

1 **THE QR FACTORIZATION FOR BANDED-PLUS-SEMISEPARABLE**
 2 **MATRICES IS COMPUTABLE IN LINEAR COMPLEXITY**

3 TAO CHEN* AND SHEEHAN OLVER*

4 **Abstract.** We show that each stage of the QR factorization of banded-plus-semiseparable matr-
 5 ices computed using Householder reflections has a specific structured perturbation. This theoretical
 6 result enables the design of linear-complexity algorithms for QR factorization and for solving the
 7 associated linear systems. Numerical experiments validate the optimal linear complexity and demon-
 8 strate substantial speedups compared with existing hierarchical approaches. The algorithms have
 9 been implemented in an open-source Julia package, providing an efficient and accessible platform for
 10 practical use.

11 **Key words.** banded-plus-semiseparable matrices, QR factorization, linear complexity, struc-
 12 tured matrices, direct solvers

13 **AMS subject classifications.** 65F05, 65F50, 15A23, 65Y20

14 **1. Introduction.**

15 Do we want to allow for complex numbers?

16 What to say about QR algorithm?

17 Add a nullspace problem coming from the ultraspherical spectral method?

18 Does it generalise to rectangular?

19 Say something about inverses of banded matrices being banded-plus-
 semiseparable?

20 Add stability plot

21 Banded-plus-semiseparable (BPS) matrices, expressible as

$$22 \quad A = \underbrace{B}_{\text{banded}} + \underbrace{\text{tril}(UV^T, -1)}_{\text{lower semiseparable, rank } r} + \underbrace{\text{triu}(WS^T, 1)}_{\text{upper semiseparable, rank } p} \in \mathbb{R}^{n \times n},$$

Change all the T
to \top

23 arise in numerous applications from PDEs with non-local interactions [12] to signal
 24 processing, control theory, and eigenvalue problems [18]. Their structure requires only
 25 $O(n)$ storage which invites the development of $O(n)$ algorithms, a goal successfully
 26 achieved for iterations of the QR algorithm for symmetric semiseparable systems [17],
 27 and for solving linear systems with *diagonal*-plus-semiseparable matrices [9]. However,
 28 generalizing these results to the case where the banded part B is a genuine banded
 29 matrix, rather than merely a diagonal one, presents significant algorithmic challenges.

This is a bizarre
reference and de-
scription. Is it AI
generated?

30 Clarify exact relationship between prior work and ours

Add citations to
Arieh Iserles W-
systems papers

31 Pioneering work established $O(n)$ solvers for sequentially semiseparable matr-
 32 ices [2] and later for the banded-plus-semiseparable case via ULV factorization [3].
 33 Banded-plus-semiseparable matrices can be viewed as hierarchically semiseparable
 34 (HSS) matrices, and solvers using HSS structure is a well-developed area [4, 1, 20, 13].
 35 A parallel line of research extensively developed the theory and algorithms for semisep-
 36 arable and quasimseparable matrices, including implicit QR algorithms for *symmetric*
 37 semiseparable matrices [17], structure-preserving analyses [10, 6, 5], approaches lever-

What's a sequen-
tially semisepa-
rable matrix?

*Department of Mathematics, Imperial College London, London, UK (t.chen24@imperial.ac.uk, s.olver@imperial.ac.uk).

aging rational Krylov techniques [11, 19], and alternative representations [18, 7]. Despite these advances, a clear theoretical guarantee that the standard QR factorization preserves the BPS structure has been missing, with most existing solvers relying on more complex ULV or intricate Givens-based schemes [14, 3, 16, 8]. A special case of BPS matrices are almost banded matrices which were used in [15] to represent discretisations of differential equations using the ultraspherical spectral method. An optimal complexity adaptive QR factorization was introduced, which also gives an optimal complexity QR factorization for BPS matrices with now lower semisparable part ($r = 0$). It also introduced an optimal complexity back-substitution for upper-triangular BPS matrices, an algorithm we also use.

In this paper, we close this theoretical gap. We prove that the QR factorization of a BPS matrix yields a factor matrix F , which is the matrix containing both R and the Householder reflectors encoding Q , that is itself BPS, with precisely characterized lower rank r , upper rank $r + p$, and bandwidths l and $l + m$. This pivotal result, proven via an inductive framework involving a new class of Householder-Modified BPS Matrices (HMBPSM), which enables the design of an $O(n)$ QR factorization. Furthermore, it facilitates a complete direct solver: applying Q^T and performing backward substitution on the structured factor R are also achieved in linear time. Our work thus provides a unified, QR-based, end-to-end $O(n)$ solution for BPS systems, backed by a rigorous structure-preservation theorem.

The rest of this paper is organized as follows. Section 2 presents our main theoretical contributions: the definitions, the core lemma on structure preservation under Householder transformations, and the main theorem with its proof. Section 3 details the resulting $O(n)$ algorithms for QR factorization, application of Q^T , and backward substitution. Section 4 presents numerical experiments that confirm the linear complexity and demonstrate performance advantages. We conclude in Section 5 with a discussion of future work.

65 2. Main results.

66 **2.1. Problem Formulation and Notation.** Before we start, an important
 67 notation will be: for a matrix M , let $M[i : j, m : n]$ represent the submatrix of M
 68 from row i to row j and from column m to column n . When $i = j$ or $m = n$, the
 69 notation will be simplified as $M[i, m : n]$ or $M[i : j, m]$. We also adopt the convention
 70 that writing **end** in an index(such as $M[i : \text{end}, :]$) to indicates the last valid index in
 71 that dimension.
 72
 73 **DEFINITION 2.1.** *A banded-plus-semiseparable matrix (BPS) with lower-semisepa-*

74 *rank r , upper-semiseparable rank p , lower-bandwidth l and upper-bandwidth m is*
 $A \in \text{BPS}_{(r,p),(l,u)}^{n \times n} \subset \mathbb{R}^{n \times n}$ *such that*

$$75 \quad A = B + \text{tril}(UV^T, -1) + \text{triu}(WS^T, 1)$$

76 *where $U, V \in \mathbb{R}^{n \times r}$, $W, S \in \mathbb{R}^{n \times p}$, and $B = (b_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ is a banded matrix*
 $77 \quad \text{satisfying } b_{ij} = 0 \text{ for } i - j > l \text{ or } j - i > m.$

78 Define vectors $\bar{\mathbf{u}}_i = U[i, :]^T \in \mathbb{R}^r$, $\bar{\mathbf{v}}_i = V[i, :]^T \in \mathbb{R}^r$, $\bar{\mathbf{w}}_i = W[i, :]^T \in \mathbb{R}^p$, and
 $79 \quad \bar{\mathbf{s}}_i = S[i, :]^T \in \mathbb{R}^p$ for $i = 1, \dots, n$. The matrix A can then be expressed element-wise

80 as:

$$81 \quad (2.1) \quad A = \begin{bmatrix} b_{11} & \bar{\mathbf{w}}_1^T \bar{\mathbf{s}}_2 + b_{12} & \cdots & \bar{\mathbf{w}}_1^T \bar{\mathbf{s}}_n + b_{1n} \\ \bar{\mathbf{u}}_2^T \bar{\mathbf{v}}_1 + b_{21} & b_{22} & \cdots & \bar{\mathbf{w}}_2^T \bar{\mathbf{s}}_n + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\mathbf{u}}_n^T \bar{\mathbf{v}}_1 + b_{n1} & \bar{\mathbf{u}}_n^T \bar{\mathbf{v}}_2 + b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

82 Applying the QR factorization to A yields a factor matrix F , whose upper triangular part stores the matrix R and whose lower triangular part contains the Householder reflection vectors \mathbf{y} generated during the factorization. We will demonstrate that F itself retains a banded-plus-semiseparable structure. Specifically, its lower semiseparable part has rank r , its upper semiseparable part has rank $r+p$, its lower bandwidth is l , and its upper bandwidth is $l+m$.

88 Before proceeding with the detailed proof, let us clarify the precise structure of
89 the factor matrix F obtained from the QR factorization. In this work, following the
90 convention of LAPack , we employ a compact representation that stores the complete
91 information of the QR factorization in a single matrix: add citation

$$92 \quad (2.2) \quad F = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ y_{2,1} & r_{22} & r_{23} & \cdots & r_{2n} \\ y_{3,1} & y_{3,2} & r_{33} & \cdots & r_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & y_{n,3} & \cdots & r_{nn} \end{bmatrix}$$

93 where: the upper triangular part (including the main diagonal) of F stores the el-
94 ements of the upper triangular matrix R , i.e.: $R = \text{triu}(F)$, and the strictly lower
95 triangular part (excluding the main diagonal) of F stores the last $n-k$ elements of
96 (rescaled) Householder reflection vectors \mathbf{y}_k generated at each step.

97 More specifically, at the k -th Householder transformation step ($k = 1, 2, \dots, n-1$),
98 we construct a reflection vector \mathbf{y}_k to eliminate the subdiagonal entries of the k -th
99 column. This vector takes the form:

$$100 \quad (2.3) \quad \mathbf{y}_k = \begin{cases} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ y_{k+1,k} \\ \vdots \\ y_{n,k} \end{bmatrix} & \left. \right\} k-1 \text{ zeros} \\ & \left. \right\} n-k \text{ elements} \end{cases}$$

101 Following the LAPack format, we normalize \mathbf{y}_k such that its first nonzero element
102 (the k -th element) equals 1. Consequently, we only need to store the elements from
103 position $k+1$ to n of this vector, which are placed in the k -th column of F , from row
104 $k+1$ to n .

105 The advantage of this representation is that it compactly stores the information of
106 both the orthogonal matrix Q (via the Householder vectors) and the upper triangular
107 matrix R within a single matrix F . The central result of this paper will demonstrate
108 that for a banded-plus-semiseparable matrix A , this factor matrix F itself maintains
109 a banded-plus-semiseparable structure.

110 It is important to note that with this normalization convention (where the first
 111 nonzero element of each Householder vector \mathbf{y}_k is 1), the full Householder transforma-
 112 tion at the k -th step is given by $I - \tau_k \mathbf{y}_k \mathbf{y}_k^T$, where τ_k is a scalar coefficient. Therefore,
 113 in addition to the factor matrix F , a vector $\boldsymbol{\tau} = (\tau_1, \tau_2, \dots, \tau_{n-1})^T$ is required to com-
 114 pletely represent the QR factorization. The orthogonal matrix Q can be reconstructed
 115 as the product $Q = (I - \tau_1 \mathbf{y}_1 \mathbf{y}_1^T)(I - \tau_2 \mathbf{y}_2 \mathbf{y}_2^T) \cdots (I - \tau_{n-1} \mathbf{y}_{n-1} \mathbf{y}_{n-1}^T)$.

116 Throughout our analysis, we will focus on the structure of the factor matrix
 117 F , while acknowledging that the complete QR representation consists of the pair
 118 $(F, \boldsymbol{\tau})$. Our main theorem establishes that F maintains the banded-plus-semiseparable
 119 structure; the scaling coefficients τ_k can be stored separately without affecting the
 120 structural properties of the algorithm.

121 We proceed to prove this by induction. First, we introduce two key definitions
 122 and a pivotal lemma.

123 **2.2. Core Definitions and a Key Lemma.** While the final factor matrix of a
 124 QR factorization is a BPS matrix, at intermediate stages it has a specific structured
 125 perturbation. Here we describe this structure in terms of a linear space that, at each
 126 stage, the principle submatrix lies in:

127 **DEFINITION 2.2** (BPS Factors Perturbation Matrix). *Given*

$$128 \quad A = B + \text{tril}(UV^T, -1) + \text{triu}(WS^T, 1) \in \text{BPS}_{(r,p),(l,u)}^{n \times n}$$

129 *a BPS Factors Perturbation Matrix (BPSFPM) is in the vector space*

$$130 \quad \mathcal{P}(A) := \left\{ UQS^T + UKU^TA + UE + XS^T + YU^TA + Z : \right. \\ 131 \quad Q \in \mathbb{R}^{r \times p}, K \in \mathbb{R}^{r \times r}, \\ 132 \quad E = [E_s \in \mathbb{R}^{r \times \min(l+m,n)}, \mathbf{0}] \in \mathbb{R}^{r \times n}, \\ 133 \quad X = \begin{bmatrix} X_s \in \mathbb{R}^{\min(l,n) \times p} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n \times p}, \\ 134 \quad Y = \begin{bmatrix} Y_s \in \mathbb{R}^{\min(l,n) \times r} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n \times r}, \\ 135 \quad Z = \begin{bmatrix} Z_s \in \mathbb{R}^{\min(l,n) \times \min(l+m,n)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n \times n} \left. \right\} \subset \mathbb{R}^{n \times n}.$$

136 In addition, we need to describe the structure of the upper-triangular part in
 137 terms of a structured vector:

138 **DEFINITION 2.3** (BPS Factors Vector). *Given $A \in \text{BPS}_{(r,p),(l,u)}^{n \times n}$ a BPS Factors
 139 Vector (BPSFV) is a row vector in the vector space*

$$140 \quad \mathcal{V}(A) := \left\{ \mathbf{d}^T + \boldsymbol{\alpha}^T(S^T[:, 2:n]) + \boldsymbol{\beta}^T((U^TA)[:, 2:n]) : \right. \\ 141 \quad \mathbf{d} = \begin{bmatrix} \mathbf{d}_s \in \mathbb{R}^{\min(l+m,n-1)} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n-1}, \boldsymbol{\alpha} \in \mathbb{R}^p, \boldsymbol{\beta} \in \mathbb{R}^r \left. \right\} \subset \mathbb{R}^{1 \times (n-1)}.$$

142 **LEMMA 2.4.** *Given $A \in \text{BPS}_{(r,p),(l,u)}^{n \times n}$ and $P \in \mathcal{P}(A)$, suppose a Householder
 143 transformation is applied to $A + P$ to eliminate the subdiagonal entries of its first
 144 column, yielding $\tilde{C} = (I - \tau \mathbf{y} \mathbf{y}^T)(A + P)$. Then the following hold:*

1 we change
140 umm vector?

141

142

143

144

145

146

147

148

149

150

151

152

153

154

155

156

157

- 145 1. The principal submatrix satisfies $\tilde{C}[2 : n, 2 : n] = A[2 : n, 2 : n] + \tilde{P}$ for
 146 $\tilde{P} \in \mathcal{P}(A[2 : n, 2 : n]).$
 147 2. The first row satisfies $\tilde{C}[1, 2 : n] \in \mathcal{V}(A).$

148 *Proof.*

149 I'll start updating the proof tomorrow

150 Let us introduce the necessary notation:

- 151 • $A = B + \text{tril}(UV^T, -1) + \text{triu}(WS^T, 1)$, where $U = (\mathbf{u}_1, \dots, \mathbf{u}_r) \in \mathbb{R}^{n \times r}$, $V =$
 152 $(\mathbf{v}_1, \dots, \mathbf{v}_r) \in \mathbb{R}^{n \times r}$, $W = (\mathbf{w}_1, \dots, \mathbf{w}_p) \in \mathbb{R}^{n \times p}$, and $S = (\mathbf{s}_1, \dots, \mathbf{s}_p) \in \mathbb{R}^{n \times p}$.
 153 Here $\mathbf{u}_i = (u_1^{(i)}, \dots, u_n^{(i)})^T \in \mathbb{R}^n$ and $\mathbf{v}_i = (v_1^{(i)}, \dots, v_n^{(i)})^T \in \mathbb{R}^n$ for $i = 1, \dots, r$;
 154 $\mathbf{w}_i = (w_1^{(i)}, \dots, w_n^{(i)})^T \in \mathbb{R}^n$ and $\mathbf{s}_i = (s_1^{(i)}, \dots, s_n^{(i)})^T \in \mathbb{R}^n$ for $i = 1, \dots, p$. Also,
 155 $B = (b_{ij})_{i,j=1}^n \in \mathbb{R}^{n,n}$ with $b_{ij} = 0$ if $i - j > l$ or $j - i > m$.
- 156 • $C = A + UQST^T + UKU^TA + UE + XST^T + YUT^TA + Z$, where Q, K, E, X, Y, Z
 157 are as in Definition [Definition 2.2](#).
- 158 • $\tilde{C} = (I - \tau\mathbf{y}\mathbf{y}^T)C$, where the Householder vector \mathbf{y} can be expressed as $\mathbf{y} =$
 159 $\mathbf{e}_1 + U^{(2)}\bar{\mathbf{k}} + \mathbf{b}$. Here, $\mathbf{e}_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^n$, $U^{(2)} \in \mathbb{R}^{n \times r}$ satisfies $U^{(2)}[1, :] =$
 160 $\mathbf{0}$ and $U^{(2)}[2 : n, :] = U[2 : n, :]$, $\bar{\mathbf{k}} \in \mathbb{R}^r$, $\mathbf{b} = (0, b_2, \dots, b_{\min(l+1, n)}, 0, \dots, 0)^T \in$
 161 \mathbb{R}^n , and τ is a coefficient found to satisfy the definition of a Householder
 162 transformation.

163 Let $\bar{\mathbf{u}}_1 = (u_1^{(1)}, \dots, u_1^{(r)})^T \in \mathbb{R}^r$. We can write:

$$164 \quad \underbrace{\mathbf{e}_1^T A}_{\min(m+1, n) \text{ nonzero entries}} = \underbrace{\mathbf{d}_1^T}_{\in \mathbb{R}^{1 \times p}} + \underbrace{\bar{\mathbf{w}}_1^T}_{\in \mathbb{R}^{1 \times p}} S^T,$$

165 where $\mathbf{d}_1 = B[1, :]^T \in \mathbb{R}^n$, $\bar{\mathbf{w}}_1 = (w_1^{(1)}, \dots, w_1^{(p)})^T \in \mathbb{R}^p$, and

$$166 \quad \underbrace{\mathbf{b}^T A}_{\min(l+m+1, n) \text{ nonzero entries}} = \underbrace{\bar{\mathbf{d}}^T}_{\mathbf{b}^T W \in \mathbb{R}^{1 \times p}} + \underbrace{\mathbf{f}^T}_{\mathbf{b}^T W \in \mathbb{R}^{1 \times p}} S^T.$$

167 Define the auxiliary vectors:

$$168 \quad (2.4) \quad \mathbf{c}_1 = Q^T U^T \mathbf{y} \in \mathbb{R}^p$$

$$169 \quad (2.5) \quad \mathbf{c}_2 = K^T U^T \mathbf{y} \in \mathbb{R}^r$$

$$170 \quad (2.6) \quad \mathbf{c}_3 = U^T \mathbf{y} \in \mathbb{R}^r$$

$$171 \quad (2.7) \quad \mathbf{c}_4 = X^T \mathbf{y} \in \mathbb{R}^p$$

$$172 \quad (2.8) \quad \mathbf{c}_5 = Y^T \mathbf{y} \in \mathbb{R}^r$$

$$173 \quad (2.9) \quad \mathbf{c}_6 = Z^T \mathbf{y} \in \mathbb{R}^n, \quad \text{which has the form } \mathbf{c}_6 = \begin{bmatrix} \mathbf{c}_{6s} \\ \mathbf{0} \end{bmatrix} \text{ with } \mathbf{c}_{6s} \in \mathbb{R}^{\min(l+m, n)}.$$

174 Also, let $\mathbf{x}^{(1)} = X[1, :]^T \in \mathbb{R}^p$, $\mathbf{y}^{(1)} = Y[1, :]^T \in \mathbb{R}^r$, and $\mathbf{z}^{(1)} = Z[1, :]^T \in \mathbb{R}^n$.

175 We now compute $(I - \tau\mathbf{y}\mathbf{y}^T)C$ by distributing the transformation over each term
 176 in the definition of C .

177 (i) **Transformation of A :** Substituting the expressions $\mathbf{y} = \mathbf{e}_1 + U^{(2)}\bar{\mathbf{k}} + \mathbf{b}$,

178 $\mathbf{e}_1^T A = \mathbf{d}_1^T + \bar{\mathbf{w}}_1^T S^T$, and $\mathbf{b}^T A = \bar{\mathbf{d}}^T + \mathbf{f}^T S^T$, we obtain:

$$(2.10) \quad \begin{aligned} (I - \tau \mathbf{y} \mathbf{y}^T) A &= A + \mathbf{e}_1 \left[\underbrace{(-\tau \mathbf{d}_1^T - \tau \bar{\mathbf{d}}^T)}_{\min(l+m+1,n) \text{ nonzero entries}} + \underbrace{(-\tau \bar{\mathbf{w}}_1^T - \tau \mathbf{f}^T)}_{\in \mathbb{R}^{1 \times p}} S^T \right. \\ &\quad \left. + (-\tau \bar{\mathbf{k}}^T) U^{(2)T} A \right] + U^{(2)} \underbrace{(-\tau \bar{\mathbf{k}} \bar{\mathbf{w}}_1^T - \tau \bar{\mathbf{k}} \mathbf{f}^T)}_{\in \mathbb{R}^{r \times p}} S^T \\ &\quad + U^{(2)} \underbrace{(-\tau \bar{\mathbf{k}} \bar{\mathbf{k}}^T)}_{\in \mathbb{R}^{r \times r}} U^{(2)T} A + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{d}_1^T - \tau \bar{\mathbf{k}} \bar{\mathbf{d}}^T) \\ &\quad \left. + (-\tau \bar{\mathbf{b}} \bar{\mathbf{w}}_1^T - \tau \mathbf{b} \mathbf{f}^T) S^T + (-\tau \bar{\mathbf{b}} \bar{\mathbf{k}}^T) U^{(2)T} A + (-\tau \bar{\mathbf{b}} \mathbf{d}_1^T - \tau \bar{\mathbf{b}} \bar{\mathbf{d}}^T) \right]. \end{aligned}$$

180 Dropping the first column, we see that the first row of the term in brackets is in $\mathcal{V}(A)$.

181 Dropping the first row and column of the remaining terms are in $\mathcal{P}(A[2:n, 2:n])$.

182 **(ii) Transformation of UQS^T :** Substituting the expressions $\mathbf{y}^T U Q = \mathbf{c}_1^T$, $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, and $U = \mathbf{e}_1 \bar{\mathbf{u}}_1^T + U^{(2)}$ where $\bar{\mathbf{u}}_1 = (u_1^{(1)}, \dots, u_1^{(r)}) \in \mathbb{R}^r$, we obtain

$$(2.11) \quad (I - \tau \mathbf{y} \mathbf{y}^T) U Q S^T = \mathbf{e}_1 (\bar{\mathbf{u}}_1^T Q - \tau \mathbf{c}_1^T) S^T + U^{(2)} (Q - \tau \bar{\mathbf{k}} \mathbf{c}_1^T) S^T + (-\tau \mathbf{b} \mathbf{c}_1^T) S^T.$$

185 **(iii) Transformation of $UKU^T A$:** Substituting the expressions $\mathbf{y}^T UK = \mathbf{c}_2^T$,

186 $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, $U = \mathbf{e}_1 \bar{\mathbf{u}}_1^T + U^{(2)}$, and $\mathbf{e}_1^T A = \mathbf{d}_1^T + \bar{\mathbf{w}}_1^T S^T$, we obtain

$$(2.12) \quad \begin{aligned} (I - \tau \mathbf{y} \mathbf{y}^T) UKU^T A &= \mathbf{e}_1 (\bar{\mathbf{u}}_1^T K \bar{\mathbf{u}}_1 \mathbf{d}_1^T - \tau \mathbf{c}_2^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T) + \mathbf{e}_1 (\bar{\mathbf{u}}_1^T K \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T - \tau \mathbf{c}_2^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + \mathbf{e}_1 (\bar{\mathbf{u}}_1^T K - \tau \mathbf{c}_2^T) U^{(2)T} A \\ &\quad + U^{(2)} (K \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T - \tau \bar{\mathbf{k}} \mathbf{c}_2^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + U^{(2)} (K - \tau \bar{\mathbf{k}} \mathbf{c}_2^T) U^{(2)T} A + U^{(2)} (K \bar{\mathbf{u}}_1 \mathbf{d}_1^T - \tau \bar{\mathbf{k}} \mathbf{c}_2^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T) \\ &\quad + (-\tau \mathbf{b} \mathbf{c}_2^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + (-\tau \mathbf{b} \mathbf{c}_2^T) U^{(2)T} A + (-\tau \mathbf{b} \mathbf{c}_2^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T). \end{aligned}$$

188 **(iv) Transformation of UE :** Substituting the expressions $\mathbf{y}^T U = \mathbf{c}_3^T$, $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, and $U = \mathbf{e}_1 \bar{\mathbf{u}}_1^T + U^{(2)}$, we obtain

$$(2.13) \quad (I - \tau \mathbf{y} \mathbf{y}^T) UE = \mathbf{e}_1 (\bar{\mathbf{u}}_1^T E - \tau \mathbf{c}_3^T E) + U^{(2)} (E - \tau \bar{\mathbf{k}} \mathbf{c}_3^T E) + (-\tau \mathbf{b} \mathbf{c}_3^T E).$$

191 **(v) Transformation of XS^T :** Substituting the expressions $\mathbf{y}^T X = \mathbf{c}_4^T$ and
192 $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, we obtain

$$(2.14) \quad (I - \tau \mathbf{y} \mathbf{y}^T) XS^T = \mathbf{e}_1 (-\tau \mathbf{c}_4^T) S^T + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{c}_4^T) S^T + (X - \tau \mathbf{b} \mathbf{c}_4^T) S^T.$$

194 **(vi) Transformation of $YU^T A$:** Substituting the expressions $\mathbf{y}^T Y = \mathbf{c}_5^T$, $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, $U = \mathbf{e}_1 \bar{\mathbf{u}}_1^T + U^{(2)}$, and $\mathbf{e}_1^T A = \mathbf{d}_1^T + \bar{\mathbf{w}}_1^T S^T$, we obtain

$$(2.15) \quad \begin{aligned} (I - \tau \mathbf{y} \mathbf{y}^T) YU^T A &= \mathbf{e}_1 (-\tau \mathbf{c}_5^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T) + \mathbf{e}_1 (-\tau \mathbf{c}_5^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + \mathbf{e}_1 (-\tau \mathbf{c}_5^T) U^{(2)T} A \\ &\quad + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{c}_5^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{c}_5^T) U^{(2)T} A + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{c}_5^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T) \\ &\quad + (Y \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T - \tau \mathbf{b} \mathbf{c}_5^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + (Y - \tau \mathbf{b} \mathbf{c}_5^T) U^{(2)T} A + (Y \bar{\mathbf{u}}_1 \mathbf{d}_1^T - \tau \mathbf{b} \mathbf{c}_5^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T). \end{aligned}$$

197 **(vii) Transformation of Z :** Substituting the expressions $\mathbf{y}^T Z = \mathbf{c}_6^T$ and $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, we obtain

$$(2.16) \quad (I - \tau \mathbf{y} \mathbf{y}^T) Z = \mathbf{e}_1 (-\tau \mathbf{c}_6^T) + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{c}_6^T) + (Z - \tau \mathbf{b} \mathbf{c}_6^T).$$

Combining equations (2.10) through (2.16), we can now identify the structure of the resulting matrix \tilde{C} .

Firstly, the submatrix $\tilde{C}[2 : n, 2 : n]$ satisfies:

$$(2.17) \quad \tilde{C}[2 : n, 2 : n] = \tilde{A} + \tilde{U}\tilde{Q}\tilde{S}^T + \tilde{U}\tilde{K}\tilde{U}^T\tilde{A} + \tilde{U}\tilde{E} + \tilde{X}\tilde{S}^T + \tilde{Y}\tilde{U}^T\tilde{A} + \tilde{Z},$$

where

$$\tilde{A} = A[2 : n, 2 : n]$$

$$\tilde{U} = U[2 : n, :]$$

$$\tilde{S} = S[2 : n, :]$$

and the updated modification matrices are given by:

$$\tilde{Q} = -\tau\bar{\mathbf{k}}\bar{\mathbf{w}}_1^T - \tau\bar{\mathbf{k}}\mathbf{f}^T + Q - \tau\bar{\mathbf{k}}\mathbf{c}_1^T + K\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T - \tau\bar{\mathbf{k}}\mathbf{c}_2^T\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T - \tau\bar{\mathbf{k}}\mathbf{c}_4^T - \tau\bar{\mathbf{k}}\mathbf{c}_5^T\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T,$$

$$\tilde{K} = -\tau\bar{\mathbf{k}}\bar{\mathbf{k}}^T + K - \tau\bar{\mathbf{k}}\mathbf{c}_2^T - \tau\bar{\mathbf{k}}\mathbf{c}_5^T,$$

$$\tilde{E} = [\tilde{E}_s, \mathbf{0}] \in \mathbb{R}^{r \times (n-1)}, \quad \text{with}$$

$$\begin{aligned} \tilde{E}_s = & (-\tau\bar{\mathbf{k}}\mathbf{d}_1^T - \tau\bar{\mathbf{k}}\bar{\mathbf{d}}^T + K\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\bar{\mathbf{k}}\mathbf{c}_2^T\bar{\mathbf{u}}_1\mathbf{d}_1^T + E - \tau\bar{\mathbf{k}}\mathbf{c}_3^T E \\ & - \tau\bar{\mathbf{k}}\mathbf{c}_5^T\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\bar{\mathbf{k}}\mathbf{c}_6^T)[:, 2 : \min(l+m+1, n)], \end{aligned}$$

$$\tilde{X} = \begin{bmatrix} \tilde{X}_s \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n-1) \times p}, \quad \text{with}$$

$$\begin{aligned} \tilde{X}_s = & (-\tau\mathbf{b}\bar{\mathbf{w}}_1^T - \tau\mathbf{b}\mathbf{f}^T - \tau\mathbf{b}\mathbf{c}_1^T - \tau\mathbf{b}\mathbf{c}_2^T\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T + X - \tau\mathbf{b}\mathbf{c}_4^T \\ & + Y\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T - \tau\mathbf{b}\mathbf{c}_5^T\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T)[2 : \min(l+1, n), :], \end{aligned}$$

$$\tilde{Y} = \begin{bmatrix} \tilde{Y}_s \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n-1) \times r}, \quad \text{with}$$

$$\tilde{Y}_s = (-\tau\mathbf{b}\bar{\mathbf{k}}^T - \tau\mathbf{b}\mathbf{c}_2^T + Y - \tau\mathbf{b}\mathbf{c}_5^T)[2 : \min(l+1, n), :],$$

$$\tilde{Z} = \begin{bmatrix} \tilde{Z}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n-1) \times (n-1)}, \quad \text{with}$$

$$\begin{aligned} \tilde{Z}_s = & (-\tau\mathbf{b}\mathbf{d}_1^T - \tau\bar{\mathbf{b}}\bar{\mathbf{d}}^T - \tau\mathbf{b}\mathbf{c}_2^T\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\mathbf{b}\mathbf{c}_3^T E + Y\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\mathbf{b}\mathbf{c}_5^T\bar{\mathbf{u}}_1\mathbf{d}_1^T \\ & + Z - \tau\mathbf{b}\mathbf{c}_6^T)[2 : \min(l+1, n), 2 : \min(l+m+1, n)]. \end{aligned}$$

The forms of $\tilde{Q}, \tilde{K}, \tilde{E}, \tilde{X}, \tilde{Y}, \tilde{Z}$ confirm that $\tilde{C}[2 : n, 2 : n]$ is an HMBPSM related to $A[2 : n, 2 : n]$, thus establishing the first part of the lemma.

Secondly, the first row of the transformed matrix, $\tilde{C}[1, 2 : n]$, can be expressed as:

$$(2.18) \quad \tilde{C}[1, 2 : n] = \hat{\mathbf{d}}^T + \hat{\boldsymbol{\alpha}}^T(S^T[:, 2 : n]) + \hat{\boldsymbol{\beta}}^T((U^{(2)T}A)[:, 2 : n]),$$

where

$$\hat{\mathbf{d}} = \begin{bmatrix} \hat{\mathbf{d}}_s \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n-1}, \quad \text{with}$$

$$\begin{aligned} \hat{\mathbf{d}}_s = & (\mathbf{d}_1^T - \tau\mathbf{d}_1^T - \tau\bar{\mathbf{d}}^T + \bar{\mathbf{u}}_1^T K\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\mathbf{c}_2^T\bar{\mathbf{u}}_1\mathbf{d}_1^T + \bar{\mathbf{u}}_1^T E - \tau\mathbf{c}_3^T E \\ & + \mathbf{y}^{(1)T}\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\mathbf{c}_5^T\bar{\mathbf{u}}_1\mathbf{d}_1^T + \mathbf{z}^{(1)T} - \tau\mathbf{c}_6^T)^T[2 : \min(l+m+1, n)], \\ \hat{\boldsymbol{\alpha}} = & (\bar{\mathbf{w}}_1^T - \tau\bar{\mathbf{w}}_1^T - \tau\mathbf{f}^T + \bar{\mathbf{u}}_1^T Q - \tau\mathbf{c}_1^T + \bar{\mathbf{u}}_1^T K\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T - \tau\mathbf{c}_2^T\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T \end{aligned}$$

$$\begin{aligned} & + \mathbf{x}^{(1)T} - \tau \mathbf{c}_4^T + \mathbf{y}^{(1)T} \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T - \tau \mathbf{c}_5^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T)^T \in \mathbb{R}^p, \\ & \hat{\boldsymbol{\beta}} = (-\tau \bar{\mathbf{k}}^T + \bar{\mathbf{u}}_1^T K - \tau \mathbf{c}_2^T + \mathbf{y}^{(1)T} - \tau \mathbf{c}_5^T)^T \in \mathbb{R}^r. \end{aligned}$$

Noting that $U^{(2)} = U - \mathbf{e}_1 \bar{\mathbf{u}}_1^T$ and $\mathbf{e}_1^T A = \mathbf{d}_1^T + \bar{\mathbf{w}}_1^T S^T$, we have $U^{(2)T} A = U^T A - \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T S^T - \bar{\mathbf{u}}_1 \mathbf{d}_1^T$. Substituting this yields an alternative expression:

$$(2.19) \quad \tilde{C}[1, 2 : n] = \tilde{\mathbf{d}}^T + \tilde{\boldsymbol{\alpha}}^T (S^T[:, 2 : n]) + \tilde{\boldsymbol{\beta}}^T ((U^T A)[:, 2 : n]),$$

where

$$\begin{aligned} \tilde{\mathbf{d}} &= \begin{bmatrix} \tilde{\mathbf{d}}_s \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n-1}, \quad \text{with} \\ \tilde{\mathbf{d}}_s &= (\mathbf{d}_1^T - \tau \mathbf{d}_1^T - \tau \bar{\mathbf{d}}^T + \bar{\mathbf{u}}_1^T E - \tau \mathbf{c}_3^T E + \mathbf{z}^{(1)T} - \tau \mathbf{c}_6^T + \tau \bar{\mathbf{k}}^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T)^T [2 : \min(l+m+1, n)], \\ \tilde{\boldsymbol{\alpha}} &= (\bar{\mathbf{w}}_1^T - \tau \bar{\mathbf{w}}_1^T - \tau \mathbf{f}^T + \bar{\mathbf{u}}_1^T Q - \tau \mathbf{c}_1^T + \mathbf{x}^{(1)T} - \tau \mathbf{c}_4^T + \tau \bar{\mathbf{k}}^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T)^T \in \mathbb{R}^p, \\ \tilde{\boldsymbol{\beta}} &= (-\tau \bar{\mathbf{k}}^T + \bar{\mathbf{u}}_1^T K - \tau \mathbf{c}_2^T + \mathbf{y}^{(1)T} - \tau \mathbf{c}_5^T)^T \in \mathbb{R}^r. \end{aligned}$$

This confirms that $\tilde{C}[1, 2 : n]$ is an HMBPSV related to A , completing the proof of the lemma. \square

2.3. Main Theorem and its Proof. Equipped with Lemma 2.4, we now state and prove the main theorem concerning the structure of the QR factor matrix F .

THEOREM 2.5. *After applying the QR factorization to a banded-plus-semiseparable matrix A (as expressed in Eq. (2.1)) with lower semiseparable rank r , upper semiseparable rank p , lower bandwidth l , and upper bandwidth m , the resulting factor matrix F is also a banded-plus-semiseparable matrix. Specifically:*

- Its lower semiseparable part has rank r .
- Its upper semiseparable part has rank $r+p$.
- Its banded part has lower bandwidth l and upper bandwidth $l+m$.

Proof. The QR factorization is computed by performing a sequence of $n-1$ Householder transformations, eliminating the subdiagonal entries of A column by column.

Let $A^{(i)}$ denote the matrix after the i -th Householder transformation, with $A^{(0)} = A$. Define $A_i = A[i : n, i : n]$, $U_i = U[i : n, :]$, $V_i = V[i : n, :]$, $S_i = S[i : n, :]$, and $W_i = W[i : n, :]$. Let $\bar{\mathbf{u}}_i = (u_i^{(1)}, \dots, u_i^{(r)})^T$ and $\bar{\mathbf{w}}_i = (w_i^{(1)}, \dots, w_i^{(p)})^T$. Let $\bar{S} = U^T A \in \mathbb{R}^{r \times n}$.

We prove by induction that after the j -th transformation ($0 \leq j < n$):

1. The submatrix $A^{(j)}[j+1 : n, j+1 : n]$ is an HMBPSM related to A_{j+1} .
2. The j -th row of the final factor F , $F[j, j+1 : n]$, is an HMBPSV related to A_j .
3. The j -th column of F below the diagonal, $F[j+1 : n, j]$, has the form $(U \bar{\mathbf{k}}_{j+1})[j+1 : n] + \mathbf{b}_{j+1}[2 : n+1-j]$, where $\bar{\mathbf{k}}_{j+1} \in \mathbb{R}^r$ and $\mathbf{b}_{j+1} \in \mathbb{R}^{n+1-j}$ is non-zero only in its first $\min(l+1, n+1-j)$ entries.

Base Case (j=0): Initially, $A^{(0)} = A$ is trivially an HMBPSM (with $Q, K, E, X, Y, Z = \mathbf{0}$) related to $A_1 = A$.

Inductive Step: Assume the induction hypothesis holds for j . That is, $A^{(j)}[j+1 : n, j+1 : n]$ is an HMBPSM related to A_{j+1} , and for $i = 1, \dots, j$:

$$(2.20) \quad F[i+1 : n, i] = (U \bar{\mathbf{k}}_{i+1})[i+1 : n] + \mathbf{b}_{i+1}[2 : n+1-i],$$

272

273 (2.21) $F[i, i+1 : n] = \tilde{\mathbf{d}}_{i+1} + (\tilde{\boldsymbol{\alpha}}_{i+1}^T S^T)[i+1 : n] + (\tilde{\boldsymbol{\beta}}_{i+1}^T \bar{S})[i+1 : n],$

274 with $\tilde{\boldsymbol{\alpha}}_{i+1} \in \mathbb{R}^p$, $\tilde{\boldsymbol{\beta}}_{i+1} \in \mathbb{R}^r$, and $\tilde{\mathbf{d}}_{i+1} \in \mathbb{R}^{n-i}$ non-zero only in its first $\min(l+m, n-i)$
275 entries.

276 Furthermore, assume:

277 (2.22)
$$\begin{aligned} A^{(j)}[j+1 : n, j+1 : n] &= A_{j+1} + U_{j+1} Q_{j+1} S_{j+1}^T + U_{j+1} K_{j+1} U_{j+1}^T A_{j+1} \\ &\quad + U_{j+1} E_{j+1} + X_{j+1} S_{j+1}^T + Y_{j+1} U_{j+1}^T A_{j+1} + Z_{j+1}, \end{aligned}$$

278 where the modification matrices $Q_{j+1}, K_{j+1}, E_{j+1}, X_{j+1}, Y_{j+1}, Z_{j+1}$ possess the sparsity
279 patterns specified in Definition 2.1.

280 If $j < n - 1$, we now perform the $(j+1)$ -th Householder transformation on this
281 HMBPSM. Let \mathbf{y}_{j+1} be the corresponding Householder vector (unlike the form given
282 in (2.3), here \mathbf{y}_{j+1} is a vector of length $n-j$ applied to the corresponding submatrix,
283 and we will follow this convention until the end of section 3.1). It can be expressed as
284 $\mathbf{y}_{j+1} = \mathbf{e}_{j+1} + U^{(j+2)} \bar{\mathbf{k}}_{j+2} + \mathbf{b}_{j+2}$, where $\mathbf{e}_{j+1} \in \mathbb{R}^{n-j}$ is the first standard basis vector,
285 $U^{(j+2)} \in \mathbb{R}^{(n-j) \times r}$ satisfies $U^{(j+2)}[1, :] = \mathbf{0}$ and $U^{(j+2)}[2 : n-j, :] = U[j+2 : n, :],$
286 $\bar{\mathbf{k}}_{j+2} \in \mathbb{R}^r$, and $\mathbf{b}_{j+2} \in \mathbb{R}^{n-j}$ is non-zero only in its first $\min(l+1, n-j)$ entries. This
287 vector defines the $(j+1)$ -th column of F :

288 (2.23) $F[j+2 : n, j+1] = (U \bar{\mathbf{k}}_{j+2})[j+2 : n] + \mathbf{b}_{j+2}[2 : n-j].$

289 We now apply Lemma Lemma 2.4 to the HMBPSM $C = A^{(j)}[j+1 : n, j+1 : n]$,
290 which is related to A_{j+1} . The Householder transformation $(I - \tau_{j+1} \mathbf{y}_{j+1} \mathbf{y}_{j+1}^T)$ is
291 applied to C , here τ_{j+1} is a coefficient found to satisfy the definition of a Householder
292 transformation.

293 From the lemma, the resulting submatrix $A^{(j+1)}[j+2 : n, j+2 : n]$ is an HMBPSM
294 related to A_{j+2} . Its structure is given by equations analogous to (2.17), with updated
295 modification matrices $Q_{j+2}, K_{j+2}, E_{j+2}, X_{j+2}, Y_{j+2}, Z_{j+2}$, which retain the required
296 sparsity patterns. This satisfies condition 1 for $j+1$.

297 Furthermore, the lemma states that $A^{(j+1)}[j+1, j+2 : n]$ is an HMBPSV related
298 to A_{j+1} . This row becomes $F[j+1, j+2 : n]$ in the final factor matrix. Following
299 the derivation (2.19) in the lemma, and using the relation

300 (2.24)
$$U_{j+1}^T A_{j+1}[:, 2 : n-j] = (U^T A - \sum_{t=1}^j \bar{\mathbf{u}}_t \bar{\mathbf{w}}_t^T S^T)[:, j+2 : n] - \sum_{t=\max(j-m+2, 1)}^j \bar{\mathbf{u}}_t (B[t, j+2 : n]),$$

301 we can express this row in the form:

302 (2.25) $F[j+1, j+2 : n] = \tilde{\mathbf{d}}_{j+2}^T + (\tilde{\boldsymbol{\alpha}}_{j+2}^T S^T)[j+2 : n] + (\tilde{\boldsymbol{\beta}}_{j+2}^T \bar{S})[j+2 : n],$

303 where $\tilde{\mathbf{d}}_{j+2}$ only nonzero in the first $\min(l+m, n-j-1)$ entries, $\tilde{\boldsymbol{\alpha}}_{j+2} \in \mathbb{R}^p$,
304 and $\tilde{\boldsymbol{\beta}}_{j+2} \in \mathbb{R}^r$. This satisfies condition 2 for $j+1$. Condition 3 for $j+1$ is already
305 established by (2.23).

306 By the principle of induction, the hypotheses hold for all $j = 0, \dots, n-1$.

307 Upon completion of all $n-1$ transformations, the factor matrix F is fully determined.
308 Aggregating the results from (2.20) and (2.21), we conclude that F can be
309 written in the form:

310 (2.26) $F = B_F + \text{tril}(U \bar{K}^T, -1) + \text{triu}([\bar{A}, \bar{B}] [S, \bar{S}^T]^T, 1),$

311 where

- 312 • $\bar{K} \in \mathbb{R}^{n \times r}$ is defined by $\bar{K}[i, :] = \bar{\mathbf{k}}_{i+1}^T$ for $i = 1, \dots, n-1$ and $\bar{K}[n, :] = \mathbf{0}$.
- 313 • $\bar{A} \in \mathbb{R}^{n \times p}$ is defined by $\bar{A}[i, :] = \bar{\mathbf{a}}_{i+1}^T$ for $i = 1, \dots, n-1$ and $\bar{A}[n, :] = \mathbf{0}$.
- 314 • $\bar{B} \in \mathbb{R}^{n \times r}$ is defined by $\bar{B}[i, :] = \bar{\beta}_{i+1}^T$ for $i = 1, \dots, n-1$ and $\bar{B}[n, :] = \mathbf{0}$.
- 315 • B_F is a banded matrix with lower bandwidth l and upper bandwidth $l+m$,
defined by:

$$317 B_F[i, j] = \begin{cases} A^{(i)}[i, j], & i = j < n \\ A^{(n-1)}[n, n], & i = j = n \\ \mathbf{b}_{j+1}[i - j + 1], & 0 < i - j \leq l \\ \bar{\mathbf{d}}_{i+1}[j - i], & 0 < j - i \leq l + m \\ 0, & \text{otherwise.} \end{cases}$$

318 The representation in (2.26) explicitly shows that F is a banded-plus-semiseparable
319 matrix with a lower semiseparable rank of r , an upper semiseparable rank of $r+p$, a
320 lower bandwidth of l , and an upper bandwidth of $l+m$. This completes the proof. \square

321 3. Algorithms.

322 **3.1. Fast QR factorization for BPS Matrices.** Based on the structure-
323 preserving theorem proven in Section 2, we now present the detailed $O(n)$ algorithm
324 for computing the QR factorization of a banded-plus-semiseparable matrix. The al-
325 gorithm exploits the proven fact that the factor matrix F maintains a BPS structure.

326 Make an algorithm environment. Possibly split into sub-algorithms

327 ALGORITHM 3.1 (Fast QR).

328 *This algorithm computes the QR factorization of a BPS matrix $A = B + \text{tril}(UV^T, -1) +$
329 $\text{triu}(WST^T, 1)$, producing the structured factor matrix F (which contains both R and
330 the Householder vectors) and the scalar coefficients τ , in $O(n)$ operations.*

331 Input:

- 332 • B : Banded matrix with lower bandwidth l , upper bandwidth m .
- 333 • $U, V \in \mathbb{R}^{n \times r}$: Generators for the lower semiseparable part of rank r .
- 334 • $W, S \in \mathbb{R}^{n \times p}$: Generators for the upper semiseparable part of rank p .

335 Output:

- 336 • F : The structured factor matrix, maintained as a BPS matrix with lower
337 rank r , upper rank $r+p$, lower bandwidth l , and upper bandwidth $l+m$. It
338 is stored via its components:
 - 339 – B_F : The updated banded part.
 - 340 – $\bar{K} \in \mathbb{R}^{n \times r}$: Generator for the lower semiseparable part of F .
 - 341 – $\bar{A} \in \mathbb{R}^{n \times p}, \bar{B} \in \mathbb{R}^{n \times r}$: Generators for the upper semiseparable part of F .
- 342 • $\tau \in \mathbb{R}^n$: The scalar coefficients for the Householder transformations.

343 Procedure:

344 1. Initialization:

- 345 • Set $A^{(0)} = A$. The initial state is an Householder-Modified BPS Matrix with
346 modification matrices Q, K, E, X, Y, Z all set to zero. This corresponds to the
347 original BPS matrix A .
- 348 • Initialize the output components $B_F, \bar{K}, \bar{A}, \bar{B}$ to zero matrices of their re-
349 spective sizes. Also compute $\bar{S} = U^T A$. Note: The matrix \bar{S} can be computed
350 initially in $O(n(r+p))$ time due to the structure of A .

351 2. For $k = 1$ to $n-1$, eliminate the subdiagonal entries of the k th 352 column:

353 We process the submatrix $A^{(k-1)}[k : n, k : n]$, which is an HMBPSM related to
 354 $A_k = A[k : n, k : n]$.

355 (a) **Form the Householder vector \mathbf{y}_{k+1} :**

- 356 • Extract the first column of the current HMBPSM.
- 357 • According to the inductive proof of Theorem 2.5, the vector \mathbf{y}_k has the specific
 358 form:

$$359 \quad \mathbf{y}_k = \mathbf{e}_k + U^{(k+1)}\bar{\mathbf{k}}_{k+1} + \mathbf{b}_{k+1}$$

- 360 • Based on the definition of a Householder transformation and the structure of
 361 our current HMBPSM, we can obtain $\bar{\mathbf{k}}_{k+1}$, \mathbf{b}_{k+1} , τ_k , and the diagonal entry
 362 generated in the current column (denoted as $A^{(k)}[k, k]$), in $O(1)$.
- 363 • Set the k -th component to τ as τ_k

364 (b) **Store the k -th column of F :**

- 365 • The subdiagonal part of this column is given by $\mathbf{y}_k[2 : end]$. From its structure,
 366 we have:

$$367 \quad F[k+1 : n, k] = (U\bar{\mathbf{k}}_{k+1})[k+1 : n] + \mathbf{b}_{k+1}[2 : n - k + 1]$$

- 368 • Store $\bar{\mathbf{k}}_{k+1}^T$ as the k -th row of \bar{K} .
- 369 • The vector $\mathbf{b}_{k+1}[2 : end]$, which is non-zero only in its first $\min(l, n - k - 1)$
 370 entries, is stored in the corresponding subdiagonal positions of the banded part
 371 B_F .
- 372 • Set the diagonal part $B_F[k, k]$ to $A^{(k)}[k, k]$, which was obtained in the previous
 373 step.

374 (c) **Compute and store the k -th row of F :**

- 375 • This row, $F[k, k+1 : n]$, is the first row of the transformed submatrix after the
 376 Householder reflection is applied. It is an Householder-Modified BPS Vector:

$$377 \quad F[k, k+1 : n] = \tilde{\mathbf{d}}_{k+1}^T + (\tilde{\boldsymbol{\alpha}}_{k+1}^T S^T)[k+1 : n] + (\tilde{\boldsymbol{\beta}}_{k+1}^T \bar{S})[k+1 : n]$$

- 378 • The vectors $\tilde{\boldsymbol{\alpha}}_{k+1} \in \mathbb{R}^p$ and $\tilde{\boldsymbol{\beta}}_{k+1} \in \mathbb{R}^r$ are computed based on the proof in
 379 theorem 2.5. Store $\tilde{\boldsymbol{\alpha}}_{k+1}^T$ and $\tilde{\boldsymbol{\beta}}_{k+1}^T$ as the k -th rows of \bar{A} and \bar{B} , respectively.
- 380 • The vector $\tilde{\mathbf{d}}_{k+2}$, which is non-zero only in its first $\min(l + m, n - k)$ entries,
 381 is stored in the corresponding superdiagonal entries of the banded part B_F .

382 (d) **Update the remaining submatrix (Implicit HMBPSM update):**

- 383 • Update matrices Q, K, E, X, Y, Z as derived in the proof of Lemma 2.4.
- 384 • Actually, we only need to store and update the nonzero parts of E, X, Y, Z ,
 385 which are E_s, X_s, Y_s, Z_s , and they require $O(1)$ storage only.
- 386 • These updates consist of low-rank operations and manipulations of matrices
 387 with limited non-zero rows/columns, which can be done in $O(1)$ time.

388 3. **Final step ($k = n$):**

- 389 • The last diagonal element $F[n, n] = A^{(n-1)}[n, n]$ is simply the last remaining
 390 1-by-1 submatrix after the $n - 1$ Householder transformations. Store it in
 391 $B_F[n, n]$.

392 4. **Output:**

- 393 • The complete QR factorization is represented by the structured factor matrix
 394 F , defined as $B_F + \text{tril}(U\bar{K}^T, -1) + \text{triu}([\bar{A}, \bar{B}][S, \bar{S}^T]^T, 1)$, and the vector τ .

395 **Complexity Analysis.** The algorithm runs for $n - 1$ steps. The cost per step
 396 can be expressed as a polynomial in term of r, p, l , and m . Since these are constants
 397 independent of n , the total complexity is $O(n)$. The memory footprint is also $O(n)$,
 398 as we store only the generators and banded components.

399 REMARK 3.1. To maintain the $O(1)$ per-step complexity in Algorithm 3.1, two
400 key quantities must be computed efficiently during the Householder updates:

- 401 • **Inner product matrix $U_k^T U_k$:** The computation of intermediate vectors
402 $\mathbf{c}_1, \dots, \mathbf{c}_6$ requires evaluating expressions like $U_k^T \mathbf{y}_k = U_k^T (\mathbf{e}_k + U^{(k+1)} \bar{\mathbf{k}}_{k+1} +$
403 $\mathbf{b}_{k+1})$, which involves $U_k^T U^{(k+1)}$ that is equal to $U_{k+1}^T U_{k+1}$. we precompute a
404 lookup table:

$$405 \quad UU_lookup[k] = U[k : n, :]^T U[k : n, :] \quad \text{for } k = 1, \dots, n$$

406 This can be computed in $O(nr^2)$ time via a backward accumulation.

- 407 • **Partial sum $\sum_{t=1}^j \bar{\mathbf{u}}_t \bar{\mathbf{w}}_t^T$:** The update of the upper triangular part in equa-
408 tion (2.24) requires this sum. We precompute:

$$409 \quad UV_lookup[j] = \sum_{t=1}^j \bar{\mathbf{u}}_t \bar{\mathbf{w}}_t^T \quad \text{for } j = 1, \dots, n-1$$

410 This is computed in $O(nrp)$ time via forward accumulation.

411 Both precomputations require $O(n)$ total time and enable $O(1)$ access to the required
412 quantities at each step of the factorization, thus preserving the overall $O(n)$ complex-
413 ity.

414 **3.2. Fast Solver for BPS Matrices.** Theorem 2.5 not only enables an efficient
415 QR factorization but also facilitates a complete direct solver for linear systems of the
416 form $A\mathbf{x} = \mathbf{b}$, where A is a banded-plus-semiseparable matrix. The solver consists of
417 two phases after the QR factorization $A = QR$: 1. Application of Q^T to the right-
418 hand side vector \mathbf{b} to form $\mathbf{c} = Q^T \mathbf{b}$. 2. Solution of the upper triangular system
419 $R\mathbf{x} = \mathbf{c}$ via backward substitution.

420 We now present $O(n)$ algorithms for both phases, leveraging the structured rep-
421 resentation of the factorization output by Algorithm 3.1.

422 **3.2.1. Fast Application of Q^T .** The orthogonal matrix Q is represented as a
423 product of Householder transformations:

$$424 \quad Q = (I - \tau_1 \mathbf{y}_1 \mathbf{y}_1^T)(I - \tau_2 \mathbf{y}_2 \mathbf{y}_2^T) \cdots (I - \tau_{n-1} \mathbf{y}_{n-1} \mathbf{y}_{n-1}^T).$$

425 Applying Q^T to a vector \mathbf{b} thus requires computing:

$$426 \quad Q^T \mathbf{b} = (I - \tau_{n-1} \mathbf{y}_{n-1} \mathbf{y}_{n-1}^T) \cdots (I - \tau_2 \mathbf{y}_2 \mathbf{y}_2^T)(I - \tau_1 \mathbf{y}_1 \mathbf{y}_1^T)\mathbf{b}.$$

427 The Householder vectors \mathbf{y}_k are stored in the factor matrix F according to the
428 normalization convention established in Section 2:

$$429 \quad \mathbf{y}_k[j] = \begin{cases} 0, & j < k \\ 1, & j = k \\ F[j, k], & j > k \end{cases} \quad \text{for } k = 1, \dots, n-1.$$

430 From Theorem 2.5, the factor matrix F admits the BPS representation:

$$431 \quad (3.1) \quad F = B_F + \text{tril}(U_F V_F^T, -1) + \text{triu}(W_F S_F^T, 1),$$

432 where $U_F, V_F \in \mathbb{R}^{n \times r}$, $W_F, S_F \in \mathbb{R}^{n \times (r+p)}$, and B_F is banded with lower bandwidth
433 l and upper bandwidth $l+m$.

434 This structure implies that each Householder vector \mathbf{y}_k can be expressed as:

$$435 \quad (3.2) \quad \mathbf{y}_k = \bar{\mathbf{e}}_k + U_F^{(k+1)} \bar{\mathbf{v}}_k + \mathbf{d}_k,$$

436 where:

- 437 • $\bar{\mathbf{e}}_k \in \mathbb{R}^n$ is the k -th standard basis vector,
 438 • $U_F^{(k+1)} \in \mathbb{R}^{n \times r}$ satisfies $U_F^{(k+1)}[1:k,:] = \mathbf{0}$ and $U_F^{(k+1)}[k+1:n,:] = U_F[k+1:n,:]$,
 439 • $\bar{\mathbf{v}}_k = V_F[k,:]^T \in \mathbb{R}^r$,
 440 • $\mathbf{d}_k \in \mathbb{R}^n$ is non-zero only in positions $k+1$ to $\min(k+l, n)$, with $\mathbf{d}_k[j] = B_F[j,k]$ for $j = k+1, \dots, \min(k+l, n)$.

443 Algorithm 3.2.1 exploits this structure to compute $Q^T \mathbf{b}$ in $O(n)$ operations by
 444 maintaining a compressed representation of the intermediate vectors throughout the
 445 transformation process.

Algorithm 3.2.1 Fast Application of Q^T to a Vector

- 1: **Input:** Factor matrix F in BPS form: $F = B_F + \text{tril}(U_F V_F^T, -1) + \text{triu}(W_F S_F^T, 1)$;
 coefficient vector $\tau = [\tau_1, \dots, \tau_{n-1}, 0]^T \in \mathbb{R}^n$; right-hand side vector $\mathbf{b} \in \mathbb{R}^n$
 - 2: **Output:** $\mathbf{c} = Q^T \mathbf{b} \in \mathbb{R}^n$
 - 3: Initialize:
 - $O \leftarrow \mathbf{0}_{n \times r}$: Storage for accumulated low-rank updates
 - $G \leftarrow \mathbf{0}_{n \times (l+1)}$: Storage for banded component updates
 - $\mathbf{h} \leftarrow \mathbf{0}_r$: Accumulator for semiseparable component
 - Let \mathbf{o}_i denote the i -th column of O
 - Let \mathbf{g}_i denote the i -th column of G
 - 4: Express initial vector: $\mathbf{b}^{(0)} = \mathbf{b} + U_F^{(1)} \mathbf{h} + \sum_{i=1}^r \mathbf{o}_i + \sum_{i=1}^{l+1} \mathbf{g}_i$
 - 5: **for** $k = 1$ to $n-1$ **do**
 - 6: Compute inner product: $c \leftarrow \mathbf{y}_k^T \mathbf{b}^{(k-1)}$ (exploit BPS structure of \mathbf{y}_k and
 precompute some lookup tables for for $O(1)$ computation)
 - 7: Update low-rank storage: $O[k,:] \leftarrow U_F[k,:] \odot \mathbf{h}^T$ (element-wise multiplication)
 - 8: Update semiseparable accumulator: $\mathbf{h} \leftarrow \mathbf{h} - \tau_k c \cdot V_F[k,:]^T$
 - 9: Update banded component:
 - 10: $G[k, 1] \leftarrow -\tau_k c$ (diagonal contribution)
 - 11: **for** $t = 1$ to $\min(l, n-k)$ **do**
 - 12: $G[k+t, t+1] \leftarrow -\tau_k c \cdot B_F[k+t, k]$ (subdiagonal contributions)
 - 13: **end for**
 - 14: Current representation: $\mathbf{b}^{(k)} = \mathbf{b} + U_F^{(k+1)} \mathbf{h} + \sum_{i=1}^r \mathbf{o}_i + \sum_{i=1}^{l+1} \mathbf{g}_i$
 - 15: **end for**
 - 16: Compute final result explicitly: $\mathbf{c} \leftarrow \mathbf{b} + U_F^{(n)} \mathbf{h} + \sum_{i=1}^r \mathbf{o}_i + \sum_{i=1}^{l+1} \mathbf{g}_i$
 - 17: **return** \mathbf{c}
-

446 THEOREM 3.1. Algorithm 3.2.1 correctly computes $\mathbf{c} = Q^T \mathbf{b}$ in $O(n)$ operations.

447 *Proof.* The proof proceeds by induction on the transformation steps. Let $\mathbf{b}^{(0)} = \mathbf{b}$
 448 and assume that after $k-1$ steps, the algorithm maintains the representation:

449
$$\mathbf{b}^{(k-1)} = \mathbf{b} + U_F^{(k)} \mathbf{h}^{(k-1)} + \sum_{i=1}^r \mathbf{o}_i^{(k-1)} + \sum_{i=1}^{l+1} \mathbf{g}_i^{(k-1)},$$

450 where the superscripts on \mathbf{h} , \mathbf{o} , and \mathbf{g} denote the state after the $(k-1)$ -th iteration.
 451 The k -th Householder transformation gives:

452
$$\mathbf{b}^{(k)} = (I - \tau_k \mathbf{y}_k \mathbf{y}_k^T) \mathbf{b}^{(k-1)} = \mathbf{b}^{(k-1)} - \tau_k (\mathbf{y}_k^T \mathbf{b}^{(k-1)}) \mathbf{y}_k.$$

453 Substituting the structured form of \mathbf{y}_k from (3.2) and the inductive representa-

454 tion:

$$\begin{aligned}
 455 \quad \mathbf{b}^{(k)} &= \mathbf{b} + U_F^{(k)} \mathbf{h}^{(k-1)} + \sum_{i=1}^r \mathbf{o}_i^{(k-1)} + \sum_{i=1}^{l+1} \mathbf{g}_i^{(k-1)} \\
 456 \quad &\quad - \tau_k c(\bar{\mathbf{e}}_k + U_F^{(k+1)} \bar{\mathbf{v}}_k + \mathbf{d}_k) \\
 457 \quad &= \mathbf{b} + U_F^{(k+1)} (\mathbf{h}^{(k-1)} - \tau_k c \bar{\mathbf{v}}_k) \\
 458 \quad &\quad + \left(\sum_{i=1}^r \mathbf{o}_i^{(k-1)} + (U_F^{(k)} - U_F^{(k+1)}) \mathbf{h}^{(k-1)} \right) \\
 459 \quad &\quad + \left(\mathbf{g}_1^{(k-1)} - \tau_k c \bar{\mathbf{e}}_k \right) + \left(\sum_{i=2}^{l+1} \mathbf{g}_i^{(k-1)} - \tau_k c \mathbf{d}_k \right).
 \end{aligned}$$

460 The algorithm updates precisely these components:

- $\mathbf{h}^{(k)} = \mathbf{h}^{(k-1)} - \tau_k c \bar{\mathbf{v}}_k$,
- $O[k, :] = U_F[k, :] \odot \mathbf{h}^{(k-1)T}$ captures $(U_F^{(k)} - U_F^{(k+1)}) \mathbf{h}^{(k-1)}$,
- Banded updates in G capture the remaining terms.

464 Thus, the representation is maintained correctly throughout all $n - 1$ steps. Each
465 step requires $O(1)$ operations due to the constant-bounded parameters r, p, l, m , yielding
466 overall $O(n)$ complexity. \square

467 **3.2.2. Fast Backward Substitution.** After computing $\mathbf{c} = Q^T \mathbf{b}$, we solve the
468 upper triangular system $R\mathbf{x} = \mathbf{c}$, where $R = \text{triu}(F)$ inherits the BPS structure of F .
469 Specifically, the upper triangular part of F satisfies:

$$470 \quad R = B_R + \text{triu}(W_F S_F^T, 1),$$

471 where $B_R = \text{triu}(B_F)$ is the upper triangular part of the banded component, maintaining upper bandwidth $l + m$.

473 Algorithm 3.2.2, which is equivalent to the one introduced in [15], exploits this
474 structure to perform backward substitution in $O(n)$ operations by maintaining a running sum for the semiseparable contributions.

476 **THEOREM 3.2.** *Algorithm 3.2.2 solves $R\mathbf{x} = \mathbf{c}$ in $O(n)$ operations.*

477 *Proof.* For completeness we include the proof from [15]. The algorithm implements standard backward substitution while exploiting the structure of R . For each
478 index j from n down to 1, the equation:

$$480 \quad R[j, j]x_j + \sum_{k=j+1}^n R[j, k]x_k = c_j$$

481 is solved for x_j .

482 The key insight is that the off-diagonal entries $R[j, k]$ for $k > j$ can be decomposed
483 as:

$$484 \quad R[j, k] = B_R[j, k] + W_F[j, :] \cdot S_F[k, :]^T.$$

485 The banded contributions $B_R[j, k]$ are non-zero only for $k = j + 1, \dots, \min(j +$
486 $l + m, n)$, requiring $O(1)$ operations per row. The semiseparable contributions are
487 accumulated in the vector \mathbf{s} , which stores:

$$488 \quad \mathbf{s} = \sum_{i=j+1}^n S_F[i, :]^T x_i.$$

Algorithm 3.2.2 Fast Backward Substitution for Structured R

```

1: Input: Upper triangular matrix  $R = \text{triu}(F)$  in structured form; transformed
   right-hand side  $\mathbf{c} \in \mathbb{R}^n$ 
2: Output: Solution  $\mathbf{x} \in \mathbb{R}^n$  satisfying  $R\mathbf{x} = \mathbf{c}$ 
3: Initialize:
   •  $\mathbf{x} \leftarrow \mathbf{0}_n$ : solution vector
   •  $\mathbf{s} \leftarrow \mathbf{0}_{r+p}$ : Accumulator for semiseparable contributions
4: for  $j = n$  down to 1 do
5:   Initialize residual:  $\text{res} \leftarrow 0$ 
6:   Add semiseparable contribution:  $\text{res} \leftarrow \text{res} + W_F[j, :] \cdot \mathbf{s}$ 
7:   Add banded contributions:
8:   for  $k = j + 1$  to  $\min(j + l + m, n)$  do
9:      $\text{res} \leftarrow \text{res} + B_R[j, k] \cdot \mathbf{x}[k]$ 
10:  end for
11:  Solve for  $x_j$ :  $\mathbf{x}[j] \leftarrow (\mathbf{c}[j] - \text{res}) / B_R[j, j]$ 
12:  Update semiseparable accumulator:  $\mathbf{s} \leftarrow \mathbf{s} + S_F[j, :]^T \cdot \mathbf{x}[j]$ 
13: end for
14: return  $\mathbf{x}$ 

```

product? I₄₈₀
 \top to . I₄₉₀
 At step j , the product $W_F[j, :] \cdot \mathbf{s}$ thus captures all semiseparable contributions
 from previously computed solution components. After computing x_j , the accumulator
 is updated to include its contribution.

I₄₉₁ Each iteration requires $O(1)$ operations, yielding overall $O(n)$ complexity. The
 correctness follows by induction from $j = n$ down to 1. \square

I₄₉₂ **3.2.3. Overall Solver Complexity.** Combining the QR factorization (Algorithm 3.1), the fast application of Q^T (Algorithm 3.2.1), and the fast backward substitution (Algorithm 3.2.2) yields a complete direct solver for BPS linear systems with $O(n)$ complexity.

I₄₉₃ COROLLARY 3.3. *For a banded-plus-semiseparable matrix $A \in \mathbb{R}^{n \times n}$ with constant-
 bounded ranks and bandwidths, the linear system $A\mathbf{x} = \mathbf{b}$ can be solved in $O(n)$ operations
 using the QR-based approach.*

I₅₀₁ Proof. Algorithm 3.1 computes the QR factorization in $O(n)$ operations. Algorithm 3.2.1 applies Q^T in $O(n)$ operations. Algorithm 3.2.2 solves the triangular system in $O(n)$ operations. The overall complexity is therefore linear in the problem size n . \square

I₅₀₅ **4. Numerical results.** To validate the theoretical complexity and demonstrate
 the practical efficiency of our proposed algorithms, we implemented the fast QR
 factorization and the complete linear solver in Julia. The implementation is publicly
 available in the SemiseparableMatrices.jl package¹, providing an open-source
 resource for the scientific computing community. All numerical tests use banded-
 plus-semiseparable matrices with fixed structural parameters $l = 4$, $m = 5$, $r = 2$,
¹ $p = 3$ to isolate the scaling behavior with respect to the matrix size n . Computations
 were carried out on a MacBook Air equipped with an Apple M2 chip (8-core CPU, 8
 GB RAM), without GPU acceleration or access to external computing resources.

Make this a citation

¹<https://github.com/JuliaLinearAlgebra/SemiseparableMatrices.jl>

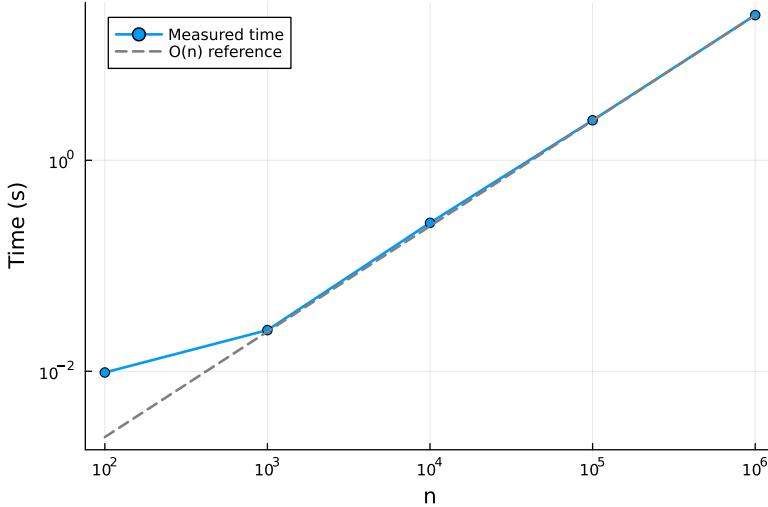


FIG. 1. Log-log plot of the total solver time (QR factorization + application of Q^T + backward substitution) versus matrix size n . The dashed reference line has slope 1, indicating ideal linear scaling.

Add different parameters for ranks and bands

Add more ticks on the y-axis

514 **4.1. Linear Complexity Verification.** Figure 1 demonstrates the linear time
 515 complexity of our complete solver for banded-plus-semiseparable linear systems. The
 516 total execution time, encompassing all three phases (QR factorization, application of
 517 Q^T , and backward substitution), scales as $O(n)$ across five orders of magnitude, from
 518 $n = 100$ to $n = 10^6$. The close alignment with the reference line of slope 1 confirms
 519 the complexity analysis in Section 3.

520 **4.2. Comparison with HODLR QR.** We compare our fast QR factorization
 521 against the state-of-the-art HODLR (Hierarchically Off-Diagonal Low-Rank) QR im-
 522 plementation from the hm-toolbox [13]. The hm-toolbox provides efficient MATLAB
 523 routines for various structured matrices, including HODLR and HSS matrices, and
 524 represents one of the most mature implementations for hierarchical matrix computa-
 525 tions.

526 Figure 2 shows the execution times for QR factorization of BPS matrices using
 527 both approaches. Our algorithm demonstrates superior scaling for larger matrix sizes.
 528 This performance advantage stems from several factors:

- 529 • **Specialized structure exploitation:** Our algorithm is specifically designed
 530 for the banded-plus-semiseparable structure, avoiding the overhead of general
 531 hierarchical representations.
- 532 • **Reduced Computational Overhead:** By working directly with the semisep-■
 533 arable generators rather than building a hierarchical representation, we avoid
 534 the logarithmic factors inherent in tree-based approaches.

535 The performance gap widens with increasing n , confirming that our method is
 536 particularly well-suited for large-scale problems. For $n = 150,000$, our implementation
 537 achieves approximately 7× speedup over the HODLR approach, demonstrating the
 538 practical benefits of our specialized algorithm.

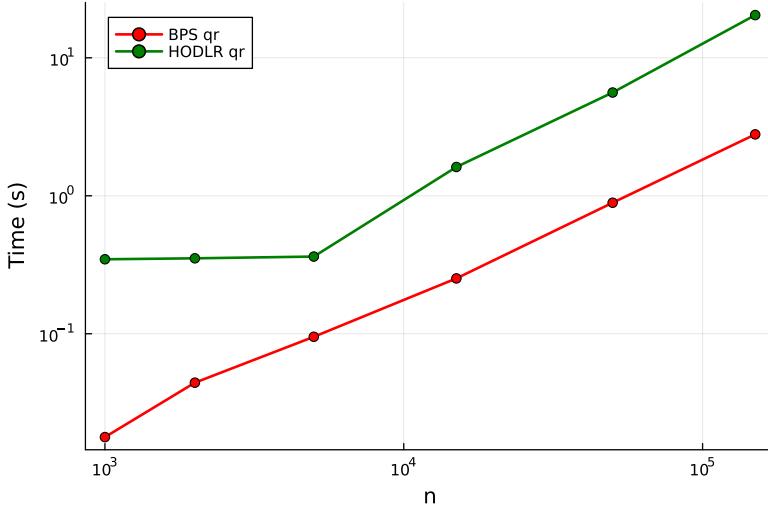


FIG. 2. Comparison of QR factorization times between our fast BPS QR algorithm and the HODLR QR implementation from [13]. Both algorithms operate on banded-plus-semiseparable matrices with parameters $l = 4$, $m = 5$, $r = 2$, $p = 3$.

Use different markers so clearer when black-and-white.

Go to $n = 10^6$

539 **5. Conclusions.** In this paper, we have established a fundamental theoretical
 540 result for BPS matrices and developed efficient algorithms based on this foundation.
 541 Our main contribution is the proof that the QR factorization of a BPS matrix pre-
 542 serves the banded-plus-semiseparable structure, with precisely characterized ranks
 543 and bandwidths in the resulting factor matrix. This theoretical insight enabled the
 544 design of a complete $O(n)$ direct solver for BPS linear systems, comprising:

- 545 • A structure-preserving QR factorization algorithm (Algorithm 3.1)
- 546 • An efficient $O(n)$ application of Q^T (Algorithm 3.2.1)
- 547 • A fast backward substitution routine (Algorithm 3.2.2)

548 The numerical experiments confirm the linear scaling of our approach and demon-
 549 strate significant performance advantages over existing HODLR-based methods. Our
 550 implementation in the SemiseparableMatrices.jl package provides the scientific com-
 551 puting community with efficient, open-source tools for working with this important
 552 class of structured matrices.

553 **Future Work.** A compelling extension involves applying our methodology to
 554 specific blocked banded matrices arising in *hp*-FEM [12]. These have optimal com-
 555 plexity so-called reverse Cholesky factorizations (Cholesky from the bottom right
 556 instead of the top left) for positive definite problems. One of our motivations for
 557 the present work is developing an optimal complexity QL factorization for these spe-
 558 cial block banded matrices. The key challenge is generalizing our framework to *block*
 559 banded-plus-semiseparable matrices while maintaining $O(N)$ complexity. The pri-
 560 mary difficulty lies in applying Householder transformations from one block to sub-
 561 sequent blocks in $O(n)$ time (where n is block size and N the total size), rather than
 562 $O(n^3)$. While our current framework doesn't directly apply, the core insight of struc-
 563 ture preservation provides a promising foundation for this challenging extension.

564 Fix capitalization

565

REFERENCES

- [1] S. CHANDRASEKARAN, P. DEWILDE, M. GU, W. LYONS, AND T. PALS, *A fast solver for hss representations via sparse matrices*, SIAM Journal on Matrix Analysis and Applications, 29 (2007), pp. 67–81.
- [2] S. CHANDRASEKARAN, P. DEWILDE, M. GU, T. PALS, AND A.-J. VAN DER VEEN, *Fast stable solver for sequentially semi-separable linear systems of equations*, in International Conference on High-Performance Computing, Springer, 2002, pp. 545–554.
- [3] S. CHANDRASEKARAN AND M. GU, *Fast and stable algorithms for banded plus semiseparable systems of linear equations*, SIAM Journal on Matrix Analysis and Applications, 25 (2003), pp. 373–384.
- [4] S. CHANDRASEKARAN, M. GU, AND T. PALS, *A fast ulv decomposition solver for hierarchically semiseparable representations*, SIAM Journal on Matrix Analysis and Applications, 28 (2006), pp. 603–622.
- [5] S. DELVAUX AND M. VAN BAREL, *Rank structures preserved by the qr-algorithm: the singular case*, Journal of Computational and Applied Mathematics, 189 (2006), pp. 157–178.
- [6] S. DELVAUX AND M. VAN BAREL, *Structures preserved by the qr-algorithm*, Journal of Computational and Applied Mathematics, 187 (2006), pp. 29–40.
- [7] S. DELVAUX AND M. VAN BAREL, *A givens-weight representation for rank structured matrices*, SIAM Journal on Matrix Analysis and Applications, 29 (2008), pp. 1147–1170.
- [8] S. DELVAUX AND M. VAN BAREL, *A qr-based solver for rank structured matrices*, SIAM Journal on Matrix Analysis and Applications, 30 (2008), pp. 464–490.
- [9] Y. EIDELMAN AND I. GOHBERG, *Inversion formulas and linear complexity algorithm for diagonal plus semiseparable matrices*, Computers & Mathematics with Applications, 33 (1997), pp. 69–79.
- [10] Y. EIDELMAN, I. GOHBERG, AND V. OLSHEVSKY, *The qr iteration method for hermitian quasiseparable matrices of an arbitrary order*, Linear Algebra and its Applications, 404 (2005), pp. 305–324.
- [11] D. FASINO, *Rational krylov matrices and qr steps on hermitian diagonal-plus-semiseparable matrices*, Numerical linear algebra with applications, 12 (2005), pp. 743–754.
- [12] K. KNOOK, S. OLVER, AND I. PAPADOPOULOS, *Quasi-optimal complexity hp-fem for poisson on a rectangle*, arXiv preprint arXiv:2402.11299, (2024).
- [13] S. MASSEI, L. ROBOL, AND D. KRESSNER, *hm-toolbox: Matlab software for hodlr and hss matrices*, SIAM Journal on Scientific Computing, 42 (2020), pp. C43–C68.
- [14] N. MASTRONARDI, S. CHANDRASEKARAN, AND S. VAN HUFFEL, *Fast and stable two-way algorithm for diagonal plus semi-separable systems of linear equations*, Numerical linear algebra with applications, 8 (2001), pp. 7–12.
- [15] S. OLVER AND A. TOWNSEND, *A fast and well-conditioned spectral method*, siam REVIEW, 55 (2013), pp. 462–489.
- [16] E. VAN CAMP, N. MASTRONARDI, AND M. VAN BAREL, *Two fast algorithms for solving diagonal-plus-semiseparable linear systems*, Journal of Computational and Applied Mathematics, 164 (2004), pp. 731–747.
- [17] R. VANDEBRIL, M. VAN BAREL, AND N. MASTRONARDI, *An implicit qr algorithm for symmetric semiseparable matrices*, Numerical Linear Algebra with Applications, 12 (2005), pp. 625–658.
- [18] R. VANDEBRIL, M. VAN BAREL, AND N. MASTRONARDI, *A note on the representation and definition of semiseparable matrices*, Numerical Linear Algebra with Applications, 12 (2005), pp. 839–858.
- [19] R. VANDEBRIL, M. VAN BAREL, AND N. MASTRONARDI, *Rational qr-iteration without inversion*, Numerische Mathematik, 110 (2008), pp. 561–575.
- [20] J. XIA, S. CHANDRASEKARAN, M. GU, AND X. S. LI, *Fast algorithms for hierarchically semiseparable matrices*, Numerical Linear Algebra with Applications, 17 (2010), pp. 953–976.