Chapter 2 : Linear Programming (LP)

2.1. Introduction

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective functions. It is a subject consisting of a set of linear equalities and/or inequalities known as constraints. Linear programming is a mathematical technique which involves the allocation of limited resources in an optimal manner, on the basis of a given criterion of optimality.

In this chapter, properties of Linear Programming Problems (LPP) have been discussed. The graphical method of solving an LPP is applicable where two variables are involved. The most widely used method for solving LPP problems consisting of any number of variables is called simplex method, developed by G. Dantzig in 1947 and made generally available in 1951.

2.2. Formulation of LP problems

The procedure for mathematical formulation of a LPP consists of the following steps:

Step 1 To write down the decision variables of the problem.

Step 2 To formulate the objective function to be optimized (maximized or minimized) as a linear function of the decision variables.

Step 3 To formulate the other conditions of the problems such as resource limitation, market constraints, interrelations between variables, etc. as linear inequations or equations in terms of the decision variables.

Step 4 To add the non-negativity constraints from the considerations so that the negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraints and the non-negative restrictions together form a Linear Programming Problem (LPP).

2.2.1. General formulation of LPP

The general formulation of the LPP can be stated as follows:

In order to find the values of n decision variables x_1 x_2 … x_n to maximize or minimize the objective function.

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{1}$$

and also satisfy *m*-constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ldots b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ldots b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \ldots b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ldots b_m \end{array}\right\} \tag{2}$$

where constraints may be in the form of inequality $\leq$ or $\geq$ or even in the form of an equation (=) and finally satisfy the non-negative restriction

$$x_1 \geq 0, x_2 \geq 0 \ldots x_n \geq 0 \qquad (3)$$

2.2.2. Matrix form of LP problem

The LPP can be expressed in the matrix form as follows:

Maximize or

Minimize Z = cx $\longrightarrow$ Objective function

Subject to, Ax ($\leq = \geq$) b Constraint equation

b > 0, $x \geq 0$ Non-negativity restrictions

where, $x = (x_1\, x_2 \cdots x_n)$

$c = (c_1\, c_2 \ldots c_n)$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m x1} \qquad A = \begin{pmatrix} a_{11}\, a_{12} & \cdots a_{1n} \\ a_{21}\, a_{22} & \cdots a_{2n} \\ \vdots & \\ a_{m1}\, a_{m2} & \cdots a_{mn} \end{pmatrix}_{m x n}$$

Example 2.1 : A manufacturer produces two types of models M_1 and M_2. Each model of the type M_1 requires 4 hours of grinding and 2 hours of polishing; whereas each model of the type M_2 requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. Profit on M_1 model is \$3.00 and on model M_2 is \$4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week?

Solution

Decision variables: Let x_1 and x_2 be the number of units of M_1 and M_2 models.

Objective function: Since the profit on both the models are given, we have to maximize the profit viz.

$$\text{Max } Z = 3x_1 + 4x_2$$

Constraints: There are two constraints — one for grinding and the other for polishing.

Numbers of hours available on each grinder for one week is 40. There are 2 grinders. Hence the manufacturer does not have more than 2 × 40 = 80 hours of grinding. M1 requires 4 hours of grinding jfand M2 requires 2 hours of grinding.

The grinding constraint is given by

$$4x_1 + 2x_2 \leq 80$$

Since there are 3 polishers, the available time for polishing in a week is given by 3 × 60 = 180 hours of polishing. M_1 requires 2 hours of polishing and M_2 requires 5 hours of polishing. Hence we have $2x1 + 5x2 \le 180$.

Finally we have,

Max $Z = 3x_1 + 4x_2$

Subject to,

$$4x_1 + 2x_2 \le 80$$
$$2x_1 + 5x_2 \le 180$$
$$x_1, x_2 \ge 0.$$

Example 2.2 A company manufactures two products A and B. These products are processed in the same machine. It takes 10 minutes to process one unit of product A and 2 minutes for each unit of product B and the machine operates for a maximum of 35 hours in a week. Product A requires 1 kg and B requires 0.5 kg of raw material per unit, the supply of which is 600 kg per week. Market constraint on product B is known to be minimum of 800 units every week. Product A costs ₹ 5 per unit and sold at ₹ 10. Product B costs ₹ 6 per unit and can be sold in the market at a unit price of ₹ 8. Determine the number of units of A and B per week to maximize the profit.

Solution

Decision variables: Let x_1 and x_2 be the number of products A and B respectively.

Objective function: Cost of product A per unit is ₹ 5 and selling price is ₹ 10 per unit.

∴ Profit on one unit of product $A = 10 - 5 =$ ₹ 5

x_1 units of product A contributes a profit of ₹ $5x_1$, profit contribution from one unit of product

$$B = 8 - 6 = ₹\ 2$$

x_2 units of product B contribute a profit of ₹ $2x_2$

The objective function is given by,

$$\text{Max } Z = 5x_1 + 2x_2$$

Constraints: Time requirement constraint is given by,

$$10x_1 + 2x_2 \le (35 \times 60)$$
$$10x_1 + 2x_2 \le 2100$$

Raw material constraint is given by,

$$x_1 + 0.5x_2 \le 600$$

Market demand of product B is 800 units every week

$$\therefore \quad x_2 \ge 800$$

The complete LPP is

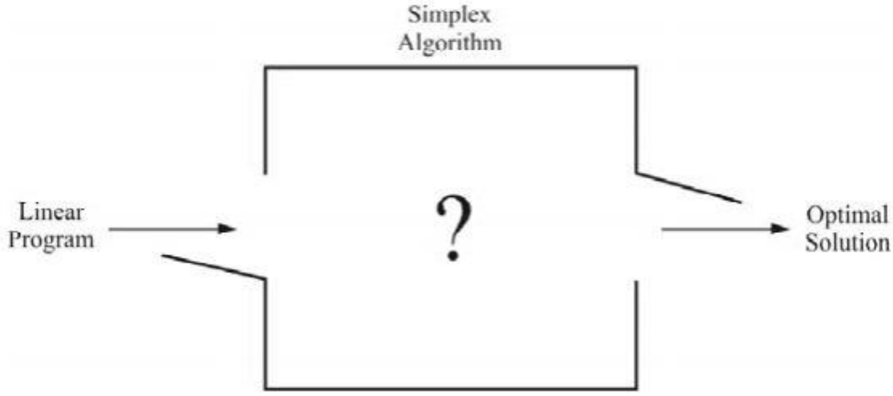
Max $Z = 5x_1 + 2x_2$

Subject to,

$$10x_1 + 2x_2 \le 2100$$
$$x_1 + 0.5x_2 \le 600$$
$$x_2 \ge 800$$
$$x_1, x_2 \ge 0.$$

2.3. Sensitivity analysis

The term sensitivity analysis often known as post-optimality analysis refers to the optimal solution of a linear programming problem, formulated using various methods. Here, you will learn how sensitivity analysis helps to solve repeatedly the real problem in a different form. Generally, these scenarios crop up as an end result of parameter changes due to the involvement of new advanced technologies and the accessibility of well-organized latest information for key (input) parameters or the ‘ what-if ’ questions. Thus, sensitivity analysis helps to produce the optimal solution of simple perturbations for the key parameters. For the optimal solutions, consider the simplex algorithm as a ‘ black box ’ which accepts the input key parameters to solve LPP as shown below:



Example 2.3 Illustrate sensitivity analysis using simplex method to solve the following LPP

Maximize $Z = 20x_1 + 10x_2$

Subject to, $x_1 + x_2 \le 3$

$3x_1 + x_2 \le 7$

And $x_1, x_2 \ge 0.$

Solution Sensitivity analysis is done after making the initial and final tableau using the simplex method. Add slack variables to convert it into equation form.

Maximize $Z = 20x_1 + 10x_2 + 0S_1 + 0S_2$

Subject to, $x_1 + x_2 + S_1 + 0S_2 = 3$

$3x_1 + x_2 + 0S_1 + S_2 = 7$

Where $x_1, x_2 \ge 0.$

To find basic feasible solution, we put $x_1 = 0$ and $x_2 = 0$. This gives $Z = 0$, $S_1 = 3$ and $S_2 = 7$. The initial table will be as follows:

Initial table

		C_j	20↓	10	0	0		
C_B	B	x_B	x_1	x_2	S_1	S_2	Min $\frac{x_B}{x_i}$	
0	S_1	3	1	1	1	0	3/1 = 3	
←0	S_2	7	3	1	0	1	7/3 = 2.33	←
	Z_j	0	0	0	0	0		
	$Z_j - C_j$		–20	–10	0	0		
			↑					

Find $\frac{x_B}{x_i}$ for each row and also find minimum for the second row. Here, $Z_j - C_j$ is maximum negative (–20). Hence, x_1 enters the basis and S_2 leaves the basis. It is shown with the help of arrows.

Key element is 3, key row is second row and key column is x_1. Now convert the key element into the entering key by dividing each element of the key row by key element using the following formula:

$$\text{New element} = \text{Old element} - \left[\frac{\text{Product of elements in the key row and the key column}}{\text{Key element}}\right]$$

The following is the first iteration table:

		C_j	20↓	10	0	0		
C_B	B	x_B	x_1	x_2	S_1	S_2		
←0	S_1	2/3	0	2/3	1	–1/3	$\frac{2}{3}/\frac{2}{3}=1$	←
20	x_1	7/3	1	1/3	0	1/3	$\frac{7}{3}/\frac{1}{3}=7$	
	Z_j	140/3	20	20/3	0	20		
	$Z_j - C_j$	–	0	–10/3	0	20/3		
				↑				

Since $Z_j - C_j$ has one value less than zero, i.e., negative value hence this is not yet the optimal solution. Value –10/3 is negative, hence x_2 enters the basis and S_1 leaves the basis. Key row is upper row.

		C_j	20	10	0	0
C_B	B	x_B	x_1	X_2	S_1	S_2
10	x_2	1	0	1	3/2	–1/2
20	x_1	2	1	0	–1/2	1/2
	Z_j	50	20	10	0	25
	$Z_j - C_j$		0	0	5	5

$Z_j - C_j \geq 0$ for all, hence optimal solution is reached, where $x_1 = 2,\ x_2 = 1, Z = 50.$

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Example 2.3 A person requires 10, 12 and 12 units of chemicals A, B and C, respectively for his garden. A liquid product contains 5, 2 and 1 units of A, B and C, respectively, per jar. A dry product contains 1, 2 and 4 units of A, B, C per carton. If the liquid product is sold for ₹ 3 per jar and the dry product is sold for ₹ 2 per carton, how many units of each product should be purchased, in order to minimize the cost and meet the requirements?

Example 2.6 A firm manufacturers 3 products A, B and C. The profits are ₹ 3, ₹ 2 and ₹ 4 respectively. The firm has 2 machines and given below is the required processing time in minutes for each machine on each product

Machines	***Product-wise processing time (min)***		
	A	***B***	***C***
M_1	4	3	5
M_2	3	2	4

Machines M_1 and M_2 have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 units of A's, 200 units of B's and 50 units of C's but not more than 150 units of A's. Set up an LPP to maximize the profit.

Example 2.8 An electric appliance company produces two products: refrigerators and ranges. Production takes place in two separate departments I and II. Refrigerators are produced in department I and ranges in department II. The company's two products are sold on a weekly basis. The weekly production cannot exceed 25 refrigerators and 35 ranges. The company regularly employs a total of 60 workers in the two departments. A refrigerator requires 2 man-weeks labour while a range requires 1 man-week labour. A refrigerator contributes a profit of ₹ 60 and a range contributes a profit of ₹ 40. How many units of refrigerators and ranges should the company produce to realize the maximum profit? Formulate the above as an LPP.