

1 **THE QR FACTORIZATION FOR BANDED-PLUS-SEMISEPARABLE**
 2 **MATRICES IS COMPUTABLE IN LINEAR COMPLEXITY**

3 TAO CHEN* AND SHEEHAN OLVER*

4 **Abstract.** We show that each stage of the QR factorization of banded-plus-semiseparable matr-
 5 ices computed using Householder reflections has a specific structured perturbation. This theoretical
 6 result enables the design of linear-complexity algorithms for QR factorization and for solving the
 7 associated linear systems. Numerical experiments validate the optimal linear complexity and demon-
 8 strate substantial speedups compared with existing hierarchical approaches. The algorithms have
 9 been implemented in an open-source Julia package, providing an efficient and accessible platform for
 10 practical use.

11 **Key words.** banded-plus-semiseparable matrices, QR factorization, linear complexity, struc-
 12 tured matrices, direct solvers

13 **AMS subject classifications.** 65F05, 65F50, 15A23, 65Y20

14 **1. Introduction.**

15 Do we want to allow for complex numbers?

16 What to say about QR algorithm?

17 Add a nullspace problem coming from the ultraspherical spectral method?

18 Does it generalise to rectangular?

19 Say something about inverses of banded matrices being banded-plus-
 semiseparable?

20 Add stability plot

21 Banded-plus-semiseparable (BPS) matrices, expressible as

$$22 \quad A = \underbrace{B}_{\text{banded}} + \underbrace{\text{tril}(UV^T, -1)}_{\text{lower semiseparable, rank } r} + \underbrace{\text{triu}(WS^T, 1)}_{\text{upper semiseparable, rank } p} \in \mathbb{R}^{n \times n},$$

Change all the T
to \top

23 arise in numerous applications from PDEs with non-local interactions [12] to signal
 24 processing, control theory, and eigenvalue problems [18]. Their structure requires only
 25 $O(n)$ storage which invites the development of $O(n)$ algorithms, a goal successfully
 26 achieved for iterations of the QR algorithm for symmetric semiseparable systems [17],
 27 and for solving linear systems with *diagonal*-plus-semiseparable matrices [9]. However,
 28 generalizing these results to the case where the banded part B is a genuine banded
 29 matrix, rather than merely a diagonal one, presents significant algorithmic challenges.

This is a bizarre
reference and de-
scription. Is it AI
generated?

30 Clarify exact relationship between prior work and ours

Add citations to
Arieh Iserles W-
systems papers

31 Pioneering work established $O(n)$ solvers for sequentially semiseparable matr-
 32 ices [2] and later for the banded-plus-semiseparable case via ULV factorization [3].
 33 Banded-plus-semiseparable matrices can be viewed as hierarchically semiseparable
 34 (HSS) matrices, and solvers using HSS structure is a well-developed area [4, 1, 20, 13].
 35 A parallel line of research extensively developed the theory and algorithms for semisep-
 36 arable and quasimseparable matrices, including implicit QR algorithms for *symmetric*
 37 semiseparable matrices [17], structure-preserving analyses [10, 6, 5], approaches lever-

What's a sequen-
tially semisepa-
rable matrix?

*Department of Mathematics, Imperial College London, London, UK (t.chen24@imperial.ac.uk, s.olver@imperial.ac.uk).

aging rational Krylov techniques [11, 19], and alternative representations [18, 7]. Despite these advances, a clear theoretical guarantee that the standard QR factorization preserves the BPS structure has been missing, with most existing solvers relying on more complex ULV or intricate Givens-based schemes [14, 3, 16, 8]. A special case of BPS matrices are almost banded matrices which were used in [15] to represent discretisations of differential equations using the ultraspherical spectral method. An optimal complexity adaptive QR factorization was introduced, which also gives an optimal complexity QR factorization for BPS matrices with now lower semisparable part ($r = 0$). It also introduced an optimal complexity back-substitution for upper-triangular BPS matrices, an algorithm we also use.

In this paper, we close this theoretical gap. We prove that the QR factorization of a BPS matrix yields a factor matrix F , which is the matrix containing both R and the Householder reflectors encoding Q , that is itself BPS, with precisely characterized lower rank r , upper rank $r + p$, and bandwidths l and $l + m$. This pivotal result, proven via an inductive framework involving a new class of Householder-Modified BPS Matrices (HMBPSM), which enables the design of an $O(n)$ QR factorization. Furthermore, it facilitates a complete direct solver: applying Q^T and performing backward substitution on the structured factor R are also achieved in linear time. Our work thus provides a unified, QR-based, end-to-end $O(n)$ solution for BPS systems, backed by a rigorous structure-preservation theorem.

The rest of this paper is organized as follows. Section 2 presents our main theoretical contributions: the definitions, the core lemma on structure preservation under Householder transformations, and the main theorem with its proof. Section 3 details the resulting $O(n)$ algorithms for QR factorization, application of Q^T , and backward substitution. Section 4 presents numerical experiments that confirm the linear complexity and demonstrate performance advantages. We conclude in Section 5 with a discussion of future work.

65 2. Main results.

66 **2.1. Problem Formulation and Notation.** Before we start, an important
 67 notation will be: for a matrix M , let $M[i : j, m : n]$ represent the submatrix of M
 68 from row i to row j and from column m to column n . When $i = j$ or $m = n$, the
 69 notation will be simplified as $M[i, m : n]$ or $M[i : j, m]$. We also adopt the convention
 70 that writing **end** in an index(such as $M[i : \text{end}, :]$) to indicates the last valid index in
 71 that dimension.
 72

73 **DEFINITION 2.1.** A banded-plus-semiseparable matrix (BPS) with lower-semisepa-
 74 ble rank r , upper-semiseparable rank p , lower-bandwidth l and upper-bandwidth m is
 75 $A \in \text{BPS}_{(r,p),(l,u)}^{n \times n} \subset \mathbb{R}^{n \times n}$ such that

$$76 A = B + \text{tril}(UV^T, -1) + \text{triu}(WS^T, 1)$$

77 where $U, V \in \mathbb{R}^{n \times r}$, $W, S \in \mathbb{R}^{n \times p}$, and $B = (b_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ is a banded matrix
 78 satisfying $b_{ij} = 0$ for $i - j > l$ or $j - i > m$.

79 Define vectors $\bar{\mathbf{u}}_i = U[i, :]^T \in \mathbb{R}^r$, $\bar{\mathbf{v}}_i = V[i, :]^T \in \mathbb{R}^r$, $\bar{\mathbf{w}}_i = W[i, :]^T \in \mathbb{R}^p$, and
 80 $\bar{\mathbf{s}}_i = S[i, :]^T \in \mathbb{R}^p$ for $i = 1, \dots, n$. The matrix A can then be expressed element-wise

80 as:

81 (2.1)
$$A = \begin{bmatrix} b_{11} & \bar{\mathbf{w}}_1^T \bar{\mathbf{s}}_2 + b_{12} & \cdots & \bar{\mathbf{w}}_1^T \bar{\mathbf{s}}_n + b_{1n} \\ \bar{\mathbf{u}}_2^T \bar{\mathbf{v}}_1 + b_{21} & b_{22} & \cdots & \bar{\mathbf{w}}_2^T \bar{\mathbf{s}}_n + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\mathbf{u}}_n^T \bar{\mathbf{v}}_1 + b_{n1} & \bar{\mathbf{u}}_n^T \bar{\mathbf{v}}_2 + b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

number
solutions that
are generated
82 Applying the QR factorization to A yields a factor matrix F , whose upper triangular part stores the matrix R and whose lower triangular part contains the Householder reflection vectors \mathbf{y} generated during the factorization. We will demonstrate that F itself retains a banded-plus-semiseparable structure. Specifically, its lower semiseparable part has rank r , its upper semiseparable part has rank $r+p$, its lower bandwidth is l , and its upper bandwidth is $l+m$.

83 Before proceeding with the detailed proof, let us clarify the precise structure of
84 the factor matrix F obtained from the QR factorization. In this work, following the
85 convention of LAPack , we employ a compact representation that stores the complete
86 information of the QR factorization in a single matrix:
87 add citation

92 (2.2)
$$F = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ y_{2,1} & r_{22} & r_{23} & \cdots & r_{2n} \\ y_{3,1} & y_{3,2} & r_{33} & \cdots & r_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & y_{n,3} & \cdots & r_{nn} \end{bmatrix}$$

93 where: the upper triangular part (including the main diagonal) of F stores the el-
94 ements of the upper triangular matrix R , i.e.: $R = \text{triu}(F)$, and the strictly lower
95 triangular part (excluding the main diagonal) of F stores the last $n - k$ elements of
96 (rescaled) Householder reflection vectors \mathbf{y}_k generated at each step.

97 More specifically, at the k -th Householder transformation step ($k = 1, 2, \dots, n-1$),
98 we construct a reflection vector \mathbf{y}_k to eliminate the subdiagonal entries of the k -th
99 column. This vector takes the form:

100 (2.3)
$$\mathbf{y}_k = \begin{cases} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ y_{k+1,k} \\ \vdots \\ y_{n,k} \end{bmatrix} & \left. \right\} n-k \text{ elements} \end{cases}$$

101 Following the LAPACK format, we normalize \mathbf{y}_k such that its first nonzero element
102 (the k -th element) equals 1. Consequently, we only need to store the elements from
103 position $k+1$ to n of this vector, which are placed in the k -th column of F , from row
104 $k+1$ to n .

105 The advantage of this representation is that it compactly stores the information of
106 both the orthogonal matrix Q (via the Householder vectors) and the upper triangular
107 matrix R within a single matrix F . The central result of this paper will demonstrate
108 that for a banded-plus-semiseparable matrix A , this factor matrix F itself maintains
109 a banded-plus-semiseparable structure.

110 It is important to note that with this normalization convention (where the first
 111 nonzero element of each Householder vector \mathbf{y}_k is 1), the full Householder transforma-
 112 tion at the k -th step is given by $I - \tau_k \mathbf{y}_k \mathbf{y}_k^T$, where τ_k is a scalar coefficient. Therefore,
 113 in addition to the factor matrix F , a vector $\boldsymbol{\tau} = (\tau_1, \tau_2, \dots, \tau_{n-1})^T$ is required to com-
 114 pletely represent the QR factorization. The orthogonal matrix Q can be reconstructed
 115 as the product $Q = (I - \tau_1 \mathbf{y}_1 \mathbf{y}_1^T)(I - \tau_2 \mathbf{y}_2 \mathbf{y}_2^T) \cdots (I - \tau_{n-1} \mathbf{y}_{n-1} \mathbf{y}_{n-1}^T)$.

116 Throughout our analysis, we will focus on the structure of the factor matrix
 117 F , while acknowledging that the complete QR representation consists of the pair
 118 $(F, \boldsymbol{\tau})$. Our main theorem establishes that F maintains the banded-plus-semiseparable
 119 structure; the scaling coefficients τ_k can be stored separately without affecting the
 120 structural properties of the algorithm.

121 We proceed to prove this by induction. First, we introduce two key definitions
 122 and a pivotal lemma.

123 **2.2. Core Definitions and a Key Lemma.** While the final factor matrix of a
 124 QR factorization is a BPS matrix, at intermediate stages it has a specific structured
 125 perturbation. Here we describe this structure in terms of a linear space that, at each
 126 stage, the perturbation to the principle submatrix lies in:

127 **DEFINITION 2.2.** *Given*

$$128 \quad A = B + \text{tril}(UV^T, -1) + \text{triu}(WS^T, 1) \in \text{BPS}_{(r,p),(l,u)}^{n \times n}$$

129 *define the vector space:*

$$\begin{aligned} 130 \quad \mathcal{P}(A) := & \left\{ UQS^T + UKU^TA + UE + XS^T + YU^TA + Z : \right. \\ 131 \quad & Q \in \mathbb{R}^{r \times p}, K \in \mathbb{R}^{r \times r}, \\ 132 \quad & E = [E_s \in \mathbb{R}^{r \times \min(l+m,n)}, \mathbf{0}] \in \mathbb{R}^{r \times n}, \\ 133 \quad & X = \begin{bmatrix} X_s \in \mathbb{R}^{\min(l,n) \times p} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n \times p}, \\ 134 \quad & Y = \begin{bmatrix} Y_s \in \mathbb{R}^{\min(l,n) \times r} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n \times r}, \\ 135 \quad & \left. Z = \begin{bmatrix} Z_s \in \mathbb{R}^{\min(l,n) \times \min(l+m,n)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n \times n} \right\} \subset \mathbb{R}^{n \times n}. \end{aligned}$$

136 In addition, we need to describe the structure of the upper-triangular part in
 137 terms of a structured vector:

138 **DEFINITION 2.3.** *Given $A \in \text{BPS}_{(r,p),(l,u)}^{n \times n}$ define the vector space*

$$\begin{aligned} 139 \quad \mathcal{V}(A) := & \left\{ \mathbf{d}^T + \boldsymbol{\alpha}^T(S^T[:, 2:n]) + \boldsymbol{\beta}^T((U^TA)[:, 2:n]) : \right. \\ 140 \quad & \mathbf{d} = \begin{bmatrix} \mathbf{d}_s \in \mathbb{R}^{\min(l+m,n-1)} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n-1}, \boldsymbol{\alpha} \in \mathbb{R}^p, \boldsymbol{\beta} \in \mathbb{R}^r \left. \right\} \subset \mathbb{R}^{1 \times (n-1)}. \end{aligned}$$

141 **LEMMA 2.4.** *Given $A \in \text{BPS}_{(r,p),(l,u)}^{n \times n}$ and $P \in \mathcal{P}(A)$, suppose a Householder
 142 transformation is applied to $A + P$ to eliminate the subdiagonal entries of its first
 143 column, yielding $\tilde{C} = (I - \tau \mathbf{y} \mathbf{y}^T)(A + P)$. Then the following hold:*

- 144 1. *The principal submatrix satisfies $\tilde{C}[2:n, 2:n] = A[2:n, 2:n] + \tilde{P}$ for
 145 $\tilde{P} \in \mathcal{P}(A[2:n, 2:n])$.*

alise so it
 't need to be
 holder

146 2. The first row satisfies $\tilde{C}[1, 2 : n] \in \mathcal{V}(A)$.

147 *Proof.*

148 I'll start updating the proof tomorrow

149 Let us introduce the necessary notation:

- 150 • $A = B + \text{tril}(UV^T, -1) + \text{triu}(WS^T, 1)$, where $U = (\mathbf{u}_1, \dots, \mathbf{u}_r) \in \mathbb{R}^{n \times r}$, $V =$
 151 $(\mathbf{v}_1, \dots, \mathbf{v}_r) \in \mathbb{R}^{n \times r}$, $W = (\mathbf{w}_1, \dots, \mathbf{w}_p) \in \mathbb{R}^{n \times p}$, and $S = (\mathbf{s}_1, \dots, \mathbf{s}_p) \in \mathbb{R}^{n \times p}$.
 152 Here $\mathbf{u}_i = (u_1^{(i)}, \dots, u_n^{(i)})^T \in \mathbb{R}^n$ and $\mathbf{v}_i = (v_1^{(i)}, \dots, v_n^{(i)})^T \in \mathbb{R}^n$ for $i = 1, \dots, r$;
 153 $\mathbf{w}_i = (w_1^{(i)}, \dots, w_n^{(i)})^T \in \mathbb{R}^n$ and $\mathbf{s}_i = (s_1^{(i)}, \dots, s_n^{(i)})^T \in \mathbb{R}^n$ for $i = 1, \dots, p$. Also,
 154 $B = (b_{ij})_{i,j=1}^n \in \mathbb{R}^{n,n}$ with $b_{ij} = 0$ if $i - j > l$ or $j - i > m$.
 155 • $C = A + UQST + UKUTA + UE + XST + YUTA + Z$, where Q, K, E, X, Y, Z
 156 are as in Definition [Definition 2.2](#).
 157 • $\tilde{C} = (I - \tau\mathbf{y}\mathbf{y}^T)C$, where the Householder vector \mathbf{y} can be expressed as $\mathbf{y} =$
 158 $\mathbf{e}_1 + U^{(2)}\bar{\mathbf{k}} + \mathbf{b}$. Here, $\mathbf{e}_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^n$, $U^{(2)} \in \mathbb{R}^{n \times r}$ satisfies $U^{(2)}[1, :] =$
 159 $\mathbf{0}$ and $U^{(2)}[2 : n, :] = U[2 : n, :]$, $\bar{\mathbf{k}} \in \mathbb{R}^r$, $\mathbf{b} = (0, b_2, \dots, b_{\min(l+1, n)}, 0, \dots, 0)^T \in$
 160 \mathbb{R}^n , and τ is a coefficient found to satisfy the definition of a Householder
 161 transformation.

162 Let $\bar{\mathbf{u}}_1 = (u_1^{(1)}, \dots, u_1^{(r)})^T \in \mathbb{R}^r$. We can write:

$$163 \quad \mathbf{e}_1^T A = \underbrace{\mathbf{d}_1^T}_{\min(m+1, n) \text{ nonzero entries}} + \underbrace{\bar{\mathbf{w}}_1^T}_{\in \mathbb{R}^{1 \times p}} S^T,$$

164 where $\mathbf{d}_1 = B[1, :]^T \in \mathbb{R}^n$, $\bar{\mathbf{w}}_1 = (w_1^{(1)}, \dots, w_1^{(p)})^T \in \mathbb{R}^p$, and

$$165 \quad \mathbf{b}^T A = \underbrace{\bar{\mathbf{d}}^T}_{\min(l+m+1, n) \text{ nonzero entries}} + \underbrace{\mathbf{f}^T}_{\mathbf{b}^T W \in \mathbb{R}^{1 \times p}} S^T.$$

166 Define the auxiliary vectors:

$$167 \quad (2.4) \quad \mathbf{c}_1 = Q^T U^T \mathbf{y} \in \mathbb{R}^p$$

$$168 \quad (2.5) \quad \mathbf{c}_2 = K^T U^T \mathbf{y} \in \mathbb{R}^r$$

$$169 \quad (2.6) \quad \mathbf{c}_3 = U^T \mathbf{y} \in \mathbb{R}^r$$

$$170 \quad (2.7) \quad \mathbf{c}_4 = X^T \mathbf{y} \in \mathbb{R}^p$$

$$171 \quad (2.8) \quad \mathbf{c}_5 = Y^T \mathbf{y} \in \mathbb{R}^r$$

$$172 \quad (2.9) \quad \mathbf{c}_6 = Z^T \mathbf{y} \in \mathbb{R}^n, \quad \text{which has the form } \mathbf{c}_6 = \begin{bmatrix} \mathbf{c}_{6s} \\ \mathbf{0} \end{bmatrix} \text{ with } \mathbf{c}_{6s} \in \mathbb{R}^{\min(l+m, n)}.$$

173 Also, let $\mathbf{x}^{(1)} = X[1, :]^T \in \mathbb{R}^p$, $\mathbf{y}^{(1)} = Y[1, :]^T \in \mathbb{R}^r$, and $\mathbf{z}^{(1)} = Z[1, :]^T \in \mathbb{R}^n$.

174 We now compute $(I - \tau\mathbf{y}\mathbf{y}^T)C$ by distributing the transformation over each term
 175 in the definition of C .

176 (i) **Transformation of A :** Substituting the expressions $\mathbf{y} = \mathbf{e}_1 + U^{(2)}\bar{\mathbf{k}} + \mathbf{b}$,

177 $\mathbf{e}_1^T A = \mathbf{d}_1^T + \bar{\mathbf{w}}_1^T S^T$, and $\mathbf{b}^T A = \bar{\mathbf{d}}^T + \mathbf{f}^T S^T$, we obtain:

$$(2.10) \quad \begin{aligned} (I - \tau \mathbf{y} \mathbf{y}^T) A &= A + \mathbf{e}_1 \left[\underbrace{(-\tau \mathbf{d}_1^T - \tau \bar{\mathbf{d}}^T)}_{\min(l+m+1,n) \text{ nonzero entries}} + \underbrace{(-\tau \bar{\mathbf{w}}_1^T - \tau \mathbf{f}^T)}_{\in \mathbb{R}^{1 \times p}} S^T \right. \\ &\quad \left. + \underbrace{(-\tau \bar{\mathbf{k}}^T)}_{\in \mathbb{R}^{1 \times r}} U^{(2)T} A \right] + U^{(2)} \underbrace{(-\tau \bar{\mathbf{k}} \bar{\mathbf{w}}_1^T - \tau \bar{\mathbf{k}} \mathbf{f}^T)}_{\in \mathbb{R}^{r \times p}} S^T \\ &\quad + U^{(2)} \underbrace{(-\tau \bar{\mathbf{k}} \bar{\mathbf{k}}^T)}_{\in \mathbb{R}^{r \times r}} U^{(2)T} A + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{d}_1^T - \tau \bar{\mathbf{k}} \bar{\mathbf{d}}^T) \\ &\quad \left. + (-\tau \bar{\mathbf{b}} \bar{\mathbf{w}}_1^T - \tau \mathbf{b} \mathbf{f}^T) S^T + (-\tau \bar{\mathbf{b}} \bar{\mathbf{k}}^T) U^{(2)T} A + (-\tau \mathbf{b} \mathbf{d}_1^T - \tau \bar{\mathbf{b}} \bar{\mathbf{d}}^T) \right]. \end{aligned}$$

179 Dropping the first column, we see that the first row of the term in brackets is in $\mathcal{V}(A)$.

180 Dropping the first row and column of the remaining terms are in $\mathcal{P}(A[2:n, 2:n])$.

181 **(ii) Transformation of UQS^T :** Substituting the expressions $\mathbf{y}^T U Q = \mathbf{c}_1^T$, $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, and $U = \mathbf{e}_1 \bar{\mathbf{u}}_1^T + U^{(2)}$ where $\bar{\mathbf{u}}_1 = (u_1^{(1)}, \dots, u_1^{(r)}) \in \mathbb{R}^r$, we obtain

$$(2.11) \quad (I - \tau \mathbf{y} \mathbf{y}^T) U Q S^T = \mathbf{e}_1 (\bar{\mathbf{u}}_1^T Q - \tau \mathbf{c}_1^T) S^T + U^{(2)} (Q - \tau \bar{\mathbf{k}} \mathbf{c}_1^T) S^T + (-\tau \mathbf{b} \mathbf{c}_1^T) S^T.$$

184 **(iii) Transformation of $UKU^T A$:** Substituting the expressions $\mathbf{y}^T UK = \mathbf{c}_2^T$,
185 $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, $U = \mathbf{e}_1 \bar{\mathbf{u}}_1^T + U^{(2)}$, and $\mathbf{e}_1^T A = \mathbf{d}_1^T + \bar{\mathbf{w}}_1^T S^T$, we obtain

$$(2.12) \quad \begin{aligned} (I - \tau \mathbf{y} \mathbf{y}^T) UKU^T A &= \mathbf{e}_1 (\bar{\mathbf{u}}_1^T K \bar{\mathbf{u}}_1 \mathbf{d}_1^T - \tau \mathbf{c}_2^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T) + \mathbf{e}_1 (\bar{\mathbf{u}}_1^T K \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T - \tau \mathbf{c}_2^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + \mathbf{e}_1 (\bar{\mathbf{u}}_1^T K - \tau \mathbf{c}_2^T) U^{(2)T} A \\ &\quad + U^{(2)} (K \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T - \tau \bar{\mathbf{k}} \mathbf{c}_2^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + U^{(2)} (K - \tau \bar{\mathbf{k}} \mathbf{c}_2^T) U^{(2)T} A + U^{(2)} (K \bar{\mathbf{u}}_1 \mathbf{d}_1^T - \tau \bar{\mathbf{k}} \mathbf{c}_2^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T) \\ &\quad + (-\tau \mathbf{b} \mathbf{c}_2^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + (-\tau \mathbf{b} \mathbf{c}_2^T) U^{(2)T} A + (-\tau \mathbf{b} \mathbf{c}_2^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T). \end{aligned}$$

187 **(iv) Transformation of UE :** Substituting the expressions $\mathbf{y}^T U = \mathbf{c}_3^T$, $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, and $U = \mathbf{e}_1 \bar{\mathbf{u}}_1^T + U^{(2)}$, we obtain

$$(2.13) \quad (I - \tau \mathbf{y} \mathbf{y}^T) UE = \mathbf{e}_1 (\bar{\mathbf{u}}_1^T E - \tau \mathbf{c}_3^T E) + U^{(2)} (E - \tau \bar{\mathbf{k}} \mathbf{c}_3^T E) + (-\tau \mathbf{b} \mathbf{c}_3^T E).$$

190 **(v) Transformation of XS^T :** Substituting the expressions $\mathbf{y}^T X = \mathbf{c}_4^T$ and
191 $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, we obtain

$$(2.14) \quad (I - \tau \mathbf{y} \mathbf{y}^T) XS^T = \mathbf{e}_1 (-\tau \mathbf{c}_4^T) S^T + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{c}_4^T) S^T + (X - \tau \mathbf{b} \mathbf{c}_4^T) S^T.$$

193 **(vi) Transformation of $YU^T A$:** Substituting the expressions $\mathbf{y}^T Y = \mathbf{c}_5^T$, $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, $U = \mathbf{e}_1 \bar{\mathbf{u}}_1^T + U^{(2)}$, and $\mathbf{e}_1^T A = \mathbf{d}_1^T + \bar{\mathbf{w}}_1^T S^T$, we obtain

$$(2.15) \quad \begin{aligned} (I - \tau \mathbf{y} \mathbf{y}^T) YU^T A &= \mathbf{e}_1 (-\tau \mathbf{c}_5^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T) + \mathbf{e}_1 (-\tau \mathbf{c}_5^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + \mathbf{e}_1 (-\tau \mathbf{c}_5^T) U^{(2)T} A \\ &\quad + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{c}_5^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{c}_5^T) U^{(2)T} A + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{c}_5^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T) \\ &\quad + (Y \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T - \tau \mathbf{b} \mathbf{c}_5^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T) S^T + (Y - \tau \mathbf{b} \mathbf{c}_5^T) U^{(2)T} A + (Y \bar{\mathbf{u}}_1 \mathbf{d}_1^T - \tau \mathbf{b} \mathbf{c}_5^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T). \end{aligned}$$

196 **(vii) Transformation of Z :** Substituting the expressions $\mathbf{y}^T Z = \mathbf{c}_6^T$ and $\mathbf{y} = \mathbf{e}_1 + U^{(2)} \bar{\mathbf{k}} + \mathbf{b}$, we obtain

$$(2.16) \quad (I - \tau \mathbf{y} \mathbf{y}^T) Z = \mathbf{e}_1 (-\tau \mathbf{c}_6^T) + U^{(2)} (-\tau \bar{\mathbf{k}} \mathbf{c}_6^T) + (Z - \tau \mathbf{b} \mathbf{c}_6^T).$$

Combining equations (2.10) through (2.16), we can now identify the structure of the resulting matrix \tilde{C} .

Firstly, the submatrix $\tilde{C}[2 : n, 2 : n]$ satisfies:

$$(2.17) \quad \tilde{C}[2 : n, 2 : n] = \tilde{A} + \tilde{U}\tilde{Q}\tilde{S}^T + \tilde{U}\tilde{K}\tilde{U}^T\tilde{A} + \tilde{U}\tilde{E} + \tilde{X}\tilde{S}^T + \tilde{Y}\tilde{U}^T\tilde{A} + \tilde{Z},$$

where

$$\tilde{A} = A[2 : n, 2 : n]$$

$$\tilde{U} = U[2 : n, :]$$

$$\tilde{S} = S[2 : n, :]$$

and the updated modification matrices are given by:

$$\tilde{Q} = -\tau\bar{\mathbf{k}}\bar{\mathbf{w}}_1^T - \tau\bar{\mathbf{k}}\mathbf{f}^T + Q - \tau\bar{\mathbf{k}}\mathbf{c}_1^T + K\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T - \tau\bar{\mathbf{k}}\mathbf{c}_2^T\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T - \tau\bar{\mathbf{k}}\mathbf{c}_4^T - \tau\bar{\mathbf{k}}\mathbf{c}_5^T\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T,$$

$$\tilde{K} = -\tau\bar{\mathbf{k}}\bar{\mathbf{k}}^T + K - \tau\bar{\mathbf{k}}\mathbf{c}_2^T - \tau\bar{\mathbf{k}}\mathbf{c}_5^T,$$

$$\tilde{E} = [\tilde{E}_s, \mathbf{0}] \in \mathbb{R}^{r \times (n-1)}, \quad \text{with}$$

$$\begin{aligned} \tilde{E}_s &= (-\tau\bar{\mathbf{k}}\mathbf{d}_1^T - \tau\bar{\mathbf{k}}\bar{\mathbf{d}}^T + K\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\bar{\mathbf{k}}\mathbf{c}_2^T\bar{\mathbf{u}}_1\mathbf{d}_1^T + E - \tau\bar{\mathbf{k}}\mathbf{c}_3^T E \\ &\quad - \tau\bar{\mathbf{k}}\mathbf{c}_5^T\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\bar{\mathbf{k}}\mathbf{c}_6^T)[:, 2 : \min(l+m+1, n)], \end{aligned}$$

$$\tilde{X} = \begin{bmatrix} \tilde{X}_s \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n-1) \times p}, \quad \text{with}$$

$$\begin{aligned} \tilde{X}_s &= (-\tau\mathbf{b}\bar{\mathbf{w}}_1^T - \tau\mathbf{b}\mathbf{f}^T - \tau\mathbf{b}\mathbf{c}_1^T - \tau\mathbf{b}\mathbf{c}_2^T\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T + X - \tau\mathbf{b}\mathbf{c}_4^T \\ &\quad + Y\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T - \tau\mathbf{b}\mathbf{c}_5^T\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T)[2 : \min(l+1, n), :], \end{aligned}$$

$$\tilde{Y} = \begin{bmatrix} \tilde{Y}_s \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n-1) \times r}, \quad \text{with}$$

$$\tilde{Y}_s = (-\tau\mathbf{b}\bar{\mathbf{k}}^T - \tau\mathbf{b}\mathbf{c}_2^T + Y - \tau\mathbf{b}\mathbf{c}_5^T)[2 : \min(l+1, n), :],$$

$$\tilde{Z} = \begin{bmatrix} \tilde{Z}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n-1) \times (n-1)}, \quad \text{with}$$

$$\begin{aligned} \tilde{Z}_s &= (-\tau\mathbf{b}\mathbf{d}_1^T - \tau\bar{\mathbf{b}}\bar{\mathbf{d}}^T - \tau\mathbf{b}\mathbf{c}_2^T\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\mathbf{b}\mathbf{c}_3^T E + Y\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\mathbf{b}\mathbf{c}_5^T\bar{\mathbf{u}}_1\mathbf{d}_1^T \\ &\quad + Z - \tau\mathbf{b}\mathbf{c}_6^T)[2 : \min(l+1, n), 2 : \min(l+m+1, n)]. \end{aligned}$$

The forms of $\tilde{Q}, \tilde{K}, \tilde{E}, \tilde{X}, \tilde{Y}, \tilde{Z}$ confirm that $\tilde{C}[2 : n, 2 : n]$ is an HMBPSM related to $A[2 : n, 2 : n]$, thus establishing the first part of the lemma.

Secondly, the first row of the transformed matrix, $\tilde{C}[1, 2 : n]$, can be expressed as:

$$(2.18) \quad \tilde{C}[1, 2 : n] = \hat{\mathbf{d}}^T + \hat{\boldsymbol{\alpha}}^T(S^T[:, 2 : n]) + \hat{\boldsymbol{\beta}}^T((U^{(2)T}A)[:, 2 : n]),$$

where

$$\hat{\mathbf{d}} = \begin{bmatrix} \hat{\mathbf{d}}_s \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n-1}, \quad \text{with}$$

$$\begin{aligned} \hat{\mathbf{d}}_s &= (\mathbf{d}_1^T - \tau\mathbf{d}_1^T - \tau\bar{\mathbf{d}}^T + \bar{\mathbf{u}}_1^T K\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\mathbf{c}_2^T\bar{\mathbf{u}}_1\mathbf{d}_1^T + \bar{\mathbf{u}}_1^T E - \tau\mathbf{c}_3^T E \\ &\quad + \mathbf{y}^{(1)T}\bar{\mathbf{u}}_1\mathbf{d}_1^T - \tau\mathbf{c}_5^T\bar{\mathbf{u}}_1\mathbf{d}_1^T + \mathbf{z}^{(1)T} - \tau\mathbf{c}_6^T)^T[2 : \min(l+m+1, n)], \end{aligned}$$

$$\hat{\boldsymbol{\alpha}} = (\bar{\mathbf{w}}_1^T - \tau\bar{\mathbf{w}}_1^T - \tau\mathbf{f}^T + \bar{\mathbf{u}}_1^T Q - \tau\mathbf{c}_1^T + \bar{\mathbf{u}}_1^T K\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T - \tau\mathbf{c}_2^T\bar{\mathbf{u}}_1\bar{\mathbf{w}}_1^T$$

231 $+ \mathbf{x}^{(1)T} - \tau \mathbf{c}_4^T + \mathbf{y}^{(1)T} \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T - \tau \mathbf{c}_5^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T)^T \in \mathbb{R}^p,$

232 $\hat{\boldsymbol{\beta}} = (-\tau \bar{\mathbf{k}}^T + \bar{\mathbf{u}}_1^T K - \tau \mathbf{c}_2^T + \mathbf{y}^{(1)T} - \tau \mathbf{c}_5^T)^T \in \mathbb{R}^r.$

233 Noting that $U^{(2)} = U - \mathbf{e}_1 \bar{\mathbf{u}}_1^T$ and $\mathbf{e}_1^T A = \mathbf{d}_1^T + \bar{\mathbf{w}}_1^T S^T$, we have $U^{(2)T} A =$
 234 $U^T A - \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T S^T - \bar{\mathbf{u}}_1 \mathbf{d}_1^T$. Substituting this yields an alternative expression:

235 (2.19) $\tilde{C}[1, 2 : n] = \tilde{\mathbf{d}}^T + \tilde{\boldsymbol{\alpha}}^T (S^T[:, 2 : n]) + \tilde{\boldsymbol{\beta}}^T ((U^T A)[:, 2 : n]),$

236 where

237 $\tilde{\mathbf{d}} = \begin{bmatrix} \tilde{\mathbf{d}}_s \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n-1}, \quad \text{with}$

238 $\tilde{\mathbf{d}}_s = (\mathbf{d}_1^T - \tau \mathbf{d}_1^T - \tau \bar{\mathbf{d}}^T + \bar{\mathbf{u}}_1^T E - \tau \mathbf{c}_3^T E + \mathbf{z}^{(1)T} - \tau \mathbf{c}_6^T + \tau \bar{\mathbf{k}}^T \bar{\mathbf{u}}_1 \mathbf{d}_1^T)^T [2 : \min(l+m+1, n)],$

239 $\tilde{\boldsymbol{\alpha}} = (\bar{\mathbf{w}}_1^T - \tau \bar{\mathbf{w}}_1^T - \tau \mathbf{f}^T + \bar{\mathbf{u}}_1^T Q - \tau \mathbf{c}_1^T + \mathbf{x}^{(1)T} - \tau \mathbf{c}_4^T + \tau \bar{\mathbf{k}}^T \bar{\mathbf{u}}_1 \bar{\mathbf{w}}_1^T)^T \in \mathbb{R}^p,$

240 $\tilde{\boldsymbol{\beta}} = (-\tau \bar{\mathbf{k}}^T + \bar{\mathbf{u}}_1^T K - \tau \mathbf{c}_2^T + \mathbf{y}^{(1)T} - \tau \mathbf{c}_5^T)^T \in \mathbb{R}^r.$ ■

241 This confirms that $\tilde{C}[1, 2 : n]$ is an HMBPSV related to A , completing the proof
 242 of the lemma. □

243 **2.3. Main Theorem and its Proof.** Equipped with Lemma 2.4, we now state
 244 and prove the main theorem concerning the structure of the QR factor matrix F .

245 THEOREM 2.5. *After applying the QR factorization to a banded-plus-semiseparable* ■
 246 *matrix A (as expressed in Eq. (2.1)) with lower semiseparable rank r , upper semisep-*
 247 *arable rank p , lower bandwidth l , and upper bandwidth m , the resulting factor matrix*
 248 *F is also a banded-plus-semiseparable matrix. Specifically:*

- 249 • Its lower semiseparable part has rank r .
- 250 • Its upper semiseparable part has rank $r+p$.
- 251 • Its banded part has lower bandwidth l and upper bandwidth $l+m$.

252 *Proof.* The QR factorization is computed by performing a sequence of $n-1$
 253 Householder transformations, eliminating the subdiagonal entries of A column by
 254 column.

255 Let $A^{(i)}$ denote the matrix after the i -th Householder transformation, with $A^{(0)} =$
 256 A . Define $A_i = A[i : n, i : n]$, $U_i = U[i : n, :]$, $V_i = V[i : n, :]$, $S_i = S[i : n, :]$,
 257 and $W_i = W[i : n, :]$. Let $\bar{\mathbf{u}}_i = (u_i^{(1)}, \dots, u_i^{(r)})^T$ and $\bar{\mathbf{w}}_i = (w_i^{(1)}, \dots, w_i^{(p)})^T$. Let
 258 $\bar{S} = U^T A \in \mathbb{R}^{r \times n}$.

259 We prove by induction that after the j -th transformation ($0 \leq j < n$):

- 260 1. The submatrix $A^{(j)}[j+1 : n, j+1 : n]$ is an HMBPSM related to A_{j+1} .
- 261 2. The j -th row of the final factor F , $F[j, j+1 : n]$, is an HMBPSV related to
 262 A_j .
- 263 3. The j -th column of F below the diagonal, $F[j+1 : n, j]$, has the form
 264 $(U \bar{\mathbf{k}}_{j+1})[j+1 : n] + \mathbf{b}_{j+1}[2 : n+1-j]$, where $\bar{\mathbf{k}}_{j+1} \in \mathbb{R}^r$ and $\mathbf{b}_{j+1} \in \mathbb{R}^{n+1-j}$
 265 is non-zero only in its first $\min(l+1, n+1-j)$ entries.

266 **Base Case (j=0):** Initially, $A^{(0)} = A$ is trivially an HMBPSM (with $Q, K, E, X, Y, Z =$ ■
 267 $\mathbf{0}$) related to $A_1 = A$.

268 **Inductive Step:** Assume the induction hypothesis holds for j . That is, $A^{(j)}[j +$
 269 $1 : n, j+1 : n]$ is an HMBPSM related to A_{j+1} , and for $i = 1, \dots, j$:

270 (2.20) $F[i+1 : n, i] = (U \bar{\mathbf{k}}_{i+1})[i+1 : n] + \mathbf{b}_{i+1}[2 : n+1-i],$

271

272 (2.21) $F[i, i+1 : n] = \tilde{\mathbf{d}}_{i+1} + (\tilde{\boldsymbol{\alpha}}_{i+1}^T S^T)[i+1 : n] + (\tilde{\boldsymbol{\beta}}_{i+1}^T \bar{S})[i+1 : n],$

273 with $\tilde{\boldsymbol{\alpha}}_{i+1} \in \mathbb{R}^p$, $\tilde{\boldsymbol{\beta}}_{i+1} \in \mathbb{R}^r$, and $\tilde{\mathbf{d}}_{i+1} \in \mathbb{R}^{n-i}$ non-zero only in its first $\min(l+m, n-i)$
274 entries.

275 Furthermore, assume:

276 (2.22)
$$\begin{aligned} A^{(j)}[j+1 : n, j+1 : n] &= A_{j+1} + U_{j+1} Q_{j+1} S_{j+1}^T + U_{j+1} K_{j+1} U_{j+1}^T A_{j+1} \\ &\quad + U_{j+1} E_{j+1} + X_{j+1} S_{j+1}^T + Y_{j+1} U_{j+1}^T A_{j+1} + Z_{j+1}, \end{aligned}$$

277 where the modification matrices $Q_{j+1}, K_{j+1}, E_{j+1}, X_{j+1}, Y_{j+1}, Z_{j+1}$ possess the sparsity
278 patterns specified in Definition 2.1.

279 If $j < n-1$, we now perform the $(j+1)$ -th Householder transformation on this
280 HMBPSM. Let \mathbf{y}_{j+1} be the corresponding Householder vector (unlike the form given
281 in (2.3), here \mathbf{y}_{j+1} is a vector of length $n-j$ applied to the corresponding submatrix,
282 and we will follow this convention until the end of section 3.1). It can be expressed as
283 $\mathbf{y}_{j+1} = \mathbf{e}_{j+1} + U^{(j+2)} \bar{\mathbf{k}}_{j+2} + \mathbf{b}_{j+2}$, where $\mathbf{e}_{j+1} \in \mathbb{R}^{n-j}$ is the first standard basis vector,
284 $U^{(j+2)} \in \mathbb{R}^{(n-j) \times r}$ satisfies $U^{(j+2)}[1, :] = \mathbf{0}$ and $U^{(j+2)}[2 : n-j, :] = U[j+2 : n, :],$
285 $\bar{\mathbf{k}}_{j+2} \in \mathbb{R}^r$, and $\mathbf{b}_{j+2} \in \mathbb{R}^{n-j}$ is non-zero only in its first $\min(l+1, n-j)$ entries. This
286 vector defines the $(j+1)$ -th column of F :

287 (2.23) $F[j+2 : n, j+1] = (U \bar{\mathbf{k}}_{j+2})[j+2 : n] + \mathbf{b}_{j+2}[2 : n-j].$

288 We now apply Lemma Lemma 2.4 to the HMBPSM $C = A^{(j)}[j+1 : n, j+1 : n]$,
289 which is related to A_{j+1} . The Householder transformation $(I - \tau_{j+1} \mathbf{y}_{j+1} \mathbf{y}_{j+1}^T)$ is
290 applied to C , here τ_{j+1} is a coefficient found to satisfy the definition of a Householder
291 transformation.

292 From the lemma, the resulting submatrix $A^{(j+1)}[j+2 : n, j+2 : n]$ is an HMBPSM
293 related to A_{j+2} . Its structure is given by equations analogous to (2.17), with updated
294 modification matrices $Q_{j+2}, K_{j+2}, E_{j+2}, X_{j+2}, Y_{j+2}, Z_{j+2}$, which retain the required
295 sparsity patterns. This satisfies condition 1 for $j+1$.

296 Furthermore, the lemma states that $A^{(j+1)}[j+1, j+2 : n]$ is an HMBPSV related
297 to A_{j+1} . This row becomes $F[j+1, j+2 : n]$ in the final factor matrix. Following
298 the derivation (2.19) in the lemma, and using the relation

299 (2.24)
$$U_{j+1}^T A_{j+1}[:, 2 : n-j] = (U^T A - \sum_{t=1}^j \bar{\mathbf{u}}_t \bar{\mathbf{w}}_t^T S^T)[:, j+2 : n] - \sum_{t=\max(j-m+2, 1)}^j \bar{\mathbf{u}}_t (B[t, j+2 : n]),$$

300 we can express this row in the form:

301 (2.25) $F[j+1, j+2 : n] = \tilde{\mathbf{d}}_{j+2}^T + (\tilde{\boldsymbol{\alpha}}_{j+2}^T S^T)[j+2 : n] + (\tilde{\boldsymbol{\beta}}_{j+2}^T \bar{S})[j+2 : n],$

302 where $\tilde{\mathbf{d}}_{j+2}$ only nonzero in the first $\min(l+m, n-j-1)$ entries, $\tilde{\boldsymbol{\alpha}}_{j+2} \in \mathbb{R}^p$,
303 and $\tilde{\boldsymbol{\beta}}_{j+2} \in \mathbb{R}^r$. This satisfies condition 2 for $j+1$. Condition 3 for $j+1$ is already
304 established by (2.23).

305 By the principle of induction, the hypotheses hold for all $j = 0, \dots, n-1$.

306 Upon completion of all $n-1$ transformations, the factor matrix F is fully determined.
307 Aggregating the results from (2.20) and (2.21), we conclude that F can be
308 written in the form:

309 (2.26) $F = B_F + \text{tril}(U \bar{K}^T, -1) + \text{triu}([\bar{A}, \bar{B}] [S, \bar{S}^T]^T, 1),$

310 where

- 311 • $\bar{K} \in \mathbb{R}^{n \times r}$ is defined by $\bar{K}[i, :] = \bar{\mathbf{k}}_{i+1}^T$ for $i = 1, \dots, n-1$ and $\bar{K}[n, :] = \mathbf{0}$.
- 312 • $\bar{A} \in \mathbb{R}^{n \times p}$ is defined by $\bar{A}[i, :] = \bar{\mathbf{a}}_{i+1}^T$ for $i = 1, \dots, n-1$ and $\bar{A}[n, :] = \mathbf{0}$.
- 313 • $\bar{B} \in \mathbb{R}^{n \times r}$ is defined by $\bar{B}[i, :] = \bar{\beta}_{i+1}^T$ for $i = 1, \dots, n-1$ and $\bar{B}[n, :] = \mathbf{0}$.
- 314 • B_F is a banded matrix with lower bandwidth l and upper bandwidth $l+m$,
defined by:

$$316 \quad B_F[i, j] = \begin{cases} A^{(i)}[i, j], & i = j < n \\ A^{(n-1)}[n, n], & i = j = n \\ \mathbf{b}_{j+1}[i - j + 1], & 0 < i - j \leq l \\ \bar{\mathbf{d}}_{i+1}[j - i], & 0 < j - i \leq l + m \\ 0, & \text{otherwise.} \end{cases}$$

317 The representation in (2.26) explicitly shows that F is a banded-plus-semiseparable
318 matrix with a lower semiseparable rank of r , an upper semiseparable rank of $r+p$, a
319 lower bandwidth of l , and an upper bandwidth of $l+m$. This completes the proof. \square

320 3. Main algorithms.

321 **3.1. Fast QR factorization for BPS Matrices.** Based on the structure-
322 preserving theorem proven in Section 2, we now present the detailed $O(n)$ algorithm
323 for computing the QR factorization of a banded-plus-semiseparable matrix. The al-
324 gorithm exploits the proven fact that the factor matrix F maintains a BPS structure.

325 Make an algorithm environment. Possibly split into sub-algorithms

326 ALGORITHM 3.1 (Fast QR).

327 This algorithm computes the QR factorization of a BPS matrix $A = B + \text{tril}(UV^T, -1) +$
328 $\text{triu}(WST^T, 1)$, producing the structured factor matrix F (which contains both R and
329 the Householder vectors) and the scalar coefficients τ , in $O(n)$ operations.

330 Input:

- 331 • B : Banded matrix with lower bandwidth l , upper bandwidth m .
- 332 • $U, V \in \mathbb{R}^{n \times r}$: Generators for the lower semiseparable part of rank r .
- 333 • $W, S \in \mathbb{R}^{n \times p}$: Generators for the upper semiseparable part of rank p .

334 Output:

- 335 • F : The structured factor matrix, maintained as a BPS matrix with lower
336 rank r , upper rank $r+p$, lower bandwidth l , and upper bandwidth $l+m$. It
337 is stored via its components:
 - 338 – B_F : The updated banded part.
 - 339 – $\bar{K} \in \mathbb{R}^{n \times r}$: Generator for the lower semiseparable part of F .
 - 340 – $\bar{A} \in \mathbb{R}^{n \times p}, \bar{B} \in \mathbb{R}^{n \times r}$: Generators for the upper semiseparable part of F .
- 341 • $\tau \in \mathbb{R}^n$: The scalar coefficients for the Householder transformations.

342 Procedure:

343 1. Initialization:

- 344 • Set $A^{(0)} = A$. The initial state is an Householder-Modified BPS Matrix with
345 modification matrices Q, K, E, X, Y, Z all set to zero. This corresponds to the
346 original BPS matrix A .
- 347 • Initialize the output components $B_F, \bar{K}, \bar{A}, \bar{B}$ to zero matrices of their re-
348 spective sizes. Also compute $\bar{S} = U^T A$. Note: The matrix \bar{S} can be computed
349 initially in $O(n(r+p))$ time due to the structure of A .

350 2. For $k = 1$ to $n-1$, eliminate the subdiagonal entries of the k th 351 column:

352 We process the submatrix $A^{(k-1)}[k : n, k : n]$, which is an HMBPSM related to
 353 $A_k = A[k : n, k : n]$.

354 (a) **Form the Householder vector \mathbf{y}_{k+1} :**

- 355 • Extract the first column of the current HMBPSM.
- 356 • According to the inductive proof of Theorem 2.5, the vector \mathbf{y}_k has the specific
 357 form:

$$358 \quad \mathbf{y}_k = \mathbf{e}_k + U^{(k+1)}\bar{\mathbf{k}}_{k+1} + \mathbf{b}_{k+1}$$

- 359 • Based on the definition of a Householder transformation and the structure of
 360 our current HMBPSM, we can obtain $\bar{\mathbf{k}}_{k+1}$, \mathbf{b}_{k+1} , τ_k , and the diagonal entry
 361 generated in the current column (denoted as $A^{(k)}[k, k]$), in $O(1)$.

- 362 • Set the k -th component to τ as τ_k

363 (b) **Store the k -th column of F :**

- 364 • The subdiagonal part of this column is given by $\mathbf{y}_k[2 : end]$. From its structure,
 365 we have:

$$366 \quad F[k+1 : n, k] = (U\bar{\mathbf{k}}_{k+1})[k+1 : n] + \mathbf{b}_{k+1}[2 : n - k + 1]$$

- 367 • Store $\bar{\mathbf{k}}_{k+1}^T$ as the k -th row of \bar{K} .

- 368 • The vector $\mathbf{b}_{k+1}[2 : end]$, which is non-zero only in its first $\min(l, n - k - 1)$
 369 entries, is stored in the corresponding subdiagonal positions of the banded part
 370 B_F .

- 371 • Set the diagonal part $B_F[k, k]$ to $A^{(k)}[k, k]$, which was obtained in the previous
 372 step.

373 (c) **Compute and store the k -th row of F :**

- 374 • This row, $F[k, k+1 : n]$, is the first row of the transformed submatrix after the
 375 Householder reflection is applied. It is an Householder-Modified BPS Vector:

$$376 \quad F[k, k+1 : n] = \tilde{\mathbf{d}}_{k+1}^T + (\tilde{\boldsymbol{\alpha}}_{k+1}^T S^T)[k+1 : n] + (\tilde{\boldsymbol{\beta}}_{k+1}^T \bar{S})[k+1 : n]$$

- 377 • The vectors $\tilde{\boldsymbol{\alpha}}_{k+1} \in \mathbb{R}^p$ and $\tilde{\boldsymbol{\beta}}_{k+1} \in \mathbb{R}^r$ are computed based on the proof in
 378 theorem 2.5. Store $\tilde{\boldsymbol{\alpha}}_{k+1}^T$ and $\tilde{\boldsymbol{\beta}}_{k+1}^T$ as the k -th rows of \bar{A} and \bar{B} , respectively.
- 379 • The vector $\tilde{\mathbf{d}}_{k+2}$, which is non-zero only in its first $\min(l + m, n - k)$ entries,
 380 is stored in the corresponding superdiagonal entries of the banded part B_F .

381 (d) **Update the remaining submatrix (Implicit HMBPSM update):**

- 382 • Update matrices Q, K, E, X, Y, Z as derived in the proof of Lemma 2.4.
- 383 • Actually, we only need to store and update the nonzero parts of E, X, Y, Z ,
 384 which are E_s, X_s, Y_s, Z_s , and they require $O(1)$ storage only.
- 385 • These updates consist of low-rank operations and manipulations of matrices
 386 with limited non-zero rows/columns, which can be done in $O(1)$ time.

387 3. **Final step ($k = n$):**

- 388 • The last diagonal element $F[n, n] = A^{(n-1)}[n, n]$ is simply the last remaining
 389 1-by-1 submatrix after the $n - 1$ Householder transformations. Store it in
 390 $B_F[n, n]$.

391 4. **Output:**

- 392 • The complete QR factorization is represented by the structured factor matrix
 393 F , defined as $B_F + \text{tril}(U\bar{K}^T, -1) + \text{triu}([\bar{A}, \bar{B}][S, \bar{S}^T]^T, 1)$, and the vector τ .

394 **Complexity Analysis.** The algorithm runs for $n - 1$ steps. The cost per step
 395 can be expressed as a polynomial in term of r, p, l , and m . Since these are constants
 396 independent of n , the total complexity is $O(n)$. The memory footprint is also $O(n)$,
 397 as we store only the generators and banded components.

398 REMARK 3.1. To maintain the $O(1)$ per-step complexity in Algorithm 3.1, two
 399 key quantities must be computed efficiently during the Householder updates:

- 400 • **Inner product matrix $U_k^T U_k$:** The computation of intermediate vectors
 401 $\mathbf{c}_1, \dots, \mathbf{c}_6$ requires evaluating expressions like $U_k^T \mathbf{y}_k = U_k^T (\mathbf{e}_k + U^{(k+1)} \bar{\mathbf{k}}_{k+1} +$
 402 $\mathbf{b}_{k+1})$, which involves $U_k^T U^{(k+1)}$ that is equal to $U_{k+1}^T U_{k+1}$. we precompute a
 403 lookup table:

$$404 \quad UU_lookup[k] = U[k:n,:]^T U[k:n,:] \quad \text{for } k = 1, \dots, n$$

405 This can be computed in $O(nr^2)$ time via a backward accumulation.

- 406 • **Partial sum $\sum_{t=1}^j \bar{\mathbf{u}}_t \bar{\mathbf{w}}_t^T$:** The update of the upper triangular part in equa-
 407 tion (2.24) requires this sum. We precompute:

$$408 \quad UV_lookup[j] = \sum_{t=1}^j \bar{\mathbf{u}}_t \bar{\mathbf{w}}_t^T \quad \text{for } j = 1, \dots, n-1$$

409 This is computed in $O(nrp)$ time via forward accumulation.

410 Both precomputations require $O(n)$ total time and enable $O(1)$ access to the required
 411 quantities at each step of the factorization, thus preserving the overall $O(n)$ complex-
 412 ity.

413 **3.2. Fast Solver for BPS Matrices.** Theorem 2.5 not only enables an efficient
 414 QR factorization but also facilitates a complete direct solver for linear systems of the
 415 form $A\mathbf{x} = \mathbf{b}$, where A is a banded-plus-semiseparable matrix. The solver consists of
 416 two phases after the QR factorization $A = QR$: 1. Application of Q^T to the right-
 417 hand side vector \mathbf{b} to form $\mathbf{c} = Q^T \mathbf{b}$. 2. Solution of the upper triangular system
 418 $R\mathbf{x} = \mathbf{c}$ via backward substitution.

419 We now present $O(n)$ algorithms for both phases, leveraging the structured rep-
 420 resentation of the factorization output by Algorithm 3.1.

421 **3.2.1. Fast Application of Q^T .** The orthogonal matrix Q is represented as a
 422 product of Householder transformations:

$$423 \quad Q = (I - \tau_1 \mathbf{y}_1 \mathbf{y}_1^T)(I - \tau_2 \mathbf{y}_2 \mathbf{y}_2^T) \cdots (I - \tau_{n-1} \mathbf{y}_{n-1} \mathbf{y}_{n-1}^T).$$

424 Applying Q^T to a vector \mathbf{b} thus requires computing:

$$425 \quad Q^T \mathbf{b} = (I - \tau_{n-1} \mathbf{y}_{n-1} \mathbf{y}_{n-1}^T) \cdots (I - \tau_2 \mathbf{y}_2 \mathbf{y}_2^T)(I - \tau_1 \mathbf{y}_1 \mathbf{y}_1^T)\mathbf{b}.$$

426 The Householder vectors \mathbf{y}_k are stored in the factor matrix F according to the
 427 normalization convention established in Section 2:

$$428 \quad \mathbf{y}_k[j] = \begin{cases} 0, & j < k \\ 1, & j = k \\ F[j,k], & j > k \end{cases} \quad \text{for } k = 1, \dots, n-1.$$

429 From Theorem 2.5, the factor matrix F admits the BPS representation:

$$430 \quad (3.1) \quad F = B_F + \text{tril}(U_F V_F^T, -1) + \text{triu}(W_F S_F^T, 1),$$

431 where $U_F, V_F \in \mathbb{R}^{n \times r}$, $W_F, S_F \in \mathbb{R}^{n \times (r+p)}$, and B_F is banded with lower bandwidth
 432 l and upper bandwidth $l+m$.

433 This structure implies that each Householder vector \mathbf{y}_k can be expressed as:

$$434 \quad (3.2) \quad \mathbf{y}_k = \bar{\mathbf{e}}_k + U_F^{(k+1)} \bar{\mathbf{v}}_k + \mathbf{d}_k,$$

435 where:

- 436 • $\bar{\mathbf{e}}_k \in \mathbb{R}^n$ is the k -th standard basis vector,
 437 • $U_F^{(k+1)} \in \mathbb{R}^{n \times r}$ satisfies $U_F^{(k+1)}[1:k,:] = \mathbf{0}$ and $U_F^{(k+1)}[k+1:n,:] = U_F[k+1:n,:]$,
 438 • $\bar{\mathbf{v}}_k = V_F[k,:]^T \in \mathbb{R}^r$,
 439 • $\mathbf{d}_k \in \mathbb{R}^n$ is non-zero only in positions $k+1$ to $\min(k+l, n)$, with $\mathbf{d}_k[j] = B_F[j,k]$ for $j = k+1, \dots, \min(k+l, n)$.

440 Algorithm 3.2.1 exploits this structure to compute $Q^T \mathbf{b}$ in $O(n)$ operations by
 441 maintaining a compressed representation of the intermediate vectors throughout the
 442 transformation process.

Algorithm 3.2.1 Fast Application of Q^T to a Vector

- 1: **Input:** Factor matrix F in BPS form: $F = B_F + \text{tril}(U_F V_F^T, -1) + \text{triu}(W_F S_F^T, 1)$;
 coefficient vector $\tau = [\tau_1, \dots, \tau_{n-1}, 0]^T \in \mathbb{R}^n$; right-hand side vector $\mathbf{b} \in \mathbb{R}^n$
 - 2: **Output:** $\mathbf{c} = Q^T \mathbf{b} \in \mathbb{R}^n$
 - 3: Initialize:
 - $O \leftarrow \mathbf{0}_{n \times r}$: Storage for accumulated low-rank updates
 - $G \leftarrow \mathbf{0}_{n \times (l+1)}$: Storage for banded component updates
 - $\mathbf{h} \leftarrow \mathbf{0}_r$: Accumulator for semiseparable component
 - Let \mathbf{o}_i denote the i -th column of O
 - Let \mathbf{g}_i denote the i -th column of G
 - 4: Express initial vector: $\mathbf{b}^{(0)} = \mathbf{b} + U_F^{(1)} \mathbf{h} + \sum_{i=1}^r \mathbf{o}_i + \sum_{i=1}^{l+1} \mathbf{g}_i$
 - 5: **for** $k = 1$ to $n-1$ **do**
 - 6: Compute inner product: $c \leftarrow \mathbf{y}_k^T \mathbf{b}^{(k-1)}$ (exploit BPS structure of \mathbf{y}_k and
 precompute some lookup tables for for $O(1)$ computation)
 - 7: Update low-rank storage: $O[k,:] \leftarrow U_F[k,:] \odot \mathbf{h}^T$ (element-wise multiplication)
 - 8: Update semiseparable accumulator: $\mathbf{h} \leftarrow \mathbf{h} - \tau_k c \cdot V_F[k,:]^T$
 - 9: Update banded component:
 - 10: $G[k, 1] \leftarrow -\tau_k c$ (diagonal contribution)
 - 11: **for** $t = 1$ to $\min(l, n-k)$ **do**
 - 12: $G[k+t, t+1] \leftarrow -\tau_k c \cdot B_F[k+t, k]$ (subdiagonal contributions)
 - 13: **end for**
 - 14: Current representation: $\mathbf{b}^{(k)} = \mathbf{b} + U_F^{(k+1)} \mathbf{h} + \sum_{i=1}^r \mathbf{o}_i + \sum_{i=1}^{l+1} \mathbf{g}_i$
 - 15: **end for**
 - 16: Compute final result explicitly: $\mathbf{c} \leftarrow \mathbf{b} + U_F^{(n)} \mathbf{h} + \sum_{i=1}^r \mathbf{o}_i + \sum_{i=1}^{l+1} \mathbf{g}_i$
 - 17: **return** \mathbf{c}
-

445 THEOREM 3.1. Algorithm 3.2.1 correctly computes $\mathbf{c} = Q^T \mathbf{b}$ in $O(n)$ operations.

446 *Proof.* The proof proceeds by induction on the transformation steps. Let $\mathbf{b}^{(0)} = \mathbf{b}$
 447 and assume that after $k-1$ steps, the algorithm maintains the representation:

448
$$\mathbf{b}^{(k-1)} = \mathbf{b} + U_F^{(k)} \mathbf{h}^{(k-1)} + \sum_{i=1}^r \mathbf{o}_i^{(k-1)} + \sum_{i=1}^{l+1} \mathbf{g}_i^{(k-1)},$$

449 where the superscripts on \mathbf{h} , \mathbf{o} , and \mathbf{g} denote the state after the $(k-1)$ -th iteration.
 450 The k -th Householder transformation gives:

451
$$\mathbf{b}^{(k)} = (I - \tau_k \mathbf{y}_k \mathbf{y}_k^T) \mathbf{b}^{(k-1)} = \mathbf{b}^{(k-1)} - \tau_k (\mathbf{y}_k^T \mathbf{b}^{(k-1)}) \mathbf{y}_k.$$

452 Substituting the structured form of \mathbf{y}_k from (3.2) and the inductive representa-

453 tion:

$$\begin{aligned}
 454 \quad \mathbf{b}^{(k)} &= \mathbf{b} + U_F^{(k)} \mathbf{h}^{(k-1)} + \sum_{i=1}^r \mathbf{o}_i^{(k-1)} + \sum_{i=1}^{l+1} \mathbf{g}_i^{(k-1)} \\
 455 \quad &\quad - \tau_k c (\bar{\mathbf{e}}_k + U_F^{(k+1)} \bar{\mathbf{v}}_k + \mathbf{d}_k) \\
 456 \quad &= \mathbf{b} + U_F^{(k+1)} (\mathbf{h}^{(k-1)} - \tau_k c \bar{\mathbf{v}}_k) \\
 457 \quad &\quad + \left(\sum_{i=1}^r \mathbf{o}_i^{(k-1)} + (U_F^{(k)} - U_F^{(k+1)}) \mathbf{h}^{(k-1)} \right) \\
 458 \quad &\quad + \left(\mathbf{g}_1^{(k-1)} - \tau_k c \bar{\mathbf{e}}_k \right) + \left(\sum_{i=2}^{l+1} \mathbf{g}_i^{(k-1)} - \tau_k c \mathbf{d}_k \right).
 \end{aligned}$$

459 The algorithm updates precisely these components:

- $\mathbf{h}^{(k)} = \mathbf{h}^{(k-1)} - \tau_k c \bar{\mathbf{v}}_k$,
- $O[k, :] = U_F[k, :] \odot \mathbf{h}^{(k-1)T}$ captures $(U_F^{(k)} - U_F^{(k+1)}) \mathbf{h}^{(k-1)}$,
- Banded updates in G capture the remaining terms.

460 Thus, the representation is maintained correctly throughout all $n - 1$ steps. Each
 461 step requires $O(1)$ operations due to the constant-bounded parameters r, p, l, m , yielding
 462 overall $O(n)$ complexity. \square

463 **3.2.2. Fast Backward Substitution.** After computing $\mathbf{c} = Q^T \mathbf{b}$, we solve the
 464 upper triangular system $R\mathbf{x} = \mathbf{c}$, where $R = \text{triu}(F)$ inherits the BPS structure of F .
 465 Specifically, the upper triangular part of F satisfies:

$$466 \quad R = B_R + \text{triu}(W_F S_F^T, 1),$$

467 where $B_R = \text{triu}(B_F)$ is the upper triangular part of the banded component, maintaining upper bandwidth $l + m$.

468 Algorithm 3.2.2, which is equivalent to the one introduced in [15], exploits this structure to perform backward substitution in $O(n)$ operations by maintaining a running sum for the semiseparable contributions.

469 **THEOREM 3.2.** *Algorithm 3.2.2 solves $R\mathbf{x} = \mathbf{c}$ in $O(n)$ operations.*

470 *Proof.* For completeness we include the proof from [15]. The algorithm implements standard backward substitution while exploiting the structure of R . For each index j from n down to 1, the equation:

$$471 \quad R[j, j]x_j + \sum_{k=j+1}^n R[j, k]x_k = c_j$$

472 is solved for x_j .

473 The key insight is that the off-diagonal entries $R[j, k]$ for $k > j$ can be decomposed as:

$$474 \quad R[j, k] = B_R[j, k] + W_F[j, :] \cdot S_F[k, :]^T.$$

475 The banded contributions $B_R[j, k]$ are non-zero only for $k = j + 1, \dots, \min(j + l + m, n)$, requiring $O(1)$ operations per row. The semiseparable contributions are 476 accumulated in the vector \mathbf{s} , which stores:

$$477 \quad \mathbf{s} = \sum_{i=j+1}^n S_F[i, :]^T x_i.$$

Algorithm 3.2.2 Fast Backward Substitution for Structured R

```

1: Input: Upper triangular matrix  $R = \text{triu}(F)$  in structured form; transformed
   right-hand side  $\mathbf{c} \in \mathbb{R}^n$ 
2: Output: Solution  $\mathbf{x} \in \mathbb{R}^n$  satisfying  $R\mathbf{x} = \mathbf{c}$ 
3: Initialize:
   •  $\mathbf{x} \leftarrow \mathbf{0}_n$ : solution vector
   •  $\mathbf{s} \leftarrow \mathbf{0}_{r+p}$ : Accumulator for semiseparable contributions
4: for  $j = n$  down to 1 do
5:   Initialize residual:  $\text{res} \leftarrow 0$ 
6:   Add semiseparable contribution:  $\text{res} \leftarrow \text{res} + W_F[j, :] \cdot \mathbf{s}$ 
7:   Add banded contributions:
8:   for  $k = j + 1$  to  $\min(j + l + m, n)$  do
9:      $\text{res} \leftarrow \text{res} + B_R[j, k] \cdot \mathbf{x}[k]$ 
10:  end for
11:  Solve for  $x_j$ :  $\mathbf{x}[j] \leftarrow (\mathbf{c}[j] - \text{res}) / B_R[j, j]$ 
12:  Update semiseparable accumulator:  $\mathbf{s} \leftarrow \mathbf{s} + S_F[j, :]^T \cdot \mathbf{x}[j]$ 
13: end for
14: return  $\mathbf{x}$ 

```

product? I₄₈₃
₄₈₄ At step j , the product $W_F[j, :] \cdot \mathbf{s}$ thus captures all semiseparable contributions
₄₈₅ from previously computed solution components. After computing x_j , the accumulator
₄₈₆ is updated to include its contribution.

Each iteration requires $O(1)$ operations, yielding overall $O(n)$ complexity. The correctness follows by induction from $j = n$ down to 1. \square

3.2.3. Overall Solver Complexity. Combining the QR factorization (Algorithm 3.1), the fast application of Q^T (Algorithm 3.2.1), and the fast backward substitution (Algorithm 3.2.2) yields a complete direct solver for BPS linear systems with $O(n)$ complexity.

COROLLARY 3.3. *For a banded-plus-semiseparable matrix $A \in \mathbb{R}^{n \times n}$ with constant-
₄₉₇ bounded ranks and bandwidths, the linear system $A\mathbf{x} = \mathbf{b}$ can be solved in $O(n)$ operations
₄₉₈ using the QR-based approach.*

Proof. Algorithm 3.1 computes the QR factorization in $O(n)$ operations. Algorithm 3.2.1 applies Q^T in $O(n)$ operations. Algorithm 3.2.2 solves the triangular system in $O(n)$ operations. The overall complexity is therefore linear in the problem size n . \square

4. Fast RQ Computation for Symmetric BPS Matrices. The fast QR factorization developed in the previous section not only provides a direct linear system solver but also forms the foundation for iterative algorithms such as the QR algorithm for computing eigenvalues. A core step in the QR iteration is the formation of the RQ product. For symmetric banded-plus-semiseparable (BPS) matrices, we show that the RQ product also preserves the BPS structure, leading to the design of a linear-complexity algorithm for its fast computation.

4.1. Structure, Definitions, and a Key Lemma for the Symmetric Case.
 Consider a symmetric BPS matrix A of the form

513 (4.1)
$$A = B + \text{tril}(UV^\top, -1) + \text{triu}(VU^\top, 1),$$

514 where B is a symmetric banded matrix with lower and upper bandwidth l , and $U, V \in$
 515 $\mathbb{R}^{n \times r}$ generate the lower and upper semiseparable parts of rank r . Let $A = QR$ be its
 516 QR factorization. Since A is symmetric, we have $RQ = Q^\top AQ$, and thus RQ is also
 517 symmetric.

518 To describe the structure of intermediate matrices in the computation of RQ , we
 519 introduce definitions analogous to those in Section 2 but tailored for the symmetric
 520 case and the R factor.

521 **DEFINITION 4.1** (Householder-modified upper-triangular matrix (HMUTM)). *Given*■
 522 *an $n \times n$ upper triangular matrix R and an $n \times r$ matrix U , a matrix Γ is called a*
 523 *Householder-modified upper-triangular matrix (HMUTM) of R under U if*

524 (4.2)
$$\Gamma = R + RU\Omega U^\top + \Phi U^\top + RU\Psi + \Lambda,$$

525 where $\Omega \in \mathbb{R}^{r \times r}$; $\Phi = \begin{bmatrix} \Phi_s \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n \times r}$ with $\Phi_s \in \mathbb{R}^{\min(l,n) \times r}$; $\Psi = [\Psi_s, \mathbf{0}] \in \mathbb{R}^{r \times n}$ with
 526 $\Psi_s \in \mathbb{R}^{r \times \min(l,n)}$; and $\Lambda = \begin{bmatrix} \Lambda_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n \times n}$ with $\Lambda_s \in \mathbb{R}^{\min(l,n) \times \min(l,n)}$.

527 **DEFINITION 4.2** (Householder-modified upper-triangular vector (HMUTV)). *Given*■
 528 *an $n \times n$ upper triangular matrix R and an $n \times r$ matrix U , a vector σ of length $n - 1$*
 529 *is called a Householder-modified upper-triangular vector (HMUTV) of R under U if*

530 (4.3)
$$\sigma = \eta + (RU\mu)[2:n],$$

531 for some $\eta = \begin{bmatrix} \eta_s \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n-1}$ with $\eta_s \in \mathbb{R}^{\min(l,n)}$ and $\mu \in \mathbb{R}^r$.

532 These definitions characterize the structured perturbations that appear in the R
 533 factor when it is right-multiplied by a sequence of Householder reflectors (i.e., by Q).
 534 The following lemma is the symmetric counterpart to Lemma 2.4 and is the engine of
 535 the inductive proof.

536 **LEMMA 4.3** (Structure preservation under right-multiplication by a Householder
 537 reflector). *Given an $n \times n$ upper triangular matrix R , an $n \times r$ matrix U , and an*
 538 *HMUTM Γ of R under U , consider its right-multiplication by a Householder reflector:*
 539 $\tilde{\Gamma} = \Gamma(I - \tau\mathbf{y}\mathbf{y}^\top)$. *Assume the Householder vector has the form $\mathbf{y} = \mathbf{e}_1 + U^{(2)}\bar{\mathbf{k}} + \mathbf{b}$,*
 540 *where \mathbf{e}_1 is the first standard basis vector, $U^{(2)} \in \mathbb{R}^{n \times r}$ satisfies $U^{(2)}[1,:] = \mathbf{0}$ and*
 541 *$U^{(2)}[2:n,:] = U[2:n,:]$, $\bar{\mathbf{k}} \in \mathbb{R}^r$, and $\mathbf{b} \in \mathbb{R}^n$ is nonzero only in its entries 2 through*
 542 *$\min(l+1, n)$. Then the following hold:*

- 543 1. *The submatrix $\tilde{\Gamma}[2:n, 2:n]$ is an HMUTM of $R[2:n, 2:n]$ under $U[2:n,:]$.*
 544 2. *The vector $\tilde{\Gamma}[2:n, 1]$ is an HMUTV of R under U .*

545 *Proof.* Let $U = (\mathbf{u}_1, \dots, \mathbf{u}_r)$ where $\mathbf{u}_i = (u_1^{(i)}, \dots, u_n^{(i)})^\top$, and denote $\bar{\mathbf{u}}_1 =$
 546 $(u_1^{(1)}, \dots, u_1^{(r)})^\top \in \mathbb{R}^r$. Write $\Gamma = R + RU\Omega U^\top + \Phi U^\top + RU\Psi + \Lambda$ with matri-
 547 ces $\Omega, \Phi, \Psi, \Lambda$ having the sparsity patterns specified in Definition 4.1.

548 Define the auxiliary vectors:

549
$$\delta_1 = \Omega U^\top \mathbf{y} \in \mathbb{R}^r, \quad \delta_2 = U^\top \mathbf{y} \in \mathbb{R}^r,$$

 550
$$\delta_3 = \Psi \mathbf{y} \in \mathbb{R}^r, \quad \delta_4 = \Lambda \mathbf{y} \in \mathbb{R}^n.$$

551 Note that δ_4 has the form $\delta_4 = \begin{bmatrix} \delta_{4s} \\ \mathbf{0} \end{bmatrix}$ with $\delta_{4s} \in \mathbb{R}^{\min(l,n)}$ due to the structure of Λ .

552 Also let $\mathbf{r}^{(1)} = R[1,:]^\top$, $\boldsymbol{\phi}^{(1)} = \Phi[1,:]^\top$, $\boldsymbol{\psi}^{(1)} = \Psi[:,1]$, and $\boldsymbol{\lambda}^{(1)} = \Lambda[:,1]$, and
 553 $\boldsymbol{\gamma} = R\mathbf{b}$.

554 We compute $\tilde{\Gamma} = \Gamma - \tau\Gamma\mathbf{y}\mathbf{y}^\top$ by distributing the operation over each term in Γ .

555 (i) **Transformation of R :** Writing $\mathbf{y} = \mathbf{e}_1 + U^{(2)}\bar{\mathbf{k}} + \mathbf{b}$ and $R\mathbf{b} = \boldsymbol{\gamma}$ where $\boldsymbol{\gamma}$ is
 556 nonzero only in its first $\min(l+1, n)$ entries, we obtain:

$$\begin{aligned} 557 \quad (4.4) \quad R(I - \tau\mathbf{y}\mathbf{y}^\top) &= R + R(-\tau\mathbf{e}_1)\mathbf{e}_1^\top + RU^{(2)}(-\tau\bar{\mathbf{k}})\mathbf{e}_1^\top + (-\tau\boldsymbol{\gamma})\mathbf{e}_1^\top \\ &\quad - \tau R\mathbf{e}_1\bar{\mathbf{k}}^\top U^{(2)\top} - \tau R\mathbf{e}_1\mathbf{b}^\top + RU^{(2)}(-\tau\bar{\mathbf{k}}\bar{\mathbf{k}}^\top)U^{(2)\top} \\ &\quad + (-\tau\boldsymbol{\gamma}\bar{\mathbf{k}}^\top)U^{(2)\top} + RU^{(2)}(-\tau\bar{\mathbf{k}}\mathbf{b}^\top) + (-\tau\boldsymbol{\gamma}\mathbf{b}^\top). \end{aligned}$$

558 (ii) **Transformation of $RU\Omega U^\top$:** Using $\mathbf{y} = \mathbf{e}_1 + U^{(2)}\bar{\mathbf{k}} + \mathbf{b}$, $\Omega U^\top \mathbf{y} = \boldsymbol{\delta}_1$, and
 559 $U = \mathbf{e}_1\bar{\mathbf{u}}_1^\top + U^{(2)}$, we get:

$$\begin{aligned} 560 \quad (4.5) \quad RU\Omega U^\top(I - \tau\mathbf{y}\mathbf{y}^\top) &= R\mathbf{e}_1(\bar{\mathbf{u}}_1^\top\Omega\bar{\mathbf{u}}_1 - \tau\bar{\mathbf{u}}_1^\top\boldsymbol{\delta}_1)\mathbf{e}_1^\top + RU^{(2)}(\Omega\bar{\mathbf{u}}_1 - \tau\boldsymbol{\delta}_1)\mathbf{e}_1^\top \\ &\quad + R\mathbf{e}_1\bar{\mathbf{u}}_1^\top\Omega U^{(2)\top} - \tau R\mathbf{e}_1\bar{\mathbf{u}}_1^\top\boldsymbol{\delta}_1\bar{\mathbf{k}}^\top U^{(2)\top} - \tau R\mathbf{e}_1\bar{\mathbf{u}}_1^\top\boldsymbol{\delta}_1\mathbf{b}^\top \\ &\quad + RU^{(2)}(\Omega - \tau\boldsymbol{\delta}_1\bar{\mathbf{k}}^\top)U^{(2)\top} + RU^{(2)}(-\tau\boldsymbol{\delta}_1\mathbf{b}^\top). \end{aligned}$$

561 (iii) **Transformation of ΦU^\top :** Using $\mathbf{y} = \mathbf{e}_1 + U^{(2)}\bar{\mathbf{k}} + \mathbf{b}$, $U^\top \mathbf{y} = \boldsymbol{\delta}_2$, and
 562 $U = \mathbf{e}_1\bar{\mathbf{u}}_1^\top + U^{(2)}$, we get:

$$563 \quad (4.6) \quad \Phi U^\top(I - \tau\mathbf{y}\mathbf{y}^\top) = (\Phi\bar{\mathbf{u}}_1 - \tau\Phi\boldsymbol{\delta}_2)\mathbf{e}_1^\top + (\Phi - \tau\Phi\boldsymbol{\delta}_2\bar{\mathbf{k}}^\top)U^{(2)\top} + (-\tau\Phi\boldsymbol{\delta}_2\mathbf{b}^\top).$$

564 (iv) **Transformation of $RU\Psi$:** Using $\mathbf{y} = \mathbf{e}_1 + U^{(2)}\bar{\mathbf{k}} + \mathbf{b}$, $\Psi\mathbf{y} = \boldsymbol{\delta}_3$, and
 565 $U = \mathbf{e}_1\bar{\mathbf{u}}_1^\top + U^{(2)}$, we get:

$$\begin{aligned} 566 \quad (4.7) \quad RU\Psi(I - \tau\mathbf{y}\mathbf{y}^\top) &= (-\tau R\mathbf{e}_1\bar{\mathbf{u}}_1^\top\boldsymbol{\delta}_3)\mathbf{e}_1^\top + RU^{(2)}(-\tau\boldsymbol{\delta}_3)\mathbf{e}_1^\top + R\mathbf{e}_1\bar{\mathbf{u}}_1^\top\Psi \\ &\quad - \tau R\mathbf{e}_1\bar{\mathbf{u}}_1^\top\boldsymbol{\delta}_3\bar{\mathbf{k}}^\top U^{(2)\top} - \tau R\mathbf{e}_1\bar{\mathbf{u}}_1^\top\boldsymbol{\delta}_3\mathbf{b}^\top \\ &\quad + RU^{(2)}(-\tau\boldsymbol{\delta}_3\bar{\mathbf{k}}^\top)U^{(2)\top} + RU^{(2)}(\Psi - \tau\boldsymbol{\delta}_3\mathbf{b}^\top). \end{aligned}$$

567 (v) **Transformation of Λ :** Using $\mathbf{y} = \mathbf{e}_1 + U^{(2)}\bar{\mathbf{k}} + \mathbf{b}$ and $\Lambda\mathbf{y} = \boldsymbol{\delta}_4$, we get:

$$568 \quad (4.8) \quad \Lambda(I - \tau\mathbf{y}\mathbf{y}^\top) = (-\tau\boldsymbol{\delta}_4)\mathbf{e}_1^\top + (-\tau\boldsymbol{\delta}_4\bar{\mathbf{k}}^\top)U^{(2)\top} + (\Lambda - \tau\boldsymbol{\delta}_4\mathbf{b}^\top).$$

569 Summing contributions (i)–(v), we identify the structure of $\tilde{\Gamma}$.

570 **First**, the submatrix $\tilde{\Gamma}[2:n, 2:n]$ satisfies:

$$571 \quad (4.9) \quad \tilde{\Gamma}[2:n, 2:n] = \tilde{R} + \tilde{R}\tilde{U}\tilde{\Omega}\tilde{U}^\top + \tilde{\Phi}\tilde{U}^\top + \tilde{R}\tilde{U}\tilde{\Psi} + \tilde{\Lambda},$$

572 where $\tilde{R} = R[2:n, 2:n]$, $\tilde{U} = U[2:n,:]$, and the updated matrices are:

$$573 \quad \tilde{\Omega} = -\tau\bar{\mathbf{k}}\bar{\mathbf{k}}^\top + \Omega - \tau\boldsymbol{\delta}_1\bar{\mathbf{k}}^\top - \tau\boldsymbol{\delta}_3\bar{\mathbf{k}}^\top \in \mathbb{R}^{r \times r},$$

$$574 \quad \tilde{\Phi} = \begin{bmatrix} \tilde{\Phi}_s \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n-1) \times r}, \quad \tilde{\Phi}_s = (-\tau\boldsymbol{\gamma}\bar{\mathbf{k}}^\top + \Phi - \tau\Phi\boldsymbol{\delta}_2\bar{\mathbf{k}}^\top - \tau\boldsymbol{\delta}_4\bar{\mathbf{k}}^\top)[2 : \min(l+1, n), :] \in \mathbb{R}^{\min(l, n-1) \times r},$$

$$575 \quad \tilde{\Psi} = [\tilde{\Psi}_s, \mathbf{0}] \in \mathbb{R}^{r \times (n-1)}, \quad \tilde{\Psi}_s = (-\tau\bar{\mathbf{k}}\mathbf{b}^\top - \tau\boldsymbol{\delta}_1\mathbf{b}^\top + \Psi - \tau\boldsymbol{\delta}_3\mathbf{b}^\top)[:, 2 : \min(l+1, n)] \in \mathbb{R}^{r \times \min(l, n-1)},$$

$$576 \quad \tilde{\Lambda} = \begin{bmatrix} \tilde{\Lambda}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n-1) \times (n-1)},$$

$$577 \quad \tilde{\Lambda}_s = (-\tau\boldsymbol{\gamma}\mathbf{b}^\top - \tau\Phi\boldsymbol{\delta}_2\mathbf{b}^\top + \Lambda - \tau\boldsymbol{\delta}_4\mathbf{b}^\top)[2 : \min(l+1, n), 2 : \min(l+1, n)] \in \mathbb{R}^{\min(l, n-1) \times \min(l, n-1)}. \quad \blacksquare$$

578 This confirms that $\tilde{\Gamma}[2:n, 2:n]$ is an HMUTM of \tilde{R} under \tilde{U} .

579 **Second**, the vector $\tilde{\Gamma}[2:n, 1]$ is given by:

580 (4.10)
$$\tilde{\Gamma}[2:n, 1] = \boldsymbol{\eta} + (RU\boldsymbol{\mu})[2:n],$$

581 where

582
$$\boldsymbol{\eta} = \begin{bmatrix} \boldsymbol{\eta}_s \\ \mathbf{0} \end{bmatrix}, \quad \boldsymbol{\eta}_s = (-\tau\boldsymbol{\gamma} + \Phi\bar{\mathbf{u}}_1 - \tau\Phi\boldsymbol{\delta}_2 - \tau\boldsymbol{\delta}_4 + \boldsymbol{\lambda}^{(1)})[2 : \min(l+1, n)] \in \mathbb{R}^{\min(l, n-1)},$$

583
$$\boldsymbol{\mu} = -\tau\bar{\mathbf{k}} + \Omega\bar{\mathbf{u}}_1 - \tau\boldsymbol{\delta}_1 - \tau\boldsymbol{\delta}_3 + \boldsymbol{\psi}^{(1)} \in \mathbb{R}^r.$$

584 This matches the form of an HMUTV (Definition 4.2).

585 Finally, the $(1, 1)$ entry of $\tilde{\Gamma}$ is updated as:

586 (4.11)
$$\begin{aligned} \tilde{\Gamma}[1, 1] &= R[1, 1] - \tau R[1, 1] - \tau \mathbf{r}^{(1)\top} U^{(2)} \bar{\mathbf{k}} - \tau \gamma[1] \\ &\quad + R[1, 1](\bar{\mathbf{u}}_1^\top \Omega \bar{\mathbf{u}}_1 - \tau \bar{\mathbf{u}}_1^\top \boldsymbol{\delta}_1) + \mathbf{r}^{(1)\top} U^{(2)}(\Omega \bar{\mathbf{u}}_1 - \tau \boldsymbol{\delta}_1) \\ &\quad + \boldsymbol{\phi}^{(1)\top} \bar{\mathbf{u}}_1 - \tau \boldsymbol{\phi}^{(1)\top} \boldsymbol{\delta}_2 - \tau R[1, 1] \bar{\mathbf{u}}_1^\top \boldsymbol{\delta}_3 + R[1, 1] \bar{\mathbf{u}}_1^\top \boldsymbol{\psi}^{(1)} \\ &\quad - \tau \mathbf{r}^{(1)\top} U^{(2)} \boldsymbol{\delta}_3 + \mathbf{r}^{(1)\top} U^{(2)} \boldsymbol{\psi}^{(1)} - \tau \boldsymbol{\delta}_4[1] + \Lambda[1, 1]. \end{aligned}$$

587 The updates (4.9), (4.10), and (4.11) provide the complete formulas needed to
588 advance the structured representation by one Householder transformation. \square

589 **4.2. Structure-Preserving Theorem for RQ .** Equipped with Lemma 4.3, we
590 can now state and prove the main structural result for the RQ product.

591 **THEOREM 4.4** (Structure of RQ for symmetric BPS matrices). *Let A be a symmetric
592 BPS matrix as in (4.1) with semiseparable rank r and bandwidth l , and let
593 $A = QR$ be its QR factorization. Then the matrix RQ is also a symmetric BPS
594 matrix. Specifically, it can be expressed as*

595 (4.12)
$$RQ = B_R + \text{tril}(\Theta \Delta^\top, -1) + \text{triu}(\Delta \Theta^\top, 1),$$

596 where

- 597 • $\Theta = RU \in \mathbb{R}^{n \times r}$,
- 598 • $\Delta \in \mathbb{R}^{n \times r}$ is a low-rank generator matrix,
- 599 • B_R is a symmetric banded matrix with lower and upper bandwidth l .

600 Thus, the RQ product has a lower and upper semiseparable rank of r and a lower and
601 upper bandwidth of l .

602 *Proof.* The proof proceeds by induction on the steps of applying Q (as a product
603 of Householder reflectors) to R from the right. Define $R^{(0)} = R$ and $R^{(j)} = R(I -$
604 $\tau_1 \mathbf{y}_1 \mathbf{y}_1^\top) \cdots (I - \tau_j \mathbf{y}_j \mathbf{y}_j^\top)$ for $j = 1, \dots, n-1$, so that $R^{(n-1)} = RQ$.

605 From the proof of Theorem 2.5, the Householder vectors have the form:

606 (4.13)
$$\mathbf{y}_k[k:n] = \mathbf{e}_1 + U^{(k+1)} \bar{\mathbf{k}}_{k+1} + \mathbf{b}_{k+1}, \quad k = 1, \dots, n-1,$$

607 where $\mathbf{e}_1 \in \mathbb{R}^{n-k+1}$, $U^{(k+1)} \in \mathbb{R}^{(n-k+1) \times r}$ satisfies $U^{(k+1)}[1,:] = \mathbf{0}$ and $U^{(k+1)}[2 : n-k+1,:] = U[k+1:n,:]$, $\bar{\mathbf{k}}_{k+1} \in \mathbb{R}^r$, and $\mathbf{b}_{k+1} \in \mathbb{R}^{n-k+1}$ is nonzero only in its
608 entries 2 through $\min(l+1, n-k+1)$.

609 Let $R_k = R[k:n, k:n]$ and $U_k = U[k:n,:]$. We prove by induction that for
610 $j = 0, \dots, n-1$:

611 1. $R^{(j)}[j+1:n, j+1:n]$ is an HMUTM of R_{j+1} under U_{j+1} .

613 2. $R^{(j)}[j+1:n, j]$ is an HMUTV of R_{j+1} under U_{j+1} .

614 **Base case** ($j = 0$): $R^{(0)} = R$ is trivially an HMUTM of R_1 under U_1 with all
615 perturbation matrices zero.

616 **Inductive step:** Assume the statement holds for some $j < n - 1$. That is,

(4.14)

$$617 \quad R^{(j)}[j+1:n, j+1:n] = R_{j+1} + R_{j+1}U_{j+1}\Omega_{j+1}U_{j+1}^\top + \Phi_{j+1}U_{j+1}^\top + R_{j+1}U_{j+1}\Psi_{j+1} + \Lambda_{j+1}, \blacksquare$$

618 with $\Omega_{j+1}, \Phi_{j+1}, \Psi_{j+1}, \Lambda_{j+1}$ having the required sparsity patterns.

619 We now compute $R^{(j+1)} = R^{(j)}(I - \tau_{j+1}\mathbf{y}_{j+1}\mathbf{y}_{j+1}^\top)$. Applying Lemma 4.3 with

620 $\Gamma = R^{(j)}[j+1:n, j+1:n]$, $R = R_{j+1}$, $U = U_{j+1}$, and $\mathbf{y} = \mathbf{y}_{j+1}[j+1:n]$, we obtain:

621 1. $\tilde{\Gamma}[2:n-j, 2:n-j] = R^{(j+1)}[j+2:n, j+2:n]$ is an HMUTM of R_{j+2}
622 under U_{j+2} .

623 2. $\tilde{\Gamma}[2:n-j, 1] = R^{(j+1)}[j+2:n, j+1]$ is an HMUTV of R_{j+1} under U_{j+1} .

624 The lemma also provides explicit update formulas for the parameters, showing they
625 retain the correct sparsity patterns. This completes the inductive step.

626 By induction, the structured form holds for all j . In particular, for each $j =$
627 $0, \dots, n-2$, we have an HMUTV representation:

$$628 \quad (4.15) \quad R^{(j+1)}[j+2:n, j+1] = \boldsymbol{\eta}_{j+2} + (R_{j+1}U_{j+1}\boldsymbol{\mu}_{j+2})[2:n-j].$$

629 Since $R_{j+1}U_{j+1}[2:n-j, :] = RU[j+2:n, :]$, this can be rewritten as:

$$630 \quad (4.16) \quad \begin{aligned} (RQ)[j+1, j+2:n] &= (RQ)[j+2:n, j+1] \\ &= R^{(j+1)}[j+2:n, j+1] = \boldsymbol{\eta}_{j+2} + (RU\boldsymbol{\mu}_{j+2})[j+2:n]. \end{aligned}$$

631 Let $\Delta \in \mathbb{R}^{n \times r}$ be defined by $\Delta[j, :] = \boldsymbol{\mu}_{j+1}^\top$ for $j = 1, \dots, n-1$ and $\Delta[n, :] = \mathbf{0}$. Let
632 $\Theta = RU$. Then the semiseparable part of RQ is exactly $\text{tril}(\Theta\Delta^\top, -1) + \text{triu}(\Delta\Theta^\top, 1)$.

633 The banded part B_R is constructed from the diagonal entries of $R^{(j)}$ and the
634 vectors $\boldsymbol{\eta}_{j+1}$, which are nonzero only in their first $\min(l, n-j)$ entries. By the
635 update formulas in Lemma 4.3, these entries depend only on local information (within
636 distance l from the diagonal), ensuring B_R is banded with bandwidth l . The symmetry
637 of B_R follows from the symmetry of RQ and the symmetry of the semiseparable
638 representation.

639 Thus, RQ admits the representation (4.12), confirming it is a symmetric BPS
640 matrix with the stated properties. \square

641 **4.3. Fast RQ Algorithm.** Theorem 4.4 is constructive and leads directly to a
642 fast algorithm for computing the RQ product without explicitly forming the dense
643 orthogonal matrix Q .

644 **Complexity analysis.** Step 1, the QR factorization, requires $O(n)$ operations
645 by Theorem 2.5. The computation of $\Theta = RU$ can be performed in $O(nr)$ time due
646 to the structure of R . The forward recursion in Step 3 performs a constant amount of
647 work per iteration, as all matrix operations involve matrices whose dimensions (e.g.,
648 $r \times r$, $r \times l$, $l \times l$) are independent of n . Therefore, Algorithm 4.3.1 runs in $O(n)$ time
649 and uses $O(n)$ storage.

650 **REMARK 4.1.** A subtle but crucial point in Algorithm 4.3.1 is that during the
651 forward recursion, we track only the evolving structure of the submatrix $R^{(j)}[j+1 :$
652 $n, j+1:n]$, even though the entire matrix $R^{(j)}[:, j+1:n]$ is being modified. More pre-
653 cisely, after applying the first j Householder transformations from the right, the first j
654 columns of $R^{(j)}$ have reached their final state and will not change in subsequent steps.

Algorithm 4.3.1 Fast RQ for Symmetric BPS Matrices

-
- 1: **Input:** A symmetric BPS matrix A given by its generators: symmetric banded B (bandwidth l), and $U, V \in \mathbb{R}^{n \times r}$ satisfying $A = B + \text{tril}(UV^\top, -1) + \text{triu}(VU^\top, 1)$.
- 2: **Output:** The RQ product in structured form: symmetric banded matrix B_R , and low-rank generators Θ and Δ .
- 3: 1. **Compute fast QR factorization:**
- 4: Run Algorithm 3.1 on A to obtain the structured factor matrix F and coefficients τ . Extract the structured representation of the R factor and all Householder vectors \mathbf{y}_k (with parameters $\bar{\mathbf{k}}_k$, \mathbf{b}_k).
- 5: Compute and store $\Theta = RU$ using the structured R .
- 6: 2. **Initialize HMUTM parameters:**
- 7: Set $\Omega_1 \leftarrow \mathbf{0}_{r \times r}$, $\Phi_1 \leftarrow \mathbf{0}_{n \times r}$, $\Psi_1 \leftarrow \mathbf{0}_{r \times n}$, $\Lambda_1 \leftarrow \mathbf{0}_{n \times n}$.
- 8: Initialize $\Delta \leftarrow \mathbf{0}_{n \times r}$.
- 9: 3. **Forward recursion (apply Q from the right):**
- 10: **for** $k = 1$ to $n - 1$ **do**
- 11: a. Retrieve the Householder parameters $\bar{\mathbf{k}}_{k+1}$ and \mathbf{b}_{k+1} for step k .
- 12: b. Using the update formulas from Lemma 4.3 (as derived in its proof), compute the new HMUTM parameters $(\Omega_{k+1}, \Phi_{k+1}, \Psi_{k+1}, \Lambda_{k+1})$ from $(\Omega_k, \Phi_k, \Psi_k, \Lambda_k, \bar{\mathbf{k}}_{k+1}, \mathbf{b}_{k+1})$.
- 13: c. Compute the HMUTV parameter μ_{k+1} and η_{k+1} from the relevant update formulas in the proof of Lemma 4.3.
- 14: d. Set $\Delta[k, :] \leftarrow \mu_{k+1}^\top$ and $B_R[k + 1 : \min(k + 1, n), k] \leftarrow \eta_{k+1}[k + 1 : \min(k + 1, n)]$.
- 15: e. Compute the diagonal element from the formula in the proof of Lemma 4.3 and store it in $B_R[k, k]$.
- 16: **end for**
- 17: 4. **Final State**
- 18: Set $B_R[n, n] \leftarrow R^{(n-1)}[n, n]$
- 19: 5. **Return** B_R, Θ, Δ .
-

655 However, the first j rows of $R^{(j)}$ (in columns $j + 1 : n$) are still subject to modification
 656 by later transformations. Nevertheless, because the final product RQ is known to be
 657 symmetric, these pending row entries are not independent: they must eventually equal
 658 the corresponding entries in the already-fixed columns. Thus, while they appear to be
 659 "in flux" during intermediate steps, their ultimate values are implicitly determined by
 660 symmetry. This allows the algorithm to safely disregard the explicit updating of these
 661 rows and focus solely on the lower-right submatrix, which is the only part whose future
 662 evolution is not predetermined. Lemma 4.3 guarantees that this submatrix maintains
 663 an HMUTM structure, enabling its efficient propagation. This exploitation of sym-
 664 metry is key to achieving $O(n)$ complexity, as it confines the work at each step to a
 665 submatrix with a structured representation of constant size independent of n .

666 **5. Numerical results.** To validate the theoretical complexity and demonstrate
 667 the practical efficiency of our proposed algorithms, we implemented the fast QR

factorization and the complete linear solver in Julia. The implementation is publicly available in the SemiseparableMatrices.jl package¹, providing an open-source resource for the scientific computing community. All numerical tests use banded-plus-semiseparable matrices with fixed structural parameters $l = 4$, $m = 5$, $r = 2$, $p = 3$ to isolate the scaling behavior with respect to the matrix size n . Computations were carried out on a MacBook Air equipped with an Apple M2 chip (8-core CPU, 8 GB RAM), without GPU acceleration or access to external computing resources.

Make this a citation

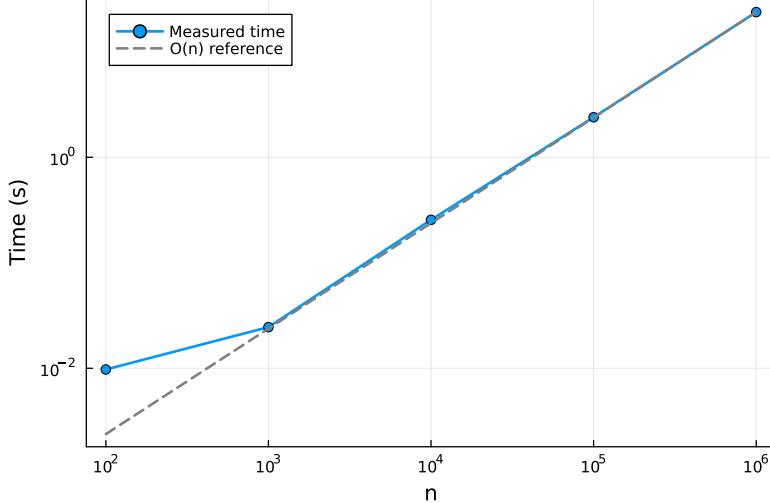


FIG. 1. Log-log plot of the total solver time (QR factorization + application of Q^T + backward substitution) versus matrix size n . The dashed reference line has slope 1, indicating ideal linear scaling.

Add different parameters for ranks and bands

Add more ticks on the y-axis

5.1. Linear Complexity Verification. Figure 1 demonstrates the linear time complexity of our complete solver for banded-plus-semiseparable linear systems. The total execution time, encompassing all three phases (QR factorization, application of Q^T , and backward substitution), scales as $O(n)$ across five orders of magnitude, from $n = 100$ to $n = 10^6$. The close alignment with the reference line of slope 1 confirms the complexity analysis in Section 3.

5.2. Comparison with HODLR QR. We compare our fast QR factorization against the state-of-the-art HODLR (Hierarchically Off-Diagonal Low-Rank) QR implementation from the hm-toolbox [13]. The hm-toolbox provides efficient MATLAB routines for various structured matrices, including HODLR and HSS matrices, and represents one of the most mature implementations for hierarchical matrix computations.

Figure 2 shows the execution times for QR factorization of BPS matrices using both approaches. Our algorithm demonstrates superior scaling for larger matrix sizes. This performance advantage stems from several factors:

¹<https://github.com/JuliaLinearAlgebra/SemiseparableMatrices.jl>

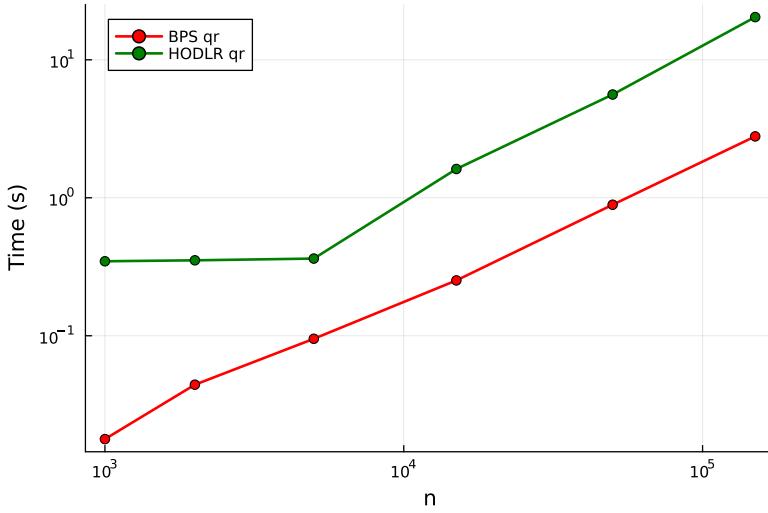


FIG. 2. Comparison of QR factorization times between our fast BPS QR algorithm and the HODLR QR implementation from [13]. Both algorithms operate on banded-plus-semiseparable matrices with parameters $l = 4$, $m = 5$, $r = 2$, $p = 3$.

Use different markers so clearer when black-and-white.

Go to $n = 10^6$

- **Specialized structure exploitation:** Our algorithm is specifically designed for the banded-plus-semiseparable structure, avoiding the overhead of general hierarchical representations.
- **Reduced Computational Overhead:** By working directly with the semisparable generators rather than building a hierarchical representation, we avoid the logarithmic factors inherent in tree-based approaches.

The performance gap widens with increasing n , confirming that our method is particularly well-suited for large-scale problems. For $n = 150,000$, our implementation achieves approximately $7\times$ speedup over the HODLR approach, demonstrating the practical benefits of our specialized algorithm.

6. Conclusions. In this paper, we have established a fundamental theoretical result for BPS matrices and developed efficient algorithms based on this foundation. Our main contribution is the proof that the QR factorization of a BPS matrix preserves the banded-plus-semiseparable structure, with precisely characterized ranks and bandwidths in the resulting factor matrix. This theoretical insight enabled the design of a complete $O(n)$ direct solver for BPS linear systems, comprising:

- A structure-preserving QR factorization algorithm (Algorithm 3.1)
- An efficient $O(n)$ application of Q^T (Algorithm 3.2.1)
- A fast backward substitution routine (Algorithm 3.2.2)

The numerical experiments confirm the linear scaling of our approach and demonstrate significant performance advantages over existing HODLR-based methods. Our implementation in the SemiseparableMatrices.jl package provides the scientific computing community with efficient, open-source tools for working with this important class of structured matrices.

Future Work. A compelling extension involves applying our methodology to specific blocked banded matrices arising in *hp*-FEM [12]. These have optimal complexity so-called reverse Cholesky factorizations (Cholesky from the bottom right instead of the top left) for positive definite problems. One of our motivations for the present work is developing an optimal complexity QL factorization for these special block banded matrices. The key challenge is generalizing our framework to *block* banded-plus-semiseparable matrices while maintaining $O(N)$ complexity. The primary difficulty lies in applying Householder transformations from one block to subsequent blocks in $O(n)$ time (where n is block size and N the total size), rather than $O(n^3)$. While our current framework doesn't directly apply, the core insight of structure preservation provides a promising foundation for this challenging extension.

Fix capitalization

726

REFERENCES

- [1] S. CHANDRASEKARAN, P. DEWILDE, M. GU, W. LYONS, AND T. PALS, *A fast solver for hss representations via sparse matrices*, SIAM Journal on Matrix Analysis and Applications, 29 (2007), pp. 67–81.

[2] S. CHANDRASEKARAN, P. DEWILDE, M. GU, T. PALS, AND A.-J. VAN DER VEEN, *Fast stable solver for sequentially semi-separable linear systems of equations*, in International Conference on High-Performance Computing, Springer, 2002, pp. 545–554.

[3] S. CHANDRASEKARAN AND M. GU, *Fast and stable algorithms for banded plus semiseparable systems of linear equations*, SIAM Journal on Matrix Analysis and Applications, 25 (2003), pp. 373–384.

[4] S. CHANDRASEKARAN, M. GU, AND T. PALS, *A fast ulv decomposition solver for hierarchically semiseparable representations*, SIAM Journal on Matrix Analysis and Applications, 28 (2006), pp. 603–622.

[5] S. DELVAUX AND M. VAN BAREL, *Rank structures preserved by the qr-algorithm: the singular case*, Journal of Computational and Applied Mathematics, 189 (2006), pp. 157–178.

[6] S. DELVAUX AND M. VAN BAREL, *Structures preserved by the qr-algorithm*, Journal of Computational and Applied Mathematics, 187 (2006), pp. 29–40.

[7] S. DELVAUX AND M. VAN BAREL, *A givens-weight representation for rank structured matrices*, SIAM Journal on Matrix Analysis and Applications, 29 (2008), pp. 1147–1170.

[8] S. DELVAUX AND M. VAN BAREL, *A qr-based solver for rank structured matrices*, SIAM Journal on Matrix Analysis and Applications, 30 (2008), pp. 464–490.

[9] Y. EIDELMAN AND I. GOHBERG, *Inversion formulas and linear complexity algorithm for diagonal plus semiseparable matrices*, Computers & Mathematics with Applications, 33 (1997), pp. 69–79.

[10] Y. EIDELMAN, I. GOHBERG, AND V. OLSHEVSKY, *The qr iteration method for hermitian quasiseparable matrices of an arbitrary order*, Linear Algebra and its Applications, 404 (2005), pp. 305–324.

[11] D. FASINO, *Rational krylov matrices and qr steps on hermitian diagonal-plus-semiseparable matrices*, Numerical linear algebra with applications, 12 (2005), pp. 743–754.

[12] K. KNOOK, S. OLVER, AND I. PAPADOPOULOS, *Quasi-optimal complexity hp-fem for poisson on a rectangle*, arXiv preprint arXiv:2402.11299, (2024).

[13] S. MASSEI, L. ROBOL, AND D. KRESSNER, *hm-toolbox: Matlab software for hodlr and hss matrices*, SIAM Journal on Scientific Computing, 42 (2020), pp. C43–C68.

[14] N. MASTRONARDI, S. CHANDRASEKARAN, AND S. VAN HUFFEL, *Fast and stable two-way algorithm for diagonal plus semi-separable systems of linear equations*, Numerical linear algebra with applications, 8 (2001), pp. 7–12.

[15] S. OLVER AND A. TOWNSEND, *A fast and well-conditioned spectral method*, siam REVIEW, 55 (2013), pp. 462–489.

[16] E. VAN CAMP, N. MASTRONARDI, AND M. VAN BAREL, *Two fast algorithms for solving diagonal-plus-semiseparable linear systems*, Journal of Computational and Applied Mathematics, 164 (2004), pp. 731–747.

[17] R. VANDERBIL, M. VAN BAREL, AND N. MASTRONARDI, *An implicit QR algorithm for symmetric semiseparable matrices*, Numerical Linear Algebra with Applications, 12 (2005), pp. 625–658.

- 770 [18] R. VANDERBIL, M. VAN BAREL, AND N. MASTRONARDI, *A note on the representation and defi-*
771 *nition of semiseparable matrices*, Numerical Linear Algebra with Applications, 12 (2005),
772 pp. 839–858.
- 773 [19] R. VANDERBIL, M. VAN BAREL, AND N. MASTRONARDI, *Rational qr-iteration without inversion*,
774 Numerische Mathematik, 110 (2008), pp. 561–575.
- 775 [20] J. XIA, S. CHANDRASEKARAN, M. GU, AND X. S. LI, *Fast algorithms for hierarchically semisep-*
776 *arable matrices*, Numerical Linear Algebra with Applications, 17 (2010), pp. 953–976.