Machine Learning Homework 2

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1. Denote $z_i = (x_i, y_i)$, $\phi_f(z_i) = I(y_i \neq f(x_i))$. Prove:

$$\begin{aligned}
&\frac{1}{2}\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i) - \mathbb{E}[\phi(z)]\right| \geq 2\epsilon\right)\\
\leq &\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i) - \frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)\right| \geq \epsilon\right)\\
\leq &2\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i) - \mathbb{E}[\phi(z)]\right| \geq \frac{\epsilon}{2}\right)
\end{aligned}$$

proof

We know that $\phi_f(z_1), ..., \phi_f(z_n), \phi_f(z_{n+1}), ..., \phi_f(z_{2n})$ are i.i.d Bernoulli $R.V.$. $\mathbb{E}(\phi_f(z)) = p$. Denote $\nu_1 = \frac{1}{n}\sum_{i=1}^{n}\phi_f(z_i)$, $\nu_2 = \frac{1}{n}\sum_{i=n+1}^{2n}\phi_f(z_i)$.

Consider the right inequality, we have

$$\sup_{\phi\in\Phi}|\nu_1 - \nu_2| \geq \epsilon \Longrightarrow \sup_{\phi\in\Phi}|\nu_1 - p| \geq \frac{\epsilon}{2} \text{ or } \sup_{\phi\in\Phi}|\nu_2 - p| \geq \frac{\epsilon}{2}$$

Therefore,

$$\begin{aligned}
&\{\sup_{\phi\in\Phi}|\nu_1 - \nu_2| \geq \epsilon\} \subset \{\sup_{\phi\in\Phi}|\nu_1 - p| \geq \frac{\epsilon}{2}\} \cup \{\sup_{\phi\in\Phi}|\nu_2 - p| \geq \frac{\epsilon}{2}\}\\
\therefore\ \mathbb{P}\left(\sup_{\phi\in\Phi}|\nu_1 - \nu_2| \geq \epsilon\right) &\leq \mathbb{P}\left(\sup_{\phi\in\Phi}|\nu_1 - p| \geq \frac{\epsilon}{2} \cup \sup_{\phi\in\Phi}|\nu 2 - p| \geq \frac{\epsilon}{2}\right)\\
&\leq \mathbb{P}\left(\sup_{\phi\in\Phi}|\nu_1 - p| \geq \frac{\epsilon}{2}\right) + \mathbb{P}\left(\sup_{\phi\in\Phi}|\nu_2 - p| \geq \frac{\epsilon}{2}\right)\\
&= 2\mathbb{P}\left(\sup_{\phi\in\Phi}|\nu_1 - p| \geq \frac{\epsilon}{2}\right)
\end{aligned}$$

Similarily, assume $f \in \Phi$ such that $\sup_{\phi\in\Phi}|\nu_1 - p| = |\nu_1' - p'|$, then

$$\begin{aligned}
&\sup_{\phi\in\Phi}|\nu_1 - p| \geq 2\epsilon \text{ and } |\nu_2' - p'| \leq \epsilon \Longrightarrow \sup_{\phi\in\Phi}|\nu_1 - \nu_2| \geq \epsilon\\
&\therefore \left\{\sup_{\phi\in\Phi}|\nu_1 - p| \geq 2\epsilon\right\} \cap \{|\nu_2' - p'| \leq \epsilon\} \subset \left\{\sup_{\phi\in\Phi}|\nu_1 - \nu_2| \geq \epsilon\right\}\\
&\therefore \mathbb{P}\left(\left\{\sup_{\phi\in\Phi}|\nu_1 - \nu_2| \geq \epsilon\right\}\right) \geq \mathbb{P}\left(\left\{\sup_{\phi\in\Phi}|\nu_1 - p| \geq 2\epsilon\right\}\right)\mathbb{P}\left(\{|\nu_2' - p'| \leq \epsilon\}\right)
\end{aligned}$$

We have an upper bound on $\mathbb{P}\left(\{|\nu_2' - p'| \geq \epsilon\}\right)$. When $n \geq \frac{\ln 2}{2\epsilon^2}$,

$$\mathbb{P}\left(\{|\nu_2' - p'| \leq \epsilon\}\right) = 1 - \mathbb{P}\left(\{|\nu_2' - p'| \geq \epsilon\}\right) \geq 1 - \exp\{-2n\epsilon^2\} \geq \frac{1}{2}$$

$$\therefore\ \mathbb{P}\left(\left\{\sup_{\phi\in\Phi} |\nu_1 - \nu_2| \geq \epsilon\right\}\right) \geq \mathbb{P}\left(\left\{\sup_{\phi\in\Phi} |\nu_1 - p| \geq 2\epsilon\right\}\right)\mathbb{P}\left(\{|\nu_2' - p'| \leq \epsilon\}\right) \geq \frac{1}{2}\mathbb{P}\left(\left\{\sup_{\phi\in\Phi} |\nu_1 - p| \geq 2\epsilon\right\}\right)$$

□