Locally-Aware Constrained Games on Networks

Guanze Peng, Juntao Chen, Shutina Liu, Tao Li and Quanyan Zhu

***Abstract*—This paper proposes a game framework to study constrained games on networks, where the players do not have complete information about all the constraints. In our framework, the information of the players regarding the constraint is quantified by their *awareness levels*. We first show that communicating the knowledge about constraints with their neighbors or increasing the connectivity of the network is equivalent to enhancing the awareness levels. And we further show that higher awareness levels result in more generalized Nash equilibria (GNE). We characterize the GNE, and we show that via the characterization, one can convert such a game into a two-layer problem, where each layer comprises an unconstrained game.**

***Index Terms*—Incomplete-information game, game on networks, constrained games, dual game.**

I. Introduction

Research on networks has been received significant attention over the last few decades, for its applications in many essential fields, such as power systems, wireless communication, and IoT ecosystems [1], [2], [3]. Game theory as a proper tool uses a bottom-up approach to address the issues such as network design and analysis of network performance, as oftentimes it is hard to coordinate across the network, especially when the network is large scale. Nevertheless, applying the game theory to networks is challenging: 1) the games played on networks are usually subject to constraints. The constraints can arise from physics, such as Kirchhoffs current law and Kirchhoffs voltage law in a circuit network; 2) the players may not be fully aware of these constraints, or the players are only aware of part of these constraints.

In this work, we propose a locally-aware constrained game on networks to captures all the key features mentioned above. We assume that the players play in a noncooperative way, and each player has the access to his neighbor's constraints. Our work can be considered as a generalization of the case where all the players do not have the knowledge of his neighbor's sets by eliminating all the connections in the network, and the case where all the players have the knowledge of his neighbor's sets by establishing a fully-connected network. We also propose the concept *awareness* to describe how knowledge the players are about the constraints. Our contributions can be summarized as the following: we fully characterize the sufficient and necessary conditions of the equilibrium solution under certain assumptions; we identify the relationship between our work and the classical constrained games, and we show that our work is a generalization of the classical constrained games; we also transform the proposed game to an equivalent dual game, which admits the form of two unconstrained games; we show that the Cournot game can be transformed to a Bertrand game using the transformation.

Our work is closely related to [4]-[5], where all the players have the same set of constraints and the set of constraints is common knowledge, which can be considered as a special case of our work. Moreover, as the equilibrium concept considered here falls in the category of generalized Nash equilibrium (GNE), our work is related to [6], [7] as well. The computation methods of GNE are studied in [8], [9], [10]. There are some recent studies on the games with asymmetric constraint information that are not on networks, which can be found in [11], [10]. The unconstrained games on networks are studied in [12], [13].

The rest of the paper is organized as the following: Section II formally defines the locally-aware constrained game. Section III reviews the results in the globally-aware constrained game. In Section IV, we focus on the locally-aware constrained game and its relation with globally-aware constrained game. Moreover, we show that under certain assumptions, a locally-aware constrained game can be decomposed into two unconstrained games. In Section V, we apply the decomposition result to enunciated the relation between Bertrand game and Cournot game. And we use two linear-quadratic game examples to validate the theoretic results. Section VI concludes the paper and points out the possible direction of future work.

II. Problem Statement

A. Game Setup

Consider a connected network, which contains N nodes (players) described by an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with $\mathcal{N} = \{1, 2, ..., N\}$ as the index set of the nodes. And the set of edges is denoted by $\mathcal{E}$ which contains the links between connected nodes. For each node $i \in \mathcal{N}$, denote the set of its neighbors by $\mathcal{N}_i$ (including himself) and the number of its neighbors by $|\mathcal{N}_i|$. Suppose a noncooperative constrained game is played among the N players. For each player $i \in \mathcal{N}$, let $\mathcal{U}_i \subseteq \mathbb{R}^{K_i}$ be the continuous compact action set, where K_i is a positive integer, and let $f_i : \prod_{i=1}^{N} \mathcal{U}_i \to \mathbb{R}$ be the cost function. Denote Player i's action by u_i. Besides, let $u = (u_j)_{j\in\mathcal{N}} \in \mathcal{U} := \prod_{j\in\mathcal{N}} \mathcal{U}_j$, and $u_{-i} = (u_j)_{j\in\mathcal{N}\setminus\{i\}} \in \mathcal{U}_{-i} := \prod_{j\in\mathcal{N}\setminus\{i\}} \mathcal{U}_j$.

The constrained game can be defined as a tuple,

$$\Xi := (\mathcal{N}, \{\mathcal{N}_i, f_i, \mathcal{U}_i, \mathcal{F}_i\}_{i\in\mathcal{N}}, \{\hat{\Omega}_m\}_{m\in\mathcal{M}}, \mathcal{A}, \mathfrak{s}), \qquad (1)$$

where

1) $\hat{\Omega}_m \subseteq \mathbb{R}^{\prod_{i=1}^{N} K_i}$ is a feasible set (constraint) which depends on the tuple of all the players' actions, where $m \in \mathcal{M} = \{1, 2, ..., M\}$;

G. Peng, S. Liu, T. Li and Q. Zhu are with Department of Electrical and Computer Engineering, New York University, NY 11201 USA {guanze.peng, sl6803, tl2636, quanyan.zhu}@nyu.edu

J. Chen is with Department of Computer and Information Science, Fordham University, New York, NY 10023 USA. Email: jchen504@fordham.edu

2) $\mathcal{A}$ is a finite set of levels of *awareness*;
3) $\mathcal{F}_i$ is subset of Ω_i, i.e., $\mathcal{F}_i \subseteq \Omega := \{\hat{\Omega}_m\}_{m\in\mathcal{M}}$, $i \in \mathcal{N}$;
4) $\mathfrak{s} : \mathcal{A} \to \sigma_{\max}(\Omega)$, is a mapping associating each level of the awareness with the information regarding the feasible sets, where $\sigma_{\max}(\cdot)$ is the largest σ-algebra operator.

We call each element of $\mathcal{A}$ *the awareness level*, and Ω_i the *awareness set* of Player i, which quantifies the information of Player i regarding the constraints. At the beginning of the game Ξ, the awareness level of each player is chosen by nature from $\mathcal{A}$, and his awareness level determines his information about the feasible sets $\hat{\Omega}_m \subseteq \mathbb{R}^{\prod_{i=1}^{N} K_i}$ via the mapping $\mathfrak{s}$. Specifically, let the awareness set $\mathcal{A}$ be

$$\mathcal{A} = \sigma_{\max}(\mathcal{M}).$$

Denote a subset of $\mathcal{M}$ by $\tilde{\mathcal{M}} \in 2^{\mathcal{M}}$. Then,

$$\mathfrak{s}(\tilde{\mathcal{M}}) = \{\hat{\Omega}_m\}_{m\in\tilde{\mathcal{M}}}.$$

In other words, players of awareness level $\tilde{\mathcal{M}}$ are aware of $\{\hat{\Omega}_m\}_{m\in\tilde{\mathcal{M}}}$.

Definition 1. *In the constrained game Ξ, if two awareness levels $\tilde{\mathcal{M}}, \hat{\mathcal{M}} \in \mathcal{A}$, satisfy*

$$\mathfrak{s}(\tilde{\mathcal{M}}) \subseteq \mathfrak{s}(\hat{\mathcal{M}}), \quad \text{or equivalently,} \quad \tilde{\mathcal{M}} \subseteq \hat{\mathcal{M}},$$

then we say the player with awareness level $\hat{\mathcal{M}}$ is more aware than the player with awareness level $\tilde{\mathcal{M}}$.

By Definition 1, the awareness level set $\mathcal{A}$ can be considered as a Boolean lattice, which is a partially ordered set. And this fact enables us to compare the awareness levels.

In a locally-aware constrained game, the players can observe the constraints of his neighbors, i.e., their constraints are locally shared. Denote the awareness level of Player i be $\tilde{\mathcal{M}}_i$, $i \in \mathcal{N}$. Then, according to the description of the game, the awareness set of Player i is

$$\mathcal{F}_i = \mathfrak{s}(\bigcup_{j\in\mathcal{N}_i} \tilde{\mathcal{M}}_j) = \left\{\hat{\Omega}_m\right\}_{m\in\tilde{\mathcal{M}}_j, j\in\mathcal{N}_i}. \tag{2}$$

It is not hard to see that observing the neighbors' constraints is equivalent to increasing the awareness level of the players, as for each player $i \in \mathcal{N}$ and every possible $\mathcal{N}_i$, we always have

$$\mathfrak{s}(\tilde{\mathcal{M}}_i) \subseteq \mathfrak{s}(\bigcup_{j\in\mathcal{N}_i} \tilde{\mathcal{M}}_j).$$

Note that the awareness framework considers a different aspect of the classical incomplete-information games. Generally speaking, the knowledge in games can be classified into three classes: the knowledge that player knows he knows, the knowledge that player knows he does not know, and the knowledge that player does not know he does not know. A large amount of work has been conducted on the information structure induced by the first two classes of knowledge, whilst the awareness framework takes the information structure induced by the third class of knowledge into consideration. In the awareness framework, the player is only aware of the constraints that he knows, and not aware of the existence of other constraints.

B. Equilibrium Concept and Existence

From the information set given in (2), each player $i \in \mathcal{N}$ faces the parameterized optimization problem given by the following:

$$\begin{aligned} \textbf{OP}_i : \quad & \min_{u_i\in\mathcal{U}_i} \quad f_i(u_i, u_{-i}) \\ & \quad \text{s.t.} \quad (u_i, u_{-i}) \in \Omega_j, \quad j \in \mathcal{N}_i, \end{aligned} \tag{3}$$

where $\Omega_j := \bigcap_{m\in\tilde{\mathcal{M}}_j} \hat{\Omega}_m$.

In this work, we suppose that the feasible set can be characterized by a constrain function $g_i : \mathcal{U} \to \mathbb{R}$, $i \in \mathcal{N}$, and of the following form

$$\Omega_i = \{(u_i, u_{-i}) : g_j(u_i, u_{-i}) \leqslant 0, j \in \mathcal{N}_i\}.$$

Therefore, (3) can be rewritten as

$$\begin{aligned} \textbf{OP}_i : \quad & \min_{u_i\in\mathcal{U}_i} \quad f_i(u_i, u_{-i}) \\ & \quad \text{s.t.} \quad g_j(u_i, u_{-i}) \leqslant 0, \quad j \in \mathcal{N}_i. \end{aligned} \tag{4}$$

Given u_{-i}, the best response of Player i is the optimal solution to (4) parameterized by u_{-i}. We denote the best response of Player i by $BR_i(u_{-i})$, which is a mapping $BR_i : \mathcal{U}_{-i} \to \mathcal{U}_i$.

A suitable equilibrium concept in constrained games is generalized Nash equilibrium (GNE) given by the following

Definition 2. *A tuple $u^* = (u_i^*, u_{-i}^*)$ is said to be a GNE solution for the game Ξ, if the following conditions are satisfied*

$$f_i(u_i^*, u_{-i}^*) \leqslant f_i(u_i, u_{-i}^*), \quad \forall\ u_i \in \Omega_i(u_{-i}^*), \quad i \in \mathcal{N},$$

where

$$\Omega_i(u_{-i}^*) := \left\{u \in \mathcal{U}_i : g_j(u, u_{-i}^*) \leqslant 0,\ j \in \mathcal{N}_i\right\}.$$

Define the global feasible set by the following

$$\tilde{\Omega} := \{u : g_i(u) \leqslant 0,\ \forall\ i \in \mathcal{N}\},$$

which consists of all the feasible solutions that satisfy all the constraint. We make the following conventional assumption which is standard in the constrained games:

Assumption 1. *(Regularity Condition) The following conditions hold throughout this work:*

a) *f_i and g_i, $i \in \mathcal{N}$ are continuously differentiable;*
c) *$\mathcal{U}_i$, $i \in \mathcal{N}$ is convex and compact;*
d) *$\tilde{\Omega}$ is nonempty;*
e) *For all $i \in \mathcal{N}$, given any $u_{-i} \in \mathcal{U}_{-i}$, **OP**$_i$ is a convex optimization problem.*

Assumption 2. *(Collective Awareness Condition) The following assumption regarding awareness is assumed to hold:*

$$\bigcup_{i\in\mathcal{N}} \tilde{\mathcal{M}}_i = \mathcal{M}.$$

Theorem 1. *Let $BR_i(u_{-i})$ be the optimal solution to **OP**$_i$, $i \in \mathcal{N}$. The following fixed point equation admits least one fixed point,*

$$BR(u) := \begin{bmatrix} BR_1(u_{-1}) \\ BR_2(u_{-2}) \\ \vdots \\ BR_N(u_{-N}) \end{bmatrix} = u, \tag{5}$$

where $BR : \mathcal{U} \to \mathcal{U}$. Moreover, there exists at least one GNE in Ξ.

Proof. The proof follows from [7], which uses the fixed point theorem developed in [14]. □

Remark 1. *Solving for GNEs in Ξ is equivalent to solving the fixed point equation (5). If Assumption 1 does not hold, then there may not exist GNE in Ξ.*

III. Globally-Aware Constrained Game

In a globally-aware constrained game, the players are aware of all the constraints in the game.

Definition 3. *A player with awareness level $\hat{\mathcal{M}}$ is said to be globally aware if*

$$\mathfrak{s}(\tilde{\mathcal{M}}) \subseteq \mathfrak{s}(\hat{\mathcal{M}}), \quad \forall\ \tilde{\mathcal{M}} \in 2^{\mathcal{M}}.$$

By definition, if the awareness of a player is $\mathcal{M}$, then he is globally aware. In a globally-aware constrained game, all the players are globally aware, i.e., each player's awareness level is $\mathcal{M}$. The tuple that represents globally-aware constrained game is given by

$$\tilde{\Xi} := (\mathcal{N}, \{\mathcal{N}_i, f_i, \mathcal{U}_i, \tilde{\mathcal{F}}_i\}_{i\in\mathcal{N}}, \{\hat{\Omega}_m\}_{m\in\mathcal{M}}, \mathcal{A}, \mathfrak{s}), \qquad (6)$$

The awareness set in $\tilde{\Xi}$ for each player $i \in \mathcal{N}$ is

$$\tilde{\mathcal{F}}_i = \{\hat{\Omega}_m\}_{m\in\mathcal{M}}.$$

Corollary 1. *There exists at least one GNE in $\tilde{\Xi}$.*

Remark 2. *We observe that a globally-aware constrained game played on arbitrary network can be considered as a locally-aware constrained game played on a fully-connected network, where a link exists between two arbitrary players.*

In the classical constrained game, for example in [4], the awareness set of Player i is given by the following:

$$\bar{\mathcal{F}}_i = \left\{ \bigcap_{m\in\mathcal{M}} \hat{\Omega}_m \right\}, \quad i \in \mathcal{N},$$

under which each player knows all the constraints. But, he is not aware of the fact that he knows all the constraints. Then, by definition, it is easy to show that the GNEs in the game defined via $\bar{\Xi} := (\mathcal{N}, \{\mathcal{N}_i, f_i, \mathcal{U}_i, \}_{i\in\mathcal{N}}, \{\bigcap_{m\in\mathcal{M}} \hat{\Omega}_m\})$, and $\tilde{\Xi}$ are the same. This says that the classical constrained game is equivalent to a globally-aware constrained game.

Remark 3. *Based on Remark 2, a classical constrained game is equivalent to a locally-aware constrained game played on a fully-connected network. Thus, the locally-aware constrained games can be seen as a generalization of the classical constrained game.*

IV. Locally-Aware Constrained Game

In this section, we study the properties of the locally-aware constrained games.

A. Connectivity of Networks, Awareness and GNEs

Consider a another undirected connected network, $\underline{\mathcal{G}} = \{\mathcal{N}, \underline{\mathcal{E}}\}$. If $\mathcal{E} \subseteq \underline{\mathcal{E}}$, we say that $\underline{\mathcal{G}}$ has stronger connectivity than $\mathcal{G}$, as $\underline{\mathcal{G}}$ contains more links.

Theorem 2. *As the connectivity of the network on which a locally-aware constrained game is played becomes stronger, the set of GNEs of that game becomes larger.*

Proof. Let a a locally-aware constrained game on network $\mathcal{G}$ and $\underline{\mathcal{G}}$, and $\underline{\mathcal{G}}$ has stronger connectivity than $\mathcal{G}$.Denote the sets of neighbors on $\underline{\mathcal{G}}$ by $\{\underline{\mathcal{N}}_i\}_{i\in\mathcal{N}}$. As there are more links connecting the players, $\mathcal{N}_i \subseteq \underline{\mathcal{N}}_i, \quad i \in \mathcal{N}$. Similar to (2), it is easy to show that $\mathcal{F}_i \subseteq \underline{\mathcal{F}}_i, \quad i \in \mathcal{N}$. Thus, on the network $\underline{\mathcal{G}}$, for each player $i \in \mathcal{N}$,

$$\begin{aligned} \underline{\mathbf{OP}}_i : \quad & \min_{u_i\in\mathcal{U}_i} \quad f_i(u_i, u_{-i}) \\ & \text{s.t.} \quad (u_i, u_{-i}) \in \Omega_j, \quad j \in \underline{\mathcal{N}}_i. \end{aligned} \qquad (7)$$

It is obvious that $\underline{\Omega}_i(u_{-i}) := \bigcap_{j\in\underline{\mathcal{N}}_i} \Omega_j \subseteq \Omega_i(u_{-i}) = \bigcap_{j\in\mathcal{N}_i} \Omega_j$, $i \in \mathcal{N}$. We proceed the rest of the proof by contradiction. Suppose that there exists u^* which is a GNE solution to Ξ, but not a GNE solution to $\underline{\Xi}$. Without loss of generality, suppose that there exists some $\tilde{u}_i^* \in \underline{\Omega}_i(u^*_{-i})$. such that,

$$f_i(u_i^*, u^*_{-i}) > f_i(\tilde{u}_i^*, u^*_{-i}).$$

However, as $\underline{\Omega}_i(u^*_{-i}) \subseteq \Omega_i(u^*_{-i})$,

$$f_i(u_i^*, u^*_{-i}) \leqslant f_i(u_i, u^*_{-i}), \quad \forall\ u_i \in \underline{\Omega}_i(u^*_{-i}).$$

By letting $u_i = \tilde{u}_i^*$, we observe that two inequalities above contradict each other. □

□

Remark 4. *The increment of connectivity of the network enhances each player's awareness, as*

$$\mathfrak{s}(\bigcup_{j\in\mathcal{N}_i} \tilde{\mathcal{M}}_j) \subseteq \mathfrak{s}(\bigcup_{j\in\underline{\mathcal{N}}_i} \tilde{\mathcal{M}}_j), \quad i \in \mathcal{N}.$$

Theorem 2 says that, as players have more awareness about the game, there are more equilibrium points, and it is easier for them to reach the consensus, i.e., an equilibrium point.

B. GNEs in Ξ and GNEs in $\tilde{\Xi}$

The following corollary presents the relation between GNEs in Ξ and GNEs in $\tilde{\Xi}$.

Corollary 2. *If $u^* = (u_i^*, u^*_{-i})$ is a GNE solution to Ξ, then it is also a GNE solution to $\tilde{\Xi}$.*

Remark 5. *As mentioned above, a globally-aware constrained game can be considered as a locally-aware constrained game played on a fully-connected network. Hence, the fully-connected network structure enhances the players to the highest possible awareness level (global awareness) in the level set $\mathcal{A}$. From this perspective, according to Remark 4, Corollary 2 is an immediate result of Theorem 2.*

Corollary 3. *If there exists a unique GNE solution to $\tilde{\Xi}$, then there also exists a unique GNE solution to Ξ. Moreover, the GNE solution to $\tilde{\Xi}$ coincides with the GNE solution to the corresponding Ξ.*

C. GNEs in Ξ and VEs in $\tilde{\Xi}$

In the constrained games, the GNEs in $\tilde{\Xi}$ are not of practical interest in the general case and have less economic justification as the concept of GNE may be far too weak to serve as a solution concept. Therefore, the variational equilibrium (VE) is proposed as a refinement of GNE [15].

Definition 4. *A tuple $u^* = (u_i^*, u_{-i}^*)$ is said to be a VE of $\tilde{\Xi}$ if u^* satisfies,*

$$F^{\mathrm{T}}(u^*)(u - u^*) \geqslant 0, \quad \forall\ u \in \tilde{\Omega},$$

where $F(u) = [\nabla_{u_1} f_1^{\mathrm{T}}(u_1, u_{-1}), ..., \nabla_{u_N} f_N^{\mathrm{T}}(u_N, u_{-N})]^{\mathrm{T}}$.

Proposition 1. *[15] Under Assumption 1, a VE of $\tilde{\Xi}$ is also a GNE of $\tilde{\Xi}$.*

A GNE in Ξ is not necessary a VE in $\tilde{\Xi}$. Consider the following counterexample in a two-player game: suppose $f_1(u_1, u_2) = \frac{1}{2}u_1^2 - u_1$, $f_2(u_1, u_2) = \frac{1}{2}u_2^2 - 2u_2$, and $\Omega_1 = \{(u_1, u_2) : u_1 + u_2 \leqslant 1\}$. The GNE solution to Ξ is $(-1, 2)$, and VE solution to $\tilde{\Xi}$ is $(0, 1)$.

D. Characterization of GNEs in Ξ

we characterize GNEs in Ξ and provide the necessary and sufficient conditions.

Lemma 1. *Let u^* be a feasible solution such that*

$$\tilde{f}(u^*; u^*) \leqslant \tilde{f}(v; u^*), \quad \forall\ v \in \Omega(u^*), \tag{8}$$

where

$$\tilde{f}(v; u) = \sum_{i \in \mathcal{N}} f_i(v_i, u_{-i}), \quad \text{and} \quad \Omega(u) = \prod_{i \in \mathcal{N}} \Omega_i(u_{-i}).$$

Then u^* is a GNE solution to Ξ.

Proof. We proceed the proof by contradiction. Suppose that u^* satisfies (8) and u^* is not a GNE solution to Ξ. Without loss of generality, suppose that there exists some $\bar{u}_i \in \Omega_i(u_{-i}^*)$ such that

$$\tilde{f}_i(u_i^*, u_{-i}^*) > \tilde{f}_i(\bar{u}_i, u_{-i}^*).$$

By adding both sides by $\sum_{j \in \mathcal{N}/\{i\}} f_j(u_j^*, u_{-j}^*)$, we have

$$\begin{aligned}\tilde{f}(u^*; u^*) &= f_i(u_i^*, u_{-i}^*) + \sum_{j \in \mathcal{N}/\{i\}} f_j(u_j^*, u_{-j}^*) \\ &> f_i(\bar{u}_i, u_{-i}^*) + \sum_{j \in \mathcal{N}/\{i\}} f_j(u_j^*, u_{-j}^*) \\ &= \tilde{f}(\bar{u}; u^*),\end{aligned}$$

where $\bar{u} = (\bar{u}_i, u_{-i}^*)$. This contradicts our assumption that u^* satisfies (8). □

□

Remark 6. *Note that the conditions in Lemma 1 also suffice to show that any u^* satisfying (8) is a GNE in $\tilde{\Xi}$ as well as shown in [4].*

Define

$$\mathcal{L}_i\left(u_i; u_{-i}; \mu_i\right) := f_i(u_i, u_{-i}) + \sum_{j \in \mathcal{N}_i} \lambda_j g_j(u_i, u_{-i}), \tag{9}$$

and

$$\mathcal{L}(v; u; \mu) := \sum_{i \in \mathcal{N}} \mathcal{L}_i(v_i; u_{-i}; \mu), \tag{10}$$

where $\mu_i := (\lambda_j)_{j \in \mathcal{N}_i}$ and $\mu = (\mu_1, \mu_2, ..., \mu_N)$.

Remark 7. *Note that the Lagrangian function (9) captures all the private information of each player i. The Lagrangian multiplier, μ_i, $i \in \mathcal{N}$, stands for the awareness level to some extent. We call μ_i the awareness variable of Player i, and the dimension of μ_i implies the awareness level of Player i.*

Theorem 3. ***(Necessary and Sufficient Condition for GNEs in** Ξ**)** *The tuple $u^* = (u_i^*, u_{-i}^*)$ is a GNE in Ξ if and only if*

$$u^* = \arg\min_{v \in \mathcal{U}} \left\{ \max_{\mu \geqslant 0} \mathcal{L}(v; u, \mu) \right\} \Big|_{v=u}. \tag{11}$$

Proof. "⇒": If u^* is a GNE solution, then u^* is feasible, i.e., $u_i^* \in \Omega_i(u_{-i}^*)$, $\forall\ i \in \mathcal{N}$. By the definition of GNE, it is equivalent to solving the following N intertwined optimizations: with fixed $u = (u_i, u_{-i})$, for each $i \in \mathcal{N}$,

$$\gamma_i^*(u) = \arg\min_{v_i \in \mathcal{U}_i} \max_{\mu_i \geqslant 0}\ f_i(v_i, u_{-i}) + \mu_i g_i(v_i, u_{-i}). \tag{12}$$

As we assume that $\mathbf{OP}_i$ is a convex optimization problem, there always exists solution to (12). To obtain GNE, we need to solve the following N fixed point equations as the following

$$\gamma_i^*(u) = u_i, \quad i \in \mathcal{N}.$$

Thus, by collecting N fixed point equations, we can write down the necessary condition equivalently and more compactly as

$$u^* = \arg\min_{v \in \mathcal{U}} \left\{ \max_{\mu \geqslant 0} \tilde{\mathcal{L}}(v; u, \mu) \right\} \Big|_{v=u}.$$

"⇐": By the definition of $\mathcal{L}$ in (10), we can rewrite the inner maximization in the sufficient condition (11) as

$$\begin{aligned}\max_{\mu \geqslant 0} \mathcal{L}(v; u; \mu) &= \max_{\mu \geqslant 0} \sum_{i \in \mathcal{N}} \mathcal{L}_i(v_i, u_{-i}, \mu_i) \\ &= \sum_{i \in \mathcal{N}} \max_{\mu_i \geqslant 0} \mathcal{L}_i(v_i, u_{-i}, \mu_i).\end{aligned} \tag{13}$$

Denote the maximizer of (13) by $\zeta_i^*(v_i, u_{-i})$, $i \in \mathcal{N}$. In order to minimize $\mathcal{L}_i(v_i, u_{-i}, \zeta_i^*(v_i, u_{-i}))$, v_i must be chosen such that $g_i(v_i, u_{-i}) \leqslant 0$. Otherwise, if $g_i(v_i, u_{-i}) > 0$ then by choosing a functional such that $\zeta_i^*(v_i, u_{-i}) > 0$, the minimax value, $\min_{v_i \in \mathcal{U}_i} \max_{\mu_i \geqslant 0} \mathcal{L}_i(v_i, u_{-i}, \mu_i)$, will go to infinity. Then, we solve for the fixed point equation

$$u^* = \arg\min_{v \in \mathcal{U}} \left\{ \sum_{i \in \mathcal{N}} \mathcal{L}_i(v_i; u_{-i}; \zeta_i^*(v_i, u_{-i}) \right\} \Big|_{v=u}.$$

Therefore, $\zeta_i^{*\mathrm{T}}(u^*)\mathbf{g}_i(u^*) = 0$. Moreover, since u^* optimizes $L(v; u; \{\zeta_i^*(v_i, u_{-i})\}_{i \in \mathcal{N}})$ as a fixed point,

$$\begin{aligned}&\mathcal{L}(u^*; u^*; \{\zeta_i^*(u_i^*, u_{-i}^*)\}_{i \in \mathcal{N}}) \\ &\qquad \leqslant \mathcal{L}(v; u^*; \{\zeta_i^*(u_i^*, u_{-i}^*)\}_{i \in \mathcal{N}}).\end{aligned}$$

That is,

$$\tilde{f}(u^*; u^*) \leqslant \tilde{f}(v; u^*), \quad \forall\ v \in \Omega(u^*).$$

By Lemma 1, the theorem follows. □

□

E. Dual Game

We aim to show that (11) are equivalent to the following:

1) **Minimization**
$$\min_{v_i \in \mathcal{U}_i} \ \mathcal{L}_i\left(v_i; u_{-i}; \mu_i\right), \quad i \in \mathcal{N}. \tag{14}$$
Let $T_i(u_{-i}, \mu_i)$, $i \in \mathcal{N}$, be the minimizing solution to (14);
2) **Fixed Point Equation**
Solve a set of fixed point equations:
$$T_i(u_{-i}, \mu_i) = u_i, \ i \in \mathcal{N}. \tag{15}$$
Denote the fixed points generated from (15) by $V(\mu) := (V_i(\mu))_{i \in \mathcal{N}}$, which are parameterized by $\mu = (\mu_1, ..., \mu_N)$;
3) **Maximization**
$$\max_{\mu_i \geqslant 0} \min_{v_i \in \mathcal{U}_i} \ \mathcal{L}_i\left(v_i; u_{-i}; \mu_i\right), \quad i \in \mathcal{N}. \tag{16}$$
Let $K_i(u_{-i})$, $i \in \mathcal{N}$, be the solution to maximizing solution to (16);
4) **Fixed Point Equation**
Solve a set of fixed point equations:
$$K_i(V_i(\mu)) = \mu_i, \ i \in \mathcal{N}. \tag{17}$$
Denote the fixed points generated from (17) by $\mu^* = (\mu_i^*)_{i \in \mathcal{N}}$. Then the equilibrium solution can be obtained by $u^* = (u_i^*)_{i \in \mathcal{N}} = (V_i(\mu^*))_{i \in \mathcal{N}}$.

Theorem 4. *Under Assumption 1,* (11) *and* (14) - (17) *generate the same fixed points.*

Proof. On closer inspection, we notice that (11) can be decomposed into a maximization, a minimization and a set of fixed point equations. Under Assumption 1, as there is no duality gap in the minimax problem [16], we can interchange the minimization and the maximization as follows:

1) **Minimization**
$$\min_{v_i} \ \mathcal{L}_i\left(v_i; u_{-i}; \mu_i\right), \quad i \in \mathcal{N}; \tag{18}$$
2) **Maximization**
$$\max_{\mu_i} \min_{v_i} \ \mathcal{L}_i\left(v_i; u_{-i}; \mu_i\right), \quad i \in \mathcal{N}; \tag{19}$$
3) **Fixed Point Equation**
Solve a set of fixed point equations:
$$T_i(K_i(u_{-i}), u_{-i}) = u_i \quad , i \in \mathcal{N}. \tag{20}$$

For any u^* obtained from (14) - (17), we only need to show that
$$T_i(K_i(u_{-i}^*), u_{-i}^*) = u_i^*, \quad i \in \mathcal{N}.$$
From (17), we know that
$$K_i(u_{-i}^*) = \mu_i^*, \quad i \in \mathcal{N}. \tag{21}$$
And from (15),
$$T_i(u_{-i}^*, \mu_i^*) = u_i^*, \tag{22}$$
By (21) and (22),
$$T_i(u_{-i}^*, \mu_i^*) = T_i(u_{-i}^*, K_i(u_{-i}^*)) = u_i^*, \quad i \in \mathcal{N}.$$

To prove the other direction. Given u^* satisfying,
$$T_i(u_{-i}^*, K_i(u_{-i}^*)) = u_i^*, \quad i \in \mathcal{N},$$
we need to show that it also satisfies Step $4'$. Let
$$\mu_i^* = K_i(u_{-i}^*), \quad i \in \mathcal{N}.$$
Then apparently,
$$T_i(u_{-i}^*, \mu_i^*) = u_i^*, \ i \in \mathcal{N} \quad \Rightarrow \quad V(\mu^*) = (V_i(\mu^*))_{i \in \mathcal{N}}.$$
And
$$K_i\left(V_{-i}(\mu^*)\right) = \mu_i^* \quad i \in \mathcal{N},$$
where $V_{-i}(\mu^*) = (V_j(\mu^*))_{j \in \mathcal{N} \setminus \{i\}}$. This completes the proof. $\square$

Remark 8. *The equivalence between* (11) *and* (14) - (17) *has an important interpretation that we can reformulate the a locally-aware constrained game as a two-layer unconstrained game. In the first layer, with fixed Lagrangian multipliers, the players play an unconstrained game using the same action as in the constrained game. In the second layer, the decision variables of the players are the Lagrangian multipliers. This result is in contrast to the results in [4] and [17], where the authors show a globally-aware constrained game can be decomposed into a lower-level modified Nash game with no coupled constraints, and a higher-level optimization problem.*

Remark 9. *The game induced by the fixed point equation* (17) *can be considered as an awareness game, where the action is the awareness variable. After solving for the equilibrium, we can separate the constraint into two classes: inactive constraint and active constraint. If Player i is more aware and has an additional Lagrange multiplier corresponding to an inactive constrain, then the equilibrium set stays the same as the inactive constraint do not affect the equilibra. If Player i is has an additional Lagrange multiplier corresponding to an active constrain $\hat{\Omega}_m$, then*
$$\{(BR_i[u_{-i}], u_{-i})\} \bigcap \hat{\Omega}_m \subseteq \left\{ \left(\widetilde{BR}_i[u_{-i}], u_{-i}\right) \right\} \bigcap \hat{\Omega}_m.$$
where $\widetilde{BR}_i$ is the best response after adding the additional Lagrange multiplier. In a nutshell, solving for GNEs in the new game formed by adding an additional Lagrange multiplier using Theorem 3 is a necessary condition to the original game. This coincides with the result of Theorem 2.

For the purpose of discussion of the relation between this work and [4], we consider the following case, where each player has the same constraint, i.e.,
$$g(u_i, u_{-i}) := g_i(u_i, u_{-i}), \quad i \in \mathcal{N}.$$
If we pose the following
$$\mu_i \ = \ \mu_j, \quad \forall \ i, j \in \mathcal{N},$$
in addition to (14) - (17), we notice that that (14) - (17) are a necessary condition to Theorem 3 in [4], which characterized the VEs in a globally-aware constrained game.

Remark 10. *In [4], the players share the same Lagrange multipliers. The interpretation of players sharing the same*

Lagrange multipliers is the following: if the players share the same Lagrange multipliers, then they have the global awareness about the constraint and this awareness is common knowledge. Using the common knowledge of the global awareness, the players know they could potentially select a better equilibrium from GNEs. And they agree to pose an additional variational inequality leading to the VE.

V. Application

A. Cournot Game and Bertrand Game

Cournot Game, where players do not have access to the other player's constrain, is special case of the locally-aware constrained games. Let u_i denote the production level of firm i, $i \in \mathcal{N} = \{1, 2\}$. Denote the price of the commodity by p. In addition, we assume a linear structure of the market demand curve:

$$p = \alpha - \beta \sum_{i \in \mathcal{N}} u_i,$$

where α and β are publicly known positive constants.

Assume that both firms have a quadratic production cost and different market demand. Then, the Cournot game between two firms can be formulated as follows: for each $i \in \mathcal{N}$,

$$\begin{aligned} \text{Firm } i: \quad & \max_{u_i \geqslant 0} \quad -k_i u_i^2 + p u_i \\ & \text{s.t.} \quad u_1 + u_2 \geqslant q_i, \end{aligned}$$

where q_i and k_i are positive constants. Now, we review the Bertrand game in the setting that products are differentiated. We denote by p_1 and p_2 the price of product 1 and product 2, respectively. The demands from firm 1 and firm 2 are $u_1 = \delta - \eta(p_1 - p_2)$, $u_2 = \delta - \eta(p_2 - p_1)$, where δ and η are positive constants. The cost function for firm i, $i \in \mathcal{N} = \{1, 2\}$, is

$$d_i(p_i, p_{-i}) = c_i \left(\delta - \eta(p_i - p_{-i})\right)^2 - \left(\delta - \eta(p_i - p_{-i})\right) p_i,$$

where c_i is the marginal cost. Then, the Bertrand game can be formulated as follows: for $i \in \mathcal{N}$,

$$\text{Firm } i: \quad \min_{p_i \geqslant 0} \; d_i(p_i, p_{-i}). \tag{23}$$

Note that BG is an unconstrained quadratic pricing game.

Following (14) - (17), we observe that the dual game of a Cournot game is of the same form as a Bertrand game.

B. Linear-Quadratic Games

1) Unique GNE Case: Consider a linear-quadratic games with linear constraint. For each player $i \in \mathcal{N} = \{1, 2\}$, the cost function is $f_i(u_i, u_{-i}) = \alpha u_i - \frac{1}{2}u_i^2 + \phi u_{-i} u_i$, and the constraint are $\hat{\Omega}_1 = \{u_1 - c_1 u_2 - d_1 \leqslant 0\}$ and $\hat{\Omega}_2 = \{u_2 - c_2 u_1 - d_2 \leqslant 0\}$. We assume that $\alpha > 0$ and $\phi > 1$. Furthermore, we assume that $c_i > \phi$, $i \in \mathcal{N}$.

As shown in Fig.1, when there is only one GNE in $\tilde{\Xi}$, then under different awareness level structure, the corresponding Ξ always admits the same GNE as $\tilde{\Xi}$.

2) Non-Unique GNE Case: When these are multiple GNE solutions to $\tilde{\Xi}$, consider the following case: $f_1(u_1, u_2) = u_1^2 + 2u_1 u_2 + 2u_2^2$, $f_2(u_1, u_2) = u_1^2 + 2u_1 u_2 + 2u_2^2$, $\hat{\Omega}_1 = \{(u_1, u_2) : u_1 + u_2 \leqslant 3\}$, and $\hat{\Omega}_2 = \{(u_1, u_2) : 2u_1 + u_2 \leqslant -5\}$.

As shown in Fig.2, as the awareness level of the players increase, the set of GNEs becomes larger.

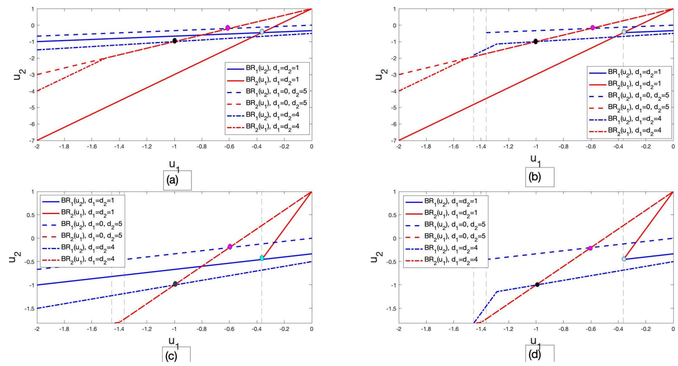

Fig. 1: An example of unique GNE in $\tilde{\Xi}$, where $\alpha = 1$, $\phi = 2$, $c_1 = 3$, and $c_2 = 4$: (a) depicts the case where Player 1 is aware of $\hat{\Omega}_1$, and Player 2 is aware of $\hat{\Omega}_2$; (b) depicts the case where Player 1 is aware of $\hat{\Omega}_1$ and $\hat{\Omega}_2$, and Player 2 is aware of $\hat{\Omega}_2$; (c) depicts the case where Player 1 is aware of $\hat{\Omega}_1$, and Player 2 is aware of $\hat{\Omega}_1$ and $\hat{\Omega}_2$; (d) depicts the case where both Player 1 and Player 2 are aware of $\hat{\Omega}_1$ and $\hat{\Omega}_2$.

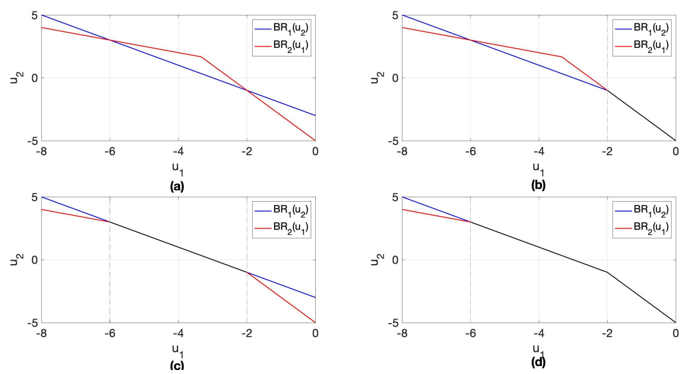

Fig. 2: An example of multiple GNEs in $\tilde{\Xi}$: The intersection of best response curves or the black line stands for the GNEs. (a) depicts the case where Player 1 is aware of $\hat{\Omega}_1$, and Player 2 is aware of $\hat{\Omega}_2$; (b) depicts the case where Player 1 is aware of $\hat{\Omega}_1$ and $\hat{\Omega}_2$, and Player 2 is aware of $\hat{\Omega}_2$; (c) depicts the case where Player 1 is aware of $\hat{\Omega}_1$, and Player 2 is aware of $\hat{\Omega}_1$ and $\hat{\Omega}_2$; (d) depicts the case where both Player 1 and Player 2 are aware of $\hat{\Omega}_1$ and $\hat{\Omega}_2$.

VI. Conclusions and Future Work

A. Conclusions

In this work, we have studied a class of constrained games on networks where the players have asymmetric information regarding the constraint. We have adopted a new concept, *awareness*, to quantify the players' knowledge about the constraint. We have shown that the increment of network connectivity is equivalent to the enhancement of the awareness level, and when the connectivity of the network increases, the game admits more GNEs. And we have shown that the constrained game considered in this work can be decomposed into two unconstrained games.

B. Future Work

1) Network Design: In a network design problem, the network designer not only needs to design the connectivity, but also needs to design the allocation of awareness.

2) Awareness as an Option: In this work, we assume that the awareness levels of the players are chosen by nature. We can also explore the case where the awareness level can be chosen by the players, i.e., the players can choose to ignore or be aware of the constraint.

References

[1] V. Kurian, J. Chen, and Q. Zhu, "Electric power dependent dynamic tariffs for water distribution systems," in *Proceedings of the 3rd International Workshop on Cyber-Physical Systems for Smart Water Networks*, 2017, pp. 35–38.

[2] M. H. Manshaei, Q. Zhu, T. Alpcan, T. Bacşar, and J.-P. Hubaux, "Game theory meets network security and privacy," *ACM Computing Surveys (CSUR)*, vol. 45, no. 3, pp. 1–39, 2013.

[3] J. Chen, C. Touati, and Q. Zhu, "Optimal secure two-layer iot network design," *IEEE Transactions on Control of Network Systems*, vol. 7, no. 1, pp. 398–409, 2019.

[4] L. Pavel, "An extension of duality to a game-theoretic framework," *Automatica*, vol. 43, no. 2, pp. 226–237, 2007.

[5] A. Charnes, "Constrained games and linear programming," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 39, no. 7, p. 639, 1953.

[6] F. Facchinei, A. Fischer, and V. Piccialli, "On generalized nash games and variational inequalities," *Operations Research Letters*, vol. 35, no. 2, pp. 159–164, 2007.

[7] G. Debreu, "A social equilibrium existence theorem," *Proceedings of the National Academy of Sciences*, vol. 38, no. 10, pp. 886–893, 1952.

[8] D. Paccagnan, B. Gentile, F. Parise, M. Kamgarpour, and J. Lygeros, "Distributed computation of generalized nash equilibria in quadratic aggregative games with affine coupling constraints," in *2016 IEEE 55th Conference on Decision and Control (CDC)*. IEEE, 2016, pp. 6123–6128.

[9] F. Facchinei and S. Sagratella, "On the computation of all solutions of jointly convex generalized nash equilibrium problems," *Optimization Letters*, vol. 5, no. 3, pp. 531–547, 2011.

[10] P. Yi and L. Pavel, "Asynchronous distributed algorithms for seeking generalized nash equilibria under full and partial-decision information," *IEEE transactions on cybernetics*, vol. 50, no. 6, pp. 2514–2526, 2019.

[11] L. Pavel, "Distributed gne seeking under partial-decision information over networks via a doubly-augmented operator splitting approach," *IEEE Transactions on Automatic Control*, vol. 65, no. 4, pp. 1584–1597, 2019.

[12] H. Ebel and S. Bornholdt, "Coevolutionary games on networks," *Physical Review E*, vol. 66, no. 5, p. 056118, 2002.

[13] M. O. Jackson and Y. Zenou, "Games on networks," in *Handbook of game theory with economic applications*. Elsevier, 2015, vol. 4, pp. 95–163.

[14] S. Eilenberg and D. Montgomery, "Fixed point theorems for multi-valued transformations," *American Journal of mathematics*, vol. 68, no. 2, pp. 214–222, 1946.

[15] A. A. Kulkarni and U. V. Shanbhag, "On the variational equilibrium as a refinement of the generalized nash equilibrium," *Automatica*, vol. 48, no. 1, pp. 45–55, 2012.

[16] S. Boyd, S. P. Boyd, and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[17] T. Başar and G. J. Olsder, *Dynamic noncooperative game theory*. SIAM, 1998.