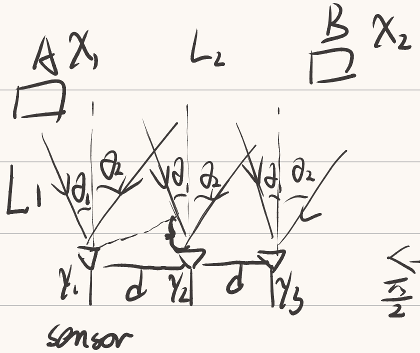

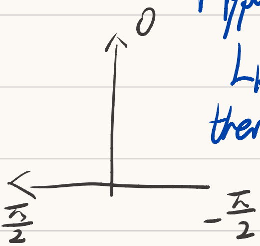

Hypothesis I:
$L_k >> d$,
then plane wavefront

$$x_1 = A_1 e^{ift} \qquad x_2 = A_2 e^{ift}$$

$$y_1 = r_1 x_1 + r_2 x_2 \qquad r_1 = e^{if\Delta t_1} = e^{if\frac{L_1}{c}} \qquad r_2 = e^{if\frac{L_2}{c}}$$

$$y_2 = r_1 x_1 \phi_1 + r_2 x_2 \phi_2 \qquad \phi_1 = e^{if\frac{d\sin\theta_1}{c}}, \ \phi_2 = e^{if\frac{d\sin\theta_2}{c}}$$

$$y_3 = r_1 x_1 \phi_1^2 + r_2 x_2 \phi_2^2$$

$$\Downarrow$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \\ \phi_1^2 & \phi_2^2 \end{bmatrix} \begin{bmatrix} r_1 x_1 \\ r_2 x_2 \end{bmatrix}$$

$$Y = \Phi X$$

$$\underset{m \times N}{\begin{bmatrix} y_1[1] & y_1[2] & \cdots & y_1[N] \\ y_2[1] & y_2[2] & \cdots & y_2[N] \\ \vdots & \vdots & & \vdots \\ y_m[1] & y_m[2] & \cdots & y_m[N] \end{bmatrix}} = \underset{m \times n}{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \phi_1 & \phi_2 & \cdots & \phi_n \\ \vdots & \vdots & & \vdots \\ \phi_1^{m-1} & \phi_2^{m-1} & \cdots & \phi_n^{m-1} \end{bmatrix}} \underset{n \times N}{\begin{bmatrix} V_1 x_1[1] & V_1 x_1[2] & \cdots & V_1 x_1[N] \\ V_2 x_2[1] & V_2 x_2[2] & \cdots & V_2 x_2[N] \\ \vdots & \vdots & & \vdots \\ V_n x_n[1] & V_n x_n[2] & \cdots & V_n x_n[N] \end{bmatrix}}$$

m sensors, n sources, N sample points

$$\underset{\text{known}}{\underset{\downarrow}{Y}} = \underset{\text{to be solved}}{\underset{\downarrow}{\phi}} X \longrightarrow \text{unknown}$$

3 sensors, 2 sources (for every sample point)

$y_1 . y_2 . y_3$ known

Can we find a set of nonzero scalar $\{c_1 . c_2 . c_3\}$ holds

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0 \quad ?$$

then $c_1(r_1 x_1 + r_2 x_2) + c_2(r_1 x_1 \phi_1 + r_2 x_2 \phi_2) + c_3(r_1 x_1 \phi_1^2 + r_2 x_2 \phi_2^2) = 0$

$$(c_1 + c_2 \phi_1 + c_3 \phi_1^2) r_1 x_1 + (c_1 + c_2 \phi_2 + c_3 \phi_2^2) r_2 x_2 = 0$$

Hypothesis II:

x_1, x_2 are linear independent

then $\begin{cases} c_1 + c_2 \phi_1 + c_3 \phi_1^2 = 0 \\ c_1 + c_2 \phi_2 + c_3 \phi_2^2 = 0 \end{cases} \Rightarrow [c_1 \ c_2 \ c_3] \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \\ \phi_1^2 & \phi_2^2 \end{bmatrix} = 0$

How to find c_1, c_2, c_3 ?

$$\begin{bmatrix} y_1[1] & y_2[1] & y_3[1] \\ y_1[2] & y_2[2] & y_3[2] \\ \vdots & \vdots & \vdots \\ y_1[N] & y_2[N] & y_3[N] \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

overdetermined homogeneous linear system $Ax = 0$

solve it with SVD ...

then we have c_1, c_2, c_3

$$\phi_1, \phi_2 = \frac{-c_2 \pm \sqrt{c_2^2 - 4c_1c_3}}{2 \cdot c_3} \Rightarrow \theta_1, \theta_2$$

(<3 sources)

We can get exact solutions when there're only 3 sensors.

but how can we solve the general case with equation of high degree?

Test value method

Denote $P(\theta) = \dfrac{1}{\sum_{i=1}^{m-n} c_1 + c_2\phi_i + c_3\phi_i^2 + \cdots + c_m\phi_i^{m-1}}$

$\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$

when $P(\theta) \to \infty$ (maximum)

the corresponding θ is the solution