Homework1

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Question 1:

There is r, v, $x \sim N(0,1)$ Define function $\Phi(u) = P(x \geq u)$

Find elementary function f, g s.t $g(u) \leq \Phi(u) \leq f(u)$

Answer:

The question is to find f, g s.t

$$-\frac{g'(u)}{\varphi(u)} \leq 1 \text{ and } \lim_{u\to+\infty}\frac{g'(u)}{\varphi(u)} = 1, \quad -\frac{f'(u)}{\varphi(u)} \geq 1 \text{ and } \lim_{u\to+\infty}\frac{f'(u)}{\varphi(u)} = 1 \text{ Where } \varphi(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}$$

By partial integration we can easily find that:

$$\int e^{-\frac{u^2}{2}}du = -\frac{1}{u}e^{-\frac{u^2}{2}} - \int \frac{1}{u^2}e^{-\frac{u^2}{2}}du$$

So $f(u) = \frac{1}{\sqrt{2\pi}u}e^{-\frac{u^2}{2}}$, we can check that $-\frac{f'(u)}{\varphi(u)} = 1 + \frac{1}{u^2} \to 1$ and > 1

Just to make it a little smaller than we can find g(u), for example $g(u) = \frac{1}{\sqrt{2\pi}\ (u+1)}e^{-\frac{u^2}{2}}$

Then $-\frac{g'(u)}{\varphi(u)} = \frac{u}{u+1} + \frac{1}{(u+1)^2} = \frac{u^2+u+1}{u^2+2u+1} \to 1$ and < 1

And when $u \to -\infty$ obviously $f(u) = 1 - g(-u)$ and $g(u) = 1 - f(-u)$

So we have：

$$f(u) = \begin{cases} 1 + \dfrac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}(u-1)}, & u \to -\infty \\ \dfrac{1}{\sqrt{2\pi}u}e^{-\frac{u^2}{2}}, & u \to +\infty \end{cases}$$

$$g(u) = \begin{cases} 1 + \dfrac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}u}, & u \to -\infty \\ \dfrac{1}{\sqrt{2\pi}(u+1)}e^{-\frac{u^2}{2}}, & u \to +\infty \end{cases}$$

Question 2:

Given r.v. $x, x_1 \dots x_n$, Bernoulli, $E[x] = p$, Then $\forall\delta > 0, P\left(\frac{1}{n}\sum_{i=1}^{n} x_i - p \geq \delta\right) = ?$

Answer:

Assume $\hat{x} = \sum_{i=1}^{n} x_i$, then $M_{\hat{x}}(t) = M_x(t)^n = (1 - p + pe^t)^n$,

Then $P\left(\frac{1}{n}\sum_{i=1}^{n} x_i - p \geq \delta\right) = P(\hat{x} \geq n(p+\delta)) \leq \inf_{t>0} e^{-tn(p+\delta)}(1 - p + pe^t)^n$

And that is to minimize $-t(p+\delta) + \ln(1 - p + pe^t) \Rightarrow \frac{pe^t}{1 - p + pe^t} = p + \delta$

Then $e^t = \frac{(1-p)(p+\delta)}{p(1-p-\delta)}, \qquad t = \ln\frac{(1-p)(p+\delta)}{p(1-p-\delta)}$

and the probability is $\exp\{-n(p+\delta)\ln\frac{(1-p)(p+\delta)}{p(1-p-\delta)} + n\ln\frac{1-p}{1-p-\delta}\}$

That is $\exp\left\{-n(p+\delta)\ln\frac{(1-p)(p+\delta)}{p(1-p-\delta)} + n(p+\delta)\ln\frac{1-p}{1-p-\delta} + n(1-p-\delta)\ln\frac{1-p}{1-p-\delta}\right\}$

$= \exp\left\{-n(p+\delta)\ln\frac{p+\delta}{p} - n(1-p-\delta)\ln\frac{1-p-\delta}{1-p}\right\}$

$= \exp\{-nD_B(p+\delta||p)\}$

Further more, assume $f(\delta) = D_B(p+\delta||p) - 2\delta^2$, then $f(0) = 0$ and:

$f'(\delta) = \ln\left(\frac{p+\delta}{p}\right) - \ln\left(\frac{1-p-\delta}{1-p}\right) - 4\delta \Rightarrow f'(0) = 0$

$f''(\delta) = \frac{2}{(p+\delta)(1-p-\delta)} - 4 \geq 0$

So $f(\delta)$ reaches its lowest point at $(0,0) \Rightarrow f(\delta) \geq 0 \Rightarrow D_B(p+\delta||p) \geq 2\delta^2$

So the probability $\leq \exp(-2n\delta^2)$

Question 3:

Prove Hoeffding Inequality:

$$x_1 \dots x_n \text{ independent } x_i \in [a_i, b_i], Ex_i = \mu_i$$

$$P\left(\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n} \mu_i \geq \varepsilon\right) \leq e^{-2n^2\varepsilon^2} / \sum_{i=1}^{n} (a_i - b_i)^\wedge 2$$

$First, we\ need\ to\ prove\ that\ E[e^{t(x-\mu)}] \leq \exp\{\frac{1}{8}t^2\ (b-a)^2\}$

By Jensen's inequality we can see that $E[e^{t(x-\mu)}]$ reaches the largest value when x is only distributed on a and b, and we can assume a=0 for convenience, now assume P(x=b)=p then we need to prove that $E\left[e^{t(x-\mu)}\right] - \frac{1}{8}t^2b^2 = (1-p)e^{-tpb} + pe^{t(b-pb)} - \exp\{\frac{1}{8}t^2b^2\}$

Assume $f(x) = \ln\left[(1-p)e^{-px} + pe^{(1-p)x}\right] - \frac{1}{8}x^2 \Rightarrow f(0) = 0$

Then $f'(x) = \frac{p(1-p)(e^x - 1)}{pe^x + 1 - p} - \frac{1}{4}x \Rightarrow f'(0) = 0$

And $f''(x) = \frac{1-p}{p}\frac{e^t}{\left(e^t + \frac{1}{p} - 1\right)^2} - \frac{1}{4}$

Obviously when $e^t = \frac{1}{p} - 1$ $f''(x)$ reaches its maximum value and it is 0 $\Rightarrow f''(x) \leq 0$

So when $x \geq 0$ $f(x) \leq 0 \Rightarrow f(tb) \leq 0 \Rightarrow E[e^{t(x-\mu)}] \leq \exp\{\frac{1}{8}t^2\ (b-a)^2\}$

Then the left part of the original inequality is:

$$P\left(\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n} \mu_i \geq \varepsilon\right) = P\left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \mu_i \geq n\varepsilon\right) \leq \inf_{t>0} e^{-tn\varepsilon} E\left[e^{t\left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \mu_i\right)}\right]$$

$$\leq \inf_{t>0} e^{-tn\varepsilon} \prod_{i=1}^{n} E\left[e^{t(x_i-\mu_i)}\right]$$

$$\leq \inf_{t>0} \exp\{-tn\varepsilon + \frac{1}{8}t^2 \sum_{i=1}^{n} (b_i - a_i)^2\}$$

$$= \exp\{-\frac{2n^2\varepsilon^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\}$$

And this is the right part of the inequality.