Note 1

Lai Zehua 2014012668

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Definition. *X is called σ^2-subgaussian if*

$$\log E[e^{\lambda(X-EX)}] \le \frac{1}{2}\lambda^2\sigma^2 = MGF(N(0,\delta^2))$$

Chernoff Bound. *If X is σ^2-subgaussian, then,*

$$Pr[X > EX + t] \le e^{-\frac{t^2}{2\sigma^2}}$$

Proof. Same. □

Hoeffding. *If $a \le X \le b$ then X is $\frac{1}{4}(b-a)^2$-subgaussian.*

Proof. Let $\log E[e^{\lambda(X-EX)}] = \phi(\lambda)$, then

$$\begin{aligned}
\phi'(\lambda) &= \frac{E[(X-EX)e^{\lambda(X-EX)}]}{E[e^{\lambda(X-EX)}]} \\
\phi''(\lambda) &= \frac{E[(X-EX)^2e^{\lambda(X-EX)}]}{E[e^{\lambda(X-EX)}]} - \frac{E[(X-EX)e^{\lambda(X-EX)}]}{E[e^{\lambda(X-EX)}]^2} \\
&\le \frac{E[(X-EX)^2e^{\lambda(X-EX)}]}{E[e^{\lambda(X-EX)}]} \\
&\le \frac{1}{4}(a-b)^2
\end{aligned}$$

□

Azuma. *X_t be r.v. $E[X_t|\mathcal{F}_{t-1}] = X_{t-1}$(martingale), $\Delta_t = X_t - X_{t-1}$. If $a_t \le \Delta_t \le b_t$, then X_t is $\frac{1}{4}\sum_{i=1}^{t}(b_t-a_t)^2$-subgaussian.*

Proof. $EX_t = 0$

$$\begin{aligned}
&Ee^{\lambda X_t} \\
=&Ee^{\lambda(X_{t-1}+\Delta_t)} \\
=&E[E[e^{\lambda(X_{t-1}+\Delta_t)}|\mathcal{F}_{t-1}]] \\
=&E[e^{\lambda X_{t-1}}E[e^{\lambda\Delta_t}|\mathcal{F}_{t-1}]] \\
\le&E[e^{\lambda X_{t-1}}]e^{\frac{1}{8}\lambda^2(b_t-a_t)^2}
\end{aligned}$$

□

McDiarmid. *$X_1, X_2, ..., X_n$ independent r.v. $f(x_1, x_2, ..., x_n)$*

$$D_i f = sup_{x_{-i}} sup_{x,y}|f(x_{-i},x) - f(x_{-i},y)|$$

then $f(X_1, X_2, ..., X_n)$ is $\frac{1}{4}\sum_{i=1}^{n} D_i f^2$-subgaussian.[1]

[1] Probability in High Dimension (Princeton).

Proof. Let $Z_i = E[f(X_1, ..., X_n)|X_1, ..., X_i]$, then $f - E[f] = \sum_{i=1}^{n} Z_i - Z_{i-1}$, then

$$E[Z_i|X_1, ..., X_{i-1}] = E[f(X_1, ..., X_n)|X_1, ..., X_{i-1}] = Z_{i-1}$$

Z_i is martingale (Doob martingale). □

Draw with/without replacement. $E[e^{\lambda(X_1+...+X_n)}] \leq E[e^{\lambda(X_1'+...+X_n')}]$.

Proof. Taylor expansion: $E[(X_1 + ... + X_n)^i] \leq E[(X_1' + ... + X_n')^i]$ □