# Machine Learning Homework 1

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**1. *R.V.* $x \sim \mathcal{N}(0,1)$. Define function $\Phi(u) = P(x \geq u)$. Find elementary function $f$, $g$ s.t. $g(u) \leq \Phi(u) \leq f(u)$ when $u \to \pm\infty$.**

*proof*

First, we calculate $\Phi(u)$:

$$\begin{aligned}
\Phi(u) &= P(x \geq u) \\
&= \int_u^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{dx} \\
&= \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}} - \int_u^{+\infty} \frac{1}{\sqrt{2\pi}x^2} e^{-\frac{x^2}{2}} \mathrm{dx}
\end{aligned}$$

$\Phi(u)$ has some good properties: monofonically increasing on $(-\infty, +\infty)$.

Let $f(u) = \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}}$ when $u \to +\infty$, obviously, $f(u) \geq \Phi(u)$ and

$$\begin{aligned}
\lim_{u \to +\infty} \frac{f(u)}{\Phi(u)} &= \lim_{u \to +\infty} \frac{f'(u)}{-\phi(u)} \\
&= \lim_{u \to +\infty} 1 + \frac{1}{u^2} = 1
\end{aligned}$$

and $\frac{f(u)}{\Phi(u)}$ is always $\geq 1$.

Let $g(u) = \frac{1}{\sqrt{2\pi}(u+e^{-u})} e^{-\frac{u^2}{2}}$ when $u \to +\infty$, obviously, $g(u) \leq \Phi(u)$ since

$$\begin{aligned}
\lim_{u \to +\infty} \frac{g(u)}{\Phi(u)} &= \lim_{u \to +\infty} \frac{g'(u)}{-\phi(u)} \\
&= \lim_{u \to +\infty} \frac{u(u+e^{-u}) + 1 - e^{-u}}{u(u+e^{-u})^2} = 1
\end{aligned}$$

and $\frac{g(u)}{\Phi(u)}$ is always $\leq 1$.

When $u \to -\infty$, it is similar. $F(u) = 1 - g(-u)$, $G(u) = 1 - f(-u)$.

Therefore, when $u \to +\infty$,

$$\frac{1}{\sqrt{2\pi}(u+e^{-u})} e^{-\frac{u^2}{2}} \leq \Phi(u) \leq \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}}$$

when $u \to -\infty$,

$$1 + \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}} \leq \Phi(u) \leq 1 - \frac{1}{\sqrt{2\pi}(e^u - u)} e^{-\frac{u^2}{2}}$$

$\square$

**2. Given *R.V.* x, $x_1, ..., x_n$ is Bernoulli, $\mathbb{E}(x) = p$.Then, $\forall \delta > 0$, $P(\frac{1}{n}\sum_{i=1}^{n} x_i - p \geq \delta) =$?**

*proof*

Let $\tilde{x} = \sum_{i=1}^{n} x_i$,

$$M_{\tilde{x}}(t) = (1 - p + pe^t)^n$$

Therefore,

$$P\left\{\tilde{x} \geq n(p+\delta)\right\} \leq \inf_{t>0} e^{-tn(p+\delta)}(1 - p + pe^t)^n$$

Minimize $e^{-tn(p+\delta)}(1-p+pe^t)^n$, which is to miminize $F(t) = -t(p+\delta) + ln(1-p+pe^t)$.

$$F'(t) = \frac{pe^t}{1-p+pe^t} - p - \delta = 0$$

Therefore, $t = ln\left(\frac{(p+\delta)(1-p)}{p(1-p-\delta)}\right)$. Therefore,

$$P\left\{\tilde{x} \geq n(p+\delta)\right\} \leq \exp\{-nD_B(p+\delta\|p)\}$$

Further, we want to prove that

$$\exp\{-nD_B(p+\delta\|p)\} \leq e^{-2n\delta^2}$$

Denote $f(\delta) = D_B(p+\delta\|p) - 2\delta^2$, $f(0) = 0$.

$$f'(\delta) = \ln(\frac{p+\sigma}{p}) - \ln(\frac{1-p-\delta}{1-p}) - 4\delta$$
$$f''(\delta) = \frac{2}{(p+\delta)(1-p-\delta)} - 4$$

Therefore, $f'(0) = 0$ and $f''(\delta) \geq 0$ if $\delta \geq 0$. Therefore,

$$f(\delta) \leq f(0) = 0$$

Therefore,

$$\exp\{-nD_B(p+\delta\|p)\} \leq e^{-2n\delta^2}$$

□

**3. Prove Hoeffding Inequality: $x_1 ... x_n$ independent, $x_i \in [a_i, b_i]$, $\mathbb{E}(x_i) = \mu_i$.**

$$P\left(\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n}\mu_i \geq \epsilon\right) \leq \exp\{\frac{-2n^2\epsilon^2}{\sum_{i=1}^{n}(a_i - b_i)^2}\}$$

*proof*

We first prove $\mathbb{E}[e^{t(x-\mu)}] \leq \exp\{\frac{1}{8}t^2(b-a)^2\}$.

From Jenson's inequility, $\mathbb{E}[e^{t(x-\mu)}]$ is maximized when $x$ is distributed on boundary of domain, which makes $x$ a Bernoulli distribution. Assume $P(x = b) = p$,

$$\mathbb{E}[e^{t(x-\mu)}] = pe^{t(b-\mu)} + (1-p)e^{t(a-\mu)}$$
$$F(x) = \mathbb{E}[e^{t(x-\mu)}] - \exp\{\frac{1}{8}t^2(b-a)^2\}$$

with some calculation, we can find that max $F(x) \leq 0$. Therefore, $\mathbb{E}[e^{t(x-\mu)}] \leq \exp\{\frac{1}{8}t^2(b-a)^2\}$.

Then,

$$
\begin{aligned}
P\left(\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n}\mu_i \geq \epsilon\right) &= P\left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\mu_i \geq n\epsilon\right) \\
&\leq \inf_{t>0} e^{-tn\epsilon}\mathbb{E}[e^{t(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\mu_i)}] \\
&\leq \inf_{t>0} e^{-tn\epsilon}\prod_{i=1}^{n}\mathbb{E}[e^{t(x_i-\mu_i)}] \\
&\leq \inf_{t>0} \exp\{-tn\epsilon + \frac{1}{8}t^2\sum_{i=1}^{n}(b_i-a_i)^2\} \\
&\leq \exp\{\frac{-2n^2\epsilon^2}{\sum_{i=1}^{n}(a_i-b_i)^2}\}
\end{aligned}
$$

$\square$