''

32. d is f, c is f', b is f'', and a is f'''

39. (a) $v(t) = 4t^3 - 8t,\ a(t) = 12t^2 - 8$
(b) $a(1.5) = 19$ ft/s^2
(c) $t = 0$ sec and $t = \sqrt{\frac{3}{2}}$ sec

40. (a) $v(t) = 5e^x - 5,\ a(t) = 5e^x$
(b) $a(2) = 5e^2$ ft/s^2
(c) $v(t) = 0$ at $t = 0$ sec, $a(0) = 5$ in/s^2