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AMATH 351
Homework 3
Due July 13, 2023

Problem 1: Existence and uniqueness: first order ODEs. For the following IVP, determine the longest possible interval on which a unique solution is guaranteed to exist *without solving the IVP.*

1. $\ln(t)y' + y = \cot(t), \quad y(2) = 3$

   Solution. First we must put this ODE in standard form. We can do so by dividing each term by $\ln(t)$.
   $$y' + \frac{1}{\ln(t)}y = \frac{\cot(t)}{\ln(t)} \tag{1}$$
   We have that $p(x) = 1/\ln(t)$ and $q(x) = \cot(t)/\ln(t)$.

   $p(x)$ is continuous over the intervals $(0, 1)$ and $(1, \infty)$ - everywhere where ln is defined and does not equal zero.

   For $q(x) = \cot(t)/\ln(t)$, the numerator is continuous when $\tan(t) \neq 0$, so over the intervals $(0, \pi)$ and $(\pi, 2\pi)$, etc. The denominator is continuous over the intervals $(0, 1)$ and $(1, \infty)$. Therefore, $q(x)$ is continuous over $(0, 1), (1, \pi), (\pi, 2\pi)$.

   Since the interval must contain $t_0 = 2$, we must look for some subinterval of $(1, \infty)$. Since $q(t)$ is continuous over the interval $(1, \pi)$, our longest possible continuous sub-interval that also contains t_0 is $\boxed{(1, \pi)}$

For the following ODE, state where in the ty-plane the conditions on f and $\partial f/\partial y$ for existence and uniqueness of solutions are satisfied.

2. $y' = \dfrac{t - y}{2t + 5y}$

Note: I am *not* asking you to find a specific rectangle in the ty-plane on which the Theorem holds. Some useful facts that you can use when solving this problem:

- Multivariate linear functions are continuous. A multivariate function $h(x, y)$ is linear if it takes the form
  $$h(x, y) = ax + by + c$$
  for some constants a, b, and c. (*Note:* this also applies to linear functions of a single variable)
- The quotient of two continuous functions is also a continuous function, as long as its denominator is not equal to zero.
- If $h(x)$ is a continuous univariate function, and $g(x, y)$ is a continuous multivariate function, then their *function composition* $h\big(g(x, y)\big)$ is also continuous.

Solution. First, we will identify the interval on which $f(t, y)$ is continuous. We know that multivariate linear functions are continuous, and since both the numerator and the denominator of f are multivariate linear functions, they are both continuous. We also know that the quotient of two continuous functions is also continuous, except for where its denominator is equal to zero. For this function, we have that the denominator equals zero when $2t+5y = 0$ which means that the denominator equals zero along the line $y = (-2/5)t$. Therefore, f is continuous everywhere except along the line $y = (-2/5)t$.
Next, we will compute $\dfrac{\partial f}{\partial y}$ using the quotient rule.
For this ODE, we have that $\dfrac{\partial f}{\partial y} = \dfrac{(2t + 5y)(-1) - (t - y)(5)}{(2t + 5y)^2} = \dfrac{-2t - 5y - 5t + 5y}{(2t + 5y)^2} = \dfrac{-7t}{(2t + 5y)^2}$

Now, we must identify the interval along which $\partial f/\partial y$ is continuous. We see that the numerator is a univariate linear function, so it is continuous. We also see that the denominator is a function composition where we can have $h(t) = t^2$ and $g(t, y) = 2t + 5y$, then $h(g(t, y)) = (2t + 5y)^2$. Since both $h(t)$ and $g(t, y)$ are continuous, we have that $h(g(t, y))$ is also continuous, therefore the denominator is also continuous. We know that the quotient of two continuous functions is continuous except for where the denominator equals zero, so we can say that $\partial f/\partial y$ is continuous except for when $(2t + 5y)^2 = 0$, which is also along the line $y = (-2/5)t$.

Therefore, since both $f(t, y)$ and $\partial f/\partial y$ are continuous everywhere except on the line $y = (-2/5)t$, the conditions for existence and uniqueness are satisfied when $\boxed{y < (-2/5)t \text{ and } y > (-2/5)t}$.

Problem 2: Homogeneous ODEs with constant coefficients. Find the solution of the following IVPs. Include a plot of the solution, and describe the behavior of the solution as $t \to \infty$:

1. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \; y'(0) = -1$

Solution. First, we will identify the constants a, b, c. For this ODE, we have $a = 1, b = 4, c = 3$. Next, we will find the roots of the characteristic equation. For this ode, we have $\lambda^2 + 4\lambda + 3 = 0$.

$$\lambda^2 + 4\lambda + 3 = 0$$
$$(\lambda + 3)(\lambda + 1) = 0$$
$$\lambda_1 = -1, \lambda_2 = -3$$

From this, we have that $y_1(t) = e^{-t}$ and $y_2(t) = e^{-3t}$, and applying the principle of superposition, we have that the general solution is $y(t) = c_1 e^{-t} + c_2 e^{-3t}$.

Now, we can use the initial conditions to solve for c_1 and c_2.
Note that $y'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$.

Plugging in the first initial condition $y(0) = 2$ we have: $2 = c_1 e^0 + c_2 e^0 \Rightarrow 2 = c_1 + c_2$.

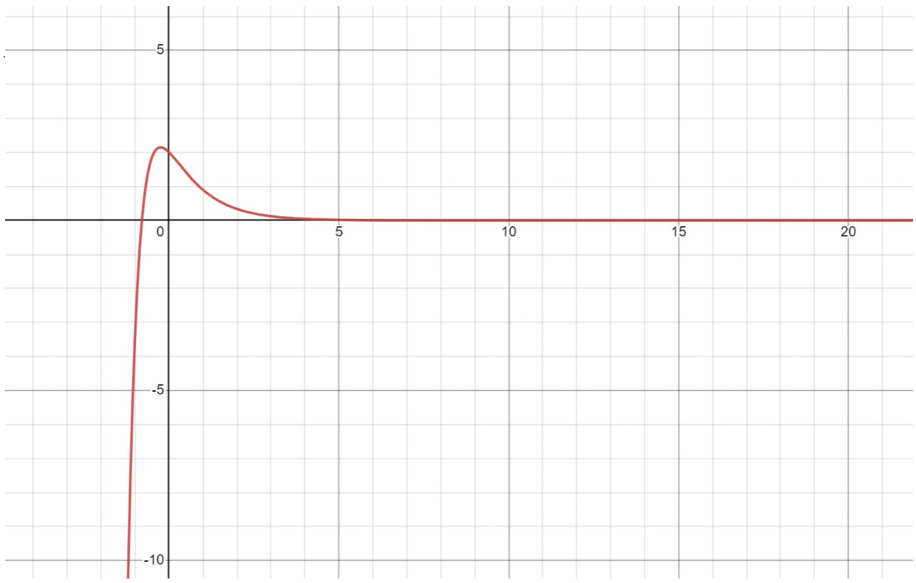
Plugging in the second initial condition $y'(0) = -1$ we have: $-1 = -c_1 e^0 - 3c_2 e^0 \Rightarrow$ $-1 = -c_1 - 3c_2$.

We can solve this system with substitution: $2 = c_1 + c_2 \Rightarrow c_1 = 2 - c_2$. Plugging this into the second equation, $-1 = -(2 - c_2) - 3c_2 \Rightarrow -1 = -2 - c_2 \Rightarrow c_2 = -1/2$. Then, it follows by either equation that $c_1 = 5/2$.

Thus, the solution to this IVP is $\boxed{y(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}}$

As t approached infinity, $y(t)$ goes to zero. We can rewrite $y(t)$ as $\frac{5}{2e^t} - \frac{1}{2e^{3t}}$. We know that e^t approaches infinity as t approaches infinity, and since the e^t term is in the denominator, $y(t)$ will go to zero.

Here is a plot of the solution:



2. $y'' + 3y' = 0, \quad y(0) = -2, \; y'(0) = 3$

Solution. First, we will identify the constants a, b, c. For this ODE, we have $a = 1, b = 3, c = 0$. Next, we will find the roots of the characteristic equation. For this ODE, we have $\lambda^2 + 3\lambda = 0$.

$$\lambda^2 + 3\lambda = 0$$
$$\lambda(\lambda + 3) = 0$$
$$\lambda_1 = 0, \lambda_2 = -3$$

From this, we have that $y_1(x) = 1$ and $y_2(x) = e^{-3x}$, and applying the principle of superposition, we have that the general solution is $y(x) = c_1 + c_2 e^{-3x}$.

Now, we can use the initial conditions to solve for c_1 and c_2. Note that $y'(x) = -3c_2 e^{-3x}$.

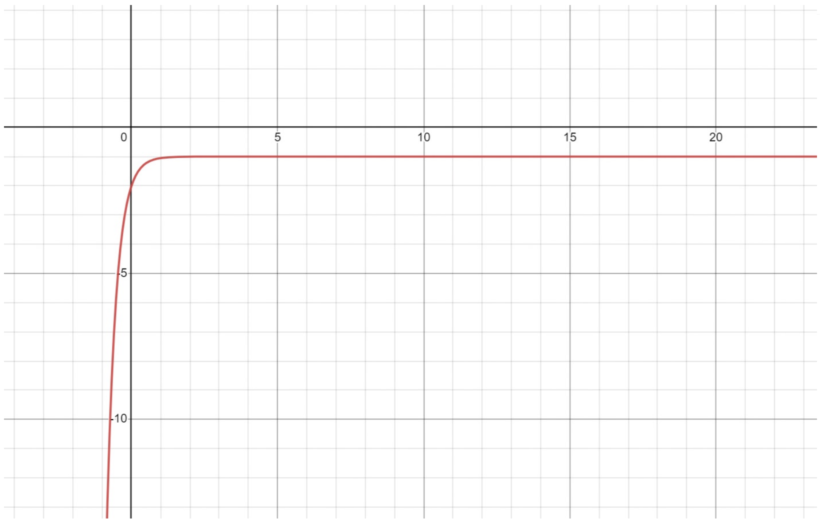
Plugging in the first initial condition $y(0) = -2$ we have: $-2 = c_1 + c_2 e^{-3(0)} \Rightarrow -2 = c_1 + c_2$.

Plugging in the second initial condition $y'(0) = 3$ we have: $3 = -3c_2 e^{-3(0)} \Rightarrow 3 = -3c_2$. We can solve this directly to obtain that $c_2 = -1$. Then, it follows from the first equation that $c_1 = -1$.

Thus, the solution to this IVP is $\boxed{y(x) = -1 - e^{-3x}}$

As x approaches infinity, $y(x)$ approaches -1. We can rewrite $y(x)$ as $-1 - 1/e^{3x}$ and we know e^{3x} approaches infinity as t approaches infinity, and since it's in the denominator, the $1/e^{3x}$ term will approach zero. Since -1 is a constant, its value doesn't change with t, so we can see that as t approaches infinity, the $1/e^{3x}$ term will approach zero so $y(x)$ approaches -1.

Here is a plot of the solution:



3. $y'' + 6y' + 9y = 0, \quad y(1) = 3, \; y'(1) = 1$

Solution. First, we will identify the constants a, b, c. For this ODE, we have $a = 1, b = 6, c = 9$. Next, we will find the roots of the characteristic equation. For this ODE, we have $\lambda^2 + 6\lambda + 9 = 0$.

$$\begin{aligned} \lambda^2 + 6\lambda + 9 &= 0 \\ (\lambda + 3)^2 &= 0 \\ \lambda_{1,3} &= -3 \end{aligned}$$

From this, we can say that $y_1(x) = e^{-3x}$ is one solution. However, since we have a repeated root, we must apply the reduction of order method to obtain two linearly independent solutions. Reduction of order tells us that $y_2 = xe^{\lambda_1 x} \Rightarrow y_2 = xe^{-3x}$. Now, we can apply the principle of superposition and obtain our general solution: $y(x) = c_1 e^{-3x} + c_2 xe^{-3x}$

Now, we can use our initial conditions to solve for c_1 and c_2. Plugging in the first initial condition $y(1) = 3$ gives us: $3 = c_1 e^{-3} + c - 2(1)e^{-3} \Rightarrow 3 = (c_1 + c_2)e^{-3}$.

Now we will plug in our second initial condition $y'(1) = 1$. Note that $y'(x) = -3c_1 e^{-3x} + c_2 e^{-3x} + -3c_2 xe^{-3x}$. Then, we have that $1 = -3c_1 e^{-3} + c_2 e^{-3} - 3c_2 e^{-3} \Rightarrow 1 = -3c_1 e^{-3} - 2c_2 e^{-3}$ $\Rightarrow 1 = (-3c_1 - 2c_2)e^{-3}$.

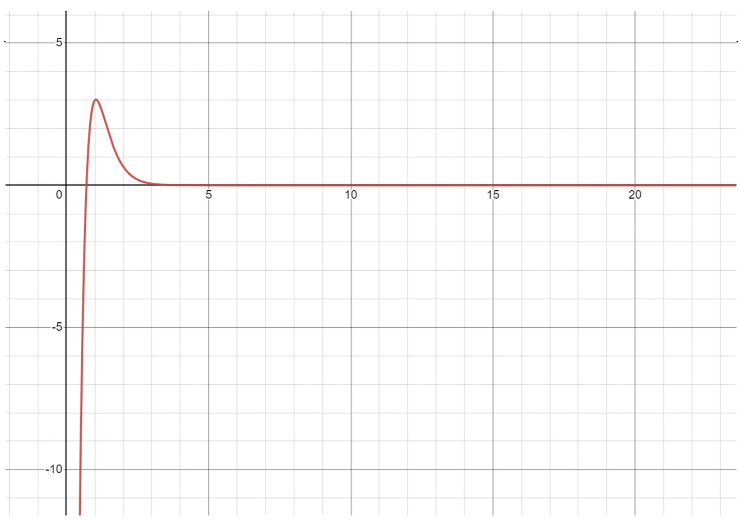
We can then solve this system via substitution. Rearranging the first equation gives $c_1 + c_2 = 3e^3 \Rightarrow c_1 = 3e^3 - c_2$. Similarly, we can rearrange the second equation: $-3c_1 - 2c_2 = e^3$. Now substituting the first into the second, we have:

$$\begin{aligned} e^3 &= -3c_1 - 2c_2 \\ &= -3(3e^3 - c_2) - 2c_2 \\ &= -9e^3 + 3c_2 - 2c_2 \\ &= -9e^3 + c_2 \\ &\Rightarrow c_2 = 10e^3 \end{aligned}$$

It follows from either equation that $c_1 = -7e^3$.

Then, the final solution to this IVP is $\boxed{y(x) = -7e^3 e^{-3x} + 10e^3 xe^{-3x}}$

As x approaches infinity, the first term will go to zero since e^{-3x} approaches zero as x goes to infinity. The second term will also go to zero - although x is a factor, e^{-3x} is exponential while x is linear, so the behavior of the e^{-3x} factor is dominant so this term also goes to zero. Overall, since both terms approach zero as t goes to infinity, y approaches zero as x goes to infinity.



Problem 3: Existence and uniqueness: second order linear ODEs. For the following IVP, determine the longest possible interval on which a unique solution is guaranteed to exist without finding the solution to the IVP. $(x-2)y'' + y' + (x-2)\tan(x)y = 0, \quad y(3) = 1, \; y'(3) = 2.$

Solution. First, we need to put this ODE in standard form by dividing each term by $(x-2)$:

$$y'' + \frac{1}{x-2}y' + \tan(x)y = 0$$

From this ODE, we see that $p(x) = \frac{1}{x-2}$ and $q(x) = \tan(x)$ and $g(x) = 0$

$g(x)$ is just a constant, so it is certainly continuous everywhere - we can say that it's continuous over the interval $(-\infty, \infty)$.

$p(x)$ is continuous everywhere but $x = 2$, so we can write that it's continuous over the intervals $(-\infty, 2)$ and $(2, \infty)$.

$q(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$ is continuous wherever $\cos(x) \neq 0$, for instance, over the intervals $(0, \pi/2)$, $(\pi/2, 3\pi/2)$, etc.

We need to identify the longest possible interval that contains t_0 and where both p and q are continuous. Since $3 > 2$ we need to choose the intersection between $(2, \infty)$ and the region where $\tan(x)$ is continuous. The largest such region where this IVP is guaranteed to have a unique solution is over the open interval $\boxed{(2, 3\pi/2)}$.

Problem 4: Linear independence & the Wronskian. Compute the Wronskian of the following pairs of functions, and state whether or not they are linearly independent.

1. $y_1(t) = e^{\lambda t}\cos(kt), \quad y_2(t) = e^{\lambda t}\sin(kt)$, where $k > 0$.

   Solution. To compute the Wronskian, we use the following formula: $W(f,g) = f(t)g'(t) - f'(t)g(t)$ where $f(t) = y_1(t)$ and $g(x) = y_2(t)$.
   Note that $y_1'(t) = \lambda e^{\lambda t}\cos(kt) - ke^{\lambda t}\sin(kt)$ and $y_2'(t) = \lambda e^{\lambda t}\sin(kt) + ke^{\lambda t}\cos(kt)$

$$\begin{aligned}
W(f,g) &= f(x)g'(x) - f'(x)g(x)\\
&= y_1y_2' - y_1'y_2\\
&= (e^{\lambda t}\cos(kt))(\lambda e^{\lambda t}\sin(kt) + ke^{\lambda t}\cos(kt)) - (\lambda e^{\lambda t}\cos(kt) - ke^{\lambda t}\sin(kt))(e^{\lambda t}\sin(kt))\\
&= (\lambda e^{2\lambda t}\cos(kt)\sin(kt)) + (ke^{2\lambda t}\cos(kt)\cos(kt)) - (\lambda e^{2\lambda t}\cos(kt)\sin(kt)) + ke^{2\lambda t}\sin(kt)\sin(kt)\\
&= (\lambda e^{2\lambda t}\cos(kt)\sin(kt)) + (ke^{2\lambda t}\cos^2(kt)) - (\lambda e^{2\lambda t}\cos(kt)\sin(kt)) + ke^{2\lambda t}\sin^2(kt))\\
&= e^{2\lambda t}(\lambda\cos(kt)\sin(kt) + k\cos^2(t) - \lambda\cos(kt)\sin(kt) + k\sin^2(kt))\\
&= ke^{2\lambda t}(\cos^2(kt) + \sin^2(kt))\\
&= \boxed{ke^{2\lambda t}}
\end{aligned}$$

   We know $k > 0$ and $e^{2\lambda t} \neq 0$ therefore the Wronskian $\neq 0$ so y_1 and y_2 are linearly independent.

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2. $y_1(t) = t^\lambda\cos\big(k\ln(t)\big), \quad y_2(t) = t^\lambda\sin\big(k\ln(t)\big)$, where $t > 0$ and $k > 0$.

   Solution. To compute the Wronskian, we use the following formula: $W(f,g) = f(t)g'(t) - f'(t)g(t)$ where $f(t) = y_1(t)$ and $g(x) = y_2(t)$.
   Note that $y_1'(x) = \lambda t^{\lambda-1}\cos(k\ln(t)) - t^\lambda\sin(k\ln(t))(k/t)$ and $y_2'(x) = \lambda t^{\lambda-1}\sin(k\ln(t)) + t^\lambda\cos(k\ln(t))(k/t)$

$$\begin{aligned}
W(f,g) &= f(x)g'(x) - f'(x)g(x)\\
&= y_1y_2' - y_1'y_2\\
&= (t^\lambda\cos\big(k\ln(t)\big))(\lambda t^{\lambda-1}\sin(k\ln(t)) + kt^{\lambda-1}\cos(k\ln(t))) -\\
&\quad\ (\lambda t^{\lambda-1}\cos(k\ln(t)) - kt^{\lambda-1}\sin(k\ln(t)))(t^\lambda\sin\big(k\ln(t)\big))\\
&= \lambda t^{2\lambda-1}\cos(k\ln(t))\sin(k\ln(t)) + kt^{2\lambda-1}\cos^2(k\ln(t)) - \lambda t^{2\lambda-1}\cos(k\ln(t))\sin(k\ln(t))\\
&\quad + kt^{2\lambda-1}\sin^2(k\ln(t))\\
&= kt^{2\lambda-1}(\sin^2(k\ln(t)) + \cos^2(k\ln(t)))\\
&= \boxed{kt^{2\lambda-1}}
\end{aligned}$$

   We know $k > 0$ and $t > 0$, and so $t^{2\lambda-1} \neq 0$ therefore the Wronskian $\neq 0$ so y_1 and y_2 are linearly independent.

Problem 5: Abel's theorem: computing the Wronskian of solutions. For each of the following ODEs, find the Wronskian of any two solutions of the equation *without solving the equation.*

1. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, where ν is a constant.
   Note: This ODE is known as *Bessel's equation.*

   Solution. First we must put the ODE in standard form, so we will divide each term by x^2.

$$y'' + \frac{1}{x}y' + \frac{x^2 - \nu^2}{x^2}y = 0$$

   From this, we have that $p(x) = 1/x$, so we can calculate the Wronskian as follows:

$$\begin{aligned} W &= ce^{-\int p(x)dx} \\ &= ce^{-\int (1/x)dx} \\ &= \boxed{ce^{-\ln(x)}} \end{aligned}$$

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2. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$, where α is a constant.
   Note: This ODE is known as *Legendre's equation.*

   Solution. First, we must put the ODE in standard form, so we will divide each term by $(1 - x^2)$:

$$y'' - \frac{2x}{(1 - x^2)}y' + \frac{\alpha(\alpha + 1)}{1 - x^2}y = 0$$

   From this we have that $p(x) = -\frac{2x}{(1-x^2)}$ so we can calculate the Wronskian as follows:

$$\begin{aligned} W &= ce^{-\int p(x)dx} \\ &= ce^{-\int -\frac{2x}{(1-x^2)}dx} \\ &= ce^{\int \frac{2x}{(1-x^2)}dx} \\ &= \boxed{ce^{-\ln(1-x^2)}} \end{aligned}$$
