A Summations

When an algorithm contains an iterative control construct such as a **while** or **for** loop, we can express its running time as the sum of the times spent on each execution of the body of the loop. For example, we found in Section 2.2 that the jth iteration of insertion sort took time proportional to j in the worst case. By adding up the time spent on each iteration, we obtained the summation (or series)

$$\sum_{j=2}^{n} j \ .$$

When we evaluated this summation, we attained a bound of $\Theta(n^2)$ on the worst-case running time of the algorithm. This example illustrates why you should know how to manipulate and bound summations.

Section A.1 lists several basic formulas involving summations. Section A.2 offers useful techniques for bounding summations. We present the formulas in Section A.1 without proof, though proofs for some of them appear in Section A.2 to illustrate the methods of that section. You can find most of the other proofs in any calculus text.

A.1 Summation formulas and properties

Given a sequence $a_1, a_2, \ldots, a_n$ of numbers, where n is a nonnegative integer, we can write the finite sum $a_1 + a_2 + \cdots + a_n$ as

$$\sum_{k=1}^{n} a_k \ .$$

If $n = 0$, the value of the summation is defined to be 0. The value of a finite series is always well defined, and we can add its terms in any order.

Given an infinite sequence $a_1, a_2, \ldots$ of numbers, we can write the infinite sum $a_1 + a_2 + \cdots$ as

$$\sum_{k=1}^{\infty} a_k \ ,$$

which we interpret to mean

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k \ .$$

If the limit does not exist, the series ***diverges***; otherwise, it ***converges***. The terms of a convergent series cannot always be added in any order. We can, however, rearrange the terms of an ***absolutely convergent series***, that is, a series $\sum_{k=1}^{\infty} a_k$ for which the series $\sum_{k=1}^{\infty} |a_k|$ also converges.

Linearity

For any real number c and any finite sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$\sum_{k=1}^{n} (ca_k + b_k) = c \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \ .$$

The linearity property also applies to infinite convergent series.

We can exploit the linearity property to manipulate summations incorporating asymptotic notation. For example,

$$\sum_{k=1}^{n} \Theta(f(k)) = \Theta\left(\sum_{k=1}^{n} f(k)\right) \ .$$

In this equation, the Θ-notation on the left-hand side applies to the variable k, but on the right-hand side, it applies to n. We can also apply such manipulations to infinite convergent series.

Arithmetic series

The summation

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n \ ,$$

is an ***arithmetic series*** and has the value

$$\sum_{k=1}^{n} k = \frac{1}{2} n(n+1) \tag{A.1}$$

$$= \Theta(n^2) \ . \tag{A.2}$$

Sums of squares and cubes

We have the following summations of squares and cubes:

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} , \tag{A.3}$$

$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} . \tag{A.4}$$

Geometric series

For real $x \neq 1$, the summation

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

is a ***geometric*** or ***exponential series*** and has the value

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} . \tag{A.5}$$

When the summation is infinite and $|x| < 1$, we have the infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} . \tag{A.6}$$

Harmonic series

For positive integers n, the nth ***harmonic number*** is

$$\begin{aligned} H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \\ &= \sum_{k=1}^{n} \frac{1}{k} \\ &= \ln n + O(1) . \end{aligned} \tag{A.7}$$

(We shall prove a related bound in Section A.2.)

Integrating and differentiating series

By integrating or differentiating the formulas above, additional formulas arise. For example, by differentiating both sides of the infinite geometric series (A.6) and multiplying by x, we get

$$\sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2} \tag{A.8}$$

for $|x| < 1$.

Telescoping series

For any sequence $a_0, a_1, \ldots, a_n$,

$$\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0 , \tag{A.9}$$

since each of the terms $a_1, a_2, \ldots, a_{n-1}$ is added in exactly once and subtracted out exactly once. We say that the sum ***telescopes***. Similarly,

$$\sum_{k=0}^{n-1} (a_k - a_{k+1}) = a_0 - a_n .$$

As an example of a telescoping sum, consider the series

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} .$$

Since we can rewrite each term as

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} ,$$

we get

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{1}{k(k+1)} &= \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= 1 - \frac{1}{n} . \end{aligned}$$

Products

We can write the finite product $a_1 a_2 \cdots a_n$ as

$$\prod_{k=1}^{n} a_k .$$

If $n = 0$, the value of the product is defined to be 1. We can convert a formula with a product to a formula with a summation by using the identity

$$\lg \left(\prod_{k=1}^{n} a_k \right) = \sum_{k=1}^{n} \lg a_k .$$

Exercises

A.1-1
Find a simple formula for $\sum_{k=1}^{n}(2k-1)$.

A.1-2 ★
Show that $\sum_{k=1}^{n} 1/(2k-1) = \ln(\sqrt{n}) + O(1)$ by manipulating the harmonic series.

A.1-3
Show that $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ for $0 < |x| < 1$.

A.1-4 ★
Show that $\sum_{k=0}^{\infty}(k-1)/2^k = 0$.

A.1-5 ★
Evaluate the sum $\sum_{k=1}^{\infty}(2k+1)x^{2k}$.

A.1-6
Prove that $\sum_{k=1}^{n} O(f_k(i)) = O\bigl(\sum_{k=1}^{n} f_k(i)\bigr)$ by using the linearity property of summations.

A.1-7
Evaluate the product $\prod_{k=1}^{n} 2 \cdot 4^k$.

A.1-8 ★
Evaluate the product $\prod_{k=2}^{n}(1 - 1/k^2)$.

A.2 Bounding summations

We have many techniques at our disposal for bounding the summations that describe the running times of algorithms. Here are some of the most frequently used methods.

Mathematical induction

The most basic way to evaluate a series is to use mathematical induction. As an example, let us prove that the arithmetic series $\sum_{k=1}^{n} k$ evaluates to $\frac{1}{2}n(n+1)$. We can easily verify this assertion for $n = 1$. We make the inductive assumption that

it holds for n, and we prove that it holds for $n + 1$. We have

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\ &= \frac{1}{2}n(n+1) + (n+1) \\ &= \frac{1}{2}(n+1)(n+2) .\end{aligned}$$

You don't always need to guess the exact value of a summation in order to use mathematical induction. Instead, you can use induction to prove a bound on a summation. As an example, let us prove that the geometric series $\sum_{k=0}^{n} 3^k$ is $O(3^n)$. More specifically, let us prove that $\sum_{k=0}^{n} 3^k \le c3^n$ for some constant c. For the initial condition $n = 0$, we have $\sum_{k=0}^{0} 3^k = 1 \le c \cdot 1$ as long as $c \ge 1$. Assuming that the bound holds for n, let us prove that it holds for $n + 1$. We have

$$\begin{aligned}\sum_{k=0}^{n+1} 3^k &= \sum_{k=0}^{n} 3^k + 3^{n+1} \\ &\le c3^n + 3^{n+1} \qquad \text{(by the inductive hypothesis)} \\ &= \left(\frac{1}{3} + \frac{1}{c}\right) c3^{n+1} \\ &\le c3^{n+1}\end{aligned}$$

as long as $(1/3 + 1/c) \le 1$ or, equivalently, $c \ge 3/2$. Thus, $\sum_{k=0}^{n} 3^k = O(3^n)$, as we wished to show.

We have to be careful when we use asymptotic notation to prove bounds by induction. Consider the following fallacious proof that $\sum_{k=1}^{n} k = O(n)$. Certainly, $\sum_{k=1}^{1} k = O(1)$. Assuming that the bound holds for n, we now prove it for $n + 1$:

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\ &= O(n) + (n+1) \qquad \Longleftarrow \textit{wrong!!} \\ &= O(n+1) .\end{aligned}$$

The bug in the argument is that the "constant" hidden by the "big-oh" grows with n and thus is not constant. We have not shown that the same constant works for *all* n.

Bounding the terms

We can sometimes obtain a good upper bound on a series by bounding each term of the series, and it often suffices to use the largest term to bound the others. For

example, a quick upper bound on the arithmetic series (A.1) is

$$\begin{aligned}\sum_{k=1}^{n} k &\le \sum_{k=1}^{n} n \\ &= n^2 .\end{aligned}$$

In general, for a series $\sum_{k=1}^{n} a_k$, if we let $a_{\max} = \max_{1 \le k \le n} a_k$, then

$$\sum_{k=1}^{n} a_k \le n \cdot a_{\max} .$$

The technique of bounding each term in a series by the largest term is a weak method when the series can in fact be bounded by a geometric series. Given the series $\sum_{k=0}^{n} a_k$, suppose that $a_{k+1}/a_k \le r$ for all $k \ge 0$, where $0 < r < 1$ is a constant. We can bound the sum by an infinite decreasing geometric series, since $a_k \le a_0 r^k$, and thus

$$\begin{aligned}\sum_{k=0}^{n} a_k &\le \sum_{k=0}^{\infty} a_0 r^k \\ &= a_0 \sum_{k=0}^{\infty} r^k \\ &= a_0 \frac{1}{1-r} .\end{aligned}$$

We can apply this method to bound the summation $\sum_{k=1}^{\infty} (k/3^k)$. In order to start the summation at $k = 0$, we rewrite it as $\sum_{k=0}^{\infty} ((k+1)/3^{k+1})$. The first term ($a_0$) is $1/3$, and the ratio (r) of consecutive terms is

$$\begin{aligned}\frac{(k+2)/3^{k+2}}{(k+1)/3^{k+1}} &= \frac{1}{3} \cdot \frac{k+2}{k+1} \\ &\le \frac{2}{3}\end{aligned}$$

for all $k \ge 0$. Thus, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{k}{3^k} &= \sum_{k=0}^{\infty} \frac{k+1}{3^{k+1}} \\ &\le \frac{1}{3} \cdot \frac{1}{1-2/3} \\ &= 1 .\end{aligned}$$

A common bug in applying this method is to show that the ratio of consecutive terms is less than 1 and then to assume that the summation is bounded by a geometric series. An example is the infinite harmonic series, which diverges since

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k} &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}\\ &= \lim_{n\to\infty}\Theta(\lg n)\\ &= \infty\,.\end{aligned}$$

The ratio of the $(k+1)$st and kth terms in this series is $k/(k+1) < 1$, but the series is not bounded by a decreasing geometric series. To bound a series by a geometric series, we must show that there is an $r < 1$, which is a *constant*, such that the ratio of all pairs of consecutive terms never exceeds r. In the harmonic series, no such r exists because the ratio becomes arbitrarily close to 1.

Splitting summations

One way to obtain bounds on a difficult summation is to express the series as the sum of two or more series by partitioning the range of the index and then to bound each of the resulting series. For example, suppose we try to find a lower bound on the arithmetic series $\sum_{k=1}^{n} k$, which we have already seen has an upper bound of n^2. We might attempt to bound each term in the summation by the smallest term, but since that term is 1, we get a lower bound of n for the summation—far off from our upper bound of n^2.

We can obtain a better lower bound by first splitting the summation. Assume for convenience that n is even. We have

$$\begin{aligned}\sum_{k=1}^{n} k &= \sum_{k=1}^{n/2} k + \sum_{k=n/2+1}^{n} k\\ &\ge \sum_{k=1}^{n/2} 0 + \sum_{k=n/2+1}^{n} (n/2)\\ &= (n/2)^2\\ &= \Omega(n^2)\,,\end{aligned}$$

which is an asymptotically tight bound, since $\sum_{k=1}^{n} k = O(n^2)$.

For a summation arising from the analysis of an algorithm, we can often split the summation and ignore a constant number of the initial terms. Generally, this technique applies when each term a_k in a summation $\sum_{k=0}^{n} a_k$ is independent of n.

Then for any constant $k_0 > 0$, we can write

$$\begin{aligned}\sum_{k=0}^{n} a_k &= \sum_{k=0}^{k_0-1} a_k + \sum_{k=k_0}^{n} a_k \\ &= \Theta(1) + \sum_{k=k_0}^{n} a_k ,\end{aligned}$$

since the initial terms of the summation are all constant and there are a constant number of them. We can then use other methods to bound $\sum_{k=k_0}^{n} a_k$. This technique applies to infinite summations as well. For example, to find an asymptotic upper bound on

$$\sum_{k=0}^{\infty} \frac{k^2}{2^k} ,$$

we observe that the ratio of consecutive terms is

$$\begin{aligned}\frac{(k+1)^2/2^{k+1}}{k^2/2^k} &= \frac{(k+1)^2}{2k^2} \\ &\le \frac{8}{9}\end{aligned}$$

if $k \ge 3$. Thus, the summation can be split into

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{k^2}{2^k} &= \sum_{k=0}^{2} \frac{k^2}{2^k} + \sum_{k=3}^{\infty} \frac{k^2}{2^k} \\ &\le \sum_{k=0}^{2} \frac{k^2}{2^k} + \frac{9}{8} \sum_{k=0}^{\infty} \left(\frac{8}{9}\right)^k \\ &= O(1) ,\end{aligned}$$

since the first summation has a constant number of terms and the second summation is a decreasing geometric series.

The technique of splitting summations can help us determine asymptotic bounds in much more difficult situations. For example, we can obtain a bound of $O(\lg n)$ on the harmonic series (A.7):

$$H_n = \sum_{k=1}^{n} \frac{1}{k} .$$

We do so by splitting the range 1 to n into $\lfloor \lg n \rfloor + 1$ pieces and upper-bounding the contribution of each piece by 1. For $i = 0, 1, \ldots, \lfloor \lg n \rfloor$, the ith piece consists

of the terms starting at $1/2^i$ and going up to but not including $1/2^{i+1}$. The last piece might contain terms not in the original harmonic series, and thus we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} &\le \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i-1} \frac{1}{2^i + j} \\
&\le \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i-1} \frac{1}{2^i} \\
&= \sum_{i=0}^{\lfloor \lg n \rfloor} 1 \\
&\le \lg n + 1 . \qquad \text{(A.10)}
\end{aligned}$$

Approximation by integrals

When a summation has the form $\sum_{k=m}^{n} f(k)$, where $f(k)$ is a monotonically increasing function, we can approximate it by integrals:

$$\int_{m-1}^{n} f(x)\,dx \le \sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)\,dx . \qquad \text{(A.11)}$$

Figure A.1 justifies this approximation. The summation is represented as the area of the rectangles in the figure, and the integral is the shaded region under the curve. When $f(k)$ is a monotonically decreasing function, we can use a similar method to provide the bounds

$$\int_{m}^{n+1} f(x)\,dx \le \sum_{k=m}^{n} f(k) \le \int_{m-1}^{n} f(x)\,dx . \qquad \text{(A.12)}$$

The integral approximation (A.12) gives a tight estimate for the nth harmonic number. For a lower bound, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} &\ge \int_{1}^{n+1} \frac{dx}{x} \\
&= \ln(n+1) . \qquad \text{(A.13)}
\end{aligned}$$

For the upper bound, we derive the inequality

$$\begin{aligned}
\sum_{k=2}^{n} \frac{1}{k} &\le \int_{1}^{n} \frac{dx}{x} \\
&= \ln n ,
\end{aligned}$$

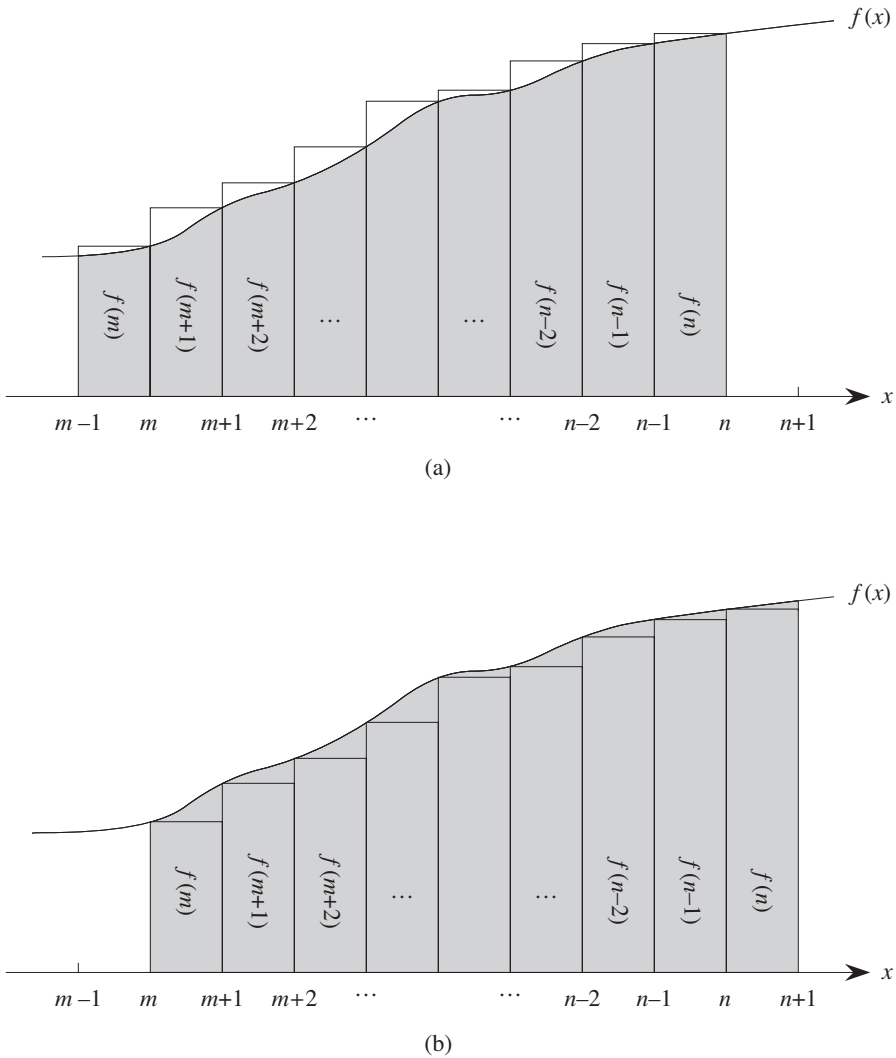

Figure A.1 Approximation of $\sum_{k=m}^{n} f(k)$ by integrals. The area of each rectangle is shown within the rectangle, and the total rectangle area represents the value of the summation. The integral is represented by the shaded area under the curve. By comparing areas in **(a)**, we get $\int_{m-1}^{n} f(x)\,dx \le \sum_{k=m}^{n} f(k)$, and then by shifting the rectangles one unit to the right, we get $\sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)\,dx$ in **(b)**.

which yields the bound

$$\sum_{k=1}^{n} \frac{1}{k} \le \ln n + 1 \; . \tag{A.14}$$

Exercises

A.2-1
Show that $\sum_{k=1}^{n} 1/k^2$ is bounded above by a constant.

A.2-2
Find an asymptotic upper bound on the summation

$$\sum_{k=0}^{\lfloor \lg n \rfloor} \lceil n/2^k \rceil \; .$$

A.2-3
Show that the nth harmonic number is $\Omega(\lg n)$ by splitting the summation.

A.2-4
Approximate $\sum_{k=1}^{n} k^3$ with an integral.

A.2-5
Why didn't we use the integral approximation (A.12) directly on $\sum_{k=1}^{n} 1/k$ to obtain an upper bound on the nth harmonic number?

Problems

A-1 Bounding summations
Give asymptotically tight bounds on the following summations. Assume that $r \ge 0$ and $s \ge 0$ are constants.

a. $\displaystyle\sum_{k=1}^{n} k^r$.

b. $\displaystyle\sum_{k=1}^{n} \lg^s k$.

c. $\displaystyle\sum_{k=1}^{n} k^r \lg^s k$.

Appendix notes

Knuth [209] provides an excellent reference for the material presented here. You can find basic properties of series in any good calculus book, such as Apostol [18] or Thomas et al. [334].