① SVD with Julia

U factor:
3-element Array{Float64,2}:

$$\begin{array}{ccc}
\begin{pmatrix}-0.339287\\-0.473523\\-0.81281\end{pmatrix} & \begin{pmatrix}0.742665\\-0.665163\\0.0775016\end{pmatrix} & \begin{pmatrix}-0.57735\\-0.57735\\0.57735\end{pmatrix} \\
u_1 & u_2 & u_3
\end{array}$$

singular values:
3-element Array{Float64,1}:
3.253087102270064
1.1905563006612325
1.812986607347358e-16

Vt factor:
3-element Array{Float64,2}:

$$\begin{array}{cl}
v_1^T & (\ -0.354155 \quad -0.749574 \quad -0.395419 \quad 0.0 \quad -0.395419\ ) \\
v_2^T & (\ 0.688894 \quad 0.195291 \quad -0.493603 \quad 0.0 \quad -0.493603\ ) \\
v_3^T & (\ 0.632456 \quad -0.632456 \quad 0.316228 \quad 0.0 \quad 0.316228\ )
\end{array}$$

a) $rk(A) = 2$ since A has 2 nonzero singular values

b) Basis for col(A) : $\{u_1, u_2\}$
Basis for row(A) : $\{v_1, v_2\}$

c) singular vectors in null(A^T) : u_3 ($\in$ in the orthogonal complement of col(A))

d) $v_3 \in$ null(A) but it's not a basis. $rk(A) = 2$ and $m=5$ so nullity(A) $= m - rk = 3$, so we'd need 3 vectors at least to have a spanning set, and we don't even have that so we definitely don't have a basis.

e) The fourth entry in each row of A is zero. This means any vector in row(A) will also have its 4^{th} entry be zero, thus the basis vectors for row(A) must also have this property, which is why v_1^T and v_2^T have their 4^{th} entry be 0. As for v_3, it's probably due to the floating-point error of the third singular value.

f) $\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$
$= (3.253087)\, u_1 v_1^T + (1.1905563)\, u_2 v_2^T$ two rank 1 matrices

g) A is 3x5 so it's a linear map from $\mathbb{R}^5 \to \mathbb{R}^3$. The sphere exists in the domain ($\mathbb{R}^5$) so it's 4 dimensional. Since A is rank 2, the sphere is sent to a hyperellipse with 2 semiaxes so its 2 dimensional ($\in \mathbb{R}^3$)

h) σ_1 (3.253087) and σ_2 (1.1905563)

② More SVD

$$|\det(A)| = |\det(U\Sigma V^T)|$$
$$= |\det(U)\det(\Sigma)\det(V^T)|$$

The determinant of an orthonormal matrix is ± 1
If U is orthonormal, then $U^TU = I$ so $\det(I) = \det(U^T)\det(U)$
$\det(I) = 1$ so $\det(U)$ and $\det(U^T)$ must equal ± 1.

So: $|\det(U)\det(\Sigma)\det(V^T)| = |\det(\Sigma)|$
$= \sigma_1 \sigma_2 \dots \sigma_n$

A is nxn so Σ is diagonal and the det of a diagonal matrix is the product of the diagonal entries.

Additionally, I also thought about this in terms of change of basis. We know that $A = U\Sigma V^T$, so if we apply A to a vector x, we have that $Ax = U\Sigma V^T x$. V^T is a change of basis matrix and changing basis is simply changing coordinate system so the determinant shouldn't change. Then, Σ represents the action of A in the V basis so the determinant of that transformation is $\det(\Sigma)$. Then, applying U takes us back to std. basis, which doesn't change the determinant, so the overall determinant is $\det(\Sigma) = \sigma_1 \sigma_2 \dots \sigma_n$.

b) singular values are eigvals of A^TA

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Eigenvalues of A^TA are: $0, 1, 3$

so the singular values of A are $1, \sqrt{3}$ by Lemma 8.1.4

$$\Rightarrow \Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_0 : \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = v_3 \quad E_1 : \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = v_2 \quad E_3 : \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} = v_1$$

$$\text{so } V^T = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$\frac{Av_1}{\sigma_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} = \frac{\begin{pmatrix} 3/\sqrt{6} \\ 3/\sqrt{6} \end{pmatrix}}{\sqrt{3}} = \begin{pmatrix} 3/\sqrt{18} \\ 3/\sqrt{18} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = u_1$$

$$\frac{Av_2}{\sigma_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = u_2$$

$$\text{so } A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

c) (i) False.

we know the singular values of A are the positive square roots of the r positive eigenvalues of AA^T by Lemma 8.1.5. By lemma 8.1.4, we know that the singular values of A^T are the positive square roots of the r positive eigenvectors of AA^T.
Thus, the singular values of A = the singular values of A^T.

(ii) True.

($\Rightarrow$) If all singular values $=1 \rightarrow \|Ax\| = \|x\|$

$A = U\Sigma V^T$ and A is $n \times n$ so $\Sigma = I$
Show $\|Ax\| = \|x\|$

$$\|U\Sigma V^T x\| = \|\Sigma V^T x\|$$ since applying an orthonormal matrix won't change its norm

$$= \|\Sigma y\|$$ Let $y = V^T x$

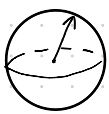
$$= \|y\|$$ since $\Sigma = I$

$$= \|V^T x\|$$

$$= \|x\|$$ since V^T is orthonormal

($\Leftarrow$) If $\|Ax\| = \|x\| \rightarrow$ All singular values are 1

A is $n \times n$ so it sends the unit sphere to an ellipse also in $\mathbb{R}^n$



If Ax sends x to a vector of the same norm then there is no stretching or scaling going on, so then the semiaxes of the hyperellipse would also be 1, indicating that the singular values are all 1.

Algebraically: $\|Ax\| = \|U\Sigma V^T x\| = \|x\|$

$$\|U\Sigma V^T x\| = \|x\| \Rightarrow \|U\Sigma V^T x\|^2 = \|x\|^2$$

$$(U\Sigma V^T x)^T (U\Sigma V^T x) = x^T x$$

$$x^T V \Sigma^T \cancel{U^T U} \Sigma V^T x = x^T x$$

$$x^T V \Sigma^T \Sigma V^T x = x^T x = x^T I x$$

$$x^T V \Sigma^2 V^T x = x^T I x$$ (Σ is diagonal so $\Sigma^T \Sigma = \Sigma^2$)

$$V \Sigma^2 V^T = I$$

$$\Sigma^2 = V^T I V = V^T V$$

$$\Sigma^2 = I$$

We know singular values must be positive and A is full rank so $\sigma_1 \dots \sigma_n = 1$.

(iii) True.

($\Rightarrow$) Suppose A_1 and A_2 have the same singular values. Then, $A_1 = U_1 \Sigma V_1^T$ and $A_2 = U_2 \Sigma V_2^T$ where A_1 and A_2 have the Σ matrix since they have the same singular values.

Since U_1, V_1, U_2, V_2 are orthonormal, $U_1^T U_1 = I$ and $V_1^T V_1 = I$ and same for U_2 and V_2.

So, A_1 and A_2 can be rearranged as such:

$$(A_1 = U_1 \Sigma V_1^T)\, V_1 \qquad\qquad (A_2 = U_2 \Sigma V_2^T)\, V_2$$

$$U^T\, (A_1 V_1 = U_1 \Sigma) \qquad\qquad U_2^T\, (A_2 V_2 = U_2 \Sigma)$$

$$U^T A_1 V_1 = \Sigma \qquad\qquad U_2^T A_2 V_2 = \Sigma$$

Now if we substitute Σ into the two original equations, we have:

$$A_1 = U_1 \,(U_2^T A_2 V_2)\, V_1^T$$

Let $Q_1 = U_1 U_2^T$, $Q_2 = V_2 V_1^T$. We know Q_1 and Q_2 are orthonormal since the product of two orthonormal matrices is orthonormal:

Proof: Suppose A and B are arbitrary orthonormal matrices. Then, $A^T A = I$ and $B^T B = I$. If we compute $(AB)^T (AB)$, we see that:

$$\begin{aligned}(AB)^T (AB) &= B^T A^T A B \\ &= B^T I_n B \\ &= I_n\end{aligned}$$

Thus, AB is orthonormal. Since A and B were arbitrary orthonormal matrices, we've shown that the product of any two orthonormal matrices is orthonormal.

Thus, if A_1 and A_2 have the same singular values then $A_1 = Q_1 A_2 Q_2$ for some orthonormal matrices Q_1 and Q_2.

($\Leftarrow$) $A_1 = Q_1 A_2 Q_2$ $\rightarrow$ A_1 and A_2 have same singular values where Q_1 and Q_2 are orthonormal matrices

If $A_1^T A$ and $A_2^T A$ have the same eigenvalues then A_1 and A_2 have the same singular values

So we need to show that $A_1^T A$ and $A_2^T A$ have the same eigenvalue matrix

$A_1 = Q_1 A_2 Q_2 \qquad A_2 = Q_1^T A_1 Q^T$

$$
\begin{aligned}
A_1^T A_1 &= (Q_1 A_2 Q_2)^T (Q_1 A_2 Q_2) \\
&= Q_2^T A_2^T \cancel{Q_1^T Q_1} A_2 Q_2 \\
&= Q_2^T A_2^T A_2 Q_2 \\
&= Q_2^{-1} A_2^T A_2 Q_2 \qquad (\text{since } Q \text{ is orthonormal so } Q_2^T = Q_2^{-1})
\end{aligned}
$$

$\Rightarrow A_1^T A_1 \sim A_2^T A_2 \quad \Rightarrow$ $A_1^T A$ and $A_2^T A$ have the same eigenvalues

Since the singular values of a matrix A are the positive square roots of the r positive eigenvalues of $A^T A$, A_1 and A_2 have the same singular values since $A_1^T A_1$ and $A_2^T A_2$ have the same eigenvalues.

③ Best-Fit Planes / PCA

SVD of A:
$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/\sqrt{2} & 1/2 \\ 0 & 1/2 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & -1/2 & 0 & 1/2 \\ -1/\sqrt{2} & -1/2 & 0 & 1/2 \end{bmatrix}^T$$

$$A^T = \begin{bmatrix} 0 & 1/2 & 1/\sqrt{2} & 1/2 \\ 0 & 1/2 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & -1/2 & 0 & 1/2 \\ -1/\sqrt{2} & -1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T$$

a) $L = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$P = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

b) rank 1 decomposition:

$$A = 2\sqrt{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix}$$
$$+ \sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \end{pmatrix}$$

Approx of $A_1 = 2\sqrt{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Approx of $A_2 =$

$$2\sqrt{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

c) $\text{proj}_{(\ell)} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Basis for ℓ is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ - it's orthonormal so $u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$uu^T \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0\ 1\ 0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ first col of A_1

$$uu^T \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ second col of A_1

$$uu^T \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$ third col of A_1

$$uu^T \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$$ fourth col of A_1

d) $P = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ so A is orthonormal

so AA^T is the projection matrix:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Proj}_P \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$ third col of A_2

e)
$$\begin{aligned} \| A - A_1 \|_2 &= \| (\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T) - \sigma_1 u_1 v_1^T \|_2 \\ &= \| \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T \|_2 \\ &= \sigma_2 \\ &= 2 \end{aligned}$$

f) $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \| A - B \|_2 = \left\| \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 1 & 1 & -1 & -1 \end{bmatrix} \right\|_2 = 2\sqrt{2} > 2$

(computed using Wolfram Alpha)

Math 318 Homework 9

University of Washington

Attributions: Many of the problems below are inspired by problems in *Introduction to Linear Algebra, Fifth Edition* by Gilbert Strang. In these cases, the problem number from Strang's book is written at the start of the question. The star problems are adapted from the chapters in the book *Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra* by Jiří Matoušek.

(1) **SVD in Julia.** Using the command svd(A) in Julia you can compute the SVD of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 \end{bmatrix}.$$

```
julia> using LinearAlgebra

julia> A = [1 1 0 0 0; 0 1 1 0 1; 1 2 1 0 1]
3-element Array{Int64,2}:
 1  1  0  0  0
 0  1  1  0  1
 1  2  1  0  1

julia> svd(A)
SVD{Float64,Float64,Array{Float64,2}}
U factor:
3-element Array{Float64,2}:
 -0.339287   0.742665   -0.57735
 -0.473523  -0.665163   -0.57735
 -0.81281    0.0775016   0.57735

singular values:
3-element Array{Float64,1}:
 3.253087102270064
 1.1905563006612325
 1.812986607347358e-16

Vt factor:
3-element Array{Float64,2}:
 -0.354155  -0.749574  -0.395419  0.0  -0.395419
  0.688894   0.195291  -0.493603  0.0  -0.493603
  0.632456  -0.632456   0.316228  0.0   0.316228
```

Mark the singular vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{v}_1, \mathbf{v}_2 \ldots$ that you see in the above SVD. You will need to show the grader what you marked. You can use these labels in answering the following questions and do not need to write down the vectors in the about output explicitly.

(a) What is the rank of A? Why?
(b) Find an orthonormal basis for the rowspace of A and columnspace of A.
(c) Which singular vectors lie in the nullspace of $A^\intercal$?
(d) Are some of the above singular vectors in the nullspace of A? If yes, which ones? Do we get a basis for the nullspace of A?

(e) Explain why the fourth column of $V^\top$ is all zero.
(f) Write down the rank one decomposition of A. How many rank one matrices are there in the decomposition?
(g) What are the dimensions of the unit sphere and hyperellipse in the domain and codomain of the linear transformation given by A such that the hyperellipse is the image of the sphere?
(h) What are the semiaxes of the hyperellipse?

(2) **More SVD**
(a) Use the SVD to argue that for any square matrix A of size $n \times n$, $|\det(A)| = \sigma_1\sigma_2\cdots\sigma_n$.
(b) (7.2 # 4) Compute the SVD of the following matrix (please do this by hand and not using software).

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Draw the unit sphere in the domain and its image hyperellipse in the codomain and mark all the left and right singular vectors and $\sigma_i\mathbf{u}_i$.
(c) Are the following true or false? Explain.
(i) A and $A^\top$ can have different singular values.
(ii) Let $A \in \mathbb{R}^{n\times n}$ and $\text{rank}(A) = n$. Then $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$ if and only if all singular values of A equal 1.
(iii) Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be matrices of equal rank. Then A_1 and A_2 have the same singular values if and only if there exist orthonormal matrices Q_1, Q_2 such that $A_1 = Q_1 A_2 Q_2$.

(3) **Best fit planes/PCA** (7.3 #4)
Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 1 & 1 & -1 & -1 \end{bmatrix}.$$

(a) Compute the best fitting line L and best fitting plane P to the four columns of A. Express L and P as the span of vectors.
(b) Compute the rank one decomposition of A, and the approximations A_1 and A_2.
(c) Check that the columns of A_1 are the projections of the columns of A on L.
(d) Check that the third column of A_2 is the projection of the third column of A on P.
(e) Compute the 2-norm of $A - A_1$.
(f) Construct a rank one matrix B of your choice and the same size as A. Check that B is not closer to A than A_1.

(4) **Codes**
Consider the set $C = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\} \subset (\mathbb{Z}_2)^3$.
(a) (i) Argue that C is a linear code, i.e., C is a subspace of $(\mathbb{Z}_2)^3$.
(ii) Find a basis for C.
(iii) Write C as the kernel of a matrix P and verify that $P\mathbf{x} = 0$ for all $\mathbf{x} \in C$. *Notice that* $\mathbf{x} \notin C$ *if and only if* $P\mathbf{x} \neq 0$. *This a handy way to show that something is not a code word.*
(b) A code C is said to be k-separated if $d(C) = k$.
(i) Draw the three-dimensional unit cube and locate the elements of the above code C among its corners.

(ii) Use your picture to verify that C is 2-separated.
Hint: Think about what k-separated means in terms of how many edges of the cube you need to walk along to get from any given code word to another.

(c) We saw the Hamming code in class. It lives in $(\mathbb{Z}_2)^7$.
(i) Argue that $\mathbf{w} = 0111101$ is not in the Hamming code.
(ii) Find the unique decoding of $\mathbf{w}$ in $(\mathbb{Z}_2)^4$.