Lecture 2 Concentration Inequality

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1. Concentration Inequality

Chernoff bound

$x_1 ... x_n$ independent, $\in [0, 1]$, $\mathbb{E}\{x_i\} = p$. Then,

$$P(\frac{1}{n}\sum x_i - \mathbb{E}[\frac{1}{n}\sum x_i] \geq \epsilon) \leq \exp\{-nD_B(p + \epsilon \| p)\} \leq \exp\{-2n\epsilon^2\}$$

Hoeffding Inequility

$x_1 ... x_n$ independent, $\in [a_i, b_i]$, $\mathbb{E}\{x_i\} = p$. Then,

$$P(\frac{1}{n}\sum x_i - \mathbb{E}[\frac{1}{n}\sum x_i] \geq \epsilon) \leq \exp\{-\frac{2n^2\epsilon^2}{\sum(b_i - a_i)^2}\}$$

? without independence, is concentration exists?

2. Martingale

Definition

$R.V.\ S_0 ... S_n ...$, $\forall i$, (fair game)

$$\mathbb{E}[S_i | S_{i-1}, ... S_0] = S_{i-1}$$

Denote $X_i = S_i - S_{i-1}$ martingale difference, then we have **Azuma's Inequality**:

$x_1 ... x_n$ martingale difference, $|x_i| \leq C_i$, $S_0 = 0$,$\mathbb{E}\{x_i\} = p$. Then,

$$P(\frac{1}{n}\sum x_i - \mathbb{E}[\frac{1}{n}\sum x_i] \geq \epsilon) \leq \exp\{-\frac{2n^2\epsilon^2}{\sum C_i^2}\}$$

Super Martingale

$R.V.\ S_0 ... S_n ...$, $\forall i$,

$$\mathbb{E}[S_i | S_{i-1}, ... S_0] \leq S_{i-1}$$

Then, $x_1 ... x_n$ super martingale difference, $|x_i| \leq C_i$, $S_0 = 0$,$\mathbb{E}\{x_i\} = p$. Then,

$$P(\frac{1}{n}\sum x_i - \mathbb{E}[\frac{1}{n}\sum x_i] \geq \epsilon) \leq \exp\{-\frac{2n^2\epsilon^2}{\sum C_i^2}\}$$

3. McDiermid Lemma

$x_1 ... x_n$ independent, function f satisfies that $\forall i$, $\forall x_1 ... x_n; x_1, ..., x_i', ..., x_n$, then

$$|f(x_1, ..., x_n) - f(x_1, ..., x_i', ..., x_n)| \leq c_i$$

is called stable.

If f is stable, then

$$\mathbb{P}\left\{f(x_1, ..., x_n) - \mathbb{E}[f(x_1, ..., x_n)] \geq \epsilon\right\} \leq \exp\{-\frac{2\epsilon^2}{\sum C_i^2}\}$$

4. Draw without replacement

$a_1, ..., a_N \in \{0, 1\}$ uniformly random draw n numbers from $a_1, ..., a_N$, denote $x_1, ..., x_n$. Consider $P(\frac{1}{n}\sum x_i - \mathbb{E}[\frac{1}{n}\sum x_i] \geq \epsilon)$

1. Draw with replacement: $x_1, ..., x_n$ independent, Chernoff bound holds.
2. Draw without replacement: $x_1, ..., x_n$ not independent, Chernoff bound holds. Actually, this is more concentrated.