# Machine Learning Homework 8

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*2017/11/23*

**Problem 1.** *Suppose $\mathcal{A}$ has uniform stability $\beta$ with respect to loss $l$ and $l \le M$, then with probability $1-\delta$,*

$$R(\mathcal{A}(S)) \le R_{\text{emp}}(\mathcal{A}(S)) + \beta + (n\beta + M)\sqrt{\frac{2\log\frac{1}{\delta}}{n}}$$

*Proof.* Denote $f(S) = R(\mathcal{A}(S)) - R_{\text{emp}}(\mathcal{A}(S))$. The theorem is equivalent to

$$\mathbb{P}[f(S) \ge \beta + \epsilon] \le \exp(-\frac{n\epsilon^2}{2(n\beta+M)^2})$$

$$\begin{aligned} E_S[R_{\text{emp}}(\mathcal{A}(S))] &= E_S\left[\frac{1}{n}\sum_{i=1}^n l(\mathcal{A}(S), z_i)\right] \\ &= \frac{1}{n}\sum_{i=1}^n E_{S,z_i'}\left[l(\mathcal{A}(S^i), z_i')\right] \end{aligned}$$

also,

$$E_S[R(\mathcal{A}(S))] = E_{S,z_i'} l(\mathcal{A}(S), z_i')$$

By the definition of uniform stability, we know that

$$|l(\mathcal{A}(S), z_i') - l(\mathcal{A}(S^i), z_i')| \le \beta$$

Therefore,

$$E_S[f(S)] \le \beta$$

Also,

$$\begin{aligned} |f(S) - f(S^i)| &= |R(\mathcal{A}(S)) - R_{\text{emp}}(\mathcal{A}(S)) - R(\mathcal{A}(S^i)) + R_{\text{emp}}(\mathcal{A}(S^i))| \\ &\le |R(\mathcal{A}(S)) - R(\mathcal{A}(S^i))| + \frac{1}{n}\sum_{i=1}^n |l(\mathcal{A}(S), z_i) - l(\mathcal{A}(S^i), z_i)| \\ &\le \beta + \frac{n-1}{n}\beta + \frac{2M}{n} \\ &\le 2(\beta + \frac{M}{n}) \end{aligned}$$

Then, by Mcdiarmid's Lemma,

$$\mathbb{P}[f(S) \ge E_S[f(S)] + \epsilon] \le \exp(-\frac{n\epsilon^2}{2(n\beta+M)^2})$$

Therefore, by the fact that $E_S[f(S)] \le \beta$

$$\mathbb{P}[f(S) \ge \beta + \epsilon] \le \exp(-\frac{n\epsilon^2}{2(n\beta+M)^2})$$

$\square$