

CS101 Algorithms and Data Structures  
Fall 2023  
Homework 3

Due date: 23:59, October 29th, 2023

1. Please write your solutions in English.
2. Submit your solutions to [gradescope.com](https://gradescope.com).
3. Set your FULL name to your Chinese name and your STUDENT ID correctly in Account Settings.
4. If you want to submit a handwritten version, scan it clearly. **CamScanner** is recommended.
5. When submitting, match your solutions to the problems correctly.
6. No late submission will be accepted.
7. Violations to any of the above may result in zero points.

**1. (6 points) Sorting practice**

Given an array:

1, 4, 7, 3, 5, 2, 8, 6

- (a) (2') Run Insertion Sort for this array. Write down the array
- after each**
- outer iteration.

```
for(int k = 1; k < n; k++){  
    for(int j = k; j > 0; j--){  
        if( array[j - 1] > array[j] )  
            swap(array[j - 1], array[j]);  
        else  
            break;  
    }  
    print(array);  
}
```

**Solution:**

```
1, 4, 7, 3, 5, 2, 8, 6  
1, 4, 7, 3, 5, 2, 8, 6  
1, 3, 4, 7, 5, 2, 8, 6  
1, 3, 4, 5, 7, 2, 8, 6  
1, 2, 3, 4, 5, 7, 8, 6  
1, 2, 3, 4, 5, 7, 8, 6  
1, 2, 3, 4, 5, 6, 7, 8
```

- (b) (2') Run Flagged Bubble Sort for this array. Write down the array
- after each**
- outer iteration.

```
for(int i = n-1; i > 0; i--){  
    int max_t = array[0];  
    bool sorted = true;  
    for(int j = 1; j <= i; j++){  
        if(array[j] < max_t){  
            array[j-1] = array[j];  
            sorted = false;  
        }  
        else{  
            array[j-1] = max_t;  
            max_t = array[j];  
        }  
    }  
    array[i] = max_t;  
    print(array);  
}
```

```
    if(sorted) break;  
}
```

**Solution:**

1, 4, 3, 5, 2, 7, 6, 8  
1, 3, 4, 2, 5, 6, 7, 8  
1, 3, 2, 4, 5, 6, 7, 8  
1, 2, 3, 4, 5, 6, 7, 8

- (c) (2') Run Merge Sort for this array. Write down the array **after each** merge and underline the sub-array being merged.

**Solution:**

1, 4, 3, 7, 2, 5, 6, 8  
1, 3, 4, 7, 2, 5, 6, 8  
1, 2, 3, 4, 5, 6, 7, 8

**2. (6 points) Which Sort?**

Given a sequence

$$A = \langle 5, 4, 8, 7, 6, 2, 1, 3, 7, 9 \rangle,$$

we have performed different sorting algorithms and printed the intermediate results. Note that the steps below are **not** necessarily consecutive steps in the algorithm, but they are guaranteed to be in the correct order.

For each group of steps, guess ( $\checkmark$ ) what the algorithm is. The algorithm might be one among the following choices:

- Insertion-sort, implemented in the way that avoids swapping elements
- Flagged Bubble-sort
- Merge-sort

(a) (2')

$$\langle 4, 5, 7, 8, 6, 2, 1, 3, 7, 9 \rangle,$$

$$\langle 2, 4, 5, 6, 7, 8, 1, 3, 7, 9 \rangle,$$

$$\langle 1, 2, 3, 4, 5, 6, 7, 8, 7, 9 \rangle.$$

☒ **Insertion-sort**   ☐ Bubble-sort   ☐ Merge-sort

(b) (2')

$$\langle 4, 5, 6, 7, 8, 2, 1, 3, 7, 9 \rangle,$$

$$\langle 4, 5, 6, 7, 8, 1, 2, 3, 7, 9 \rangle,$$

$$\langle 1, 2, 3, 4, 5, 6, 7, 7, 8, 9 \rangle.$$

☐ Insertion-sort   ☐ Bubble-sort   ☒ **Merge-sort**

(c) (2')

$$\langle 4, 5, 7, 6, 8, 2, 1, 3, 7, 9 \rangle,$$

$$\langle 4, 5, 7, 6, 2, 1, 3, 7, 8, 9 \rangle,$$

$$\langle 4, 5, 6, 2, 1, 3, 7, 7, 8, 9 \rangle.$$

☐ Insertion-sort   ☒ **Bubble-sort**   ☐ Merge-sort

**3. (8 points) Multiple Choices**

Each question has **one or more** correct answer(s). Select all the correct answer(s). For each question, you will get 0 points if you select one or more wrong answers, but you will get 1 point if you select a non-empty subset of the correct answers.

Write your answers in the following table.

(a)	(b)	(c)	(d)
AB(C)	ABC	D	BC

(a) (2') which of the sorts are in-place sorting algorithm?

**A. Insertion-sort**

**B. Bubble-sort**

C. Merge-sort

D. None of all

(b) (2') Which of the following situations are **true** for an array of  $n$  random numbers?

**A. The number of inversions in this array can be found by applying a recursive algorithm adapted from merge-sort in  $\Theta(n \log n)$  time.**

**B. If it has no more than  $n$  inversions, it can be sorted in  $O(n)$  time.**

**C. If it has exactly  $n(n-1)/2$  inversions, it can be sorted in  $O(n)$  time.**

D. If the array is  $\langle 8, 6, 3, 7, 4 \rangle$ , there are 6 inversions.

**Solution:**

A. One popular way to compute the number of inversions is to use a divide-and-conquer algorithm, which is quite similar to merge-sort and runs in  $\Theta(n \log n)$  time.

C. If the array has exactly  $n(n-1)/2$  inversions, it is reversely sorted so that it can be sorted simply by a *reverse* operation, which is  $O(n)$ .

(c) (2') Which of the following statements are true?

A. For an array of length  $n$ , in the  $k$ -th iteration of insertion-sort, finding a correct position for a new element to be inserted at takes  $\Theta(k)$  time. If we use *binary-search* instead (which takes  $\Theta(\log k)$  time), it is possible to optimize the total running time to  $\Theta(n \log n)$ .

B. A sorting algorithm is stable if its worst-case time complexity is the same as its best-case time complexity.

C. Merge-sort requires  $\Theta(\log n)$  extra space when sorting an array of  $n$  elements.

**D. Given 2 sorted lists of size  $m$  and  $n$  respectively, and we want to merge them to one sorted list by mergesort. Then in the worst case, we need  $m + n - 1$  comparisons.**

**Solution:**

A. We still need  $\Theta(k)$  time to move elements so the total running time is still  $\Theta(n^2)$ .

D. The worst case is that all elements in two arrays are compared once.

- (d) (2') Choose the situation where **insertion sort** is the most suitable among insertion sort, bubble sort and merge sort. Your choice should be the one that satisfies all the special constraints and is most efficient.
- A. Sorting an array of coordinates of points  $\langle (x_1, y_1), \dots, (x_n, y_n) \rangle$  on a 2d plane in ascending order of the  $x$  coordinate, while preserving the original order of the  $y$  coordinate for any pair of elements  $(x_i, y_i), (x_j, y_j)$  with  $x_i = x_j$ .
  - B. Sorting an array that is *almost* sorted with only  $n/2$  inversions.**
  - C. Sorting an array on an embedded system with quite limited memory. You may only use  $\Theta(1)$  extra space, but a high time cost is acceptable.**
  - D. Given a database with a large number of records. You need to sort the record with high efficiency.

**Solution:**

- B. Here using insertion sort is  $\Theta(n)$ .
- C. We need to use in-place sort, and insertion sort is better than bubble sort in practice.

**4. (12 points) SmallSum**

Given an array  $\langle a_1, \dots, a_n \rangle$ , Let

$$f_i = \sum_{j=1}^{i-1} a_j \mathbb{I}(a_j < a_i)$$

Then, we define SmallSum:

$$\text{SmallSum} = \sum_{i=1}^n f_i$$

For example, for array  $\langle 1, 3, 4, 2, 5 \rangle$ :

- for element 1:  $f_1 = 0$
- for element 3:  $f_2 = 1$
- for element 4:  $f_3 = 1 + 3 = 4$
- for element 2:  $f_4 = 1$
- for element 5:  $f_5 = 1 + 3 + 4 + 2 = 10$

So the SmallSum is  $0 + 1 + 4 + 1 + 10 = 16$ .

Most of you can come up with the  $O(n^2)$  solution, but let's think it in another way.

(a) (1') For the example above, for an element  $a_k$ , how many times does it add to the sum?

- A.  $\sum_{i=1}^k \mathbb{I}(a_k < a_i)$
- B.  $\sum_{i=1}^k \mathbb{I}(a_k \geq a_i)$
- C.  $\sum_{i=k+1}^n \mathbb{I}(a_k < a_i)$**
- D.  $\sum_{i=k+1}^n \mathbb{I}(a_k \geq a_i)$

(b) (8') Fill in the blanks in the code snippet below. (Hint: relate this algorithm to counting inversions.)

Consider alternating the Enhance Merge Sort algorithm to solve the problem.

The solution has 2 functions:

```
int split(int arr[], int left, int right){
    if(left == right) return 0;
    int mid = -----;
    int result = split(arr, left, mid) + split(arr, mid + 1, right);
    return result + merge(arr, left, mid, right);
}
```

The following function calculates the contribution of  $\langle a_{\text{left}}, \dots, a_{\text{mid}} \rangle$  to the sum.

```
int merge(int arr[], int left, int mid, int right){
    int length = right - left + 1;
    int* temp = new int[length];
    int i = 0;
    int p1 = left;
    int p2 = mid + 1;
```

```

int result = 0;
while(p1 <= mid && p2 <= right){
    if(arr[p1]<arr[p2]){
        // Add arr[p1]'s contribution to final answer.
        result += _____;
        // move the elements of arr into temp.
        // (for the following three lines, you may not fill all of
        them.)

        _____;
        _____;
        _____;
    }
    else{
        // move the elements of arr into temp.
        // (for the following three lines, you may not fill all of
        them.)

        _____;
        _____;
        _____;
    }
}
while(_____){
    temp[i++] = arr[p1++];
}
while(_____){
    temp[i++] = arr[p2++];
}
for (int j = 0; j < length ; j++){
    _____;
}
delete[] temp;
return result;
}

```

**Solution:**

```

int split(int arr[], int left, int right){
    if(left == right) return 0;
    int mid = left + (right - left) / 2;
    int result = split(arr, left, mid) + split(arr, mid + 1, right);
    return result + merge(arr, left, mid, right);
}

int merge(int arr[], int left, int mid, int right){
    int length = right - left + 1;
    int* temp = new int[length];
    int i = 0;

```



```
int p1 = left;
int p2 = mid + 1;
int result = 0;
while(p1 <= mid && p2 <= right){
    if(arr[p1] < arr[p2]){
        result += arr[p1] * (right - p2 + 1);
        temp[i++] = arr[p1++];
    }
    else{
        temp[i++] = arr[p2++];
    }
}
while(p1 <= mid){
    temp[i++] = arr[p1++];
}
while(p2 <= right){
    temp[i++] = arr[p2++];
}
for (int j = 0; j < length; j++){
    arr[left+j] = temp[j];
}
delete[] temp;
return result;
}
```

- (c) (3') What's the time complexity of our algorithm? Please answer in the form of  $\Theta(\cdot)$  and prove your answer.

**Solution:** Let  $T(n)$  be the running time of the algorithm, then we have

$$T(n) = \begin{cases} 2T(n/2) + \Theta(n), & n > 1, \\ \Theta(1), & n = 1. \end{cases}$$

From this we conclude that  $T(n) = \Theta(n \log(n))$ .